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ORIGINAL PAPER

Efficient Difference Schemes for the Caputo-Tempered Fractional Diffusion Equations Based on Polynomial Interpolation

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Abstract
The tempered fractional calculus has been successfully applied for depicting the time evolution of a system describing non-Markovian diffusion particles. The related governing equations are a series of partial differential equations with tempered fractional derivatives. Using the polynomial interpolation technique, in this paper, we present three efficient numerical formulas, namely the tempered L1 formula, the tempered L1-2 formula, and the tempered L2-1_σ formula, to approximate the Caputo-tempered fractional derivative of order $\alpha \in (0, 1)$. The truncation error of the tempered L1 formula is of order $2-\alpha$, and the tempered L1-2 formula and L2-1_σ formula are of order $3-\alpha$. As an application, we construct implicit schemes and implicit ADI schemes for one-dimensional and two-dimensional time-tempered fractional diffusion equations, respectively. Furthermore, the unconditional stability and convergence of two developed difference schemes with tempered L1 and L2-1_σ formulas are proved by the Fourier analysis method. Finally, we provide several numerical examples to demonstrate the correctness and effectiveness of the theoretical analysis.

Keywords Caputo-tempered fractional derivative · Polynomial interpolation · Implicit ADI schemes · Stability

Mathematics Subject Classification 65M06

1 Introduction

The tempered fractional calculus is a generalization of the fractional calculus, and the definition contains both the weak singular kernel and the exponential kernel. It has particular mathematical properties and plays an important role in the actual mathematical

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physics model. The tempered fractional calculus describes the transition between normal and anomalous diffusions or some anomalous diffusions in finite time or bounded space domain. As we know, the anomalous diffusion phenomena can be seen frequently in the nature world, and the continuous time random walk (CTRW) model has been proved to be a useful tool that describes this phenomenon well [28]. The process of the CTRW model, which is non-Markovian is usually depicted by the waiting time probability density function (PDF) and the jump length PDF. For the CTRW model with tempered power law waiting time, the PDF of diffusion particles obeys the tempered fractional diffusion equation [13, 16, 40]

$$u_t(\mathbf{x},t) = \kappa_\alpha \, {}_0\mathrm{D}_t^{1-\alpha,\lambda}(\Delta u(\mathbf{x},t)) - \lambda u(\mathbf{x},t), \tag{1}$$

where $u(\mathbf{x},t)$ denotes the probability density of searching a particle in position $\mathbf{x}$ at time t, $\kappa_\alpha > 0$ is the diffusion coefficient, the standard Laplace operator $\Delta = \sum\limits_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}, d = 1, 2$, and ${}_0\mathrm{D}_t^{1-\alpha,\lambda}$ $(0 < \alpha < 1)$ represents the Riemann–Liouville tempered fractional derivative operator. The Riemann–Liouville tempered fractional derivative of order $\beta\,(n-1 < \beta < n)$ gives [18, 31]

$${}_0\mathrm{D}_t^{\beta,\lambda}u(\mathbf{x},t) = \frac{\mathrm{e}^{-\lambda t}}{\Gamma(n-\beta)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{\mathrm{e}^{\lambda s}u(\mathbf{x},s)}{(t-s)^{1+\beta-n}}\mathrm{d}s.$$

Using the properties of the fractional derivative [21], Eq. (1) can be rewritten as

$${}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(\mathbf{x},t) = \kappa_\alpha \Delta u(\mathbf{x},t), \tag{2}$$

where ${}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}$ denotes the Caputo-tempered fractional derivative [21, 32, 37]

$${}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(t) = \frac{\mathrm{e}^{-\lambda t}}{\Gamma(1-\alpha)} \int_0^t \frac{(\mathrm{e}^{\lambda s}u(s))'}{(t-s)^\alpha}\mathrm{d}s, \ 0 < \alpha < 1. \tag{3}$$

The more detailed background and application of model (1), we refer to [16, 28] or Appendix A. For the equivalence of models (1) and (2), see [36].

In this paper, without loss of generality, we consider the following time fractional subdiffusion equation:

$${}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(\mathbf{x},t) = \Delta u(\mathbf{x},t) + f(\mathbf{x},t), \ \mathbf{x} \in \Omega, t \in (0,T]$$

with the initial condition

$$u(\mathbf{x},0) = \phi(\mathbf{x}), \ \mathbf{x} \in \Omega$$

and a Dirichlet boundary condition

$$u(\mathbf{x},t) = \psi(\mathbf{x},t), \ \mathbf{x} \in \partial\Omega, t \in (0,T],$$

where $\Omega \in R^d$ is a bounded domain in $\mathbb{R}^d, d = 1, 2$ with the boundary $\partial\Omega$, and f, ϕ, ψ are given functions.

There are several numerical approaches to the Caputo fractional derivative. The first technique is the piecewise interpolation polynomials, such as the L1 approximation [10, 11, 23, 29, 35, 42], the L1-2 approximation [14, 20], and the L2-1_σ approximation [1] and so on. Another widely used technique is the shifted Grünwald–Letnikov approach [2, 6, 19, 26, 30]. The main idea is to approximate the Riemann–Liouville derivative by the Grünwald–Letnikov formula. The third used technique is the so called Lubich's approach which is introduced by [25]. The main advantages of above-mentioned approaches are simplicity and efficiency. More importantly, the error estimates can be analyzed when above-mentioned approaches are used to approximate the time fractional diffusion equations [22, 38]. More applications of above-mentioned approaches to solve the various kinds of fractional order partial differential equations, see [15, 24, 34] and references, therein.

Recently, there has been a vast interest in the numerical solutions of tempered fractional differential equations due to their wide applications [2, 4, 7, 9, 16, 27, 37, 40, 41]. Several numerical methods, such as the Grünwald–Letnikov approach and the Lubich's approach, have been used to approximate the tempered fractional derivatives. In [2], Baeumera and Meerschaert presented the finite difference scheme and the particle tracking method for the space-tempered fractional diffusion equation with drift. Applying the spectral method, Zayernouri et al. [39] obtained the eigenfunctions of the tempered fractional Sturm–Liouville problem. Based on the weighted and shifted Grünwald difference (WSGD) operator, Li and Deng [19] constructed a series of high-order numerical forms for the space-tempered fractional diffusion equation, and proved the stability and convergence by the matrix analysis. Using the weighted and shifted Lubich difference operator to approximate the time-tempered fractional derivative, Sun et al. [36] analyzed some local discontinuous Galerkin schemes for a time-tempered fractional diffusion equation. The implicit numerical scheme of the tempered fractional Black–Scholes equation is provided by Zhang et al. [43], and the corresponding theoretical analysis is given. Using the Lubich's approach for the time-tempered fractional derivative, Chen and Deng [5] derived a high-order algorithm for a time–space fractional Feynman–Kac equation. Dehghan et al. [8] proposed a numerical scheme that is convergent with the second-order accuracy in time for the space–time tempered fractional diffusion–wave equation, and the unconditional stability and convergence of the developed method are also proved. Deng and Zhang [9] developed the finite difference/finite element schemes for a time fractional Feynman–Kac equation. Ding and Li [12] designed a high-order numerical algorithm to solve the space-tempered fractional convection equation with Riemann–Liouville fractional derivatives, and the rigorous stability and convergence analysis of the algorithm are also given. So far, the application of the piecewise interpolation polynomial technique to solve the tempered fractional diffusion equation has not been intensively investigated.

In this paper, we are interested in developing efficient and accurate numerical approximations for the time-tempered fractional diffusion equation. In particular, one of the challenges of this problem lies in the existence of the singular kernel and the smooth kernel. To overcome the difficulty, we discrete the tempered Caputo fractional derivative by transforming it to the conventional Caputo fractional derivative. In Sect. 2, we present three kinds of quadrature formulas for the Caputo-tempered fractional derivative of order $\alpha \in (0, 1)$. The tempered L1 formula with the order $2-\alpha$, the tempered L1-2 and L2-1_σ formulas with the order $3-\alpha$. And the corresponding analysis of truncation errors of three formulas are discussed in detail. The presented discrete formulas are applied to solve the time-tempered fractional diffusion equations of one dimension and two dimension in Sects. 3 and 4, respectively. Besides, the unconditional stability and convergence of difference schemes with tempered L1 and L2-1_σ formulas are proved by the Fourier method. In Sect. 5, the numerical results in the examples illustrate the effectiveness of the three proposed discrete methods. Finally, we make a brief conclusion in Sect. 6.

2 Interpolation Formulas of Caputo-Tempered Fractional Derivative

In this section, we are concerned with the efficient numerical discretization of the Caputo-tempered fractional derivative. For a given positive integer N, let $\{t_k\}_{k=0}^{N}$ be an equidistant partition of $[0, T]$, and denote $t_k = k\tau, t_{k+1/2} = (t_k + t_{k+1})/2$, where $\tau = T/N$ is the time step size.

2.1 Tempered L1 Formula

To discrete the Caputo-tempered fractional derivative (3), we introduce the following notations:

$$\delta_t v_{k-\frac{1}{2}} = \frac{v(t_k) - v(t_{k-1})}{\tau} \ , \ \ \delta_t^2 v_k = \frac{\delta_t v_{k+\frac{1}{2}} - \delta_t v_{k-\frac{1}{2}}}{\tau},$$

then we denote the linear interpolation function of $v(t)$ as $P_{1,\ell}v(t)$ on each small interval $[t_{\ell-1}, t_\ell]\,(1 \leqslant \ell \leqslant k)$, i.e.,

$$P_{1,\ell}v(t) = v(t_{\ell-1})\frac{t_\ell - t}{\tau} + v(t_\ell)\frac{t - t_{\ell-1}}{\tau},$$

which leads to

$$(P_{1,\ell}v(t))' = \frac{v(t_\ell) - v(t_{\ell-1})}{\tau} = \delta_t v_{\ell-\frac{1}{2}},$$

and

$$v(t) - P_{1,\ell}v(t) = \frac{v''(\xi_\ell)}{2}(t - t_{\ell-1})(t - t_\ell), \ \ t \in [t_{\ell-1}, t_\ell], \ \xi_\ell \in (t_{\ell-1}, t_\ell).$$

Denoting $v(t) = \mathrm{e}^{\lambda t}u(t)$ and using the piecewise linear approximation on each cell $[t_{\ell-1}, t_\ell]$, then we can get the expression of the tempered L1 formula of the Caputo-tempered fractional derivative, as follows:

$$\begin{aligned}
{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(t)\big|_{t=t_k} &= \frac{\mathrm{e}^{-\lambda t_k}}{\Gamma(1-\alpha)}\sum_{\ell=1}^{k}\int_{t_{\ell-1}}^{t_\ell}(t_k - s)^{-\alpha}(P_{1,\ell}v(s))'\mathrm{d}s + R^k \\
&= \frac{\mathrm{e}^{-\lambda t_k}}{\Gamma(1-\alpha)}\sum_{\ell=1}^{k}\delta_t v_{\ell-\frac{1}{2}}\int_{t_{\ell-1}}^{t_\ell}(t_k - s)^{-\alpha}\mathrm{d}s + R^k \\
&= \frac{\mathrm{e}^{-\lambda t_k}\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{\ell=1}^{k}a_{k-\ell}^{(\alpha)}\delta_t v_{\ell-\frac{1}{2}} + R^k.
\end{aligned} \tag{4}$$

By simple calculation, Eq. (4) provides

$$\begin{aligned}
&{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(t)\big|_{t=t_k} \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)}u(t_k) - \sum_{\ell=1}^{k-1}\left(a_{k-\ell-1}^{(\alpha)} - a_{k-\ell}^{(\alpha)}\right)\mathrm{e}^{\lambda(t_\ell - t_k)}u(t_\ell) - a_{k-1}^{(\alpha)}\mathrm{e}^{\lambda(t_0 - t_k)}u(t_0)\right] + R^k \\
&=: \ \mathbb{D}_t^{\alpha,\lambda}u(t_k) + R^k,
\end{aligned} \tag{5}$$

where

$$R^k = {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda} u(t)\big|_{t=t_k} - \mathbb{D}_t^{\alpha,\lambda} u(t_k),$$

and

$$a_j^{(\alpha)} = (j+1)^{1-\alpha} - j^{1-\alpha}, \quad 0 \leqslant j \leqslant k-1. \tag{6}$$

We denote $\mu = \tau^\alpha \Gamma(2-\alpha)$, $U^k \approx u(t_k)$ and

$$\overline{d}_{a,m}^{(\alpha)} = a_{k-m}^{(\alpha)} \mathrm{e}^{\lambda(t_{m-1}-t_k)}, \ m = 1, 2, \cdots, k, \tag{7}$$

$$\widetilde{d}_{a,m}^{(\alpha)} = a_{k-m}^{(\alpha)} \mathrm{e}^{\lambda(t_m-t_k)}, \ m = 1, 2, \cdots, k, \tag{8}$$

then the tempered L1 approximation operator $\mathbb{D}_t^{\alpha,\lambda} U^k$ can be rewritten as the following form:

$$\mathbb{D}_t^{\alpha,\lambda} U^k = \frac{1}{\mu}\left[\widetilde{d}_{a,k}^{(\alpha)} U^k - \sum_{\ell=1}^{k-1}\left(\overline{d}_{a,\ell+1}^{(\alpha)} - \widetilde{d}_{a,\ell}^{(\alpha)}\right)U^\ell - \overline{d}_{a,1}^{(\alpha)} U^0\right]. \tag{9}$$

For coefficients $\{a_j^{(\alpha)}\}$, $\{\overline{d}_{a,m}^{(\alpha)}\}$, and $\{\widetilde{d}_{a,m}^{(\alpha)}\}$, we can obtain

Lemma 1 *For any $\alpha \in (0,1), k \geqslant 1$, the coefficients $a_j^{(\alpha)} (0 \leqslant j \leqslant k-1)$ and $\overline{d}_{a,m}^{(\alpha)}$, $\widetilde{d}_{a,m}^{(\alpha)}$ $(1 \leqslant m \leqslant k)$ defined in Eqs.* (6), (7), *and* (8) *satisfy*

(i) $a_j^{(\alpha)} > 0$;

(ii) $1 = a_0^{(\alpha)} > a_1^{(\alpha)} > a_2^{(\alpha)} > \cdots > a_{k-1}^{(\alpha)}$;

(iii) $\sum_{j=0}^{k-1} a_j^{(\alpha)} = k^{1-\alpha}$;

(iv) $\overline{d}_{a,m}^{(\alpha)} > 0$, $\widetilde{d}_{a,m}^{(\alpha)} > 0$;

(v) $\overline{d}_{a,1}^{(\alpha)} < \overline{d}_{a,2}^{(\alpha)} < \overline{d}_{a,3}^{(\alpha)} < \cdots < \overline{d}_{a,k}^{(\alpha)}$;

(vi) $\widetilde{d}_{a,1}^{(\alpha)} < \widetilde{d}_{a,2}^{(\alpha)} < \widetilde{d}_{a,3}^{(\alpha)} < \cdots < \widetilde{d}_{a,k}^{(\alpha)}$;

(vii) $\overline{d}_{a,m+1}^{(\alpha)} - \widetilde{d}_{a,m}^{(\alpha)} \leqslant a_{k-m-1}^{(\alpha)} - a_{k-m}^{(\alpha)}$, $1 \leqslant m \leqslant k-1$;

(viii) $\overline{d}_{a,1}^{(\alpha)} \leqslant a_{k-1}^{(\alpha)}$.

Proof The proof is provided in Appendix B.

Lemma 2 *Assume that $u(t) \in C^2[0, t_k]$. Then, there is the following error estimate*:

$$\left|R^k\right| \leqslant \frac{1}{2\Gamma(1-\alpha)}\left[\frac{1}{4} + \frac{\alpha}{(1-\alpha)(2-\alpha)}\right]\left[\lambda^2 \max_{t_0 \leqslant t \leqslant t_k} |u(t)| + 2\lambda \max_{t_0 \leqslant t \leqslant t_k} \left|u'(t)\right| + \max_{t_0 \leqslant t \leqslant t_k} \left|u''(t)\right|\right]\tau^{2-\alpha} \tag{10}$$

for R^k.

Proof We list the proof in Appendix C.

To see the smoothness of the solution more clearly, we also give the asymptotic expansion formula of the tempered fractional derivative. The asymptotic expansion formula of the tempered L1 approximation is derived from the asymptotic expansion of the L1 approximation. There is the asymptotic expansion of the L1 approximation of the order $3-\alpha$ [10, 11]

$$\begin{aligned}&\frac{1}{\mu}\sum_{\ell=0}^{k}\varpi_{\ell}^{(\alpha)}u^{k-\ell}\\&=\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha}u(t)\big|_{t=t_k}+\frac{\zeta(\alpha-1)}{\Gamma(2-\alpha)}(u^k)''\tau^{2-\alpha}-\left(\frac{u'(0)}{\Gamma(-\alpha)t_k{}^{1+\alpha}}\right.\\&\quad\left.-\frac{\mathrm{d}^2}{\mathrm{d}t^2}{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha}u(t)\Big|_{t=t_k}\right)\frac{\tau^2}{12}+\mathcal{O}(\tau^{3-\alpha}),\end{aligned}$$

where $u^k=u(t_k)$, $\zeta(s)$ is the Riemann zeta function, and the L1 approximation has weights $\varpi_0^{(\alpha)}=1, \varpi_k^{(\alpha)}=(k-1)^{1-\alpha}-k^{1-\alpha}$, and $\varpi_{\ell}^{(\alpha)}=(\ell-1)^{1-\alpha}-2\ell^{1-\alpha}+(\ell+1)^{1-\alpha}$ for $1\leqslant l\leqslant k-1$.

