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ORIGINAL PAPER

Superconvergence Study of the Direct Discontinuous Galerkin Method and Its Variations for Diffusion Equations

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Abstract

In this paper, we apply the Fourier analysis technique to investigate superconvergence properties of the direct disontinuous Galerkin (DDG) method (Liu and Yan in SIAM J Numer Anal 47(1):475–698, 2009), the DDG method with the interface correction (DDGIC) (Liu and Yan in Commun Comput Phys 8(3):541–564, 2010), the symmetric DDG method (Vidden and Yan in Comput Math 31(6):638–662, 2013), and the nonsymmetric DDG method (Yan in J Sci Comput 54(2):663–683, 2013). We also include the study of the interior penalty DG (IPDG) method, due to its close relation to DDG methods. Error estimates are carried out for both P^2 and P^3 polynomial approximations. By investigating the quantitative errors at the Lobatto points, we show that the DDGIC and symmetric DDG methods are superior, in the sense of obtaining $(k+2)$th superconvergence orders for both P^2 and P^3 approximations. Superconvergence order of $(k+2)$ is also observed for the IPDG method with P^3 polynomial approximations. The errors are sensitive to the choice of the numerical flux coefficient for even degree P^2 approximations, but are not for odd degree P^3 approximations. Numerical experiments are carried out at the same time and the numerical errors match well with the analytically estimated errors.

Keywords Direct discontinuous Galerkin methods · Superconvergence · Fourier analysis · Diffusion equation

Mathematics Subject Classification 65M60

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1 Introduction

In this paper, we apply the Fourier analysis tool to investigate superconvergence properties of the direct discontinuous Galerkin (DDG) method [19], the DDG method with interface correction (DDGIC) [20], the symmetric DDG method [25], and the nonsymmetric DDG method [27] for diffusion equations. We also include a study of the interior penalty discontinuous Galerkin (IPDG) method [4], since the IPDG method is closely related to DDG methods.

The DDG methods are a class of discontinuous Galerkin (DG) finite element methods designed for diffusion equations. It was first proposed in [19] by Liu and Yan following the direct weak formulation of parabolic equations. A numerical flux concept of $\widehat{(u_h)_x}$ was introduced to approximate the solution's spatial derivative u_x across element interface. Different to the local DG (LDG) method, in which auxiliary variables are introduced for the solution's spatial derivatives and rewritten of the original equation into a first-order system is required, the DDG method is based on the weak formulation of the diffusion equation and thus being solved directly.

For the original DDG method [19], suitable higher order numerical flux coefficients are hard to be identified and the accuracy loss is observed for nonuniform simulation. In [20], Liu and Yan modified the DDG method with extra interface terms added and obtained by the DDGIC method. It turns out that the DDGIC method is so far the best solver for time dependent diffusion problems. To carry out the $L^2(L^2)$ error estimate, Vidden and Yan further introduced the concept of the test function numerical flux and proposed the symmetric DDG method in [25]. The symmetric DDG method is a more suitable solver for elliptic-type PDEs. In [27], Yan also studied the nonsymmetric DDG method for nonlinear diffusion problems, in which the optimal convergence order is obtained for any degree polynomial approximations.

With lower order piecewise constant and linear approximations, DDGIC and symmetric DDG methods degenerate to the IPDG method. On the other hand, DDG methods are found to have quite a few advantages over the IPDG method for higher order P^k ($k \geqslant 2$) approximations. DDGIC polynomial solutions have been proved to satisfy strict maximum principle with at least third order of accuracy in [8], while only second order can be obtained for IPDG and LDG methods [34]. With the Fourier analysis technique, DDG methods are proved to be superconvergent on its approximation to the solution's spatial derivative u_x in [33]. No such superconvergence result is observed for the IPDG method. Recently in [17], the two interface jump conditions have been built into numerical flux definition and a uniform high order symmetric DDG method is obtained for elliptic interface problems.

Superconvergence properties of DG and LDG methods for hyperbolic and parabolic problems have been intensively investigated in the past via different approaches, such as the negative norm estimate [13, 18], by considering the problem as an initial or boundary value problem [1–3, 16], by special decomposition of error and playing with test functions in the weak formulation [6, 9, 10, 28, 29], by the Fourier analysis [11, 14, 22, 23, 15, 35], etc. However, there is few work regarding the superconvergence study of the DDG methods. In [7], the authors proved the superconvergence property of the DDGIC method for one-dimensional linear convection diffusion equations, under some suitable choice of numerical fluxes and initial discretization. In [33], superconvergence study on the moment errors of the DDGIC and the symmetric DDG methods are carried out via the Fourier analysis approach for piecewise P^2 polynomial approximations.

The Fourier analysis has been known to be a powerful technique to study the stability and error estimates for DG methods and other related schemes, especially in some cases when the standard finite element technique can not be applied. It was applied to analyse the instability of "bad" schemes in [30], to demonstrate the optimal convergence via quantitative error estimates in [31, 32, 35], and to show the superconvergence property at special points in [14, 33, 35], etc. Though the Fourier analysis is limited to linear problems with periodic boundary conditions and uniform mesh, it can be used as a guidance to problems under general settings. In this paper, we will continue adopting the Fourier analysis technique to investigate the superconvergence properties of four DDG methods for one-dimensional linear heat equation. The analysis is carried out not only for the case of piecewise P^2 polynomial approximations but also for the case of piecewise P^3 polynomial approximations, which is more challenging for Fourier type analysis. We also apply the Fourier analysis technique to study the IPDG method [4], due to its close relation to the DDG methods. By investigating the quantitative error at Lobatto points via the Fourier analysis, we show that as follows.

i) DDGIC and symmetric DDG methods are the best among the five DG methods studied. At Lobatto points, DDGIC and symmetric DDG methods are superconvergent of order $(k+2)$ with quadratic ($k=2$) and cubic ($k=3$) polynomial approximations. The errors are sensitive to the numerical flux coefficient β_1 for P^2 approximations ($\beta_1 = \frac{1}{2k(k+1)}$), but are not sensitive to the numerical flux coefficients for P^3 odd degree polynomial approximations.
ii) The IPDG method is found to be superconvergent of order $(k+2)$ at Lobatto points with P^3 polynomial approximations. For P^2 approximation, the IPDG method is superconvergent of order $(k+2)$ at the cell center, but is convergent with optimal order of $(k+1)$ at the other two Lobatto points, which are similar to DDGIC and symmetric DDG methods with $\beta_1 \neq \frac{1}{2k(k+1)}$ taken in the numerical flux.
iii) The original DDG method and nonsymmetric DDG method are convergent of optimal $(k+1)$th order at Lobbato points for P^3 polynomial approximations. For P^2 approximations and with the proper choice of $\beta_1 = \frac{1}{2k(k+1)} = \frac{1}{12}$ in the numerical flux, the errors at Lobbato points are superconvergent of $(k+2)$th order, while the errors are only convergent of order k with $\beta_1 \neq \frac{1}{2k(k+1)}$.
iv) The superconvergence and optimal convergence orders at the Lobatto points are independent of the β_0 coefficient in the numerical flux, as long as it is chosen large enough to guarantee the stability of the methods.

The paper is organized as follows: we first review the DDG method and its variations for heat equation in Sect. 2. In Sect. 3, we present the Fourier analysis procedure in detail. For each method, we carry out the quantitative error estimates at Lobatto points via the Fourier analysis and show the numerical tests errors match well with the analytical predicted errors. Conclusions are given in Sect. 4.

2 DDG Method and Its Variations

In this section, we present the scheme formulation of the DDG finite element method and its variations for the model problem

$$u_t - u_{xx} = 0, \quad x \in [0, 2\pi] \tag{1}$$

with initial condition $u(x, 0) = \sin x$ and periodic boundary condition. The exact solution to (1) is given by

$$u(x, t) = \mathrm{e}^{-t} \sin(x). \tag{2}$$

We consider to approximate heat equation (1) with DG methods in space, and couple with the strong stability preserving (SSP) Runge-Kutta (RK) method [24] in time for the fully discretization. The spatial domain $[0, 2\pi]$ is partitioned into N computational cells, i.e., $[0, 2\pi] = \bigcup_{j=1}^{N} I_j$. We denote the cell by $I_j = [x_{j-1/2}, x_{j+1/2}]$ and the cell center by $x_j = \frac{1}{2}\left(x_{j-1/2} + x_{j+1/2}\right)$, for $j = 1, \cdots, N$, where

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi.$$

We consider the uniform partition with mesh size $\Delta x = \frac{2\pi}{N}$. The piecewise polynomial solution space of DG methods is defined as

$$\mathbb{V}_h^k := \{v \in L^2[0, 2\pi] : v|_{I_j} \in P^k(I_j),\ j = 1, \cdots, N\},$$

where $P^k(I_j)$ denotes the set of polynomials of degree at most k on cell I_j. For $v_h \in \mathbb{V}_h^k$, we adopt following notations to denote the jump and average of v_h across the cell interface $x_{j+1/2}$:

$$v_h^{\pm} = v_h(x \pm 0, t), \quad [\![v_h]\!] = v_h^+ - v_h^-, \quad \{\!\{v_h\}\!\} = \frac{v_h^+ + v_h^-}{2}.$$

Now we are ready to define the DDG method and its variations.

2.1 DDG Method

Multiply the heat equation (1) with test function, integrate over cell I_j and have integration by part, and formally we obtain the DDG method as follows: find the solution $u_h \in \mathbb{V}_h^k$, such that for any test function $v_h \in \mathbb{V}_h^k$, we have

$$\int_{I_j} (u_h)_t v_h \mathrm{d}x - \widehat{(u_h)_x}(v_h)^-_{j+\frac{1}{2}} + \widehat{(u_h)_x}(v_h)^+_{j-\frac{1}{2}} + \int_{I_j} (u_h)_x (v_h)_x \mathrm{d}x = 0. \tag{3}$$

The above formulation is based on the weak formulation of the heat equation directly, thus it was named direct discontinuous Galerkin methods in [19]. Notice that the DG solution is discontinuous across cell interfaces, thus we introduce the numerical flux $\widehat{(u_h)_x}$ to approximate the solution derivative $(u_h)_x$ at the cell interfaces $x_{j\pm\frac{1}{2}},\ j = 1, \cdots, N$, which is given by

$$\widehat{(u_h)_x} = \beta_0 \frac{[\![u_h]\!]}{\Delta x} + \{\!\{(u_h)_x\}\!\} + \beta_1 \Delta x [\![(u_h)_{xx}]\!] + \beta_2 (\Delta x)^3 [\![(u_h)_{xxxx}]\!] + \cdots. \tag{4}$$

The numerical flux $\widehat{(u_h)_x}$ is uniquely defined at the cell interface and is consistent to the solution spatial derivative u_x. It involves the solution's jump, derivative average, second order derivative jump and even-th derivative higher order jump quantities at the cell interface. There exists a large group of admissible coefficient pair (β_0, β_1) that ensures the stability and convergence of the DDG method, see [19] for more details.

On the other hand, it is hard to identify admissible numerical flux coefficients for the high order terms, for example β_2 in (4). Higher order jump terms, i.e., $[\![(u_h)_{xxxx}]\!]$ is none zero only when $k \geqslant 4$, which turns out to be not essential and can be cut off from the numerical flux formula. In [20], we modified the original DDG scheme formulation (3) by adding interface correction terms that involve the test function derivative $(v_h)_x$. The interface correction terms are important to balance the solution and test functions in the bilinear form, which in return guarantee the optimal convergence and improve the capacity of the DDG method.

2.2 DDG Method with Interface Correction

Now the scheme formulation of the DDG method with interface correction (DDGIC) [20] is defined as: find the solution $u_h \in \mathbb{V}_h^k$, such that for any test function $v_h \in \mathbb{V}_h^k$, we have

$$\int_{I_j} (u_h)_t v_h \mathrm{d}x - \widehat{(u_h)_x} v_h \Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} + \int_{I_j} (u_h)_x (v_h)_x \mathrm{d}x + \frac{(v_h)_x^-}{2} [\![u_h]\!]_{j+\frac{1}{2}} + \frac{(v_h)_x^+}{2} [\![u_h]\!]_{j-\frac{1}{2}} = 0, \quad (5)$$

where we adopt the short notation of

$$\widehat{(u_h)_x} v_h \Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} := \widehat{(u_h)_x} (v_h)^-_{j+\frac{1}{2}} - \widehat{(u_h)_x} (v_h)^+_{j-\frac{1}{2}}.$$

The numerical flux is given by

$$\widehat{(u_h)_x} = \beta_0 \frac{[\![u_h]\!]}{\Delta x} + \{\!\{(u_h)_x\}\!\} + \beta_1 \Delta x [\![(u_h)_{xx}]\!]. \quad (6)$$

Notice that higher order jump terms are dropped off from (4). The numerical flux formula of the DDGIC method only involves the solution jump, derivative average and the second order derivative jump terms. With the admissible coefficient pair (β_0, β_1) chosen in (6), the DDGIC method is proved to be stable and convergent with optimal order of accuracy [20].

The DDGIC method is closely related to the classical IPDG method [4, 26]. For piecewise constant $(k = 0)$ and linear $(k = 1)$ approximations, the second derivative jump term $[\![(u_h)_{xx}]\!]$ degenerates to zero and has no contribution to the calculation of the numerical flux (6). Thus, the DDGIC method degenerates to the IPDG method with low order approximations. On the other hand, DDG methods are found to have quite a few advantages over the IPDG method and LDG method [12] for high order approximations (P^k with $k \geqslant 2$).

Numerical tests show a small fixed penalty coefficient $\beta_0 = 2$ can be applied to obtain optimal convergence for P^k $(k \leqslant 9)$ approximations with the DDGIC method. Polynomial solutions have been proved to satisfy strict maximum principle with at least third order of accuracy for the DDGIC method in [8], while only second order can be obtained for IPDG and LDG methods [34]. With the Fourier analysis technique, in [33], the solutions of the DDG methods are proved to be superconvergent on its approximation to the solution's spatial derivative u_x. No such superconvergence result is observed for the IPDG method.

2.3 Symmetric/Nonsymmetric DDG Method

To carry out the $L^2(L^2)$ error estimate, we further introduce the numerical flux concept for the test function $(v_h)_x$ and obtain a symmetric DDG method in [25]. It turns out the symmetric DDG method is a more suitable DDG method for elliptic-type PDEs. With the stiffness matrix being symmetric, fast solvers can be applied with the symmetric DDG method.

To compare with the nonsymmetric DG method of Baumann and Oden [5] and the nonsymmetric interior penalty Galerkin (NIPG) method [21], we switch the sign of interface terms to negative and obtain the nonsymmetric DDG method in [27]. Different to the accuracy loss issues in [5, 21], the nonsymmetric DDG method is found to obtain optimal order convergence for all degree polynomial approximations.

Denote $\sigma = \pm 1$, we have the symmetric ($\sigma = +1$) and nonsymmetric ($\sigma = -1$) DDG methods uniformly laid out as follows: find the solution $u_h \in \mathbb{V}_h^k$, such that for any test function $v_h \in \mathbb{V}_h^k$, we have

$$\int_{I_j} (u_h)_t v_h \mathrm{d}x - \widehat{(u_h)_x} v_h \Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} + \int_{I_j} (u_h)_x (v_h)_x \mathrm{d}x + \sigma \left(\widetilde{(v_h)_x} [\![u_h]\!]_{j+\frac{1}{2}} + \widetilde{(v_h)_x} [\![u]\!]_{j-\frac{1}{2}} \right) = 0, \tag{7}$$

where numerical fluxes of the solution and test function are taken as

$$\begin{cases} \widehat{(u_h)_x} &= \beta_{0u} \frac{[\![u_h]\!]}{\Delta x} + \{\!\{ (u_h)_x \}\!\} + \beta_1 \Delta x [\![(u_h)_{xx}]\!], \\ \widetilde{(v_h)_x} &= \beta_{0v} \frac{[\![v_h]\!]}{\Delta x} + \{\!\{ (v_h)_x \}\!\} + \beta_1 \Delta x [\![(v_h)_{xx}]\!]. \end{cases} \tag{8}$$

For the symmetric DDG method, with the notation of $\beta_0 = \beta_{0u} + \beta_{0v}$, a quadratic form restriction on the coefficient pair (β_0, β_1) is proved to lead to admissible numerical flux (8), and guarantees the optimal convergence of the method. We refer the readers to [25] for more details. For the nonsymmetric DDG method, with all interface terms involving β_1 cancelling out in the global formulation, the nonsymmetric DDG is stable with $\beta_0 = \beta_{0u} - \beta_{0v} > 0$ taken in the numerical flux, see [27]. The stability result also holds for nonlinear diffusion equations.

2.4 IPDG Method

Finally, we lay out the scheme of the classical IPDG method [4], under the format of DDG methods. We find the solution $u_h \in \mathbb{V}_h^k$, such that for any test function $v_h \in \mathbb{V}_h^k$, we have

$$\int_{I_j} (u_h)_t v_h \mathrm{d}x - \widehat{(u_h)_x} v_h \Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} + \int_{I_j} (u_h)_x (v_h)_x \mathrm{d}x + \frac{(v_h)_x^-}{2} [\![u_h]\!]_{j+\frac{1}{2}} + \frac{(v_h)_x^+}{2} [\![u_h]\!]_{j-\frac{1}{2}} = 0. \tag{9}$$

The penalty term and the derivative average are combined into the numerical flux with

$$\widehat{(u_h)_x} = \beta_0 \frac{[\![u_h]\!]}{\Delta x} + \{\!\{ (u_h)_x \}\!\}. \tag{10}$$

Notice that the original form of the IPDG method did not recognize the scale of Δx and its relationship to the jump term $[\![u_h]\!]$, thus the penalty coefficient β_0 has to be large enough to stabilize the scheme, especially with refined mesh. Now we know term $\frac{[\![u_h]\!]}{\Delta x}$ is a low order but also a leading term on its approximation to the solution derivative $(u_h)_x$. The difference between the DDGIC method and the IPDG method is that the second derivative jump term $[\![(u_h)_{xx}]\!]$ is included in the numerical flux (6) for the DDGIC method.

Up to now, we have discretized the spatial variable with various DG methods, yielding the semi-discrete scheme written as

$$(u_h)_t = L(u_h) \tag{11}$$

with L being the spatial discretization operator. To discretize the temporal variable, we use the following third-order SSP RK method [24]:

$$\begin{cases} u_h^{(1)} &= u_h^n + \Delta t L(u_h^n), \\ u_h^{(2)} &= \frac{3}{4}u_h^n + \frac{1}{4}\left(u_h^{(1)} + \Delta t L(u_h^{(1)})\right), \\ u_h^{n+1} &= \frac{1}{3}u_h^n + \frac{2}{3}\left(u_h^{(2)} + \Delta t L(u_h^{(2)})\right). \end{cases} \tag{12}$$

3 Superconvergence Study by Fourier Analysis

In this section, we first present the rewriting details of DG methods of Sect. 2 as finite difference schemes. Then, we perform the standard von Neumann Fourier analysis to these methods and symbolically calculate the errors at Lobatto points. Quantitative error estimates and numerical experiments are carried out for each method. We present the detailed results for the DDG method in Sect. 3.1, for the DDGIC method in Sect. 3.2, for the symmetric DDG method in Sect. 3.3, for the nonsymmetric DDG method in Sect. 3.4, and for the IPDG method in Sect. 3.5.

We now use the DDG method (3) as an example to demonstrate the Fourier analysis procedure. First we adopt a local basis of the solution space $\mathbb{V}_k^h$, denoted as $\phi_j^l(x)$, $l = 1, 2, \cdots, k+1$. In particular, the basis functions are chosen to be the Lagrangian polynomials based on the following $k+1$ Lobatto points in the cell I_j, i.e.,

$$x_j^l = x_j + \frac{\zeta_l}{2}\Delta x, \quad l = 1, 2, \cdots, k+1.$$

Here $\{\zeta_l\}_{l=1}^{k+1}$ are the roots of polynomial $(1-x^2)P_k'(x) = 0$, with $P_k(x)$ being the Legendre polynomial of degree k.