The tempered L1 approximation (9) has weights $\widetilde{d}_{a,k}^{(\alpha)}=\varpi_0^{(\alpha)}=1, \overline{d}_{a,1}^{(\alpha)}=-\varpi_k^{(\alpha)}/\mathrm{e}^{\lambda k\tau}$, and $\overline{d}_{a,k-\ell+1}^{(\alpha)}-\widetilde{d}_{a,k-\ell}^{(\alpha)}=-\varpi_{\ell}^{(\alpha)}/\mathrm{e}^{\lambda\ell\tau}$ $(1\leqslant\ell\leqslant k-1)$. By applying the asymptotic expansion formula for the function $\mathrm{e}^{\lambda s}u(s)$ and multiplying by $\mathrm{e}^{-\lambda t_k}$, we obtain the asymptotic expansion formula of the order $3-\alpha$ of the tempered L1 approximation as follows:

$$\begin{aligned}&\frac{1}{\mu}\left(\widetilde{d}_{a,k}^{(\alpha)}u^k-\sum_{\ell=1}^{k-1}\left(\overline{d}_{a,\ell+1}^{(\alpha)}-\widetilde{d}_{a,\ell}^{(\alpha)}\right)u^{\ell}-\overline{d}_{a,1}^{(\alpha)}u^0\right)\\&=\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(t)\big|_{t=t_k}+\frac{\zeta(\alpha-1)}{\Gamma(2-\alpha)}\left((u^k)''+2\lambda(u^k)'+\lambda^2u^k\right)\tau^{2-\alpha}\\&\quad-\left[\frac{\mathrm{e}^{-\lambda t_k}(\lambda u(0)+u'(0))}{\Gamma(-\alpha)t_k^{1+\alpha}}-\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+2\lambda\frac{\mathrm{d}}{\mathrm{d}t}+\lambda^2\right){}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(t)\Big|_{t=t_k}\right]\frac{\tau^2}{12}+\mathcal{O}(\tau^{3-\alpha}).\end{aligned}$$

2.2 Tempered L1-2 Formula

To improve the accuracy, we apply the quadratic interpolation $P_{2,\ell}v(t)$ on each cell $[t_{\ell-1},t_{\ell}](\ell\geqslant 2)$ and the linear interpolation $P_{1,1}v(t)=\delta_t v_{1/2}$ on the first cell $[t_0,t_1]$ to approximate the function $v(t)$ like [14]. For the cells $[t_{\ell-1},t_{\ell}]\,(\ell\geqslant 2)$, the quadratic interpolation function $P_{2,\ell}v(t)$ with three points $(t_{\ell-2},v(t_{\ell-2}))$, $(t_{\ell-1},v(t_{\ell-1}))$, and $(t_{\ell},v(t_{\ell}))$ is employed, i.e.,

$$\begin{aligned}P_{2,\ell}v(t)=v(t_{\ell-1})&+\frac{v(t_{\ell})-v(t_{\ell-1})}{\tau}(t-t_{\ell-1})\\&-\frac{v(t_{\ell})-2v(t_{\ell-1})+v(t_{\ell-2})}{\tau^2}\frac{(t-t_{\ell-1})(t_{\ell}-t)}{2},\end{aligned}$$

and it leads to

$$\left(P_{2,\ell}v(t)\right)' = \delta_t v_{\ell-\frac{1}{2}} + \left(\delta_t^2 v_{\ell-1}\right)\left(t - t_{\ell-\frac{1}{2}}\right), \tag{11}$$

and

$$v(t) - P_{2,\ell}v(t) = \frac{v'''\left(\eta_\ell\right)}{6}\left(t - t_{\ell-2}\right)\left(t - t_{\ell-1}\right)\left(t - t_\ell\right),\ \ t \in [t_{\ell-1}, t_\ell],\ \eta_\ell \in \left(t_{\ell-2}, t_\ell\right).$$

By virtue of Eq. (11), we can obtain

$$\begin{aligned}
&{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(t)\big|_{t=t_k}\\
&= \frac{\mathrm{e}^{-\lambda t_k}}{\Gamma(1-\alpha)}\left[\int_{t_0}^{t_1}(t_k - s)^{-\alpha}(P_{1,1}v(s))'\mathrm{d}s + \sum_{\ell=2}^{k}\int_{t_{\ell-1}}^{t_\ell}(t_k - s)^{-\alpha}(P_{2,\ell}v(s))'\mathrm{d}s\right] + T^k\\
&= \frac{\mathrm{e}^{-\lambda t_k}}{\Gamma(1-\alpha)}\left[\sum_{\ell=1}^{k}\delta_t v_{\ell-\frac{1}{2}}\int_{t_{\ell-1}}^{t_\ell}(t_k - s)^{-\alpha}\mathrm{d}s + \sum_{\ell=2}^{k}(\delta_t^2 v_{\ell-1})\int_{t_{\ell-1}}^{t_\ell}(s - t_{\ell-\frac{1}{2}})(t_k - s)^{-\alpha}\mathrm{d}s\right] + T^k\\
&= \mathbb{D}_t^{\alpha,\lambda}u(t_k) + \frac{\mathrm{e}^{-\lambda t_k}\tau^{2-\alpha}}{\Gamma(2-\alpha)}\sum_{\ell=2}^{k}b_{(k-\ell)}^{(\alpha)}\delta_t^2 v_{\ell-1} + T^k.
\end{aligned}$$

Recalling $v = \mathrm{e}^{\lambda t}u$, we have

$$\begin{aligned}
&{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(t)\big|_{t=t_k}\\
&= \frac{1}{\mu}\left[c_0^{(k,\alpha)}u(t_k) - \sum_{\ell=1}^{k-1}\left(c_{k-\ell-1}^{(k,\alpha)} - c_{k-\ell}^{(k,\alpha)}\right)\mathrm{e}^{\lambda(t_\ell - t_k)}u(t_\ell) - c_{k-1}^{(k,\alpha)}\mathrm{e}^{\lambda(t_0 - t_k)}u(t_0)\right] + T^k\\
&:= \partial_t^{\alpha,\lambda}u(t_k) + T^k,
\end{aligned} \tag{12}$$

where $T^k = {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(t)\big|_{t=t_k} - \partial_t^{\alpha,\lambda}u(t_k)$. For $k = 1$, there is $c_0^{(k,\alpha)} = a_0^{(\alpha)} = 1$; for $k \geqslant 2$, the coefficients are given as

$$c_j^{(k,\alpha)} = \begin{cases} a_0^{(\alpha)} + b_0^{(\alpha)}, & j = 0,\\ a_j^{(\alpha)} + b_j^{(\alpha)} - b_{j-1}^{(\alpha)}, & 1 \leqslant j \leqslant k-2,\\ a_j^{(\alpha)} - b_{j-1}^{(\alpha)}, & j = k-1, \end{cases} \tag{13}$$

and

$$b_j^{(\alpha)} = \frac{1}{2-\alpha}\left[(j+1)^{2-\alpha} - j^{2-\alpha}\right] - \frac{1}{2}\left[(j+1)^{1-\alpha} + j^{1-\alpha}\right],\ j \geqslant 0.$$

We call the fractional numerical differentiation formula (12) the tempered L1-2 formula, which adds a correction term $\frac{\mathrm{e}^{-\lambda t_k}\tau^{2-\alpha}}{\Gamma(2-\alpha)}\sum_{\ell=2}^{k}b_{(k-\ell)}^{(\alpha)}\delta_t^2 v_{\ell-1}$ in the tempered L1 operator $\mathbb{D}_t^{\alpha,\lambda}u(t_k)$ for $k \geqslant 2$. Setting

$$\overline{d}_{c,m}^{(k,\alpha)} = c_{k-m}^{(k,\alpha)}\mathrm{e}^{\lambda(t_{m-1} - t_k)},\ m = 1, 2, \cdots, k, \tag{14}$$

$$\widetilde{d}_{c,m}^{(k,\alpha)} = c_{k-m}^{(k,\alpha)} \mathrm{e}^{\lambda(t_m - t_k)},\ m = 1, 2, \cdots, k, \tag{15}$$

the tempered L1-2 approximation operator $\partial_t^{\alpha,\lambda} U^k$ can be rewritten as

$$\partial_t^{\alpha,\lambda} U^k = \frac{1}{\mu}\left[\widetilde{d}_{c,k}^{(k,\alpha)} U^k - \sum_{\ell=1}^{k-1}\left(\overline{d}_{c,\ell+1}^{(k,\alpha)} - \widetilde{d}_{c,\ell}^{(k,\alpha)}\right)U^\ell - \overline{d}_{c,1}^{(k,\alpha)} U^0\right]. \tag{16}$$

The tempered L1-2 approximation operator (16) reduces to the tempered L1 approximation operator (9) for $k = 1$; for $k = 2$, $c_0^{(k,\alpha)} \in (1, 3/2)$, $c_1^{(k,\alpha)} \in (-1/2, 1)$, then we can obtain $\overline{d}_{c,2}^{(k,\alpha)} - \widetilde{d}_{c,1}^{(k,\alpha)} \leqslant c_0^{(k,\alpha)} - c_1^{(k,\alpha)}$; as for $k \geqslant 3$, the properties of $\overline{d}_{c,m}^{(k,\alpha)}$ and $\widetilde{d}_{c,m}^{(k,\alpha)}$ $(1 \leqslant m \leqslant k)$ can be derived with the help of properties of $c_j^{(k,\alpha)}$, see [14] .

Lemma 3 *For any* $\alpha \in (0,1)$, $k \geqslant 3$, *the coefficients* $c_j^{(k,\alpha)}$ $(0 \leqslant j \leqslant k-1)$ *and* $\overline{d}_{c,m}^{(k,\alpha)}$, $\widetilde{d}_{c,m}^{(k,\alpha)}$ $(1 \leqslant m \leqslant k)$ *defined in Eqs.* (13), (14), *and* (15) *satisfy*

(i) $c_0^{(k,\alpha)} > \left|c_1^{(k,\alpha)}\right|$;

(ii) $c_j^{(k,\alpha)} > 0,\ j \neq 1$;

(iii) $c_2^{(k,\alpha)} \geqslant c_3^{(k,\alpha)} \geqslant \cdots \geqslant c_{k-1}^{(k,\alpha)}$;

(iv) $c_0^{(k,\alpha)} > c_2^{(k,\alpha)}$;

(v) $\overline{d}_{c,m}^{(k,\alpha)} > 0,\ \widetilde{d}_{c,m}^{(k,\alpha)} > 0,\ m \neq k-1$;

(vi) $\overline{d}_{c,1}^{(k,\alpha)} \leqslant c_{k-1}^{(k,\alpha)}$;

(vii) $\overline{d}_{c,m+1}^{(k,\alpha)} > \widetilde{d}_{c,m}^{(k,\alpha)},\ 1 \leqslant m \leqslant k-1, m \neq k-2$;

(viii) $\overline{d}_{c,m+1}^{(k,\alpha)} - \widetilde{d}_{c,m}^{(k,\alpha)} \leqslant c_{k-m-1}^{(k,\alpha)} - c_{k-m}^{(k,\alpha)},\ 1 \leqslant m \leqslant k-1, m \neq k-2$.

It is easy to check that $c_1^{(k,\alpha)} \in (-1/2, 1)$, $c_2^{(k,\alpha)} \in (0,1)$, the positive and negative of $c_1^{(k,\alpha)}$ and $c_1^{(k,\alpha)} - c_2^{(k,\alpha)}$ change with the difference α $(0 < \alpha < 1)$, which are discussed in the following lemma.

Lemma 4 *The coefficients of* $c_1^{(k,\alpha)}$ *and* $c_1^{(k,\alpha)} - c_2^{(k,\alpha)}$ *for different* k *and* $\alpha \in (0,1)$ *satisfy*

(i) $c_1^{(k,\alpha)} > 0,\ \overline{d}_{c,1}^{(k,\alpha)} \leqslant c_1^{(k,\alpha)}$, *for any* $\alpha \in (0, \alpha_1), \alpha_1 \approx 0.673\,6$ *when* $k = 2$;

(ii) $c_1^{(k,\alpha)} - c_2^{(k,\alpha)} > 0,\ \overline{d}_{c,2}^{(k,\alpha)} - \widetilde{d}_{c,1}^{(k,\alpha)} \leqslant c_1^{(k,\alpha)} - c_2^{(k,\alpha)}$, *for any* $\alpha \in (0, \alpha_2), \alpha_2 \approx 0.390\,9,\ k = 3$;

(iii) $c_1^{(k,\alpha)} - c_2^{(k,\alpha)} > 0,\ \overline{d}_{c,k-1}^{(k,\alpha)} - \widetilde{d}_{c,k-2}^{(k,\alpha)} \leqslant c_1^{(k,\alpha)} - c_2^{(k,\alpha)}$, *for any* $\alpha \in (0, \alpha_3), \alpha_3 \approx 0.373\,9,\ k > 3$;

(iv) $|c_1^{(k,\alpha)} - c_2^{(k,\alpha)}| < \frac{1}{2}$.

Proof The proof is provided in Appendix D.

Lemma 5 *Suppose that* $u(t) \in C^3[0, t_k]$, *and for the truncation error* T^k, *we can derive that*

$$\left|T^1\right| \leqslant \frac{\alpha}{2\Gamma(3-\alpha)}\left[\lambda^2 \max_{t_0 \leqslant t \leqslant t_1} |u(t)| + 2\lambda \max_{t_0 \leqslant t \leqslant t_1} \left|u'(t)\right| + \max_{t_0 \leqslant t \leqslant t_1} \left|u''(t)\right|\right]\tau^{2-\alpha}, \tag{17}$$

$$\left|T^k\right| \leqslant \frac{1}{\Gamma(1-\alpha)}\left\{ \frac{\alpha}{12}\left[\lambda^2 \max_{t_0\leqslant t\leqslant t_1} |u(t)| + 2\lambda \max_{t_0\leqslant t\leqslant t_1} \left|u'(t)\right| + \max_{t_0\leqslant t\leqslant t_1} \left|u''(t)\right|\right](t_k-t_1)^{-\alpha-1}\tau^3 \right.$$
$$+ \left[\frac{1}{12} + \frac{\alpha}{3(1-\alpha)(2-\alpha)}\left(\frac{1}{2}+\frac{1}{3-\alpha}\right)\right]\left[\lambda^3 \max_{t_0\leqslant t\leqslant t_k} |u(t)| + 3\lambda^2 \max_{t_0\leqslant t\leqslant t_k} \left|u'(t)\right| \right.$$
$$\left.\left. + 3\lambda \max_{t_0\leqslant t\leqslant t_k} \left|u''(t)\right| + \max_{t_0\leqslant t\leqslant t_k} \left|u'''(t)\right|\right]\tau^{3-\alpha}\right\},\ k \geqslant 2. \tag{18}$$

Proof The proof is provided in Appendix E.

2.3 Tempered L2-1$_\sigma$ Formula

By Alikhanov's work [1], letting $\sigma = 1-\alpha/2$, we discuss the discretization of the tempered fractional derivative at $t_{k+\sigma}\,(k=0,1,\cdots,N-1)$. The interpolation polynomial based on the quadratic interpolation on each interval $[t_{\ell-1},t_\ell]\,(1\leqslant \ell\leqslant k)$ is constructed to approximate $v(t)=\mathrm{e}^{\lambda t}u(t)$ using the three points $(t_{\ell-1},v(t_{\ell-1}))$, $(t_\ell,v(t_\ell))$, and $(t_{\ell+1},v(t_{\ell+1}))$, i.e.,

$$\Pi_{2,\ell}v(t) = v(t_\ell) - \frac{v(t_\ell)-v(t_{\ell-1})}{\tau}(t_\ell - t) - \frac{v(t_{\ell+1})-2v(t_\ell)+v(t_{\ell-1})}{\tau^2}\frac{(t_\ell-t)(t-t_{\ell-1})}{2},$$

leading to

$$(\Pi_{2,\ell}v(t))' = \delta_t v_{\ell-\frac{1}{2}} + (\delta_t^2 v_\ell)(t-t_{\ell-\frac{1}{2}}),$$

and

$$v(t) - \Pi_{2,\ell}v(t) = \frac{v'''(\gamma_\ell)}{6}(t-t_{\ell-1})(t-t_\ell)(t-t_{\ell+1}),\ t\in[t_{\ell-1},t_\ell],\gamma_\ell\in(t_{\ell-1},\ t_{\ell+1}).$$

For the last small interval $[t_k,t_{k+\sigma}]$, we use the linear interpolation in the cell $[t_k,t_{k+1}]$ at non-equidistant subdivision points $t_k, t_{k+\sigma}$, and t_{k+1}. To sum up, we have

$$\begin{aligned}
&{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}u(t)\big|_{t=t_{k+\sigma}}\\
&= \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}}{\Gamma(1-\alpha)}\sum_{\ell=1}^{k}\int_{t_{\ell-1}}^{t_\ell}\frac{v'(s)\mathrm{d}s}{(t_{k+\sigma}-s)^\alpha} + \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}}{\Gamma(1-\alpha)}\int_{t_k}^{t_{k+\sigma}}\frac{v'(s)\mathrm{d}s}{(t_{k+\sigma}-s)^\alpha}\\
&= \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}}{\Gamma(1-\alpha)}\sum_{\ell=1}^{k}\int_{t_{\ell-1}}^{t_\ell}\frac{(\Pi_{2,\ell}v(s))'\mathrm{d}s}{(t_{k+\sigma}-s)^\alpha} + \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}\delta_t v_{k+\frac{1}{2}}}{\Gamma(1-\alpha)}\int_{t_k}^{t_{k+\sigma}}\frac{\mathrm{d}s}{(t_{k+\sigma}-s)^\alpha} + T^{k+\sigma}\\
&= \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}}{\Gamma(1-\alpha)}\sum_{\ell=1}^{k}\int_{t_{\ell-1}}^{t_\ell}\frac{\delta_t v_{\ell-\frac{1}{2}} + (\delta_t^2 v_\ell)(s-t_{\ell-\frac{1}{2}})\mathrm{d}s}{(t_{k+\sigma}-s)^\alpha} + \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}\tau^{1-\alpha}}{\Gamma(2-\alpha)}\delta_t v_{k+\frac{1}{2}}\sigma^{1-\alpha} + T^{k+\sigma}\\
&= \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{\ell=1}^{k}\left(p_{k-\ell+1}^{(\alpha,\sigma)}\delta_t v_{\ell-\frac{1}{2}} + q_{k-\ell+1}^{(\alpha,\sigma)}(\delta_t v_{\ell+\frac{1}{2}} - \delta_t v_{\ell-\frac{1}{2}})\right) + p_0^{(\alpha,\sigma)}\delta_t v_{k+\frac{1}{2}}\right] + T^{k+\sigma}\\
&= \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{\ell=0}^{k}s_{k-\ell}^{(k,\alpha,\sigma)}\Big(v(t_{\ell+1})-v(t_\ell)\Big) + T^{k+\sigma},
\end{aligned}$$

where

$$p_0^{(\alpha,\sigma)} = \sigma^{1-\alpha}, \quad p_j^{(\alpha,\sigma)} = (j+\sigma)^{1-\alpha} - (j-1+\sigma)^{1-\alpha},\ j \geqslant 1,$$
$$q_j^{(\alpha,\sigma)} = \frac{1}{2-\alpha}\left[(j+\sigma)^{2-\alpha} - (j-1+\sigma)^{2-\alpha}\right] - \frac{1}{2}\left[(j+\sigma)^{1-\alpha} + (j-1+\sigma)^{1-\alpha}\right],\ j \geqslant 1.$$