In Table 1 we display the location of $\{\zeta_l\}$ for $k = 2, 3$. With the basis $\phi_j^l(x)$ chosen, the numerical solution can be represented as

$$u_h|_{I_j} = \sum_{l=1}^{k+1} u_j^l \phi_j^l(x), \quad x \in I_j. \tag{13}$$

Table 1 Lobatto points $\{\zeta_l\}$: the roots of $(1-x^2)P_k'(x) = 0$

	$\{\zeta_l\}$
$k = 2$	$\zeta_1 = -1,\ \zeta_2 = 0,\ \zeta_3 = 1$
$k = 3$	$\zeta_1 = -1,\ \zeta_2 = -\frac{1}{\sqrt{5}},\ \zeta_3 = \frac{1}{\sqrt{5}},\ \zeta_4 = 1$

After substituting (13) into (3) and inverting a local mass matrix of $(k+1)\times(k+1)$, the DDG finite element method can be rewritten in the following form:

$$\frac{\mathrm{d}\boldsymbol{u}_j}{\mathrm{d}t} = A\boldsymbol{u}_{j-1} + B\boldsymbol{u}_j + C\boldsymbol{u}_{j+1}, \tag{14}$$

where $\boldsymbol{u}_j = \left(u_j^1, u_j^2, \cdots, u_j^{k+1}\right)^{\mathrm{T}}$, and A, B and C are $(k+1)\times(k+1)$ matrices that only depend on the parameters β_0 and β_1 defined in the numerical flux (4).

Clearly, $\boldsymbol{u}_j$, the coefficients of the solution u_h inside cell I_j, is a vector of solution values at Lobatto points, since the local basis functions are Lagrange polynomials based on these points. In this way, the DG finite element scheme (14) can be considered as a finite difference scheme. However, it is not a standard finite difference scheme, since each point in the group of $k+1$ points belonging to the cell I_j obeys a different form.

In order to solve (14), we have the assumption on the solution

$$\boldsymbol{u}_j(t) = \hat{\mathbf{u}}(t)\mathrm{e}^{\mathrm{i}x_j} \tag{15}$$

with $\mathrm{i}=\sqrt{-1}$. Substituting (15) into (14), the coefficient vector $\hat{\mathbf{u}}$ satisfies the following ODE system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}(t) = G(\Delta x)\hat{\mathbf{u}}(t), \tag{16}$$

where $G(\Delta x)$ is the amplification matrix, given by

$$G(\Delta x) = A\mathrm{e}^{-\mathrm{i}\Delta x} + B + C\mathrm{e}^{\mathrm{i}\Delta x},$$

and A, B, C are taken from (14). Denote the eigenvalues of G as $\lambda_1,\ \lambda_2, \cdots, \lambda_{k+1}$, and the corresponding eigenvectors as $V_1,\ V_2, \cdots, V_{k+1}$, the general solution of the ODE system (16) can be written out as

$$\hat{\mathbf{u}}(t) = a_1\mathrm{e}^{\lambda_1 t}V_1 + a_2\mathrm{e}^{\lambda_2 t}V_2 + \cdots + a_{k+1}\mathrm{e}^{\lambda_{k+1} t}V_{k+1}, \tag{17}$$

where the coefficients $a_1, a_2, \cdots, a_{k+1}$ are determined by the initial condition

$$\hat{\mathbf{u}}(0) = \left(\mathrm{e}^{\frac{\zeta_l}{2}\Delta x}, \mathrm{e}^{\frac{\zeta_2}{2}\Delta x}, \cdots, \mathrm{e}^{\frac{\zeta_{k+1}}{2}\Delta x}\right)^{\mathrm{T}}.$$

Now we have the explicit expression for the solution of the DG method for (1). Comparing with the exact solution (2), we obtain the quantitative error estimates at the Lobatto points, which are denoted as

$$\left\|e_{r_l}\right\|_\infty = \max_{1\leqslant j\leqslant N}\left|u\left(x_j^l, t\right) - u_j^l(t)\right|, \quad l = 1, 2, \cdots, k+1. \tag{18}$$

To validate the analytical results, we numerically solve (1) with the five DG methods presented in Sect. 2 coupled with the third-order SSP RK method (12) [24] for temporal discretization. Very small time step ($\Delta t = 0.001\Delta x^2$) is taken in the numerical experiments to make sure the temporal error is negligible comparing with the spatial error. Final time is set as $T=1$.

Now we provide the error estimates by the Fourier analysis and numerical experiments for the five DG methods in the following subsections.

3.1 DDG Method

In this section we perform the Fourier analysis for the DDG method (3) with P^2 quadratic and P^3 cubic polynomial approximations. According to (4), fourth and higher order derivative jump terms, i.e., $[\![(u_h)_{xxxx}]\!]$ do not contribute to the calculation of the numerical flux. Thus, the numerical flux of the DDG method for both P^2 and P^3 cases is taken in the form of

$$\widehat{(u_h)_x} = \beta_0 \frac{[\![u]\!]}{\Delta x} + \{\!\{ (u_h)_x \}\!\} + \beta_1 \Delta x [\![(u_h)_{xx}]\!].$$

For the DDG method with P^2 approximations, the matrices $A,\ B,\ C$ of (14) are

$$A = \begin{pmatrix} \frac{9}{2}(-1+8\beta_1) & -18(-1+4\beta_1) & \frac{9}{2}(-3+2\beta_0+8\beta_1) \\ -\frac{3}{4}(-1+8\beta_1) & 3(-1+4\beta_1) & -\frac{3}{4}(-3+2\beta_0+8\beta_1) \\ \frac{3}{2}(-1+8\beta_1) & -6(-1+4\beta_1) & \frac{3}{2}(-3+2\beta_0+8\beta_1) \end{pmatrix},$$

$$B = \begin{pmatrix} -(11+9\beta_0+48\beta_1) & 16(1+6\beta_1) & -(5+3\beta_0+48\beta_1) \\ \frac{1}{2}(14+3\beta_0+24\beta_1) & -2(7+12\beta_1) & \frac{1}{2}(14+3\beta_0+24\beta_1) \\ -(5+3\beta_0+48\beta_1) & 16(1+6\beta_1) & -(11+9\beta_0+48\beta_1) \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{3}{2}(-3+2\beta_0+8\beta_1) & -6(-1+4\beta_1) & \frac{3}{2}(-1+8\beta_1) \\ -\frac{3}{4}(-3+2\beta_0+8\beta_1) & 3(-1+4\beta_1) & -\frac{3}{4}(-1+8\beta_1) \\ \frac{9}{2}(-3+2\beta_0+8\beta_1) & -18(-1+4\beta_1) & \frac{9}{2}(-1+8\beta_1) \end{pmatrix}.$$

For P^3 polynomial approximations, the matrices $A,\ B,\ C$ of (14) are

$$A = \begin{pmatrix} 8(1-20\beta_1) & 20(1-\sqrt{5}-4(1-3\sqrt{5})\beta_1) & 20(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & 16(\beta_0+20\beta_1-3) \\ \frac{-2}{\sqrt{5}}(1-20\beta_1) & \sqrt{5}(\sqrt{5}-1+4(1-3\sqrt{5})\beta_1) & -\sqrt{5}(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & \frac{-4}{\sqrt{5}}(\beta_0+20\beta_1-3) \\ \frac{2}{\sqrt{5}}(1-20\beta_1) & \sqrt{5}(1-\sqrt{5}-4(1-3\sqrt{5})\beta_1) & \sqrt{5}(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & \frac{4}{\sqrt{5}}(\beta_0+20\beta_1-3) \\ -2(1-20\beta_1) & -5(1-\sqrt{5}-4(1-3\sqrt{5})\beta_1) & -5(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & -4(\beta_0+20\beta_1-3) \end{pmatrix},$$

$$B = \begin{pmatrix} -2(8\beta_0+180\beta_1+15) & 10(1+\sqrt{5}+6(1+5\sqrt{5})\beta_1) & 10(1-\sqrt{5}+6(1-5\sqrt{5})\beta_1) & 2(2\beta_0+120\beta_1+5) \\ \frac{4\beta_0+29}{\sqrt{5}}+24\sqrt{5}\beta_1+5 & -30(1+4\beta_1) & 20(1+6\beta_1) & -\frac{4\beta_0+29}{\sqrt{5}}-24\sqrt{5}\beta_1+5 \\ -\frac{4\beta_0+29}{\sqrt{5}}-24\sqrt{5}\beta_1+5 & 20(1+6\beta_1) & -30(1+4\beta_1) & \frac{4\beta_0+29}{\sqrt{5}}+24\beta_1\sqrt{5}+5 \\ 2(2\beta_0+120\beta_1+5) & 10(1-\sqrt{5}+6(1-5\sqrt{5})\beta_1) & 10(1+\sqrt{5}+6(1+5\sqrt{5})\beta_1) & -2(8\beta_0+180\beta_1+15) \end{pmatrix},$$

$$C = \begin{pmatrix} -4(\beta_0+20\beta_1-3) & -5(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & -5(1-\sqrt{5}-4(1-3\sqrt{5})\beta_1) & -2(1-20\beta_1) \\ \frac{4}{\sqrt{5}}(\beta_0+20\beta_1-3) & \sqrt{5}(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & \sqrt{5}(1-\sqrt{5}-4(1-3\sqrt{5})\beta_1) & \frac{2}{\sqrt{5}}(1-20\beta_1) \\ \frac{-4}{\sqrt{5}}(\beta_0+20\beta_1-3) & -\sqrt{5}(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & \sqrt{5}(\sqrt{5}-1+4(1-3\sqrt{5})\beta_1) & \frac{-2}{\sqrt{5}}(1-20\beta_1) \\ 16(\beta_0+20\beta_1-3) & 20(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & 20(1-\sqrt{5}-4(1-3\sqrt{5})\beta_1) & 8(1-20\beta_1) \end{pmatrix}.$$

We investigate the superconvergence property of the DDG method with different β_1 coefficients taken in the numerical flux $\widehat{(u_h)_x}$ of (4). In particular, we study two settings of $(\beta_0, \beta_1) = (1, \frac{1}{8})$ and $(\beta_0, \beta_1) = (1, \frac{1}{12})$ for the P^2 case, and $(\beta_0, \beta_1) = (12, \frac{1}{8})$ and $(\beta_0, \beta_1) = (12, \frac{1}{24})$ for the P^3 case. The errors at Lobatto points via Fourier analysis are presented in Table 2. For the P^2 case, with $\beta_1 = \frac{1}{2k(k+1)} = \frac{1}{12}$, the errors of the DDG method at Lobatto points are superconvergent of order $k+2=4$; with $\beta_1 \neq \frac{1}{12}$, the errors of the DDG method at Lobatto points are only second order accurate. For the P^3 case, the error is observed with the optimal $k+1=4$th order of accuracy at the Lobatto points. The convergence order does not depend on the choice of β_1 for the P^3 case.

Table 2 Analytical error estimate at Lobatto points for the DDG method (3)

P^k	$\beta_1 \neq \frac{1}{2k(k+1)}$	$\beta_1 = \frac{1}{2k(k+1)}$
$k=2$	$\beta_1 = \frac{1}{8}$	$\beta_1 = \frac{1}{12}$
e_{r_1}	$\frac{1}{24}t\mathrm{e}^{-t}\Delta x^2 + O(\Delta x^4)$	$\frac{\mathrm{e}^{-t}(8t+1)}{2\,880}\Delta x^4 + O(\Delta x^6)$
e_{r_2}	$\frac{1}{24}t\mathrm{e}^{-t}\Delta x^2 + O(\Delta x^4)$	$\frac{\mathrm{e}^{-t}(16t-1)}{5\,760}\Delta x^4 + O(\Delta x^6)$
e_{r_3}	$\frac{1}{24}t\mathrm{e}^{-t}\Delta x^2 + O(\Delta x^4)$	$\frac{\mathrm{e}^{-t}(8t+1)}{2\,880}\Delta x^4 + O(\Delta x^6)$
$k=3$	$\beta_1 = \frac{1}{8}$	$\beta_1 = \frac{1}{24}$
e_{r_1}	$4.96\times10^{-4}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^5)$	$3.31\times10^{-5}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^6)$
e_{r_2}	$4.96\times10^{-4}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^5)$	$3.31\times10^{-5}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^6)$
e_{r_3}	$4.96\times10^{-4}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^5)$	$3.31\times10^{-5}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^6)$
e_{r_4}	$4.96\times10^{-4}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^5)$	$3.31\times10^{-5}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^6)$

Table 3 Errors at Lobatto points for the DDG method (3) with P^2 polynomials (β_0=1)

N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order
10	6.12E−03		6.05E−03		1.79E−04		1.79E−04	
20	1.50E−03	2.03	1.51E−03	2.00	1.12E−05	4.01	1.12E−05	4.00
40	3.77E−04	1.99	3.78E−04	2.00	6.99E−07	4.00	7.00E−07	4.00
80	9.45E−05	2.00	9.46E−05	2.00	4.37E−08	4.00	4.37E−08	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order
10	6.09E−03		6.05E−03		1.50E−04		1.49E−04	
20	1.50E−03	2.02	1.51E−03	2.00	9.23E−06	4.02	9.33E−06	4.00
40	3.77E−04	1.99	3.78E−04	2.00	5.82E−07	3.99	5.83E−07	4.00
80	9.45E−05	2.00	9.46E−05	2.00	3.64E−08	4.00	3.65E−08	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order
10	6.12E−03		6.05E−03		1.79E−04		1.79E−04	
20	1.50E−03	2.03	1.51E−03	2.00	1.12E−05	4.01	1.12E−05	4.00
40	3.77E−04	1.99	3.78E−04	2.00	6.99E−07	4.00	7.00E−07	4.00
80	9.45E−05	2.00	9.46E−05	2.00	4.37E−08	4.00	4.37E−08	4.00

Table 4 Errors at Lobatto points for the DDG method (3) with P^3 polynomials ($\beta_0 = 12$)

N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order
10	2.86E−05		2.84E−05		1.75E−06		1.90E−06	
20	1.76E−06	4.02	1.78E−06	4.00	1.15E−07	3.93	1.19E−07	4.00
40	1.11E−07	3.99	1.11E−07	4.00	7.35E−09	3.97	7.41E−09	4.00
80	6.94E−09	4.00	6.94E−09	4.00	4.61E−10	3.99	4.63E−10	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order
10	2.83E−05		2.84E−05		2.06E−06		1.90E−06	
20	1.76E−06	4.01	1.78E−06	4.00	1.22E−07	4.07	1.19E−07	4.00
40	1.11E−07	3.99	1.11E−07	4.00	7.47E−09	4.03	7.41E−09	4.00
80	6.94E−09	4.00	6.94E−09	4.00	4.63E−10	4.01	4.63E−10	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order
10	2.83E−05		2.84E−05		2.06E−06		1.90E−06	
20	1.76E−06	4.01	1.78E−06	4.00	1.22E−07	4.07	1.19E−07	4.00
40	1.11E−07	3.99	1.11E−07	4.00	7.47E−09	4.03	7.41E−09	4.00
80	6.94E−09	4.00	6.94E−09	4.00	4.63E−10	4.01	4.63E−10	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_4}	Order	e_{r_4}	Order	e_{r_4}	Order	e_{r_4}	Order
10	2.86E−05		2.84E−05		1.75E−06		1.90E−06	
20	1.76E−06	4.02	1.78E−06	4.00	1.15E−07	3.93	1.19E−07	4.00
40	1.11E−07	3.99	1.11E−07	4.00	7.35E−09	3.97	7.41E−09	4.00
80	6.94E−09	4.00	6.94E−09	4.00	4.61E−10	3.99	4.63E−10	4.00

We numerically implement the DDG method (3) for the heat equation (1), and compare the numerical solution errors at Lobatto points and the errors predicted by Fourier analysis. Tables 3 and 4 list the errors with P^2 and P^3 approximations, respectively. The numerical results agree very well with the analytical ones.

3.2 DDGIC Method

In this section we perform the Fourier analysis for the DDGIC method (5) with the numerical flux taken as

$$\widehat{(u_h)_x} = \beta_0 \frac{[\![u_h]\!]}{\Delta x} + \{\!\{(u_h)_x\}\!\} + \beta_1 \Delta x [\![(u_h)_{xx}]\!].$$

For the DDGIC method with P^2 approximation, the matrices $A,\ B,\ C$ of (14) are

$$A = \begin{pmatrix} -\frac{9}{2}(1-8\beta_1) & 18(1-4\beta_1) & \frac{9}{2}(2\beta_0+8\beta_1-\ 7) \\ \frac{3}{4}(1-8\beta_1) & -3(1-4\beta_1) & -\frac{3}{4}(2\beta_0+8\beta_1-13) \\ -\frac{3}{2}(1-8\beta_1) & 6(1-4\beta_1) & \frac{3}{2}(2\beta_0+8\beta_1-11) \end{pmatrix},$$

$$B = \begin{pmatrix} -9\beta_0-48\beta_1+7 & 16(1+6\beta_1) & -3\beta_0-48\beta_1+7 \\ \frac{1}{2}(3\beta_0+24\beta_1-1) & -2(7+12\beta_1) & \frac{1}{2}(3\beta_0+24\beta_1-1) \\ -3\beta_0-48\beta_1+7 & 16(1+6\beta_1) & -9\beta_0-48\beta_1+7 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{3}{2}(2\beta_0+8\beta_1-11) & 6(1-4\beta_1) & -\frac{3}{2}(1-8\beta_1) \\ -\frac{3}{4}(2\beta_0+8\beta_1-13) & -3(1-4\beta_1) & \frac{3}{4}(1-8\beta_1) \\ \frac{9}{2}(2\beta_0+8\beta_1-\ 7) & 18(1-4\beta_1) & -\frac{9}{2}(1-8\beta_1) \end{pmatrix}.$$

For the DDGIC method with P^3 approximations, the matrices $A,\ B,\ C$ of (14) are given as

$$A = \begin{pmatrix} 8(1-20\beta_1) & 20(1-\sqrt{5}-(4-12\sqrt{5})\beta_1) & 20(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & 4(4\beta_0+80\beta_1-27) \\ \frac{-2}{\sqrt{5}}(1-20\beta_1) & \sqrt{5}(\sqrt{5}-1-(4-12\sqrt{5})\beta_1) & -\sqrt{5}(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & \frac{-4}{\sqrt{5}}(\beta_0+20\beta_1)+\frac{51}{\sqrt{5}}+3 \\ \frac{1}{\sqrt{5}}(1-20\beta_1) & \sqrt{5}(1-\sqrt{5}-(4-12\sqrt{5})\beta_1) & \sqrt{5}(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & \frac{4}{\sqrt{5}}(\beta_0+20\beta_1)-\frac{51}{\sqrt{5}}+3 \\ -2(1-20\beta_1) & -5(1-\sqrt{5}-(4-12\sqrt{5})\beta_1) & -5(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & -4\beta_0-80\beta_1+42 \end{pmatrix},$$

$$B = \begin{pmatrix} -16\beta_0-360\beta_1+30 & 10(1+\sqrt{5}+6(1+5\sqrt{5})\beta_1)) & 10(1-\sqrt{5}+6(1-5\sqrt{5}\beta_1)) & 4(\beta_0+60\beta_1-5) \\ \frac{4\beta_0-10}{\sqrt{5}}+24\sqrt{5}\beta_1+2 & -30(1+4\beta_1) & 20(1+6\beta_1) & -\frac{4\beta_0-10}{\sqrt{5}}-24\sqrt{5}\beta_1+2 \\ -\frac{4\beta_0-10}{\sqrt{5}}-24\sqrt{5}\beta_1+2 & 20(1+6\beta_1) & -30(1+4\beta_1) & \frac{4\beta_0-10}{\sqrt{5}}+24\sqrt{5}\beta_1+2 \\ 4(\beta_0+60\beta_1-5) & 10(1-\sqrt{5}+6(1-5\sqrt{5})\beta_1) & 10(1+\sqrt{5}+6(1+5\sqrt{5})\beta_1) & -16\beta_0-360\beta_1+30 \end{pmatrix},$$

$$C = \begin{pmatrix} -4\beta_0-80\beta_1+42 & -5(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & -5(1-\sqrt{5}-4(1-3\sqrt{5})\beta_1) & -2(1-20\beta_1) \\ \frac{4\beta_0-51}{\sqrt{5}}+16\sqrt{5}\beta_1+3 & \sqrt{5}(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & \sqrt{5}(1-\sqrt{5}-4(1-3\sqrt{5})\beta_1) & \frac{1}{\sqrt{5}}(1-20\beta_1) \\ -\frac{4\beta_0-51}{\sqrt{5}}-16\sqrt{5}\beta_1+3 & -\sqrt{5}(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & -\sqrt{5}(1-\sqrt{5}-4(1-3\sqrt{5})\beta_1) & -\frac{2}{\sqrt{5}}(1-20\beta_1) \\ 4(4\beta_0+80\beta_1-27) & 20(1+\sqrt{5}-4(1+3\sqrt{5})\beta_1) & 20(1-\sqrt{5}-4(1-3\sqrt{5})\beta_1) & 8(1-20\beta_1) \end{pmatrix}.$$