And for $k=0$, we have $s_0^{(k,\alpha,\sigma)} = p_0^{(\alpha,\sigma)} = \sigma^{1-\alpha}$; for $k \geqslant 1$, we have

$$s_j^{(k,\alpha,\sigma)} = \begin{cases} p_0^{(\alpha,\sigma)} + q_1^{(\alpha,\sigma)}, & j=0, \\ p_j^{(\alpha,\sigma)} + q_{j+1}^{(\alpha,\sigma)} - q_j^{(\alpha,\sigma)}, & 1 \leqslant j \leqslant k-1, \\ p_k^{(\alpha,\sigma)} - q_k^{(\alpha,\sigma)}, & j=k. \end{cases} \tag{19}$$

Moreover, recalling $v = \mathrm{e}^{\lambda t}u$, we have

$$\begin{aligned} &{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda} u(t)\big|_{t=t_{k+\sigma}} \\ &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\Bigg[\mathrm{e}^{\lambda(t_{k+1}-t_{k+\sigma})} s_0^{(k,\alpha,\sigma)} u(t_{k+1}) - \sum_{\ell=0}^{k-1}\left(s_{k-\ell-1}^{(k,\alpha,\sigma)} - s_{k-\ell}^{(k,\alpha,\sigma)}\right)\mathrm{e}^{\lambda(t_{\ell+1}-t_{k+\sigma})} u(t_{\ell+1}) \\ &\quad - s_k^{(k,\alpha,\sigma)} \mathrm{e}^{\lambda(t_0-t_{k+\sigma})} u(t_0)\Bigg] + T^{k+\sigma} \quad := \Delta_t^{\alpha,\lambda} u(t_{k+\sigma}) + T^{k+\sigma}, \end{aligned} \tag{20}$$

where $T^{k+\sigma} = {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda} u(t)\big|_{t=t_{k+\sigma}} - \Delta_t^{\alpha,\lambda} u(t_{k+\sigma})$. Here, we call this numerical differentiation formula (20) the tempered L2-1_σ formula. If we adopt the approximate solution U^k and set

$$\overline{d}_{s,m}^{(k,\alpha,\sigma)} = s_{k-m}^{(k,\alpha,\sigma)} \mathrm{e}^{\lambda(t_m - t_{k+\sigma})},\ m = 0, 1, \cdots, k, \tag{21}$$

$$\widetilde{d}_{s,m}^{(k,\alpha,\sigma)} = s_{k-m}^{(k,\alpha,\sigma)} \mathrm{e}^{\lambda(t_{m+1} - t_{k+\sigma})},\ m = 0, 1, \cdots, k, \tag{22}$$

then the tempered L2-1_σ approximation operator $\Delta_t^{\alpha,\lambda} U^{\bar{k}}$ is equivalent to the form

$$\Delta_t^{\alpha,\lambda} U^{\bar{k}} = \frac{1}{\mu}\left[\widetilde{d}_{s,k}^{(k,\alpha,\sigma)} U^{k+1} - \sum_{\ell=0}^{k-1}\left(\overline{d}_{s,\ell+1}^{(k,\alpha,\sigma)} - \widetilde{d}_{s,\ell}^{(k,\alpha,\sigma)}\right) U^{\ell+1} - \overline{d}_{s,0}^{(k,\alpha,\sigma)} U^0\right]. \tag{23}$$

It is clear that $s_0^{(k,\alpha,\sigma)} = \sigma^{1-\alpha} > 0$ and $\overline{d}_{s,0}^{(k,\alpha,\sigma)} < s_0^{(k,\alpha,\sigma)}$, $\widetilde{d}_{s,0}^{(k,\alpha,\sigma)} > s_0^{(k,\alpha,\sigma)}$ for $k=0$. And, for $k \geqslant 1$, with the similar fashion given in [1], we can check that the properties of $\overline{d}_{s,m}^{(k,\alpha,\sigma)}$ and $\widetilde{d}_{s,m}^{(k,\alpha,\sigma)}$ $(1 \leqslant m \leqslant k)$ hold the following lemma.

Lemma 6 *For any $\alpha \in (0,1)$, $k \geqslant 1$, $s_j^{(k,\alpha,\sigma)}$ $(0 \leqslant j \leqslant k)$ defined in Eq.* (19) *and $\overline{d}_{s,m}^{(k,\alpha,\sigma)}$, $\widetilde{d}_{s,m}^{(k,\alpha,\sigma)}$ $(0 \leqslant m \leqslant k)$ defined in Eqs.* (21) *and* (22), *we have*

(i) $s_k^{(k,\alpha,\sigma)} > \frac{1-\alpha}{2}(k+\sigma)^{-\alpha} > 0$;
(ii) $s_0^{(k,\alpha,\sigma)} > s_1^{(k,\alpha,\sigma)} > s_2^{(k,\alpha,\sigma)} > \cdots > s_{k-1}^{(k,\alpha,\sigma)} > s_k^{(k,\alpha,\sigma)}$;
(iii) $\overline{d}_{s,m}^{(k,\alpha,\sigma)} > 0$, $\widetilde{d}_{s,m}^{(k,\alpha,\sigma)} > 0$;
(iv) $\overline{d}_{s,m+1}^{(k,\alpha,\sigma)} > \widetilde{d}_{s,m}^{(k,\alpha,\sigma)}$, $0 \leqslant m \leqslant k-1$;
(v) $\widetilde{d}_{s,k}^{(k,\alpha,\sigma)} > s_0^{(k,\alpha,\sigma)}$;

(vi) $\overline{d}_{s,0}^{(k,\alpha,\sigma)} < s_k^{(k,\alpha,\sigma)}$.

Lemma 7 *Suppose that* $u(t) \in C^3[0, t_{k+1}]\, (0 \leqslant k \leqslant N-1)$, *then the truncation error of the tempered* $L2$-1_σ *formula gives*

$$\begin{aligned}\left|T^{k+\sigma}\right| \leqslant & \frac{1}{3\sigma^\alpha \Gamma(1-\alpha)}\Big[\lambda^3 \max_{t_0 \leqslant t \leqslant t_{k+1}} |u(t)| + 3\lambda^2 \max_{t_0 \leqslant t \leqslant t_{k+1}} \left|u'(t)\right| \\ & + 3\lambda \max_{t_0 \leqslant t \leqslant t_{k+1}} \left|u''(t)\right| + \max_{t_0 \leqslant t \leqslant t_{k+1}} \left|u'''(t)\right|\Big]\tau^{3-\alpha} + \mathcal{O}(\tau^{3-\alpha}).\end{aligned} \tag{24}$$

Proof The proof is provided in Appendix F.

3 Implicit Schemes for One-Dimensional Time-Tempered Fractional Diffusion Equation

3.1 Derivation of Difference Schemes

For a given positive integer M, let $\{x_j\}_{j=0}^M$ be a partition of [0, L] with $x_j = jh_x$ and $h_x = L/M$. Define a mesh function $V_h = \{u | u = (u_0, u_1, \cdots, u_M)\}$, $V_h^0 = \{u | u \in V_h, u_0 = u_M = 0\}$ with the discrete L_2 norm $\|\omega^k(x)\|_2^2 = \sum_{j=1}^{M-1} h_x |\omega_j^k|^2$.

Consider the initial boundary value problem as follows:

$$\begin{cases} {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda} u(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) + f(x,t),\ x \in (0,L), t \in (0,T], \\ u(x,0) = \phi(x),\ x \in (0,L), \\ u(0,t) = \psi_1(t), u(L,t) = \psi_2(t),\ t \in (0,T], \end{cases} \tag{25}$$

where the functions f, ϕ, ψ_1, ψ_2 are given, and smooth which satisfy our numerical schemes. Denote $u_j^k = u(x_j, t_k), f_j^k = f(x_j, t_k)$. Here we consider the spatial variable is approximated by the center difference formula $\delta_x^2 u_j^k = \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h_x^2} + \mathcal{O}(h_x^2)$, and the time variable is discretized by tempered L1, L1-2, and L2-1_σ formulas, respectively. Then, the first equation of Eq. (25) at the grid points (x_j, t_k) and $(x_j, t_{k+\sigma})$ holds

$$\mathbb{D}_t^{\alpha,\lambda} u_j^k = \delta_x^2 u_j^k + f_j^k + R_j^k, \quad 1 \leqslant j \leqslant M-1, 1 \leqslant k \leqslant N, \tag{26}$$

$$\partial_t^{\alpha,\lambda} u_j^k = \delta_x^2 u_j^k + f_j^k + T_j^k, \quad 1 \leqslant j \leqslant M-1, 1 \leqslant k \leqslant N, \tag{27}$$

$$\Delta_t^{\alpha,\lambda} u_j^{\bar{k}} = \delta_x^2 u_j^{k+\sigma} + f_j^{k+\sigma} + T_j^{k+\sigma}, \quad 1 \leqslant j \leqslant M-1, 0 \leqslant k \leqslant N-1, \tag{28}$$

where $f_j^{k+\sigma} = f(x_j, t_{k+\sigma})$ and $u_j^{k+\sigma} = \left(\sigma u_j^{k+1} + (1-\sigma) u_j^k\right) + \mathcal{O}(\tau^2)$, thus the truncation error

$$|R_j^k| \leqslant C_1(\tau^{2-\alpha} + h_x^2), \quad |T_j^k| \leqslant C_2(\tau^{3-\alpha} + h_x^2), \quad |T_j^{k+\sigma}| \leqslant C_3(\tau^2 + h_x^2). \tag{29}$$

The value of the approximation solution to a function $u(x, t)$ at the grid point (x_j, t_k) is denoted by U_j^k. If the truncation error is omitted and U_j^k is adopted, then we have the difference schemes

$$\mathbb{D}_t^{\alpha,\lambda} U_j^k = \delta_x^2 U_j^k + f_j^k, \quad 1 \leqslant j \leqslant M-1, 1 \leqslant k \leqslant N, \tag{30}$$

$$\partial_t^{\alpha,\lambda} U_j^k = \delta_x^2 U_j^k + f_j^k, \quad 1 \leqslant j \leqslant M-1, 1 \leqslant k \leqslant N, \tag{31}$$

$$\Delta_t^{\alpha,\lambda} U_j^{\bar{k}} = \delta_x^2 U_j^{k+\sigma} + f_j^{k+\sigma}, \quad 1 \leqslant j \leqslant M-1, 0 \leqslant k \leqslant N-1, \tag{32}$$

where $U_j^{k+\sigma} = \sigma U_j^{k+1} + (1-\sigma) U_j^k$.

3.2 Stability Analysis

In this subsection, we will analyze the stability of numerical schemes (30) and (32) using the Fourier analysis [3, 6, 20]. Let $\tilde{U}_j^k$ be the approximate solution of Eqs. (30) and (32). Then, define $\varepsilon_j^k = U_j^k - \tilde{U}_j^k$, $1 \leqslant k \leqslant N$, $1 \leqslant j \leqslant M-1$, and the error vector $\varepsilon^k = [\varepsilon_1^k, \varepsilon_2^k, \varepsilon_3^k, \cdots, \varepsilon_{M-1}^k]^{\mathrm{T}} \in V_h^0$. To begin the error analysis, we define the grid function

$$\varepsilon^k(x) = \begin{cases} \varepsilon_j^k, & x_j - \dfrac{h_x}{2} < x \leqslant x_j + \dfrac{h_x}{2}, \ 1 \leqslant j \leqslant M-1, \\ 0, & 0 \leqslant x \leqslant \dfrac{h_x}{2} \ \text{and} \ L - \dfrac{h_x}{2} < x \leqslant L. \end{cases}$$

Therefore, $\varepsilon^k(x)$ can be expanded by the Fourier series

$$\varepsilon^k(x) = \sum_{m=-\infty}^{+\infty} \xi_m^k \mathrm{e}^{\mathrm{i}2\pi m x/L}, k = 1, 2, \cdots, N,$$

where $\xi_m^k = \frac{1}{L}\int_0^L \varepsilon^k(x) \mathrm{e}^{-\mathrm{i}2\pi m x/L} \mathrm{d}x, \mathrm{i} = \sqrt{-1}$. According to the definition of the discrete L_2 norm and Parseval's equality, there exists

$$\|\varepsilon^k\|_2^2 = \sum_{j=1}^{M-1} h_x |\varepsilon_j^k|^2 = \int_0^L |\varepsilon_j^k|^2 \mathrm{d}x = \sum_{n=-\infty}^{+\infty} |\xi_m^k|^2. \tag{33}$$

From the numerical schemes (30) and (32), we get the following error equations, respectively:

$$\widetilde{d}_{a,k}^{(\alpha)} \varepsilon_j^k - \mu \delta_x^2 \varepsilon_j^k = \sum_{\ell=1}^{k-1} \left(\overline{d}_{a,\ell+1}^{(\alpha)} - \widetilde{d}_{a,\ell}^{(\alpha)} \right) \varepsilon_j^\ell + \overline{d}_{a,1}^{(\alpha)} \varepsilon_j^0, \tag{34}$$

$$\widetilde{d}_{s,k}^{(k,\alpha,\sigma)} \varepsilon_j^{k+1} - \mu\sigma \delta_x^2 \varepsilon_j^{k+1} = \sum_{\ell=0}^{k-1} \left(\overline{d}_{s,\ell+1}^{(k,\alpha,\sigma)} - \widetilde{d}_{s,\ell}^{(k,\alpha,\sigma)} \right) \varepsilon_j^{\ell+1} + \overline{d}_{s,0}^{(k,\alpha,\sigma)} \varepsilon_j^0 + \mu(1-\sigma) \delta_x^2 \varepsilon_j^k. \tag{35}$$

Suppose that the solutions of Eqs. (34) and (35) can be written as the following form:

$$\varepsilon_j^k = \xi_m^k e^{i\beta j h_x},\ \beta = \frac{2\pi m}{L}. \tag{36}$$

The main idea of the Fourier analysis method is to check the propagation of the error ε_j^0 produced in the initial condition. Since the information is exactly given in boundary conditions, we have $\varepsilon_0^k = \varepsilon_M^k = 0$. Now, we check the propagation of the error with the development of time steps. For the stability of the difference scheme (30), we have

Theorem 1 *The difference scheme* (30) *is unconditionally stable, and we can derive that*

$$\|\varepsilon^k\|_2 \leqslant \|\varepsilon^0\|_2,\ 1 \leqslant k \leqslant N.$$

Proof Inserting Eq. (36) into Eq. (34), and combining with the Euler formula $e^{i\beta h} = \cos(\beta h) + i\sin(\beta h)$, we arrive at

$$\left[\widetilde{d}_{a,k}^{(\alpha)} + \mu\frac{4}{h_x^2}\sin^2\left(\frac{\beta h_x}{2}\right)\right]\xi_m^k = \sum_{\ell=1}^{k-1}\left(\overline{d}_{a,\ell+1}^{(\alpha)} - \widetilde{d}_{a,\ell}^{(\alpha)}\right)\xi_m^\ell + \overline{d}_{a,1}^{(\alpha)}\xi_m^0, \tag{37}$$

where $\widetilde{d}_{a,k}^{(\alpha)} = a_0^{(\alpha)} = 1$.

Suppose that $\xi_m^k (k = 1, 2, \cdots, N)$ are the solutions of Eq. (37), we will prove the following inequality by mathematical induction:

$$|\xi_m^k| \leqslant |\xi_m^0|, k = 1, 2, \cdots, N. \tag{38}$$

The inequality (38) holds obviously for $k = 1$. Now, we suppose that the inequality (38) is true for $k = 1, 2, \cdots, n-1$. Thus, for $k = n$, using Eq. (37) and (vii), (viii) of Lemma 1, we have

$$\left[1 + \mu\frac{4}{h_x^2}\sin^2\left(\frac{\beta h_x}{2}\right)\right]|\xi_m^n| \leqslant \sum_{\ell=1}^{n-1}\left(a_{n-\ell-1}^{(\alpha)} - a_{n-\ell}^{(\alpha)}\right)|\xi_m^\ell| + a_{n-1}^{(\alpha)}|\xi_m^0| \leqslant |\xi_m^0|,$$

and it yeilds

$$|\xi_m^n| \leqslant |\xi_m^0|,\quad \forall n = 1, 2, \cdots, N,$$

where the relation $\sum_{\ell=1}^{n-1}(a_{n-\ell-1}^{(\alpha)} - a_{n-\ell}^{(\alpha)}) + a_{n-1}^{(\alpha)} = 1$ is used. Applying the definition (33) and the inequality (38), we get

$$\|\varepsilon^k\|_2^2 = \sum_{m=-\infty}^{+\infty}|\xi_m^k|^2 \leqslant \sum_{m=-\infty}^{+\infty}|\xi_m^0|^2 = \|\varepsilon^0\|_2^2,$$

which means the difference scheme (30) is unconditionally stable.

With the similar approach, we have the stability of the difference scheme (32).