In Table 5, we list the analytically calculated errors at Lobatto points for the DDGIC method with different choices of β_1. We investigate two settings of $(\beta_0, \beta_1) = (3, \frac{1}{8})$ and $(\beta_0, \beta_1) = (3, \frac{1}{12})$ for the P^2 case, and $(\beta_0, \beta_1) = (12, \frac{1}{8})$ and $(\beta_0, \beta_1) = (12, \frac{1}{24})$ for the P^3 case. The errors are sensitive to the choice of β_1 for the P^2 case, but are not for the P^3 case.

With P^2 approximations and $\beta_1 = \frac{1}{2k(k+1)} = \frac{1}{12}$, the error of the DDGIC method is superconvergent of order $k + 2 = 4$ at the Lobatto points. With $\beta_1 \neq \frac{1}{2k(k+1)}$, the superconvergence order of $k + 2$ is observed at the cell center point, while only optimal order of $k + 1$ is observed for the other two Lobatto points. With P^3 polynomial approximations, the error is superconvergent of order $k + 2 = 5$ at Lobatto points for both $\beta_1 = \frac{1}{2k(k+1)}$ and $\beta_1 \neq \frac{1}{2k(k+1)}$.

Similar to the DDG method, we numerically implement the DDGIC method and compare the numerical solution errors with the errors predicted by the Fourier analysis at the Lobatto points. The two groups of errors match well, see Tables 6 and 7 for P^2 and P^3 polynomials approximations, respectively.

3.3 Symmetric DDG Method

In this section we perform the Fourier analysis error estimate for the symmetric DDG method of (7). We follow (8) as the numerical flux formula with $\beta_0 = \beta_{0u} + \beta_{0v}$.

For the symmetric DDG method and P^2 approximations, the matrices $A,\ B,\ C$ of (14) are

$$A = \begin{pmatrix} -\frac{9}{2}(1-8\beta_1) & 18(1-4\beta_1) & \frac{3}{2}(6\beta_0+64\beta_1-21) \\ \frac{3}{4}(1-8\beta_1) & -3(1-4\beta_1) & -\frac{3}{4}(2\beta_0+48\beta_1-13) \\ \frac{3}{2}(1-8\beta_1) & 6(1-4\beta_1) & \frac{3}{2}(2\beta_0+48\beta_1-11) \end{pmatrix},$$

$$B = \begin{pmatrix} -9(\beta_0+12\beta_1)+7 & 16(1+6\beta_1) & -3(\beta_0+36\beta_1)+7 \\ \frac{3}{2}(\beta_0+28\beta_1)-\frac{1}{2} & -2(7+12\beta_1) & \frac{3}{2}(\beta_0+28\beta_1)-\frac{1}{2} \\ -3(\beta_0+36\beta_1)+7 & 16(1+6\beta_1) & -9(\beta_0+12\beta_1)+7 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{3}{2}(2\beta_0+48\beta_1-11) & 6(1-4\beta_1) & \frac{3}{2}(1-8\beta_1) \\ -\frac{3}{4}(2\beta_0+48\beta_1-13) & -3(1-4\beta_1) & \frac{3}{4}(1-8\beta_1) \\ \frac{3}{2}(6\beta_0+64\beta_1-21) & 18(1-4\beta_1) & -\frac{9}{2}(1-8\beta_1) \end{pmatrix}.$$

For the symmetric DDG method and P^3 approximations, the matrices $A,\ B,\ C$ of (14) are

$$A = \begin{pmatrix} 8(1-20\beta_1) & 20(1-\sqrt{5}-(4-12\sqrt{5})\beta_1) & 20(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & 4(4\beta_0+200\beta_1-27) \\ \frac{-2}{\sqrt{5}}(1-20\beta_1) & 5-\sqrt{5}+4(\sqrt{5}-15)\beta_1 & -5-\sqrt{5}+4(\sqrt{5}+15)\beta_1 & \frac{-4\beta_0}{\sqrt{5}}-4(3+25\sqrt{5})\beta_1+\frac{51}{\sqrt{5}}+3 \\ \frac{2}{\sqrt{5}}(1-20\beta_1) & \sqrt{5}-5-4(\sqrt{5}-15)\beta_1 & 5+\sqrt{5}-4(\sqrt{5}+15)\beta_1 & \frac{4\beta_0}{\sqrt{5}}-4(3-25\sqrt{5})\beta_1-\frac{51}{\sqrt{5}}+3 \\ -2(1-20\beta_1) & 5(\sqrt{5}-1+(4-12\sqrt{5})\beta_1) & -5(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & -4\beta_0-440\beta_1+42 \end{pmatrix},$$

$$B = \begin{pmatrix} -2(8\beta_0+420\beta_1-15) & 10(1+\sqrt{5}+6(1+5\sqrt{5})\beta_1) & 10(1-\sqrt{5}+6(1-5\sqrt{5})\beta_1) & 4(\beta_0+150\beta_1-5) \\ 12(1+9\sqrt{5})\beta_1+\frac{4\beta_0}{\sqrt{5}}+2-2\sqrt{5} & -30(4\beta_1+1) & 20(6\beta_1+1) & 12(1-9\sqrt{5})\beta_1-\frac{4\beta_0}{\sqrt{5}}+2+2\sqrt{5} \\ 12(1-9\sqrt{5})\beta_1-\frac{4\beta_0}{\sqrt{5}}+2\sqrt{5}+2 & 20(6\beta_1+1) & -30(4\beta_1+1) & 12(1+9\sqrt{5})\beta_1+\frac{4\beta_0}{\sqrt{5}}-2\sqrt{5}+2 \\ 4(\beta_0+150\beta_1-5) & 10(1-\sqrt{5}+6(1-5\sqrt{5})\beta_1) & 10(1+\sqrt{5}+6(1+5\sqrt{5})\beta_1) & -2(8\beta_0+420\beta_1-15) \end{pmatrix},$$

$$C = \begin{pmatrix} -4\beta_0-440\beta_1+42 & -5(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & 5(\sqrt{5}-1+(4-12\sqrt{5})\beta_1) & -2(1-20\beta_1) \\ \frac{4\beta_0}{\sqrt{5}}-4(3-25\sqrt{5})\beta_1-\frac{51}{\sqrt{5}}+3 & 5+\sqrt{5}-4(\sqrt{5}+15)\beta_1 & \sqrt{5}-5-4(\sqrt{5}-15)\beta_1 & \frac{2}{\sqrt{5}}(1-20\beta_1) \\ \frac{-4\beta_0}{\sqrt{5}}-4(3+25\sqrt{5})\beta_1+\frac{51}{\sqrt{5}}+3 & -5-\sqrt{5}+4(\sqrt{5}+15)\beta_1 & 5-\sqrt{5}+4(\sqrt{5}-15)\beta_1 & \frac{-2}{\sqrt{5}}(1-20\beta_1) \\ 4(4\beta_0+200\beta_1-27) & 20(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & 20(1-\sqrt{5}-(4-12\sqrt{5})\beta_1) & 8(1-20\beta_1) \end{pmatrix}.$$

Analytically calculated errors at the Lobatto points are listed in Table 8 for the symmetric DDG method. For both P^2 and P^3 approximations, two choices of the β_1 coefficient are considered. We investigate the cases of $(\beta_0, \beta_1) = (2, \frac{1}{8})$ and $(\beta_0, \beta_1) = (2, \frac{1}{12})$ for P^2 polynomial approximations, and $(\beta_0, \beta_1) = (24, \frac{1}{8})$ and $(\beta_0, \beta_1) = (24, \frac{1}{24})$ for P^3 approximations. Errors and orders at Lobatto points for the symmetric DDG method are similar to the ones of the DDGIC method. Superconvergence orders are sensitive to the choice of the β_1 coefficient for P^2 approximations, but are not for P^3 approximations.

Table 5 Analytical error estimate at Lobatto points for the DDGIC method (5)

P^k	$\beta_1 \neq \frac{1}{2k(k+1)}$	$\beta_1 = \frac{1}{2k(k+1)}$
$k=2$	$\beta_1 = \frac{1}{8}$	$\beta_1 = \frac{1}{12}$
e_{r_1}	$\frac{1}{96}\mathrm{e}^{-t}\Delta x^3 + O(\Delta x^5)$	$\frac{\mathrm{e}^{-t}(4t-1)}{2\,880}\Delta x^4 + O(\Delta x^6)$
e_{r_2}	$\frac{3t+1}{960}\mathrm{e}^{-t}\Delta x^4 + O(\Delta x^6)$	$\frac{\mathrm{e}^{-t}(8t+1)}{5\,760}\Delta x^4 + O(\Delta x^6)$
e_{r_3}	$\frac{1}{96}\mathrm{e}^{-t}\Delta x^3 + O(\Delta x^5)$	$\frac{\mathrm{e}^{-t}(4t-1)}{2\,880}\Delta x^4 + O(\Delta x^6)$
$k=3$	$\beta_1 = \frac{1}{8}$	$\beta_1 = \frac{1}{24}$
e_{r_1}	$2.11\times10^{-4}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$	$2.21\times10^{-5}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$
e_{r_2}	$7.62\times10^{-5}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$	$3.04\times10^{-5}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$
e_{r_3}	$7.62\times10^{-5}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$	$3.04\times10^{-5}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$
e_{r_4}	$2.11\times10^{-4}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$	$2.21\times10^{-5}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$

For the P^2 case, with $\beta_1 = \frac{1}{2k(k+1)} = \frac{1}{12}$, the error is superconvergent of order $k+2=4$ at the Lobatto points. With $\beta_1 \neq \frac{1}{2k(k+1)}$, the error is still superconvergent of order $k+2$ at the cell center, but is convergent with optimal order of $k+1$ at other two Lobatto points. For the P^3 case, superconvergence orders of $k+2=5$ are observed at the Lobatto points for both $\beta_1 = \frac{1}{2k(k+1)}$ and $\beta_1 \neq \frac{1}{2k(k+1)}$.

We further apply the symmetric DDG method (7) for the model equation, and compare the numerical solution errors at Lobatto points with the errors estimated by the Fourier analysis. Our numerical results agree well with those predicted by Fourier analysis. In Tables 9 and 10, we list the errors of the symmetric DDG method at Lobatto points with piecewise P^2 and P^3 polynomials, respectively.

3.4 Nonsymmetric DDG Method

In this section we carry out Fourier analysis on the nonsymmetric DDG method of (7). The numerical fluxes are taken from (8) with $\beta_0 = \beta_{0u} - \beta_{0v}$. For the nonsymmetric DDG method and P^2 polynomial approximations, the matrices $A,\ B,\ C$ of (14) are

$$A = \begin{pmatrix} -\frac{9}{2}(1-8\beta_1) & 18(1-4\beta_1) & \frac{3}{2}(6\beta_0 - 16\beta_1 + 3) \\ \frac{3}{4}(1-8\beta_1) & -3(1-4\beta_1) & -\frac{3}{4}(2\beta_0 - 32\beta_1 + 7) \\ -\frac{3}{2}(1-8\beta_1) & 6(1-4\beta_1) & \frac{3}{2}(2\beta_0 - 32\beta_1 + 5) \end{pmatrix},$$

$$B = \begin{pmatrix} -9\beta_0 + 12\beta_1 - 29 & 16(6\beta_1 + 1) & -3\beta_0 + 12\beta_1 - 17 \\ \frac{1}{2}(3\beta_0 - 36\beta_1 + 29) & -2(12\beta_1 + 7) & \frac{1}{2}(3\beta_0 - 36\beta_1 + 29) \\ -3\beta_0 + 12\beta_1 - 17 & 16(6\beta_1 + 1) & -9\beta_0 + 12\beta_1 - 29 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{3}{2}(2\beta_0 - 32\beta_1 + 5) & 6(1-4\beta_1) & -\frac{3}{2}(1-8\beta_1) \\ -\frac{3}{4}(2\beta_0 - 32\beta_1 + 7) & -3(1-4\beta_1) & \frac{3}{4}(1-8\beta_1) \\ \frac{3}{2}(6\beta_0 - 16\beta_1 + 3) & 18(1-4\beta_1) & -\frac{9}{2}(1-8\beta_1) \end{pmatrix}.$$

Table 6 Errors at Lobatto points for the DDGIC method with P^2 polynomials ($\beta_0 = 3$)

N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order
10	1.00E−03		9.51E−04		6.99E−05		5.97E−05	
20	1.20E−04	3.06	1.19E−04	3.00	3.93E−06	4.15	3.73E−06	4.00
40	1.49E−05	3.01	1.49E−05	3.00	2.36E−07	4.06	2.33E−07	4.00
80	1.86E−06	3.00	1.86E−06	3.00	1.46E−08	4.01	1.46E−08	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order
10	2.39E−04		2.39E−04		9.28E−05		8.96E−05	
20	1.48E−05	4.02	1.49E−05	4.00	5.58E−06	4.06	5.60E−06	4.00
40	9.30E−07	3.99	9.33E−07	4.00	3.50E−07	4.00	3.50E−07	4.00
80	5.83E−08	4.00	5.83E−08	4.00	2.19E−08	4.00	2.19E−08	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order
10	1.00E−03		9.51E−04		6.99E−05		5.97E−05	
20	1.20E−04	3.06	1.19E−04	3.00	3.93E−06	4.15	3.73E−06	4.00
40	1.49E−05	3.01	1.49E−05	3.00	2.36E−07	4.06	2.33E−07	4.00
80	1.86E−06	3.00	1.86E−06	3.00	1.46E−08	4.01	1.46E−08	4.00

For the nonsymmetric DDG method and P^3 approximations, the matrices A, B, C of (14) are

$$A = \begin{pmatrix} 8(1-20\beta_1) & 20(1-\sqrt{5}-(4-12\sqrt{5})\beta_1) & 20(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & 4(4\beta_0-40\beta_1+3) \\ \frac{-2}{\sqrt{5}}(1-20\beta_1) & 5-\sqrt{5}+4(\sqrt{5}-15)\beta_1 & -5-\sqrt{5}+4(\sqrt{5}+15)\beta_1 & \frac{-4\beta_0}{\sqrt{5}}+4(3+17\sqrt{5})\beta_1-\frac{27}{\sqrt{5}}-3 \\ \frac{2}{\sqrt{5}}(1-20\beta_1) & \sqrt{5}-5-4(\sqrt{5}-15)\beta_1 & 5+\sqrt{5}-4(\sqrt{5}+15)\beta_1 & \frac{4\beta_0}{\sqrt{5}}+4(3-17\sqrt{5})\beta_1+\frac{27}{\sqrt{5}}-3 \\ -2(1-20\beta_1) & 5(\sqrt{5}-1+(4-12\sqrt{5})\beta_1) & -5(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & -4\beta_0+280\beta_1-18 \end{pmatrix},$$

$$B = \begin{pmatrix} -2(8\beta_0-60\beta_1+45) & 10(1+\sqrt{5}+6(1+5\sqrt{5})\beta_1) & 10(1-\sqrt{5}+6(1-5\sqrt{5})\beta_1) & 4(\beta_0-30\beta_1+10) \\ \frac{4\beta_0+68}{\sqrt{5}}-12(1+5\sqrt{5})\beta_1+8 & -30(4\beta_1+1) & 20(6\beta_1+1) & -\frac{4\beta_0+68}{\sqrt{5}}-12(1-5\sqrt{5})\beta_1+8 \\ -\frac{4\beta_0+68}{\sqrt{5}}-12(1-5\sqrt{5})\beta_1+8 & 20(6\beta_1+1) & -30(4\beta_1+1) & \frac{4\beta_0+68}{\sqrt{5}}-12(1+5\sqrt{5})\beta_1+8 \\ 4(\beta_0-30\beta_1+10) & 10(1-\sqrt{5}+6(1-5\sqrt{5})\beta_1) & 10(1+\sqrt{5}+6(1+5\sqrt{5})\beta_1) & -2(8\beta_0-60\beta_1+45) \end{pmatrix},$$

$$C = \begin{pmatrix} -4\beta_0+280\beta_1-18 & -5(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & 5(\sqrt{5}-1+(4-12\sqrt{5})\beta_1) & -2(1-20\beta_1) \\ \frac{4\beta_0+27}{\sqrt{5}}+4(3-17\sqrt{5})\beta_1-3 & 5+\sqrt{5}-4(\sqrt{5}+15)\beta_1 & \sqrt{5}-5-4(\sqrt{5}-15)\beta_1 & \frac{2}{\sqrt{5}}(1-20\beta_1) \\ -\frac{4\beta_0+27}{\sqrt{5}}+4(3+17\sqrt{5})\beta_1-3 & -5-\sqrt{5}+4(\sqrt{5}+15)\beta_1 & 5-\sqrt{5}+4(\sqrt{5}-15)\beta_1 & \frac{-2}{\sqrt{5}}(1-20\beta_1) \\ 4(4\beta_0-40\beta_1+3) & 20(1+\sqrt{5}-(4+12\sqrt{5})\beta_1) & 20(1-\sqrt{5}-(4-12\sqrt{5})\beta_1) & 8(1-20\beta_1) \end{pmatrix}.$$

Table 7 Errors at Lobatto points for the DDGIC method with P^3 polynomials ($\beta_0 = 12$)

N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order
10	8.00E−06		7.59E−06		7.86E−07		7.95E−07	
20	2.49E−07	5.01	2.37E−07	5.00	2.58E−08	4.93	2.48E−08	5.00
40	7.76E−09	5.00	7.41E−09	5.00	8.18E−10	4.98	7.76E−10	5.00
80	2.42E−10	5.00	2.32E−10	5.00	2.57E−11	4.99	2.43E−11	5.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order
10	2.74E−06		2.75E−06		1.08E−06		1.09E−06	
20	8.67E−08	4.98	8.58E−08	5.00	3.52E−08	4.94	3.42E−08	5.00
40	2.72E−09	5.00	2.68E−09	5.00	1.11E−09	4.98	1.07E−09	5.00
80	8.49E−11	5.00	8.38E−11	5.00	3.48E−11	5.00	3.34E−11	5.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order
10	2.74E−06		2.75E−06		1.08E−06		1.09E−06	
20	8.67E−08	4.98	8.58E−08	5.00	3.52E−08	4.94	3.42E−08	5.00
40	2.72E−09	5.00	2.68E−09	5.00	1.11E−09	4.98	1.07E−09	5.00
80	8.49E−11	5.00	8.38E−11	5.00	3.48E−11	5.00	3.34E−11	5.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_4}	Order	e_{r_4}	Order	e_{r_4}	Order	e_{r_4}	Order
10	8.00E−06		7.59E−06		7.86E−07		7.95E−07	
20	2.49E−07	5.01	2.37E−07	5.00	2.58E−08	4.93	2.48E−08	5.00
40	7.76E−09	5.00	7.41E−09	5.00	8.18E−10	4.98	7.76E−10	5.00
80	2.42E−10	5.00	2.32E−10	5.00	2.57E−11	4.99	2.43E−11	5.00