Theorem 2 *The difference scheme* (32) *is unconditionally stable, and we can derive that*

$$\|\varepsilon^k\|_2 \leqslant \|\varepsilon^0\|_2,\ 1 \leqslant k \leqslant N.$$

Proof From Eq. (35), we have

$$
\begin{aligned}
&\left(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)}+\frac{4\mu\sigma}{h_x^2}\sin^2\left(\frac{\beta h_x}{2}\right)\right)\xi_m^{k+1}\\
&=\sum_{\ell=0}^{k-1}\left(\overline{d}_{s,\ell+1}^{(k,\alpha,\sigma)}-\widetilde{d}_{s,\ell}^{(k,\alpha,\sigma)}\right)\xi_m^{\ell+1}+\overline{d}_{s,0}^{(k,\alpha,\sigma)}\xi_m^0-\left(\frac{4(1-\sigma)\mu}{h^2}\sin^2\left(\frac{\beta h_x}{2}\right)\right)\xi_m^k,
\end{aligned}\tag{39}
$$

where $\overline{d}_{s,k}^{(k,\alpha,\sigma)}=s_0^{(k,\alpha,\sigma)}\mathrm{e}^{\lambda(t_k-t_{k+\sigma})}$, $s_0^{(k,\alpha,\sigma)}=\sigma^{1-\alpha}>0\,(k=0)$, and $s_0^{(k,\alpha,\sigma)}=p_0^{(\alpha,\sigma)}+q_1^{(\alpha,\sigma)}>0\,(k\geqslant 1)$.

Suppose that ξ_m^{k+1} $(k=0,1,\cdots,N-1)$ are the solutions of the numerical scheme (39), we shall prove that

$$
|\xi_m^{k+1}|\leqslant|\xi_m^0|, k=0,1,\cdots,N-1. \tag{40}
$$

For $k=0$, it holds

$$
\left(s_0^{(k,\alpha,\sigma)}+\frac{4\mu\sigma}{h_x^2}\sin^2\left(\frac{\beta h_x}{2}\right)\right)\xi_m^1\leqslant\left(s_0^{(k,\alpha,\sigma)}-\frac{4(1-\sigma)\mu}{h_x^2}\sin^2\left(\frac{\beta h}{2}\right)\right)\xi_m^0\leqslant s_0^{(k,\alpha,\sigma)}\xi_m^0,
$$

and hence

$$
|\xi_m^1|\leqslant|\xi_m^0|.
$$

Now, we suppose that Eq. (40) is true for $k=0,1,\cdots,n-1$. Then, according to Eq. (39), for $k=n$, we have

$$
\begin{aligned}
&\left(s_0^{(k,\alpha,\sigma)}+\frac{4\mu\sigma}{h_x^2}\sin^2\left(\frac{\beta h_x}{2}\right)\right)|\xi_m^{n+1}|\\
&\leqslant\left[\sum_{\ell=0}^{n-1}\left(s_{n-\ell-1}^{(k,\alpha,\sigma)}-s_{n-\ell}^{(k,\alpha,\sigma)}\right)+s_k^{(k,\alpha,\sigma)}-\frac{4(1-\sigma)\mu}{h_x^2}\sin^2\left(\frac{\beta h_x}{2}\right)\right]|\xi_m^0|\leqslant s_0^{(k,\alpha,\sigma)}|\xi_m^0|.
\end{aligned}
$$

In view of the relation $\sum\limits_{\ell=0}^{n-1}\left(s_{n-\ell-1}^{(k,\alpha,\sigma)}-s_{n-\ell}^{(k,\alpha,\sigma)}\right)=s_0^{(k,\alpha,\sigma)}-s_n^{(k,\alpha,\sigma)}$, by simple argument, there exists

$$
|\xi_m^{n+1}|\leqslant\frac{1}{1+\frac{4\mu\sigma}{s_0^{(k,\alpha,\sigma)}h_x^2}\sin^2(\frac{\beta h_x}{2})}|\xi_m^0|\leqslant|\xi_m^0|,\quad\forall n=0,1,\cdots,N-1.
$$

Finally, using Eqs. (33) and (40), we obtain the difference scheme (32) is unconditionally stable.

3.3 Convergence Analysis

In this subsection, we are dedicated in studying the convergence order of the difference schemes. As an example, we give the complete proof with the tempered L1 approximate and the proofs of other difference schemes are omitted with respect to the length of this paper. Define the truncation error $\rho_j^k=u(x_j,t_k)-U_j^k=u_j^k-U_j^k,\ 1\leqslant k\leqslant N,\ 1\leqslant j\leqslant M-1$, and let the error vectors

$$\rho^k = [\rho_1^k, \rho_2^k, \rho_3^k, \cdots, \rho_{M-1}^k]^{\mathrm{T}}, \quad R^k = [R_1^k, R_2^k, R_3^k, \cdots, R_{M-1}^k]^{\mathrm{T}}, \quad 1 \leqslant k \leqslant N.$$

Set the grid functions

$$\rho^k(x) = \begin{cases} \rho_j^k, & x_j - \dfrac{h_x}{2} < x \leqslant x_j + \dfrac{h_x}{2},\ 1 \leqslant j \leqslant M-1, \\ 0, & 0 \leqslant x \leqslant \dfrac{h_x}{2} \text{ and } L - \dfrac{h_x}{2} < x \leqslant L, \end{cases}$$

$$R^k(x) = \begin{cases} R_j^k, & x_j - \dfrac{h_x}{2} < x \leqslant x_j + \dfrac{h_x}{2},\ 1 \leqslant j \leqslant M-1, \\ 0, & 0 \leqslant x \leqslant \dfrac{h_x}{2} \text{ and } L - \dfrac{h_x}{2} < x \leqslant L. \end{cases}$$

And $\rho^k(x), R^k(x)$ have the following Fourier series expansions:

$$\rho^k(x) = \sum_{m=-\infty}^{+\infty} \eta_m^k \mathrm{e}^{\mathrm{i}2\pi mx/L}, k = 1, 2, \cdots, N,$$

$$R^k(x) = \sum_{m=-\infty}^{+\infty} \gamma_m^k \mathrm{e}^{\mathrm{i}2\pi mx/L}, k = 1, 2, \cdots, N,$$

where $\eta_m^k = \frac{1}{L}\int_0^L \rho^k(x)\mathrm{e}^{-\mathrm{i}2\pi mx/L}\mathrm{d}x, \gamma_m^k = \frac{1}{L}\int_0^L R^k(x)\mathrm{e}^{-\mathrm{i}2\pi mx/L}\mathrm{d}x$. From the definition of the L_2 norm and Parseval's equality, we can obtain

$$\|\rho^k(x)\|_2^2 = \sum_{j=1}^{M-1} h_x|\rho_j^k|^2 = \int_0^L |\rho_j^k|^2\mathrm{d}x = \sum_{m=-\infty}^{+\infty} |\eta_m^k|^2, \tag{41}$$

$$\|R^k(x)\|_2^2 = \sum_{j=1}^{M-1} h_x|R_j^k|^2 = \int_0^L |R_j^k|^2\mathrm{d}x = \sum_{m=-\infty}^{+\infty} |\gamma_m^k|^2. \tag{42}$$

Subtracting numerical schemes (26) from Eq. (30) and multiplying h_x^2 on both sides of the final error equation, we arrive at

$$\left(\frac{h_x^2}{\mu}\widetilde{d}_{a,k}^{(\alpha)} + 2\right)\rho_j^k - \rho_{j+1}^k - \rho_{j-1}^k = \frac{h_x^2}{\mu}\left[\sum_{\ell=1}^{k-1}\left(\overline{d}_{a,\ell+1}^{(\alpha)} - \widetilde{d}_{a,\ell}^{(\alpha)}\right)\rho_j^\ell + \overline{d}_{a,1}^{(\alpha)}\rho_j^0\right] + h_x^2 R_j^k, \tag{43}$$

where $\rho_j^0 = 0, 1 \leqslant j \leqslant M-1$; $\rho_0^k = \rho_M^k, 1 \leqslant k \leqslant N-1$. Assume that ρ_j^k and R_j^k are

$$\rho_j^k = \eta_m^k \mathrm{e}^{\mathrm{i}\beta jh_x}, \quad R_j^k = \gamma_m^k \mathrm{e}^{\mathrm{i}\beta jh_x}, \ \ \beta = \frac{2\pi m}{L}. \tag{44}$$

Noticing that $\rho_j^0 = 0$, we obtain $\eta_m^0 = 0$. And we can deduce the following inequality from Eq. (29) and the first equality of Eq. (42):

$$\|R^k\|_2 \leqslant C_1\sqrt{L}(\tau^{2-\alpha} + h_x^2). \tag{45}$$

And from the convergence of the last series of Eq. (42), there exists a positive constant $\hat{C}$ such that [20]

$$|\gamma_m^k| \leqslant \hat{C}\tau|\gamma_m^1|, \ \ k = 1, 2, \cdots, N. \tag{46}$$

Lemma 8 *There is the following relationship between* η_m^k *and* γ_m^1*:*

$$|\eta_m^k| \leqslant (1+\tau)^k|\gamma_m^1|, \ \ k = 1, 2, \cdots, N. \tag{47}$$

Proof Inserting Eq. (44) into Eq. (43), we arrive at

$$\left[\frac{h_x^2}{\mu} + 4\sin^2\left(\frac{\beta h_x}{2}\right)\right]\eta_m^k = \frac{h_x^2}{\mu}\sum_{\ell=1}^{k-1}\left(\overline{d}_{a,\ell+1}^{(\alpha)} - \widetilde{d}_{a,\ell}^{(\alpha)}\right)\eta_m^\ell + h_x^2\gamma_m^k. \tag{48}$$

As for $k = 1$, there is

$$\left[\frac{h_x^2}{\mu} + 4\sin^2\left(\frac{\beta h_x}{2}\right)\right]\eta_m^1 = h_x^2\gamma_m^1,$$

and we can easily come to

$$|\eta_m^1| \leqslant \mu|\gamma_m^1| \leqslant |\gamma_m^1| \leqslant \hat{C}(1+\tau)|\gamma_m^1|,$$

where $\mu = \tau^\alpha\Gamma(2-\alpha) < 1$ when $0 < \alpha, \tau < 1$. We suppose that the inequality (47) is true for $k = 1, 2, \cdots, n-1$. For $k = n$, we deduce the following estimate from Eqs. (48) and (46):

$$\left[\frac{h_x^2}{\mu} + 4\sin^2\left(\frac{\beta h_x}{2}\right)\right]|\eta_m^n| \leqslant \frac{h_x^2}{\mu}\sum_{\ell=1}^{n-1}\left(\overline{d}_{a,\ell+1}^{(\alpha)} - \widetilde{d}_{a,\ell}^{(\alpha)}\right)|\eta_m^\ell| + h_x^2|\gamma_m^n|,$$

i.e.,

$$\begin{aligned}
|\eta_m^n| &\leqslant \sum_{\ell=1}^{n-1}\left(\overline{d}_{a,\ell+1}^{(\alpha)} - \widetilde{d}_{a,\ell}^{(\alpha)}\right)|\eta_m^\ell| + \mu|\gamma_m^n| \\
&\leqslant (1+\tau)^{n-1}\hat{C}|\gamma_m^1|\sum_{\ell=1}^{n-1}\left(\overline{d}_{a,\ell+1}^{(\alpha)} - \widetilde{d}_{a,\ell}^{(\alpha)}\right) + \hat{C}\tau|\gamma_m^1| \\
&\leqslant (1+\tau)^n\hat{C}|\gamma_m^1|.
\end{aligned}$$

Therefore, this lemma is proved.

Let $C_{x,t}^{4,2}$ denote the space of function $u(x,t)$ which satisfies $u(x,t) \in C^2[0,t_k] \cap C^4[0,L]$. Then, there exists the following convergence theorem.

Theorem 3 *Suppose* $u(x,t) \in C_{x,t}^{4,2}$*, then the difference scheme with the tempered* $L1$ *formula (30) is convergent with the convergence order* $\mathcal{O}(\tau^{2-\alpha} + h_x^2)$*.*

Proof Applying the definition (41)–(42), Eq. (45), and Lemma 8, we arrive at

$$\|\rho^k\|_2 \leqslant (1+\tau)^k\hat{C}\|R^1\|_2 \leqslant \mathrm{e}^{k\tau}C_1\hat{C}\sqrt{L}(\tau^{2-\alpha} + h_x^2) \leqslant \tilde{C}(\tau^{2-\alpha} + h_x^2),$$

where $\tilde{C} = \mathrm{e}^T C_1\hat{C}\sqrt{L}$. Therefore, the difference scheme (30) is L_2 convergent.

The proof of the convergence of the difference scheme with the tempered L2-1_σ formula (32) is similar to Theorem 3, so there exists the following result.

Theorem 4 *Suppose $u(x,t) \in C^{4,3}_{x,t}$, then the difference scheme with the tempered L2-1_σ formula (32) is convergent with the convergence order $\mathcal{O}(\tau^2 + h_x^2)$.*

4 Implicit ADI Schemes for Two-Dimensional Tempered Fractional Diffusion Equation

In this section, we will discretize the two-dimensional diffusion equation using tempered L1, L1-2, and L2-1_σ formulas. From the point of view of computation, we construct ADI schemes by introducing different terms of mixed derivatives similar to [42]. Consider the following two-dimensional initial boundary value problem:

$$\begin{cases} {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda} u(x,y,t) = \dfrac{\partial^2 u}{\partial x^2}(x,y,t) + \dfrac{\partial^2 u}{\partial y^2}(x,y,t) + f(x,y,t),\ (x,y) \in \Omega, t \in (0,T], \\ u(x,y,0) = \phi(x,y),\ (x,y) \in \bar{\Omega}, \\ u(x,y,t) = \psi(x,y,t),\ (x,y) \in \partial\Omega, t \in (0,T], \end{cases} \tag{49}$$

where $\Omega = (0,L) \times (0,L)$ and $\partial\Omega$ is the boundary of Ω. Here, we also assume that functions f, ϕ, ψ_1, and ψ_2 are smooth enough for our numerical schemes.

4.1 Construction of Tempered ADI Schemes

For two positive integers M_1 and M_2, denote $h_x = L/M_1, h_y = L/M_2$ with $x_j = jh_x, y_m = mh_y$. Let $\bar{\Omega}_h = \{(x_j, y_m) | 0 \leqslant j \leqslant M_1, 0 \leqslant m \leqslant M_2\}$, $\Omega_h = \bar{\Omega}_h \cap \Omega$, $\Omega_\tau = \{t_k | 0 \leqslant k \leqslant N\}$.

4.1.1 Tempered L1-ADI

For any grid function in $\bar{\Omega}_h \times \Omega_\tau$, we define the spatial difference quotients

$$\delta_x v^k_{j+\frac{1}{2},m} = \frac{v^k_{j+1,m} - v^k_{j,m}}{h_x}, \quad \delta_x^2 v^k_{j,m} = \frac{\delta_x v^k_{j+\frac{1}{2},m} - \delta_x v^k_{j-\frac{1}{2},m}}{h_x},$$
$$\delta_x^2 \delta_y^2 v^k_{j,m} = \frac{\delta_x(\delta_y^2 v^k_{j+\frac{1}{2},m}) - \delta_x(\delta_y^2 v^k_{j-\frac{1}{2},m})}{h_x}.$$

And denote the grid functions in $\bar{\Omega}_h \times \Omega_\tau$, $u^k_{j,m} = u(x_j, y_m, t_k)$, $f^k_{j,m} = f(x_j, y_m, t_k)$, $(x_j, y_m) \in \Omega_h, 0 \leqslant k \leqslant N$. Using the tempered L1 formula (5) to approximate the time derivative in the first equation of Eq. (49) at the grid point (x_j, y_m, t_k), we have

$$\mathbb{D}_t^{\alpha,\lambda} u^k_{j,m} = \delta_x^2 u^k_{j,m} + \delta_y^2 u^k_{j,m} + f^k_{j,m} + \hat{R}^k_{j,m},\ (x_j, y_m) \in \Omega_h, 1 \leqslant k \leqslant N. \tag{50}$$

Using Lemma 2 and the Taylor expansion formula, we get

$$\hat{R}^k_{j,m} = \mathcal{O}(\tau^{2-\alpha} + h_x^2 + h_y^2). \tag{51}$$

Moreover, add the mixed derivative term $\mu^2 \mathbb{D}_t^{\alpha,\lambda}\left(\delta_x^2\delta_y^2 u^k_{j,m}\right)$ to Eq. (50), and there exists

$$\mathbb{D}_t^{\alpha,\lambda}\left(u^k_{j,m} + \mu^2\delta_x^2\delta_y^2 u^k_{j,m}\right) = \delta_x^2 u^k_{j,m} + \delta_y^2 u^k_{j,m} + f^k_{j,m} + (\hat{R}_1)^k_{j,m}, \tag{52}$$

where $(\hat{R}_1)^k_{j,m} = \hat{R}^k_{j,m} + \mu^2\mathbb{D}_t^{\alpha,\lambda}\left(\delta_x^2\delta_y^2 u^k_{j,m}\right)$, $(x_j, y_m) \in \Omega_h, 1 \leqslant k \leqslant N$.

Lemma 9 *For the truncation error given in Eq.* (52), *there is*

$$|(\hat{R}_1)^k_{j,m}| = \mathcal{O}(\tau^{\min\{2\alpha, 2-\alpha\}} + h_x^2 + h_y^2). \tag{53}$$

Proof If $v(x) \in C^2[x-h, x+h]$, the Taylor's formula with the integral remainder implies

$$\frac{v(x+h) - 2v(x) + v(x-h)}{h^2} = \int_0^1 [v''(x+\xi h) + v''(x-\xi h)](1-\xi)\mathrm{d}\xi.$$

Setting $\delta_x^2\delta_y^2 u^k_{j,m} = P(x_j, y_m, t_k)$, we have

$$\begin{aligned} P(x,y,t) = \int_0^1\int_0^1 \Big[&\frac{\partial^4 u}{\partial x^2\partial y^2}(x-\xi h_x, y-\zeta h_y, t) + \frac{\partial^4 u}{\partial x^2\partial y^2}(x+\xi h_x, y-\zeta h_y, t) \\ &+ \frac{\partial^4 u}{\partial x^2\partial y^2}(x-\xi h_x, y+\zeta h_y, t) + \frac{\partial^4 u}{\partial x^2\partial y^2}(x+\xi h_x, y+\zeta h_y, t)\Big](1-\xi)(1-\zeta)\mathrm{d}\xi\mathrm{d}\zeta, \end{aligned}$$

and hence

$$\mu^2\mathbb{D}_t^{\alpha,\lambda}\left(\delta_x^2\delta_y^2 u^k_{j,m}\right) = \mu^2\left[{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}P(x_j, y_m, t_k) + \mathcal{O}(\tau^{2-\alpha})\right] = \mathcal{O}(\tau^{2\alpha}). \tag{54}$$

In addition, combining with Eq. (51), Eq. (53) immediately follows.