Table 8 Analytical error estimate at Lobatto points for the symmetric DDG method (7)

P^k	$\beta_1 \neq \frac{1}{2k(k+1)}$	$\beta_1 = \frac{1}{2k(k+1)}$
$k=2$	$\beta_1 = \frac{1}{8}$	$\beta_1 = \frac{1}{12}$
e_{r_1}	$\frac{1}{48}\mathrm{e}^{-t}\Delta x^3 + O(\Delta x^5)$	$\frac{\mathrm{e}^{-t}(4t-1)}{2\,880}\Delta x^4 + O(\Delta x^6)$
e_{r_2}	$\frac{t+2}{2\,880}\mathrm{e}^{-t}\Delta x^4 + O(\Delta x^6)$	$\frac{\mathrm{e}^{-t}(8t+1)}{5\,760}\Delta x^4 + O(\Delta x^6)$
e_{r_3}	$\frac{1}{48}\mathrm{e}^{-t}\Delta x^3 + O(\Delta x^5)$	$\frac{\mathrm{e}^{-t}(4t-1)}{2\,880}\Delta x^4 + O(\Delta x^6)$
$k=3$	$\beta_1 = \frac{1}{8}$	$\beta_1 = \frac{1}{24}$
e_{r_1}	$1.37\times10^{-4}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$	$8.84\times10^{-6}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$
e_{r_2}	$1.42\times10^{-4}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$	$2.13\times10^{-5}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$
e_{r_3}	$1.42\times10^{-4}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$	$2.13\times10^{-5}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$
e_{r_4}	$1.37\times10^{-4}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$	$8.84\times10^{-6}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$

Table 9 Errors at Lobatto points for the symmetric DDG method with P^2 polynomials ($\beta_0 = 2$)

N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order
10	1.94E−03		1.90E−03		9.90E−05		5.97E−05	
20	2.39E−04	3.02	2.38E−04	3.00	4.43E−06	4.48	3.73E−06	4.00
40	2.97E−05	3.01	2.97E−05	3.00	2.45E−07	4.18	2.33E−07	4.00
80	3.71E−06	3.00	3.71E−06	3.00	1.48E−08	4.05	1.46E−08	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order
10	6.21E−05		5.97E−05		8.68E−05		8.96E−05	
20	3.72E−06	4.06	3.73E−06	4.00	5.49E−06	3.98	5.60E−06	4.00
40	2.33E−07	4.00	2.33E−07	4.00	3.48E−07	3.98	3.50E−07	4.00
80	1.46E−08	4.00	1.46E−08	4.00	2.18E−08	3.99	2.19E−08	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order
10	1.94E−03		1.90E−03		9.90E−05		5.97E−05	
20	2.39E−04	3.02	2.38E−04	3.00	4.43E−06	4.48	3.73E−06	4.00
40	2.97E−05	3.01	2.97E−05	3.00	2.45E−07	4.18	2.33E−07	4.00
80	3.71E−06	3.00	3.71E−06	3.00	1.48E−08	4.05	1.46E−08	4.00

Table 10 Errors at Lobatto points for the symmetric DDG method with P^3 polynomials ($\beta_0 = 24$)

N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order
10	4.84E−06		4.95E−06		3.10E−07		3.18E−07	
20	1.53E−07	4.98	1.55E−07	5.00	1.01E−08	4.94	9.95E−09	5.00
40	4.81E−09	5.00	4.83E−09	5.00	3.20E−10	4.98	3.11E−10	5.00
80	1.50E−10	5.00	1.51E−10	5.00	1.01E−11	4.98	9.71E−12	5.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order
10	4.82E−06		5.12E−06		7.73E−07		7.66E−07	
20	1.59E−07	4.92	1.60E−07	5.00	2.40E−08	5.01	2.39E−08	5.00
40	5.03E−09	4.98	5.00E−09	5.00	7.50E−10	5.00	7.48E−10	5.00
80	1.58E−10	4.99	1.56E−10	5.00	2.34E−11	5.00	2.34E−11	5.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order
10	4.82E−06		5.12E−06		7.73E−07		7.66E−07	
20	1.59E−07	4.92	1.60E−07	5.00	2.40E−08	5.01	2.39E−08	5.00
40	5.03E−09	4.98	5.00E−09	5.00	7.50E−10	5.00	7.48E−10	5.00
80	1.58E−10	4.99	1.56E−10	5.00	2.34E−11	5.00	2.34E−11	5.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_4}	Order	e_{r_4}	Order	e_{r_4}	Order	e_{r_4}	Order
10	4.84E−06		4.95E−06		3.10E−07		3.18E−07	
20	1.53E−07	4.98	1.55E−07	5.00	1.01E−08	4.94	9.95E−09	5.00
40	4.81E−09	5.00	4.83E−09	5.00	3.20E−10	4.98	3.11E−10	5.00
80	1.50E−10	5.00	1.51E−10	5.00	1.01E−11	4.98	9.71E−12	5.00

In Table 11, we list the analytically calculated errors at Lobatto points for the nonsymmetric DDG method. Two choices of the β_1 coefficient of (8) are investigated. For P^2 approximations, two settings of $(\beta_0, \beta_1) = (2, \frac{1}{8})$ and $(\beta_0, \beta_1) = (2, \frac{1}{12})$ are considered. For P^3 approximations, two settings of $(\beta_0, \beta_1) = (24, \frac{1}{8})$ and $(\beta_0, \beta_1) = (24, \frac{1}{24})$ are considered. For P^2 approximations, with $\beta_1 = \frac{1}{2k(k+1)} = \frac{1}{12}$, the errors of the nonsymmetric DDG method at Lobatto points are superconvergent of order $k + 2 = 4$, while other β_1 coefficients, the accuracy of the errors drop to second order. For P^3 approximations, the error is observed with the optimal $k + 1 = 4$ order of accuracy at the Lobatto points. The convergence order does not depend on the choice of β_1 for the P^3 case.

We apply the nonsymmetric DDG scheme (7) to solve the model problem (1) numerically, and compare the errors at Lobatto points of the numerical solution and the errors predicted by Fourier analysis. In Tables 12 and 13 we list the errors of the nonsymmetric DDG solution at Lobatto points with piecewise P^2 and P^3 polynomials, respectively. The numerical results agree well with those predicted by Fourier analysis.

With the intensive study on analytical and numerical errors of DDG methods in Sects. 3.1–3.4, we observe the choice of the numerical flux coefficient β_1 has an impact on the error behavior of the numerical solution. We further explored the relation of the L^∞ errors of the four DDG methods with respect to β_0 or β_1 in Fig. 1. With different β_0 choices, we observe the errors stay the same for all DDG methods with both P^2 and P^3 polynomials, see Fig. 1a and c. For P^2 polynomial approximations, the errors of all four DDG methods are sensitive to the choice of the β_1 coefficient. The error is superconvergent with $\beta_1 = \frac{1}{12}$, while the errors stay the same with different β_1 coefficients for the P^3 case, see Fig. 1b and d. These tests further confirm the analytical and numerical results studied in previous sections.

3.5 IPDG Method

In this section, we perform the Fourier analysis for the IPDG method [4] with (10) as the numerical flux. Since the IPDG method equals to the DDGIC method with $\beta_1 = 0$ in (6), we refer to the matrices A, B, C of (14) in Sect. 3.2 for the IPDG method. In Table 14, we list the analytically calculated errors of the IPDG method at Lobatoo points. We choose the penalty coefficient $\beta_0 = 3$ for the P^2 case and $\beta_0 = 12$ for the P^3 case, which are the same settings as the DDGIC method. For P^2 polynomial approximations, the error of the IPDG solution is superconvergent of order $k + 2$ at the cell center and is optimally convergent of order $k + 1$ at the other two Lobatto points. For P^3 polynomial approximations, the error of the IPDG method is superconvergent of order $k + 2$ at all Lobatto points. In Table 15, we list numerically calculated errors of the IPDG method at Lobatto points with piecewise P^2 and P^3 approximations. The numerical results agree well with the analytical ones.

4 Conclusion

In this paper, we discuss supercovergence properties of various DDG methods for one-dimensional heat equation via Fourier analysis approach. By investigating the quantitative errors at Lobatto points, we show that as follows.

Table 11 Analytical error estimate at Lobatto points for the nonsymmetric DDG method

P^k	$\beta_1 \neq \frac{1}{2k(k+1)}$	$\beta_1 = \frac{1}{2k(k+1)}$
$k=2$	$\beta_1 = \frac{1}{8}, \quad \beta_0 = 2$	$\beta_1 = \frac{1}{12}, \quad \beta_0 = 2$
e_{r_1}	$\frac{1}{36}\mathrm{e}^{-t}t\Delta x^2 + O(\Delta x^4)$	$\frac{\mathrm{e}^{-t}(4t+1)}{2\,880}\Delta x^4 + O(\Delta x^6)$
e_{r_2}	$\frac{1}{36}\mathrm{e}^{-t}t\Delta x^2 + O(\Delta x^4)$	$\frac{\mathrm{e}^{-t}(8t-1)}{5\,760}\Delta x^4 + O(\Delta x^6)$
e_{r_3}	$\frac{1}{36}\mathrm{e}^{-t}t\Delta x^2 + O(\Delta x^4)$	$\frac{\mathrm{e}^{-t}(4t+1)}{2\,880}\Delta x^4 + O(\Delta x^6)$
$k=3$	$\beta_1 = \frac{1}{8}, \quad \beta_0 = 24$	$\beta_1 = \frac{1}{24}, \quad \beta_0 = 24$
e_{r_1}	$4.54 \times 10^{-4}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^5)$	$3.02 \times 10^{-5}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^6)$
e_{r_2}	$4.54 \times 10^{-4}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^5)$	$3.02 \times 10^{-5}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^6)$
e_{r_3}	$4.54 \times 10^{-4}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^5)$	$3.02 \times 10^{-5}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^6)$
e_{r_4}	$4.54 \times 10^{-4}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^5)$	$3.02 \times 10^{-5}\mathrm{e}^{-t}t\Delta x^4 + O(\Delta x^6)$