Letting $U^k_{j,m}$ be the approximation solution of $u(x_j, y_m, t_k)$ and omitting the truncation error in Eq. (52), we have

$$\begin{cases} \mathbb{D}_t^{\alpha,\lambda}\left(U^k_{j,m} + \mu^2\delta_x^2\delta_y^2 U^k_{j,m}\right) = \delta_x^2 U^k_{j,m} + \delta_y^2 U^k_{j,m} + f^k_{j,m}, \ (x_i, y_m) \in \Omega_h, 1 \leqslant k \leqslant N, \\ U^0_{j,m} = \phi(x_j, y_m), \quad (x_j, y_m) \in \bar{\Omega}_h, \\ U^k_{j,m} = \psi(x_j, y_m, t_k), \ (x_j, y_m) \in \partial\Omega_h, 1 \leqslant k \leqslant N. \end{cases} \tag{55}$$

Because $\mathbb{D}_t^{\alpha,\lambda}\left(\delta_x^2\delta_y^2 u^k_{j,m}\right)$ is the tempered L1 operator of ${}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}P(x_j, y_m, t_k)$, we call Eq. (55) the tempered L1-ADI scheme. According to the expression of the tempered L1 approximation operator (9) and noting that $\widetilde{d}^{(\alpha)}_{a,k} = 1$, multiplying μ on both sides of Eq. (55) and the first equation of Eq. (55) yields

$$\begin{aligned} &(I - \mu\delta_x^2)(I - \mu\delta_y^2)U^k_{j,m} \\ &= \sum_{\ell=1}^{k-1}\left(\overline{d}^{(\alpha)}_{a,\ell+1} - \widetilde{d}^{(\alpha)}_{a,\ell}\right)\left(U^\ell_{j,m} + \mu^2\delta_x^2\delta_y^2 U^\ell_{j,m}\right) + \overline{d}^{(\alpha)}_{a,1}\left(U^0_{j,m} + \mu^2\delta_x^2\delta_y^2 U^0_{j,m}\right) + \mu f^k_{j,m}, \end{aligned} \tag{56}$$

where I is the identical operator. Let $U^*_{j,m} = (I - \mu\delta_y^2)U^k_{j,m}$, $0 \leqslant j \leqslant M_1$, $1 \leqslant m \leqslant M_2 - 1$. Then, we can rewrite the ADI scheme (55) as the following two steps.

Step 1 For fixed $m = 1, 2, \cdots, M_2 - 1$, using

$$\begin{cases} (I - \mu\delta_x^2)U^*_{j,m} = \sum_{\ell=1}^{k-1} \left(\overline{d}^{(\alpha)}_{a,\ell+1} - \widetilde{d}^{(\alpha)}_{a,\ell}\right)\left(U^\ell_{j,m} + \mu^2\delta_x^2\delta_y^2 U^\ell_{j,m}\right) \\ \qquad\qquad + \overline{d}^{(\alpha)}_{a,1}\left(U^0_{j,m} + \mu^2\delta_x^2\delta_y^2 U^0_{j,m}\right) + \mu f^k_{j,m}, \quad 1 \leqslant j \leqslant M_1 - 1, \\ U^*_{0,m} = (I - \mu\delta_y^2)U^k_{0,m}, \quad U^*_{M_1,m} = (I - \mu\delta_y^2)U^k_{M_1,m}, \end{cases}$$

we get the intermediate variables $\{U^*_{j,m}\}$.

Step 2 For fixed $j = 1, 2, \cdots, M_1 - 1$, using

$$\begin{cases} (I - \mu\delta_y^2)U^k_{j,m} = U^*_{j,m}, \quad 1 \leqslant m \leqslant M_2 - 1, \\ U^k_{j,0} = \psi(x_j, y_0, t_k), \quad U^k_{j,M_2} = \psi(x_j, y_{_{M_2}}, t_k), \end{cases}$$

we get the numerical solutions $\{U^k_{j,m}\}, 0 \leqslant k \leqslant N$.

4.1.2 Tempered L1-2-ADI

If the time fractional derivative in problem (49) is approximated by the L1-2 formula (12), we have

$$\partial_t^{\alpha,\lambda} u^k_{j,m} = \delta_x^2 u^k_{j,m} + \delta_y^2 u^k_{j,m} + f^k_{j,m} + \tilde{R}^k_{j,m}, \ (x_j, y_m) \in \Omega_h, 1 \leqslant k \leqslant N. \tag{57}$$

With the help of Lemma 5 and the Taylor expansion, we have

$$\tilde{R}^k_{j,m} = \mathcal{O}(\tau^{3-\alpha} + h_x^2 + h_y^2).$$

Adding the mixed derivatives term $\mu^2(\widetilde{d}^{(k,\alpha)}_{c,k})^{-2}\partial_t^{\alpha,\lambda}\left(\delta_x^2\delta_y^2 u^k_{j,m}\right)$ to the both sides of Eq. (57), and noticing the fact that $\widetilde{d}^{(k,\alpha)}_{c,k} = c_0^{(k,\alpha)}$, there is

$$\partial_t^{\alpha,\lambda}\left(u^k_{j,m} + \mu^2(c_0^{(k,\alpha)})^{-2}\delta_x^2\delta_y^2 u^k_{j,m}\right) = \delta_x^2 u^k_{j,m} + \delta_y^2 u^k_{j,m} + f^k_{j,m} + (\hat{R}_2)^k_{j,m}, \tag{58}$$

where $(\hat{R}_2)^k_{j,m} = \tilde{R}^k_{j,m} + \mu^2(c_0^{(k,\alpha)})^{-2}\partial_t^{\alpha,\lambda}\left(\delta_x^2\delta_y^2 u^k_{j,m}\right)$, $(x_j, y_m) \in \Omega_h, 1 \leqslant k \leqslant N$. Recalling the truncation error of the small term (54), we deduce that

$$|(\hat{R}_2)^k_{j,m}| = \mathcal{O}(\tau^{2\alpha} + h_x^2 + h_y^2). \tag{59}$$

Omitting the truncation error in Eq. (58), we have the tempered L1-2-ADI scheme

$$\begin{cases} \partial_t^{\alpha,\lambda}\left(U_{j,m}^k + \mu^2 (c_0^{(k,\alpha)})^{-2}\delta_x^2\delta_y^2 U_{j,m}^k\right) = \delta_x^2 U_{j,m}^k + \delta_y^2 U_{j,m}^k + f_{j,m}^k,\ (x_i, y_m) \in \Omega_h, 1 \leqslant k \leqslant N, \\ U_{j,m}^0 = \phi(x_j, y_m), (x_j, y_m) \in \bar{\Omega}_h, \\ U_{j,m}^k = \psi(x_j, y_m, t_k),\ (x_i, y_m) \in \partial\Omega_h, 1 \leqslant k \leqslant N, \end{cases}$$

which can be rewritten as

$$\begin{aligned} &\left(I - \mu(c_0^{(k,\alpha)})^{-1}\delta_x^2\right)\left(I - \mu(c_0^{(k,\alpha)})^{-1}\delta_y^2\right)U_{j,m}^k \\ &= (c_0^{(k,\alpha)})^{-1}\Big[\sum_{\ell=1}^{k-1}\left(\overline{d}_{c,\ell+1}^{(k,\alpha)} - \widetilde{d}_{c,\ell}^{(k,\alpha)}\right)\left(U_{j,m}^\ell + \mu^2(c_0^{(k,\alpha)})^{-2}\delta_x^2\delta_y^2 U_{j,m}^\ell\right) \\ &\quad + \overline{d}_{c,1}^{(k,\alpha)}\left(U_{j,m}^0 + \mu^2(c_0^{(k,\alpha)})^{-2}\delta_x^2\delta_y^2 U_{j,m}^0\right) + \mu f_{j,m}^k\Big]. \end{aligned} \tag{60}$$

If denote $U_{j,m}^* = (I - \mu(c_0^{(k,\alpha)})^{-1}\delta_y^2)U_{j,m}^k,\ 0 \leqslant j \leqslant M_1,\ 1 \leqslant m \leqslant M_2 - 1$, then, for the view of computation, it can follow the procedure below.

Step 1 For fixed $m = 1, 2, \cdots, M_2 - 1$, computing

$$\begin{cases} \left(I - \mu(c_0^{(k,\alpha)})^{-1}\delta_x^2\right)U_{j,m}^* = (c_0^{(k,\alpha)})^{-1}\Big[\overline{d}_{c,1}^{(k,\alpha)}\left(U_{j,m}^0 + \mu^2(c_0^{(k,\alpha)})^{-2}\delta_x^2\delta_y^2 U_{j,m}^0\right) \\ \qquad + \sum_{\ell=1}^{k-1}\left(\overline{d}_{c,\ell+1}^{(k,\alpha)} - \widetilde{d}_{c,\ell}^{(k,\alpha)}\right)\left(U_{j,m}^\ell + \mu^2(c_0^{(k,\alpha)})^{-2}\delta_x^2\delta_y^2 U_{j,m}^\ell\right) + \mu f_{j,m}^k\Big],\ 1 \leqslant j \leqslant M_1 - 1, \\ U_{0,m}^* = \left(I - \mu(c_0^{(k,\alpha)})^{-1}\delta_y^2\right)U_{0,m}^k, \quad U_{M_1,m}^* = \left(I - \mu(c_0^{(k,\alpha)})^{-1}\delta_y^2\right)U_{M_1,m}^k, \end{cases}$$

we can obtain $\{U_{j,m}^*\}$.

Step 2 For fixed $j = 1, 2, \cdots, M_1 - 1$, applying

$$\begin{cases} (I - \mu(c_0^{(k,\alpha)})^{-1}\delta_y^2)U_{j,m}^k = U_{j,m}^*,\ 1 \leqslant m \leqslant M_2 - 1, \\ U_{j,0}^k = \psi(x_j, y_0, t_k), \quad U_{j,M_2}^k = \psi(x_j, y_{M_2}, t_k), \end{cases}$$

we obtain the numerical solutions $\{U_{j,m}^k\}, 0 \leqslant k \leqslant N$.

4.1.3 Tempered L2-1_σ-ADI

Unlike above two numerical ADI schemes, we consider the numerical method for the first equation in Eq. (49) at the non-integer grid point $(x_j, y_m, t_{k+\sigma})$

$${}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda} u_{j,m}^{k+\sigma} = \frac{\partial^2}{\partial x^2}u_{j,m}^{k+\sigma} + \frac{\partial^2}{\partial y^2}u_{j,m}^{k+\sigma} + f_{j,m}^{k+\sigma},\ (x_j, y_m) \in \Omega_h, 0 \leqslant k \leqslant N-1,$$

where $u_{j,m}^{k+\sigma} = u(x_j, y_m, t_{k+\sigma})$. Combining with the tempered L2-1_σ formula (20), we have

$$\Delta_t^{\alpha,\lambda} u_{j,m}^{\bar{k}} = \delta_x^2 u_{j,m}^{k+\sigma} + \delta_y^2 u_{j,m}^{k+\sigma} + f_{j,m}^{k+\sigma} + \hat{R}_{j,m}^{k+\sigma},\ (x_j, y_m) \in \Omega_h, 0 \leqslant k \leqslant N-1. \tag{61}$$

Using the Taylor expansion, we can check that

$$u_{j,m}^{k+\sigma} = \sigma u_{j,m}^{k+1} + (1-\sigma)u_{j,m}^{k} + \mathcal{O}(\tau^2),$$

which means $\hat{R}_{j,m}^{k+\sigma} = \mathcal{O}(\tau^2 + h_x^2 + h_y^2)$.

With the similar method of the tempered L1-ADI and tempered L1-2-ADI schemes, we add another term $\mu^2\sigma^2(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-2}\Delta_t^{\alpha,\lambda}\left(\delta_x^2\delta_y^2 u_{j,m}^{k+\sigma}\right)$ to Eq. (61), which yields

$$\begin{aligned}
&\Delta_t^{\alpha,\lambda}\left(u_{j,m}^{\bar{k}} + \mu^2\sigma^2(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-2}\delta_x^2\delta_y^2 u_{j,m}^{k+\sigma}\right) = \delta_x^2 u_{j,m}^{k+\sigma} + \delta_y^2 u_{j,m}^{k+\sigma} + f_{j,m}^{k+\sigma} + (\hat{R}_3)_{j,m}^{k+\sigma},\\
&(x_j, y_m) \in \Omega_h, 0 \leqslant k \leqslant N-1,
\end{aligned} \tag{62}$$

where $(\hat{R}_3)_{j,m}^{k+\sigma} = \hat{R}_{j,m}^{k+\sigma} + \mu^2\sigma^2(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-2}\Delta_t^{\alpha,\lambda}\left(\delta_x^2\delta_y^2 u_{j,m}^{k+\sigma}\right)$ with the truncation error

$$|(\hat{R}_3)_{j,m}^{k+\sigma}| = \mathcal{O}(\tau^{2\alpha} + h_x^2 + h_y^2). \tag{63}$$

Dropping the truncation error (62), we get the tempered L2-1_σ-ADI scheme

$$\begin{cases}
\Delta_t^{\alpha,\lambda}\left(U_{j,m}^{\bar{k}} + \mu^2\sigma^2(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-2}\delta_x^2\delta_y^2 U_{j,m}^{k+\sigma}\right) - \delta_x^2 U_{j,m}^{k+\sigma} - \delta_y^2 U_{j,m}^{k+\sigma} = f_{j,m}^{k+\sigma},\\
\quad (x_j, y_m) \in \Omega_h, 0 \leqslant k \leqslant N-1,\\
U_{j,m}^{0} = \phi(x_j, y_m), \qquad (x_j, y_m) \in \bar{\Omega}_h,\\
U_{j,m}^{k+\sigma} = \psi(x_j, y_m, t_{k+\sigma}), \quad (x_j, y_m) \in \partial\Omega_h, 0 \leqslant k \leqslant N-1.
\end{cases} \tag{64}$$

Taking into account the expression of the tempered L2-1_σ approximation formula (23), we can rewrite the scheme (64) as follows:

$$\begin{aligned}
&\left(I - \mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\delta_x^2\right)\left(I - \mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\delta_y^2\right)U_{j,m}^{k+1}\\
&= (\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\left[\sum_{\ell=0}^{k-1}\left(\overline{d}_{s,\ell+1}^{(k,\alpha,\sigma)} - \widetilde{d}_{s,\ell}^{(k,\alpha,\sigma)}\right)\left(U_{j,m}^{\ell+1} + \mu^2\sigma^2(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-2}\delta_x^2\delta_y^2 U_{j,m}^{\ell+1}\right)\right.\\
&\quad + \overline{d}_{s,0}^{(k,\alpha,\sigma)}\left(U_{j,m}^{0} + \mu^2\sigma^2(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-2}\delta_x^2\delta_y^2 U_{j,m}^{0}\right) + \mu f_{j,m}^{k+\sigma}\\
&\quad \left.+ \mu(1-\sigma)(\delta_x^2 U_{j,m}^{k} + \delta_y^2 U_{j,m}^{k})\right], \ (x_j, y_m) \in \Omega_h, 0 \leqslant k \leqslant N-1.
\end{aligned} \tag{65}$$

The tempered L2-1_σ-ADI scheme (65) can be summarized briefly in the following procedure.

Step 1 For fixed $m = 1, 2, \cdots, M_2 - 1$, using

$$\begin{cases} \left(I-\mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\delta_x^2\right)U_{j,m}^{*} \\ =(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\left[\sum_{\ell=0}^{k-1}\left(\overline{d}_{s,\ell+1}^{(k,\alpha,\sigma)}-\widetilde{d}_{s,\ell}^{(k,\alpha,\sigma)}\right)\left(U_{j,m}^{\ell+1}+\mu^2\sigma^2(\widetilde{d}_{s,k}^{(\alpha,k)})^{-2}\delta_x^2\delta_y^2U_{j,m}^{\ell+1}\right)\right. \\ \quad+\overline{d}_{s,0}^{(k,\alpha,\sigma)}\left(U_{j,m}^0+\mu^2\sigma^2(\widetilde{d}_{s,k}^{(\alpha,k)})^{-2}\delta_x^2\delta_y^2U_{j,m}^0\right)+\mu f_{j,m}^{k+\sigma} \\ \quad\left.+\mu(1-\sigma)(\delta_x^2U_{j,m}^k+\delta_y^2U_{j,m}^k)\right],\quad 1\leqslant j\leqslant M_1-1. \\ U_{0,m}^{*}=\left(I-\mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\delta_y^2\right)U_{0,m}^{k+1},\ \ U_{M_1,m}^{*}=\left(I-\mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\delta_y^2\right)U_{M_1,m}^{k+1}, \end{cases}$$

we get the intermediate variables $U_{j,m}^{*}=\left(I-\mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\delta_y^2\right)U_{j,m}^{k+1},\ 0\leqslant j\leqslant M_1,\ 0\leqslant k\leqslant N-1$.

Step 2 For fixed $j=1,2,\cdots,M_1-1$, using

$$\begin{cases} \left(I-\mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\delta_y^2\right)U_{j,m}^{k+1}=U_{j,m}^{*},\quad 1\leqslant m\leqslant M_2-1, \\ U_{j,0}^{k+1}=\psi(x_j,y_0,t_{k+1}),\ \ U_{j,M_2}^{k+1}=\psi(x_j,y_{_{M_2}},t_{k+1}), \end{cases}$$

we can get the numerical solutions $\{U_{j,m}^{k+1}\},0\leqslant k\leqslant N-1$.