Table 12 Errors at Lobatto points for the nonsymmetric DDG method with P^2 polynomials ($\beta_0 = 2$)

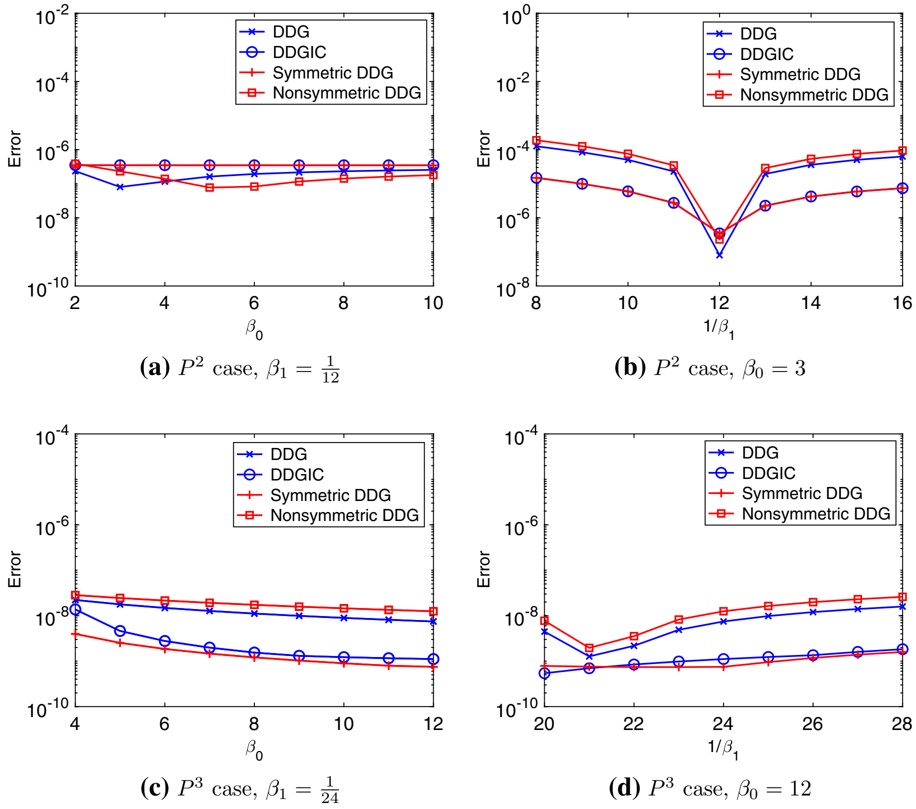
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order
10	3.91E−03		4.03E−03		9.10E−05		9.95E−05	
20	1.00E−03	1.97	1.01E−03	2.00	6.04E−06	3.91	6.22E−06	4.00
40	2.52E−04	1.99	2.52E−04	2.00	3.86E−07	3.97	3.89E−07	4.00
80	6.30E−05	2.00	6.30E−05	2.00	2.43E−08	3.99	2.43E−08	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order
10	4.03E−03		4.03E−03		6.95E−05		6.97E−05	
20	9.96E−04	2.02	1.01E−03	2.00	4.30E−06	4.02	4.35E−06	4.00
40	2.51E−04	1.99	2.52E−04	2.00	2.71E−07	3.99	2.72E−07	4.00
80	6.30E−05	2.00	6.30E−05	2.00	1.70E−08	4.00	1.70E−08	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order
10	3.91E−03		4.03E−03		9.10E−05		9.95E−05	
20	1.00E−03	1.97	1.01E−03	2.00	6.04E−06	3.91	6.22E−06	4.00
40	2.52E−04	1.99	2.52E−04	2.00	3.86E−07	3.97	3.89E−07	4.00
80	6.30E−05	2.00	6.30E−05	2.00	2.43E−08	3.99	2.43E−08	4.00

Table 13 Errors at Lobatto points for the nonsymmetric DDG method with P^3 polynomials ($\beta_0 = 24$)

N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order
10	2.50E−05		2.60E−05		1.51E−06		1.73E−06	
20	1.61E−06	3.96	1.63E−06	4.00	1.05E−07	3.85	1.08E−07	4.00
40	1.01E−07	3.99	1.02E−07	4.00	6.72E−09	3.96	6.76E−09	4.00
80	6.35E−09	4.00	6.36E−09	4.00	4.22E−10	3.99	4.23E−10	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order
10	2.55E−05		2.60E−05		1.94E−06		1.73E−06	
20	1.61E−06	3.99	1.63E−06	4.00	1.12E−07	4.11	1.08E−07	4.00
40	1.01E−07	3.99	1.02E−07	4.00	6.84E−09	4.04	6.76E−09	4.00
80	6.35E−09	4.00	6.36E−09	4.00	4.23E−10	4.01	4.23E−10	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order
10	2.55E−05		2.60E−05		1.94E−06		1.73E−06	
20	1.61E−06	3.99	1.63E−06	4.00	1.12E−07	4.11	1.08E−07	4.00
40	1.01E−07	3.99	1.02E−07	4.00	6.84E−09	4.04	6.76E−09	4.00
80	6.35E−09	4.00	6.36E−09	4.00	4.23E−10	4.01	4.23E−10	4.00
N	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{24}$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_4}	Order	e_{r_4}	Order	e_{r_4}	Order	e_{r_4}	Order
10	2.50E−05		2.60E−05		1.51E−06		1.73E−06	
20	1.61E−06	3.96	1.63E−06	4.00	1.05E−07	3.85	1.08E−07	4.00
40	1.01E−07	3.99	1.02E−07	4.00	6.72E−09	3.96	6.76E−09	4.00
80	6.35E−09	4.00	6.36E−09	4.00	4.22E−10	3.99	4.23E−10	4.00

i) For piecewise P^2 polynomials with $\beta_1 = \frac{1}{12}$ chosen in the numerical flux, the errors of DDG, DDGIC, symmetric/nonsymmetric DDG methods at Lobatto points are superconvergent of order $k+2$.

ii) For piecewise P^2 polynomials, the errors of DDGIC and symmetric DDG methods with $\beta_1 \neq \frac{1}{12}$ and the errors of the IPDG method, are of order $k+2$ superconvergence at the cell center and are of order $k+1$ optimal convergence at other Lobatto points. The errors of DDG and nonsymmetric DDG methods are of order k convergence at all Lobatto points.



(a) P^2 case, $\beta_1 = \frac{1}{12}$ **(b)** P^2 case, $\beta_0 = 3$

(c) P^3 case, $\beta_1 = \frac{1}{24}$ **(d)** P^3 case, $\beta_0 = 12$

Fig. 1 L^∞ error of the four DDG methods with P^2 polynomials (top) and P^3 polynomials (bottom) approximations and different choices of β_0 (left) and β_1 (right) coefficients in the numerical flux. $N = 40$

iii) For piecewise P^3 polynomials, the errors of IPDG, DDGIC and symmetric DDG methods are superconvergent of order $k + 2$ at all Lobatto points, while the errors of DDG and nonsymmetric DDG methods are of order $k + 1$ optimal convergence at the Lobatto points.

Numerical errors agree well with analytical ones. Fourier analysis depends on periodic boundary conditions and uniform mesh, but can serve as a guidance to problems under general settings.

Table 14 Analytical error estimate at Lobatto points for the IPDG method (9)

$\beta_0 = 3$		$\beta_0 = 12$	
	$k = 2$		$k = 3$
e_{r_1}	$\frac{1}{48}\mathrm{e}^{-t}\Delta x^3 + O(\Delta x^5)$	e_{r_1}	$2.90 \times 10^{-4}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$
e_{r_2}	$\frac{\mathrm{e}^{-t}(4t+3)}{1\,920}\Delta x^4 + O(\Delta x^6)$	e_{r_2}	$1.54 \times 10^{-4}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$
e_{r_3}	$\frac{1}{48}\mathrm{e}^{-t}\Delta x^3 + O(\Delta x^4)$	e_{r_3}	$1.54 \times 10^{-4}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$
		e_{r_4}	$2.90 \times 10^{-4}\mathrm{e}^{-t}t\Delta x^5 + O(\Delta x^6)$

Table 15 Errors of the IPDG method at Lobatto points

N	$k = 2$				$k = 3$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order	e_{r_1}	Order
10	2.04E−03		1.90E−03		1.00E−05		1.04E−05	
20	2.42E−04	3.08	2.38E−04	3.00	3.29E−07	4.93	3.26E−07	5.00
40	2.98E−05	3.02	2.97E−05	3.00	1.04E−08	4.98	1.02E−08	5.00
80	3.72E−06	3.00	3.71E−06	3.00	3.27E−10	5.00	3.19E−10	5.00
N	$k = 2$				$k = 3$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order	e_{r_2}	Order
10	2.24E−04		2.09E−04		4.95E−06		5.51E−06	
20	1.31E−05	4.09	1.31E−05	4.00	1.69E−07	4.87	1.72E−07	5.00
40	8.17E−07	4.00	8.17E−07	4.00	5.39E−09	4.97	5.38E−09	5.00
80	5.10E−08	4.00	5.10E−08	4.00	1.69E−10	4.99	1.68E−10	5.00
N	$k = 2$				$k = 3$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order	e_{r_3}	Order
10	2.04E−03		1.90E−03		4.95E−06		5.51E−06	
20	2.42E−04	3.08	2.38E−04	3.00	1.69E−07	4.87	1.72E−07	5.00
40	2.98E−05	3.02	2.97E−05	3.00	5.39E−09	4.97	5.38E−09	5.00
80	3.72E−06	3.00	3.71E−06	3.00	1.69E−10	4.99	1.68E−10	5.00
N	$k = 2$				$k = 3$			
	Numerical results		Predicted by analysis		Numerical results		Predicted by analysis	
					e_{r_4}	Order	e_{r_4}	Order
10	–	–	–	–	1.00E−05		1.04E−05	
20	–	–	–	–	3.29E−07	4.93	3.26E−07	5.00
40	–	–	–	–	1.04E−08	4.98	1.02E−08	5.00
80	–	–	–	–	3.27E−10	5.00	3.19E−10	5.00

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