4.2 Stability Analysis

Let $\tilde{U}_j^k$ be the approximate solution of Eqs. (56) and (65). Then, define $\hat{\varepsilon}_{j,m}^k=U_{j,m}^k-\tilde{U}_{j,m}^k$, $1\leqslant k\leqslant N, 1\leqslant j\leqslant M_1-1, 1\leqslant m\leqslant M_2-1$, and error vectors

$$\hat{\varepsilon}^k=\left[\hat{\varepsilon}_{1,1}^k,\hat{\varepsilon}_{1,2}^k,\cdots,\hat{\varepsilon}_{1,M_2-1}^k,\hat{\varepsilon}_{2,1}^k,\hat{\varepsilon}_{2,2}^k,\cdots,\hat{\varepsilon}_{2,M_2-1}^k,\cdots,\hat{\varepsilon}_{M_1-1,1}^k,\hat{\varepsilon}_{M_1-1,2}^k,\cdots,\hat{\varepsilon}_{M_1-1,M_2-1}^k\right]^{\mathrm{T}}.$$

Similar to the one-dimensional case, define the function $\hat{\varepsilon}^k(x,y)$ in the domain $0\leqslant x,y\leqslant L$ with the node values $\hat{\varepsilon}_{j,m}^k$ at (x_j,y_m,t_k). This means

$$\hat{\varepsilon}^k(x,y)=\begin{cases} \hat{\varepsilon}_{j,m}^k,\quad x_j-\dfrac{h_x}{2}<x\leqslant x_j+\dfrac{h_x}{2},\ y_m-\dfrac{h_y}{2}<y\leqslant y_m+\dfrac{h_y}{2}, \\ \quad 1\leqslant j\leqslant M_1-1,\ 1\leqslant m\leqslant M_2-1, \\ 0,\qquad 0\leqslant x\leqslant \dfrac{h_x}{2}\ \text{ or }\ L-\dfrac{h_x}{2}<x\leqslant L, \\ 0\leqslant y\leqslant\dfrac{h_y}{2}\ \text{ or }\ L-\dfrac{h_y}{2}<y\leqslant L. \end{cases}$$

And $\hat{\varepsilon}^k(x,y)$ can be expanded in the Fourier series

$$\hat{\varepsilon}^k(x,y)=\sum_{l_1=-\infty}^{+\infty}\sum_{l_2=-\infty}^{+\infty}\xi_{l_1,l_2}^k\mathrm{e}^{\mathrm{i}2\pi(l_1x+l_2y)/L},\ 1\leqslant k\leqslant N,$$

where

$$\xi_{l_1,l_2}^k = \frac{1}{L^2}\int_0^L\int_0^L \hat{\varepsilon}^k(x,y)\mathrm{e}^{-\mathrm{i}2\pi(l_1x+l_2y)/L}\mathrm{d}x\mathrm{d}y.$$

According to the definition of the discrete L_2 norm and Parseval's equality, there exists

$$\|\hat{\varepsilon}^k\|_2^2 = \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_xh_y|\hat{\varepsilon}_{j,m}^k|^2 = \sum_{l_1=-\infty}^{+\infty}\sum_{l_2=-\infty}^{+\infty}|\xi_{l_1,l_2}^k|^2. \tag{66}$$

From numerical schemes (56) and (65), we have the related error equations

$$\begin{aligned}&(I-\mu\delta_x^2)(I-\mu\delta_y^2)\hat{\varepsilon}_{j,m}^k\\ &=\sum_{\ell=1}^{k-1}\left(\overline{d}_{a,\ell+1}^{(\alpha)}-\widetilde{d}_{a,\ell}^{(\alpha)}\right)\left(\hat{\varepsilon}_{j,m}^{\ell}+\mu^2\delta_x^2\delta_y^2\hat{\varepsilon}_{j,m}^{\ell}\right)\\ &\quad+\overline{d}_{a,1}^{(\alpha)}\left(\hat{\varepsilon}_{j,m}^0+\mu^2\delta_x^2\delta_y^2\hat{\varepsilon}_{j,m}^0\right),\ (x_j,y_m)\in\Omega_h, 1\leqslant k\leqslant N,\end{aligned} \tag{67}$$

$$\begin{aligned}&\left(I-\mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\delta_x^2\right)\left(I-\mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\delta_y^2\right)\hat{\varepsilon}_{j,m}^{k+1}\\ &=(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\Bigg[\overline{d}_{s,0}^{(\alpha,k)}\left(\hat{\varepsilon}_{j,m}^0+\mu^2\sigma^2(\widetilde{d}_{s,k}^{(\alpha,k)})^{-2}\delta_x^2\delta_y^2\hat{\varepsilon}_{j,m}^0\right)\\ &\quad+\sum_{\ell=0}^{k-1}\left(\overline{d}_{s,\ell+1}^{(k,\alpha,\sigma)}-\widetilde{d}_{s,\ell}^{(k,\alpha,\sigma)}\right)\left(\hat{\varepsilon}_{j,m}^{\ell+1}+\mu^2\sigma^2(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-2}\delta_x^2\delta_y^2\hat{\varepsilon}_{j,m}^{\ell+1}\right)\\ &\quad+\mu(1-\sigma)(\delta_x^2\hat{\varepsilon}_{j,m}^k+\delta_y^2\hat{\varepsilon}_{j,m}^k)\Bigg],\ (x_j,y_m)\in\Omega_h,\ 0\leqslant k\leqslant N-1.\end{aligned} \tag{68}$$

Assume that the solutions of Eqs. (67) and (68) have the following form:

$$\hat{\varepsilon}_{j,m}^k = \xi_{l_1,l_2}^k\mathrm{e}^{\mathrm{i}\beta_1jh_x+\mathrm{i}\beta_2mh_y}, \tag{69}$$

where $\beta_1 = 2\pi l_1/L, \beta_2 = 2\pi l_2/L$, and there are

$$\begin{aligned}\delta_x^2\hat{\varepsilon}_{j,m}^k &= \frac{-4}{h_x^2}\sin^2\left(\frac{\beta_1h_x}{2}\right)\xi_{l_1,l_2}^k\mathrm{e}^{\mathrm{i}\beta_1jh_x+\mathrm{i}\beta_2mh_y},\\ \delta_y^2\hat{\varepsilon}_{j,m}^k &= \frac{-4}{h_y^2}\sin^2\left(\frac{\beta_2h_y}{2}\right)\xi_{l_1,l_2}^k\mathrm{e}^{\mathrm{i}\beta_1jh_x+\mathrm{i}\beta_2mh_y},\\ \delta_x^2\delta_y^2\hat{\varepsilon}_{j,m}^k &= \frac{16}{h_x^2h_y^2}\sin^2\left(\frac{\beta_1h_x}{2}\right)\sin^2\left(\frac{\beta_2h_y}{2}\right)\xi_{l_1,l_2}^k\mathrm{e}^{\mathrm{i}\beta_1jh_x+\mathrm{i}\beta_2mh_y}.\end{aligned}$$

For the stability of the tempered L1-ADI scheme, we have

Theorem 5 *The tempered L1-ADI scheme (56) is unconditionally stable, and we can obtain*

$$\|\hat{\varepsilon}^k\|_2\leqslant\|\hat{\varepsilon}^0\|_2,\ 1\leqslant k\leqslant N.$$

Proof Inserting Eq. (69) and the Euler formula into Eq. (67), the error equation (67) can be rewritten as

$$\Big(1+\mu\kappa_1\Big)\Big(1+\mu\kappa_2\Big)\xi^k_{l_1,l_2}$$
$$=\sum_{\ell=1}^{k-1}\Big(\overline{d}^{(\alpha)}_{a,\ell+1}-\widetilde{d}^{(\alpha)}_{a,\ell}\Big)(1+\mu^2\kappa_1\kappa_2)\xi^\ell_{l_1,l_2}+\overline{d}^{(\alpha)}_{a,1}(1+\mu^2\kappa_1\kappa_2)\xi^0_{l_1,l_2},(x_j,y_m)\in\Omega_h,1\leqslant k\leqslant N, \tag{70}$$

where $\kappa_1=\frac{4}{h_x{}^2}\sin^2(\frac{\beta_1 h_x}{2})$, $\kappa_2=\frac{4}{h_y{}^2}\sin^2(\frac{\beta_2 h_y}{2})$. Suppose that $\xi^k_{l_1,l_2}(k=1,2,\cdots,N)$ are the solutions of Eq. (70), then we will use the mathematical induction to proof the following inequality:

$$|\xi^k_{l_1,l_2}|\leqslant|\xi^0_{l_1,l_2}|,\ k=1,2,\cdots,N. \tag{71}$$

For $k=1$, Eq. (70) clearly deduces that

$$|\xi^1_{l_1,l_2}|=\frac{\overline{d}^{(\alpha)}_{a,1}(1+\mu^2\kappa_1\kappa_2)}{(1+\mu\kappa_1)(1+\mu\kappa_2)}|\xi^0_{l_1,l_2}|\leqslant|\xi^0_{l_1,l_2}|.$$

Now, we assume that Eq. (71) is true for $k=1,2,\cdots,n-1$. For $k=n$, there exists

$$|\xi^n_{l_1,l_2}|\leqslant\frac{(1+\mu^2\kappa_1\kappa_2)}{(1+\mu\kappa_1)(1+\mu\kappa_2)}\left[\sum_{\ell=1}^{n-1}\Big(\overline{d}^{(\alpha)}_{a,\ell+1}-\widetilde{d}^{(\alpha)}_{a,\ell}\Big)\xi^\ell_{l_1,l_2}+\overline{d}^{(\alpha)}_{a,1}\xi^0_{l_1,l_2}\right],$$
$$\leqslant|\xi^0_{l_1,l_2}|,\ (x_j,y_m)\in\Omega_h,\forall\,n=1,2,\cdots,N,$$

where the relation $\sum_{\ell=1}^{n-1}(a^{(\alpha)}_{n-\ell-1}-a^{(\alpha)}_{n-\ell})+a^{(\alpha)}_{n-1}=1$ is utilized. Thus, Eq. (71) is proved. With the help of Eqs. (66) and (71), we obtain

$$\|\hat{\varepsilon}^k\|_2^2=\sum_{l_1=-\infty}^{+\infty}\sum_{l_2=-\infty}^{+\infty}|\xi^k_{l_1,l_2}|^2\leqslant\sum_{l_1=-\infty}^{+\infty}\sum_{l_2=-\infty}^{+\infty}|\xi^0_{l_1,l_2}|^2=\|\hat{\varepsilon}^0\|_2^2,$$

which indicates that the tempered L1-ADI scheme (56) is unconditionally stable.

For the stability of the L2-1_σ-ADI difference scheme (65), we have

Theorem 6 *The tempered L2-1_σ-ADI scheme (65) is unconditionally stable, and we can obtain*

$$\|\hat{\varepsilon}^{k+1}\|_2\leqslant\|\hat{\varepsilon}^0\|_2,\ 0\leqslant k\leqslant N-1.$$

Proof Rewriting the error equation (68) as

$$(1+\mu\sigma(\widetilde{d}^{(k,\alpha,\sigma)}_{s,k})^{-1}\kappa_1)(1+\mu\sigma(\widetilde{d}^{(k,\alpha,\sigma)}_{s,k})^{-1}\kappa_2)\xi^{k+1}_{l_1,l_2}$$
$$=\frac{1}{\widetilde{d}^{(k,\alpha,\sigma)}_{s,k}}\Big[(1+\mu^2\sigma^2(\widetilde{d}^{(k,\alpha,\sigma)}_{s,k})^{-2}\kappa_1\kappa_2)\Big(\sum_{\ell=0}^{k-1}\Big(\overline{d}^{(k,\alpha,\sigma)}_{s,\ell+1}-\widetilde{d}^{(k,\alpha,\sigma)}_{s,\ell}\Big)\xi^{\ell+1}_{l_1,l_2}$$
$$+\overline{d}^{(k,\alpha,\sigma)}_{s,0}\xi^0_{l_1,l_2}\Big)-\mu(1-\sigma)(\kappa_1+\kappa_2)\xi^k_{l_1,l_2}\Big],\ (x_j,y_m)\in\Omega_h,0\leqslant k\leqslant N-1. \tag{72}$$

Supposing that $\xi^{k+1}_{l_1,l_2}(k=0,1,\cdots,N-1)$ are the solutions of Eq. (72), now we will prove the inequality

$$|\xi_{l_1,l_2}^{k+1}| \leqslant |\xi_{l_1,l_2}^{0}|, k = 0, 1, \cdots, N-1. \tag{73}$$

For $k = 0$, there is

$$|\xi_{l_1,l_2}^{1}| = \frac{\overline{d}_{s,0}^{(k,\alpha,\sigma)}(1+\mu^2\sigma^2(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-2}\kappa_1\kappa_2) - \mu(1-\sigma)(\kappa_1+\kappa_2)}{\widetilde{d}_{s,k}^{(k,\alpha,\sigma)}(1+\mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\kappa_1)(1+\mu\sigma(\widetilde{d}_{s,k}^{(k,\alpha,\sigma)})^{-1}\kappa_2)}|\xi_{l_1,l_2}^{0}| \leqslant |\xi_{l_1,l_2}^{0}|.$$

Next, we suppose that

$$|\xi_{l_1,l_2}^{k+1}| \leqslant |\xi_{l_1,l_2}^{0}|, \quad k = 0, 1, \cdots, n-1.$$

For $k = n$, using the relation $\sum_{\ell=0}^{k-1}(s_{k-\ell-1}^{(k,\alpha,\sigma)} - s_{k-\ell}^{(k,\alpha,\sigma)}) + s_k^{(k,\alpha,\sigma)} = s_0^{(k,\alpha,\sigma)}$ and $\widetilde{d}_{s,k}^{(k,\alpha,\sigma)} = s_0^{(k,\alpha,\sigma)}\mathrm{e}^{(\lambda t_{n+1} - t_{n+\sigma})} > s_0^{(k,\alpha,\sigma)}$, there exists

$$\begin{aligned}|\xi_{l_1,l_2}^{n+1}| &\leqslant \frac{(1+\mu^2\sigma^2(\widetilde{d}_{s,n}^{(k,\alpha,\sigma)})^{-2}\kappa_1\kappa_2)}{\widetilde{d}_{s,n}^{(k,\alpha,\sigma)}(1+\mu\sigma(\widetilde{d}_{s,n}^{(k,\alpha,\sigma)})^{-1}\kappa_1)(1+\mu\sigma(\widetilde{d}_{s,n}^{(k,\alpha,\sigma)})^{-1}\kappa_2)} \\ &\quad \times \left(\sum_{\ell=0}^{n-1}(s_{n-\ell-1}^{(k,\alpha,\sigma)} - s_{n-\ell}^{(k,\alpha,\sigma)}) + s_n^{(k,\alpha,\sigma)}\right)|\xi_{l_1,l_2}^{0}|, \\ &\leqslant |\xi_{l_1,l_2}^{0}|, \ (x_i, y_m) \in \Omega_h, 0 \leqslant k \leqslant N.\end{aligned}$$

Thus, according to Eqs. (73) and (66), we have $\|\hat{\varepsilon}^k\|_2^2 \leqslant \|\hat{\varepsilon}^0\|_2^2$.

4.3 Convergence Analysis

Just like the one-dimensional case, the convergence analysis of the two-dimensional case can be obtained similarly. Let $C_{x,y,t}^{4,4,2}$ denote the space of function $u(x, y, t)$ which satisfies $u(x, y, t) \in C^2[0, t_k] \cap C^4(\bar{\Omega})$, $\bar{\Omega} = [0, L] \times [0, L]$. Then, the corresponding convergence results are given as follows.

Theorem 7 *Suppose $u(x, y, t) \in C_{x,y,t}^{4,4,2}$, then the tempered L1-ADI scheme (56) is convergent with the accuracy $\mathcal{O}(\tau^{2\alpha} + h_x^2 + h_y^2)$ for $\alpha \in (0, 2/3]$, and $\mathcal{O}(\tau^{2-\alpha} + h_x^2 + h_y^2)$ for $\alpha \in (2/3, 1)$.*

Theorem 8 *Suppose $u(x, y, t) \in C_{x,y,t}^{4,4,3}$, then the tempered L2-$1_\sigma$-ADI scheme (65) is convergent with the accuracy $\mathcal{O}(\tau^{2\alpha} + h_x^2 + h_y^2)$ for $\alpha \in (0, 1)$.*

5 Numerical Experiments

5.1 Truncation Error

In this subsection, we present an example to verify the theoretical results of the three proposed tempered formulas.

Example 1 We calculate the Caputo-tempered fractional derivative of function $u(t) = \mathrm{e}^{-\lambda t} t^{4+\alpha}$ in [0, 1]. Applying the formula [21]

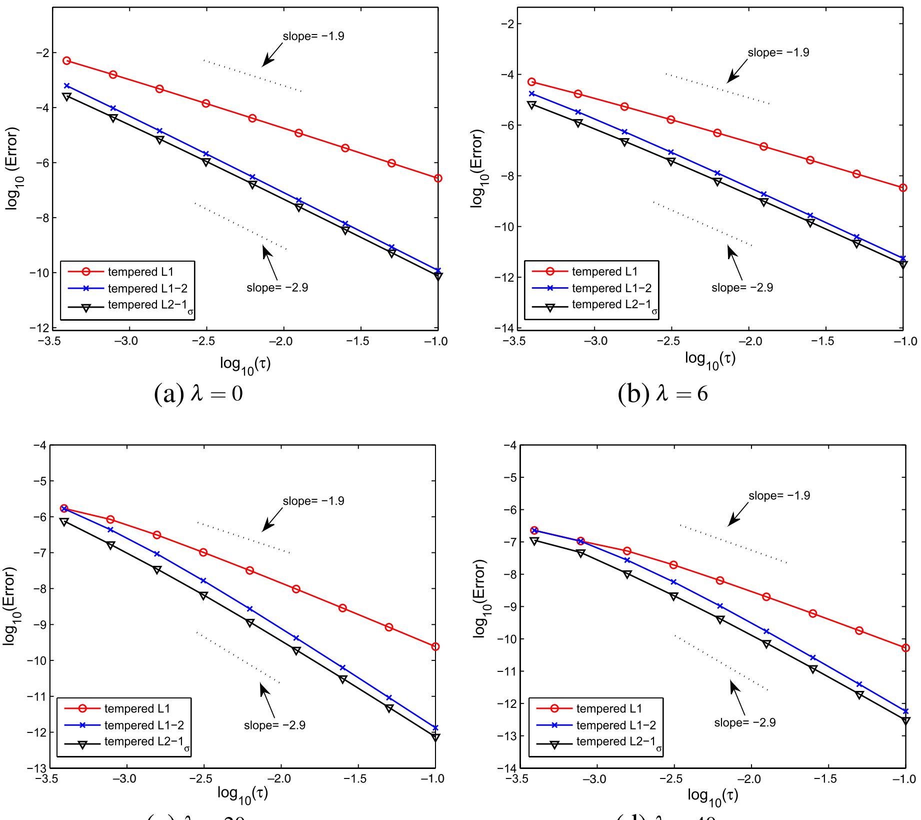
$$ {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}(\mathrm{e}^{-\lambda t} t^{\beta}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} \mathrm{e}^{-\lambda t} t^{\beta-\alpha}, $$

the exact solution of $u(t)$ is given by

$$ {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda}(\mathrm{e}^{-\lambda t} t^{4+\alpha})\big|_{t=t_k} = \frac{\Gamma(5+\alpha)}{24} \mathrm{e}^{-\lambda t} t^{4}\Big|_{t=t_k}. $$

The numerical errors and the convergence orders are calculated with the different parameters $\alpha = 0.1, 0.5, 0.9$ and $\lambda = 0, 6, 20, 40$. The pointwise maximum norm error and the corresponding convergence order are measured by $\mathrm{Error}(\tau) = \max\limits_{0 \leqslant j \leqslant N} \left| u(t_j) - U_j \right|$, $\mathrm{Order} = \log_2 \left(\frac{\mathrm{Error}(\tau)}{\mathrm{Error}(\tau/2)} \right)$, where U_j is the approximate solution of $u(t)$ at the grid point t_j.

Figures 1, 2, 3, where the time steps are 1/10, 1/20, 1/40, 1/80, 1/160, 1/320, 1/640, 1/1 280, and 1/2 560, give the maximum norm errors and convergence orders for different α, λ of the three presented discretization formulas (9), (16), and (23), respectively.



(a) $\lambda = 0$ (b) $\lambda = 6$ (c) $\lambda = 20$ (d) $\lambda = 40$

Fig. 1 The log–log plot of the maximum norm errors versus time-steps for $\alpha = 0.1$

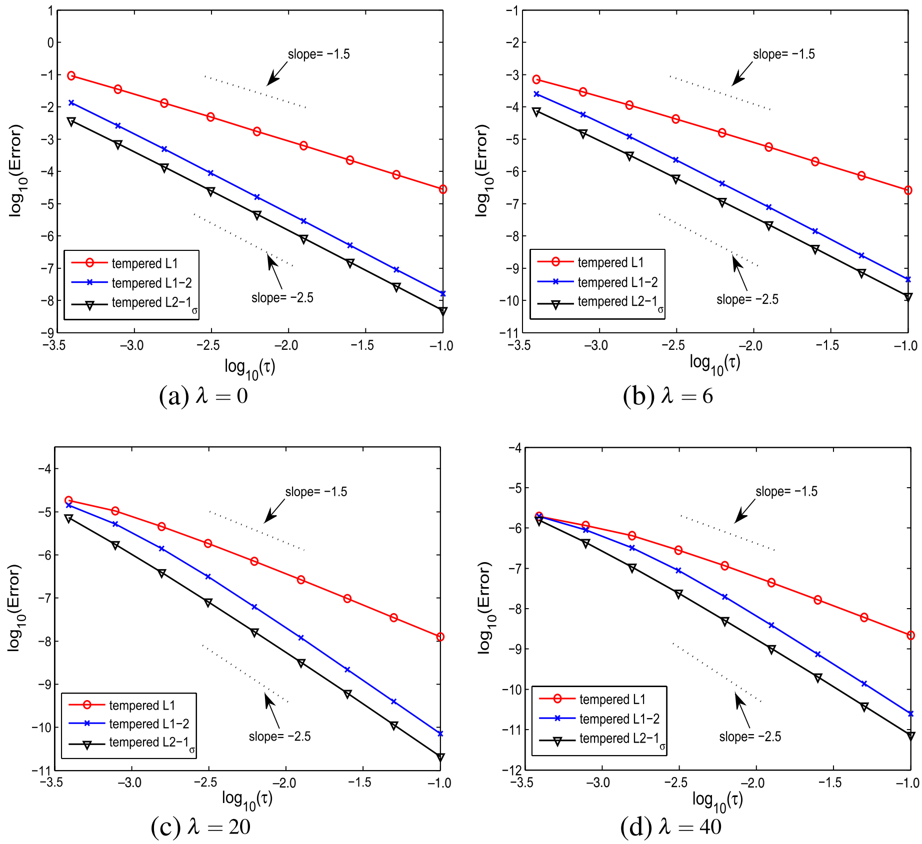


Fig. 2 The log–log plot of the maximum norm errors versus time-steps for $\alpha = 0.5$

Based on the results displayed in Figs. 1, 2, 3, we can easily observe that the convergence orders of tempered L1-2 and L2-1_σ formulas are both $3 - \alpha$, which is higher than $2 - \alpha$ order of the tempered L1 formula. These are in accordance with the theoretical analysis.

5.2 One-Dimensional Problem

To further illustrate the effectiveness of the proposed numerical formulas, we test two kinds of equations. The first equation is the initial boundary value problem of the time-tempered fractional diffusion equation and the second one is the time-tempered fractional Burgers equation.

5.2.1 Caputo-Tempered Fractional Diffusion Equation

Example 2 In this example, we solve problem (25) with $L = 1$ using the proposed numerical schemes (30)–(32). The source term $f(x, t)$ was chosen as $f(x,t) = \mathrm{e}^{-\lambda t + x} t^4 \left[\frac{\Gamma(5+\alpha)}{\Gamma(5)} - t^{\alpha}\right]$

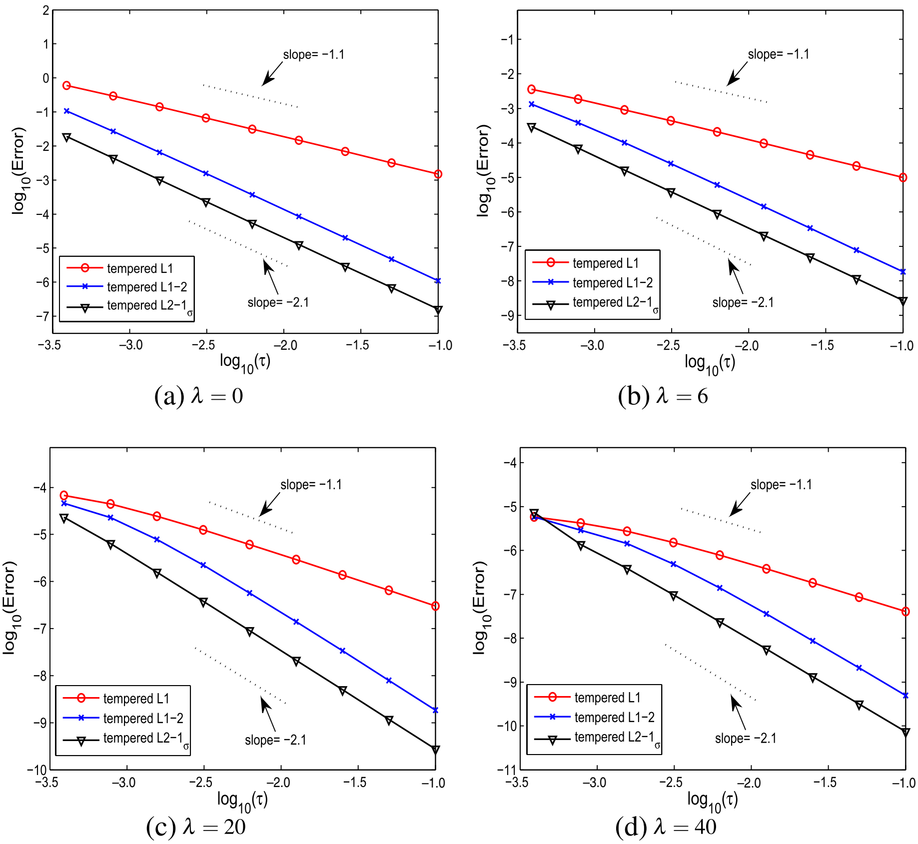


Fig. 3 The log–log plot of the maximum norm errors versus time-steps for $\alpha = 0.9$

such that the exact solution is $u(x,t) = \mathrm{e}^{-\lambda t}\mathrm{e}^{x}t^{4+\alpha}$ with $\phi(x) = 0, \psi_1(t) = \mathrm{e}^{-\lambda t}t^{4+\alpha}$, $\psi_2(t) = \mathrm{e}^{-\lambda t+1}t^{4+\alpha}$. The numerical errors can be measured by the maximum norm errors at each discrete point

$$\text{Error} = \max_{0\leqslant k\leqslant N}\left(\max_{0\leqslant j\leqslant M}\left|u(x_j,t_k) - U_j^k\right|\right).$$

To demonstrate the numerical accuracy in time direction with a small spatial step size $h = 1/2\,000$, we list the computed results in Table 1. The orders of convergence are identified with theoretical orders of convergence (abbreviated as TOC) in the last row, and the errors of the tempered L1-2 and L2-1_σ formulas are significantly smaller than that of the tempered L1 formula. Owing to the interpolation at the non-grid point $t_{k+\sigma}$, it should be noted that the order of the tempered L2-1_σ formula is $\mathcal{O}(2)$ rather than $\mathcal{O}(3-\alpha)$. Moreover, the comparisons of CPU times (in seconds) for different implicit difference schemes are exhibited in Fig. 4 of $\alpha = 0.1$ and $\lambda = 6$. From Fig. 4 we can see that the CPU times are proportional to the time steps, while the L2-1_σ formula requires more time for this problem than tempered L1 and L1-2 formulas.

Table 1 (Example 2) The maximum norm errors and the corresponding convergence orders of tempered L1, L1-2, and L2-1$_\sigma$ formulas for $\alpha = 0.5$

	τ	Tempered L1 formula		Tempered L1-2 formula		Tempered L2-1$_\sigma$ formula	
		Error	Order	Error	Order	Error	Order
$\lambda = 20$	1/10	2.886 8E−06	–	2.203 1E−06	–	8.495 4E−07	–
	1/20	1.536 8E−06	0.909 6	7.204 6E−07	1.612 5	2.220 9E−07	1.935 6
	1/40	6.580 2E−07	1.223 7	1.972 8E−07	1.868 7	4.933 2E−08	2.170 5
	1/80	2.631 7E−07	1.322 1	4.363 7E−08	2.176 6	1.048 3E−08	2.234 4
	1/160	1.005 5E−07	1.388 0	8.760 6E−09	2.316 5	2.145 8E−09	2.288 5
	1/320	3.736 8E−08	1.428 1	1.667 3E−09	2.393 5	4.369 4E−10	2.296 0
	1/640	1.365 6E−08	1.452 2	3.078 1E−10	2.437 4	9.034 7E−11	2.273 9
	1/1 280	4.936 6E−09	1.468 0	5.588 7E−11	2.461 4	1.916 3E−11	2.237 2
	1/2 560	1.771 9E−09	1.478 2	1.005 3E−11	2.474 9	4.164 5E−12	2.202 1
TOC			1.5		2.5		2.0

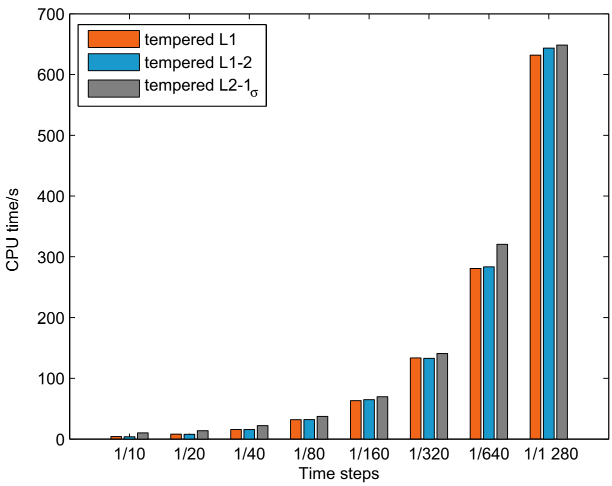


Fig. 4 The comparisons of CPU times for different implicit difference schemes with $\alpha = 0.1$, $\lambda = 6$, $h = 1/2\,000$

5.2.2 Caputo-Tempered Fractional Burgers Equation

Consider the following time-tempered fractional Burgers equation:

$$\begin{cases} {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha,\lambda} u(x,t) + u(x,t)\dfrac{\partial u}{\partial x}(x,t) = \kappa_\alpha \dfrac{\partial^2 u}{\partial x^2}(x,t) + f(x,t),\ x \in (0,L), t \in (0,T], \\ u(x,0) = u_0(x),\ x \in (0,L), \\ u(0,t) = 0, u(L,t) = 0,\ t \in (0,T], \end{cases} \tag{74}$$

where κ_α is a positive constant. And denote the difference operator

$$\delta_{\hat{x}} U_j^k = \frac{U_{j+1}^k - U_{j-1}^k}{2h}.$$

The time direction is discretized by three proposed difference formulas, and the first and second derivatives of space direction are discretized by central difference. Regarding the nonlinear advection term, we adopt the linearization technique

$$u\frac{\partial u}{\partial x}\Big|_{(x,t)=(x_j,t_k)} \approx \frac{1}{3}[2U_j^{k-1}\delta_{\hat{x}}(U_j^k) + U_j^k\delta_{\hat{x}}(U_j^{k-1})] := \mathcal{N}(U_j^k).$$

Then, the following discretization schemes can be obtained, respectively:

$$\mathbb{D}_t^{\alpha,\lambda}U_j^k + \mathcal{N}(U_j^k) = \kappa_\alpha \delta_x^2 U_j^k + f_j^k, \quad 1 \leqslant j \leqslant M-1, 1 \leqslant k \leqslant N, \tag{75}$$

$$\partial_t^{\alpha,\lambda}U_j^k + \mathcal{N}(U_j^k) = \kappa_\alpha \delta_x^2 U_j^k + f_j^k, \quad 1 \leqslant j \leqslant M-1, 1 \leqslant k \leqslant N, \tag{76}$$

$$\Delta_t^{\alpha,\lambda}U_j^{\bar{k}} + \mathcal{N}(U_j^{k+\sigma}) = \kappa_\alpha \delta_x^2 U_j^{k+\sigma} + f_j^{k+\sigma}, \quad 1 \leqslant j \leqslant M-1, 0 \leqslant k \leqslant N-1, \tag{77}$$

where $\mathcal{N}(U_j^{k+\sigma}) = \frac{1}{3}[2U_j^k\delta_{\hat{x}}(U_j^{k+\sigma}) + U_j^{k+\sigma}\delta_{\hat{x}}(U_j^k)]$, and the initial value and boundary condition are discretized as $U_j^0 = u_0(x_j),\ U_0^k = 0,\ U_M^k = 0$.

Example 3 We solve the problem (74) using the numerical schemes (75)–(77) with $L = 1$. The source term is determined by the exact solution $u(x,t) = \mathrm{e}^{-\lambda t}t^{3+\alpha}x^3(1-x)$, the initial value and boundary value are both 0.

Without loss of generality, we can get the following numerical results with $\kappa_\alpha = 1$. We check the numerical results in both time and space directions with $\lambda = 6$. The computational results about time direction are listed in Table 2 with $h = 1/2\,000$, while the corresponding results are listed in Table 3 for space direction with different ways of selecting τ of three difference schemes.

5.3 Two-Dimensional Problem

Example 4 Consider the problem (49) on $\Omega = (0,\pi)\times(0,\pi)$, $t \in (0, 1/2]$. The exact solution is $u(x,y,t) = \mathrm{e}^{-\lambda t}\sin(x)\sin(y)t^2$ with the right source term

$$f(x,y,t) = \mathrm{e}^{-\lambda t}\sin(x)\sin(y)\left[\frac{2}{\Gamma(3-\alpha)} + 2t^2\right].$$

Table 2 (Example 3) The maximum norm errors and the corresponding convergence orders of time direction of difference schemes adopting tempered L1, L1-2, and L2-1_σ formulas for $\alpha = 0.5$ and $h = 1/2\,000$

	τ	Tempered L1 formula		Tempered L1-2 formula		Tempered L2-1_σ formula	
		Error	Order	Error	Order	Error	Order
$\lambda = 6$	1/10	7.303 8E−06	–	3.051 5E−06	–	6.898 4E−06	–
	1/20	2.991 1E−06	1.288 0	7.072 3E−07	2.109 2	1.912 7E−06	1.850 6
	1/40	1.157 7E−06	1.369 4	1.453 2E−07	2.283 0	4.828 8E−07	1.985 9
	1/80	4.341 5E−07	1.415 0	2.798 8E−08	2.376 3	1.186 8E−07	2.024 7
	1/160	1.594 2E−07	1.445 3	5.211 5E−09	2.425 1	2.949 3E−08	2.008 6
	1/320	5.777 0E−08	1.464 5	9.462 5E−10	2.461 4	7.496 6E−09	1.976 1
TOC			1.5		2.5		2.0

Table 3 (Example 3) The maximum norm errors and the corresponding convergence orders of space direction of difference schemes adopting tempered L1, L1-2, and L2-1_σ formulas for $\alpha = 0.5$

	h	Tempered L1 formula		Tempered L1-2 formula		Tempered L2-1_σ formula	
		$\tau = h^{4/3}$		$\tau = h^{4/5}$		$\tau = h$	
		Error	Order	Error	Order	Error	Order
$\lambda = 6$	1/10	7.777 4E−06	–	6.829 6E−06	–	1.365 5E−04	–
	1/20	1.922 4E−06	2.016 4	2.203 7E−06	1.631 9	5.341 3E−05	1.354 1
	1/40	4.789 3E−07	2.005 0	7.240 2E−07	1.605 8	1.537 1E−05	1.797 0
	1/80	1.193 3E−07	2.004 9	2.148 6E−07	1.752 6	4.070 1E−06	1.917 1
	1/160	2.977 3E−08	2.002 9	5.884 4E−08	1.868 5	1.045 3E−06	1.961 2
	1/320	7.434 7E−09	2.001 6	1.565 6E−08	1.910 2	2.648 3E−07	1.980 7
TOC			2.0		2.0		2.0

The error in this example is measured by

$$\text{Error} = \max_{0\leqslant k\leqslant N}\left\{\max_{0\leqslant j\leqslant M_1}\left(\max_{0\leqslant m\leqslant M_2}\left|u(x_j, y_m, t_k) - U_{j,m}^k\right|\right)\right\}.$$

Tables 4 and 5 show the numerical results calculated by the three different implicit ADI schemes with different α for classic ($\lambda = 0$) and tempered ($\lambda = 6$) situation when the spatial step size $h_x = h_y = \pi/400$ are fixed, respectively. In Fig. 5, we plot the CPU times

Table 4 (Example 4) The maximum norm errors and the numerical convergence orders of tempered L1-ADI, L1-2-ADI, and L2-1_σ-ADI approximation with $\lambda = 0$

α	τ	Tempered L1-ADI		Tempered L1-2-ADI		Tempered L2-1_σ-ADI	
		Error	Order	Error	Order	Error	Order
1/2	1/10	3.493 0E−03	–	3.849 4E−03	–	4.078 6E−03	–
	1/20	1.992 5E−03	0.809 9	1.910 6E−03	1.010 5	1.953 8E−03	1.061 8
	1/40	1.084 1E−03	0.878 0	9.538 4E−04	1.002 3	9.573 8E−04	1.029 1
	1/80	5.731 5E−04	0.919 6	4.766 7E−04	1.000 7	4.738 8E−04	1.014 6
	1/160	2.974 2E−04	0.946 4	2.381 0E−04	1.001 4	2.355 5E−04	1.008 4
TOC			1		1		1
2/3	1/10	2.091 9E−03	–	2.092 0E−03	–	1.694 1E−03	–
	1/20	8.748 8E−04	1.257 7	6.066 1E−04	1.786 0	5.985 3E−04	1.501 0
	1/40	3.575 3E−04	1.291 0	2.308 0E−04	1.394 2	2.203 7E−04	1.441 5
	1/80	1.443 3E−04	1.308 7	8.944 0E−05	1.367 6	8.317 0E−05	1.405 8
	1/160	5.789 6E−05	1.317 8	3.482 4E−05	1.360 9	3.174 3E−05	1.389 6
TOC			1.33		1.33		1.33
3/4	1/10	3.869 8E−03	–	2.362 5E−03	–	1.186 2E−03	–
	1/20	1.754 5E−03	1.141 1	6.189 4E−04	1.932 5	3.622 1E−04	1.711 5
	1/40	7.812 5E−04	1.167 3	1.595 5E−04	1.955 8	1.146 7E−04	1.659 3
	1/80	3.430 7E−04	1.187 3	4.119 7E−05	1.953 4	3.716 3E−05	1.625 6
	1/160	1.493 3E−04	1.200 0	1.375 1E−05	1.583 0	1.121 0E−05	1.618 9
TOC			1.25		1.5		1.5

Table 5 (Example 4) The maximum norm errors and the numerical convergence orders of tempered L1-ADI, L1-2-ADI, and L2-1_σ-ADI approximation with $\lambda = 6$

α	τ	Tempered L1-ADI		Tempered L1-2-ADI		Tempered L2-1_σ-ADI	
		Error	Order	Error	Order	Error	Order
1/2	1/10	3.793 7E−04	–	8.745 3E−04	–	2.418 2E−04	–
	1/20	1.546 4E−04	1.294 7	3.032 4E−04	1.528 1	1.103 5E−04	1.131 8
	1/40	6.130 0E−05	1.334 9	9.084 9E−05	1.738 9	5.859 3E−05	0.913 3
	1/80	3.375 5E−05	0.860 8	3.193 4E−05	1.508 4	3.026 3E−05	0.953 1
	1/160	1.809 6E−05	0.899 5	1.591 8E−05	1.004 4	1.538 0E−05	0.976 5
TOC			1		1		1
2/3	1/10	8.315 5E−04	–	1.148 1E−03	–	1.518 3E−04	–
	1/20	3.634 4E−04	1.194 1	4.067 5E−04	1.497 0	6.366 2E−05	1.254 9
	1/40	1.543 2E−04	1.235 8	1.224 0E−04	1.732 6	1.906 2E−05	1.738 8
	1/80	6.359 4E−05	1.279 0	3.379 7E−05	1.856 6	5.205 4E−06	1.872 6
	1/160	2.579 6E−05	1.301 8	8.916 1E−06	1.922 4	1.937 1E−06	1.426 1
TOC			1.33		1.33		1.33
3/4	1/10	1.130 6E−03	–	1.296 6E−03	–	1.228 0E−04	–
	1/20	5.183 1E−04	1.125 2	4.585 2E−04	1.499 6	5.304 3E−05	1.211 1
	1/40	2.293 7E−04	1.176 1	1.373 3E−04	1.739 4	1.639 4E−05	1.694 0
	1/80	9.937 0E−05	1.206 8	3.772 7E−05	1.863 9	4.533 3E−06	1.854 5
	1/160	4.253 6E−05	1.224 1	9.909 6E−06	1.928 7	1.178 2E−06	1.944 0
TOC			1.25		1.5		1.5

of the tempered L1-ADI, tempered L1-2-ADI, and tempered L2-1_σ-ADI schemes. From Fig. 5, we observe that the CPU times of L2-1_σ-ADI scheme are bigger than two other schemes, which is almost the same as the one-dimensional case.

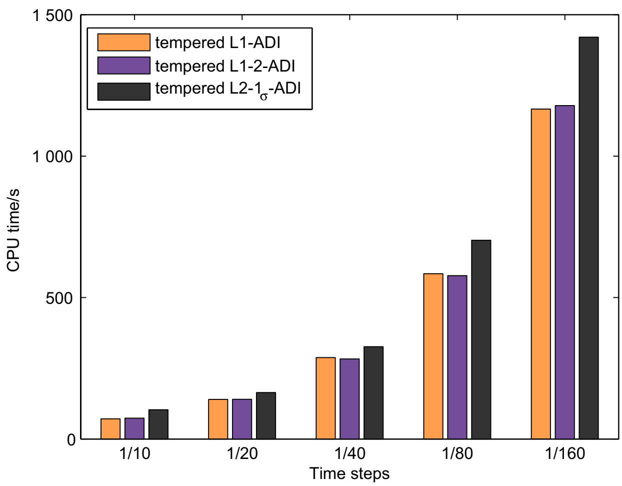


Fig. 5 The comparisons of CPU times for three different implicit ADI schemes

6 Conclusion

In this paper, we presented and analyzed the efficient difference schemes for diffusion equations with the Caputo-tempered fractional derivative. To design the difference schemes, we first proposed the tempered L1 formula for the Caputo-tempered fractional derivative of the order $\alpha \in (0, 1)$. The tempered L1 formula is constructed by using the piecewise linear interpolation on each small interval with the order $2 - \alpha$. To improve the numerical accuracy, another two fractional numerical quadrature formulas, called tempered L1-2 and L2-1_σ formulas with the order $3 - \alpha$ are presented. The tempered L1-2 formula is established by means of the quadratic interpolation approximation on each cell $[t_{\ell-1}, t_\ell]\,(\ell \geqslant 2)$, while the linear interpolation approximation is applied on the first cell $[t_0, t_1]$. The tempered L2-1_σ formula is developed by using the quadratic interpolation approximation on each cell $[t_{\ell-1}, t_\ell]\,(1 \leqslant \ell \leqslant k)$, while the linear interpolation in the cell $[t_k, t_{k+1}]$ is applied on the last non-integer grid cell $[t_k, t_{k+\sigma}]$.

We further designed the difference schemes for one- and two-dimensional fractional diffusion equations with the help of the presented interpolation formulas. We checked the stability and convergence of two proposed difference schemes by the Fourier analysis method. The key idea of our method is to examine the weighted coefficients of difference schemes. The analysis shows that the implicit numerical schemes are unconditionally stable and convergent when the tempered L1 formula and L2-1_σ formula are used. However, the rigorous theoretical analysis of numerical scheme are not obtained for the tempered L1-2 formula due to the lack of positivity of the weighting coefficients. Finally, several numerical examples are given to validate the theoretical results. As the Caputo fractional derivative, the challenges still exist due to the nonlocal property of tempered fractional derivatives [17]. We expect that a new technique will be needed to construct the fast algorithm for the considered problem.

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Appendix A: Background of Eq. (1)

In particle motion, sometimes the waiting time of particles is too long, however, too long waiting time may not be appropriate for some physical process, therefore, we need to control the waiting time of particles in the finite time domain. To overcome this shortcoming, the PDFs which are exponentially tempered power law are put forward [13, 21]. Therefore, the tempered power law waiting time leads to the time-tempered fractional derivatives, which have proved available in geophysics. And the purpose of tempered fractional differential equations is to more accurately describe the motion behavior of particles in complex dynamic systems. The CTRW model is based on the idea that the length of a given jump, as well as the waiting time elapsing between two successive jumps is drawn from a pdf $\psi(\mathbf{x}, t)$ which will be referred to as the jump pdf. For $\psi(\mathbf{x}, t)$, the jump length pdf gives $\varphi(\mathbf{x}) = \int_0^\infty \psi(\mathbf{x}, t)\mathrm{d}t$, and the waiting time pdf obeys $w(t) = \int_{R^d} \psi(\mathbf{x}, t)\mathrm{d}\mathbf{x}$. Thus, $\varphi(\mathbf{x})\mathrm{d}\mathbf{x}$ produces the probability for a jump length in the interval $(\mathbf{x}, \mathbf{x} + \mathrm{d}\mathbf{x})$ and $w(t)\mathrm{d}t$ the probability for a waiting time in the interval $(t, t + \mathrm{d}t)$. If the jump length and waiting time are independent random variables, one finds the decoupled form $\psi(\mathbf{x}, t) = \varphi(\mathbf{x})w(t)$ for the jump pdf $\psi(\mathbf{x}, t)$. The probability density function $\eta(\mathbf{x}, t)$ of the positions $\mathbf{x}$ of the particle, which have just made a jump at time t can be described through an appropriate generalized master equation [16, 33]

$$\eta(\mathbf{x}, t) = \int_{R^d} \eta(\mathbf{x}', t')\mathrm{e}^{-\lambda(t-t')}\psi(\mathbf{x} - \mathbf{x}', t - t')\mathrm{d}\mathbf{x}'\mathrm{d}t' + u(\mathbf{x}, 0)\delta(t).$$

The pdf $u(\mathbf{x}, t)$ of particle being in $\mathbf{x}$ at time t is given by

$$u(\mathbf{x}, t) = \int_0^t \eta(\mathbf{x}, t')\mathrm{e}^{-\lambda(t-t')}\Psi(t - t')\mathrm{d}t',$$

where $\Psi(t - t')$ is the probability of staying at site $\mathbf{x}$ for a time $t - t'$ after a jump

$$\Psi(t) = 1 - \int_0^t w(t')\mathrm{d}t'.$$

The pdf $u(x, t)$ in the Fourier–Laplace space takes the form

$$\widehat{\widetilde{u}}(\mathbf{k}, s) = \frac{1 - \widetilde{w}(s + \lambda)}{s + \lambda} \frac{\widehat{u}(\mathbf{k}, 0)}{1 - \widetilde{w}(s + \lambda)\widehat{\varphi}(\mathbf{k})}, \tag{A1}$$

where the Laplace and Fourier transforms are defined, respectively, by $\mathcal{L}(u(t)) = \tilde{u}(s) = \int_0^\infty u(t)\mathrm{e}^{-st}\mathrm{d}t, s \in C, Re(s) > 0$, and $\mathcal{F}(u(\mathbf{x})) = \hat{u}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int_{R^d} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}u(\mathbf{x})\mathrm{d}\mathbf{x}, \mathbf{k} = (k_1, k_2, \cdots, k_d) \in R^d$. In the Fourier space, the jump length distribution behaved [28]

$$\widehat{\varphi}(\mathbf{k}) = \mathrm{e}^{\Lambda(\mathbf{k})} \approx 1 + \Lambda(\mathbf{k}), |\mathbf{k}| \to 0,$$

where $\Lambda(\mathbf{k})$ is the characteristic exponent of $\varphi(\mathbf{x})$, the probability distribution for jumps. Taking $\Lambda(\mathbf{k}) = -C_\mu|\mathbf{k}|^2$, and a waiting time pdf of a Pareto type [28] $\widetilde{w}(s) \simeq 1 - \Gamma(1 - \alpha)\bar{\tau}^\alpha s^\alpha, 0 < \alpha < 1$, Eq. (A1) can be rewritten in the following form:

$$s\widehat{\widetilde{u}}(\mathbf{k}, s) - s\widehat{u}(\mathbf{k}, 0) = -(s + \lambda)^{1-\alpha} \frac{\sigma^\mu}{\Gamma(1 - \alpha)\bar{\tau}^\alpha} |\mathbf{k}|^\mu \widehat{\widetilde{u}}(\mathbf{k}, s) - \lambda\widehat{\widetilde{u}}(\mathbf{k}, s), \tag{A2}$$

Introducing the Riemann–Liouville tempered fractional derivative operator [18, 31]

$${}_0\mathrm{D}_t^{1-\alpha,\lambda}u(\mathbf{x}, t) = \frac{\mathrm{e}^{-\lambda t}}{\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\mathrm{e}^{\lambda s}u(\mathbf{x}, s)}{(t - s)^{1-\alpha}}\mathrm{d}s,$$

and the Fourier transform of the standard Laplace operator is $\mathcal{F}\big(\Delta u(\mathbf{x})\big) = -|\mathbf{k}|^2\mathcal{F}(u(\mathbf{x}))$, after inverting the Fourier–Laplace transform on both sides of Eq. (A2), we get that the pdf of diffusion particles obeys the tempered fractional diffusion equation [16]

$$u_t(\mathbf{x}, t) = \kappa_\alpha \, {}_0\mathrm{D}_t^{1-\alpha,\lambda}(\Delta u(\mathbf{x}, t)) - \lambda u(\mathbf{x}, t).$$

Appendix B: Proof of Lemma 1

For the proof of (i), (ii), and (iii), see the references [23, 35].

(iv) According to (i) and the definition of Eqs. (7) and (8), it is easy to see that (iv) is true.

(v) Let $g(x) = [(k-x+1)^{(1-\alpha)} - (k-x)^{1-\alpha}]\mathrm{e}^{\lambda(t_{x-1}-t_k)}$. Then, there is

$$g'(x) = \mathrm{e}^{\lambda(x-1-k)\tau}\Big[\lambda\tau\Big((k-x+1)^{1-\alpha} - (k-x)^{1-\alpha}\Big) + (\alpha-1)\Big((k-x+1)^{-\alpha} - (k-x)^{-\alpha}\Big)\Big] > 0,$$

which represents that $\overline{d}_{a,m}^{(\alpha)}$ is monotone increasing about m.

(vi) $\widetilde{d}_{a,m}^{(\alpha)}$ is also monotone increasing with the same fashion of (v).

(vii) $\overline{d}_{a,m+1}^{(\alpha)} - \widetilde{d}_{a,m}^{(\alpha)} = \Big(a_{k-m-1}^{(\alpha)} - a_{k-m}^{(\alpha)}\Big)\mathrm{e}^{\lambda(t_m-t_k)} \leqslant a_{k-m-1}^{(\alpha)} - a_{k-m}^{(\alpha)},\ \ 1 \leqslant m \leqslant k-1.$

(viii) $\overline{d}_{a,1}^{(\alpha)} = a_{k-1}^{(\alpha)}\mathrm{e}^{\lambda(t_0-t_k)} \leqslant a_{k-1}^{(\alpha)}.$

Appendix C: Proof of Lemma 2

By simple calculation, we have the truncation error as follows:

$$\begin{aligned}
R^k &= \frac{\mathrm{e}^{-\lambda t_k}}{\Gamma(1-\alpha)}\sum_{\ell=1}^{k}\int_{t_{\ell}-1}^{t_\ell}\big[v(s) - P_{1,\ell}v(s)\big]'(t_k-s)^{-\alpha}\mathrm{d}s\\
&= -\frac{\alpha\mathrm{e}^{-\lambda t_k}}{\Gamma(1-\alpha)}\sum_{\ell=1}^{k}\int_{t_{\ell}-1}^{t_\ell}\big[v(s) - P_{1,\ell}v(s)\big](t_k-s)^{-\alpha-1}\mathrm{d}s\\
&= \frac{\alpha\mathrm{e}^{-\lambda t_k}}{\Gamma(1-\alpha)}\sum_{\ell=1}^{k}\int_{t_{\ell}-1}^{t_\ell}\frac{1}{2}v''(\xi_\ell)(s-t_{\ell-1})(t_\ell-s)(t_k-s)^{-\alpha-1}\mathrm{d}s,\ \ \xi_\ell \in (t_{\ell-1}, t_\ell),
\end{aligned} \tag{C1}$$

and it yields

$$\Big|R^k\Big| \leqslant \frac{\alpha\mathrm{e}^{-\lambda t_k}}{2\Gamma(1-\alpha)}\max_{t_0\leqslant t\leqslant t_k}\big|v''(t)\big|\sum_{\ell=1}^{k}\int_{t_{\ell}-1}^{t_\ell}(s-t_{\ell-1})(t_\ell-s)(t_k-s)^{-\alpha-1}\mathrm{d}s. \tag{C2}$$

On the other hand, for the integral of the right term, we have

$$\begin{aligned}
&\sum_{\ell=1}^{k}\int_{t_{\ell}-1}^{t_\ell}(s-t_{\ell-1})(t_\ell-s)(t_k-s)^{-\alpha-1}\mathrm{d}s\\
&= \sum_{\ell=1}^{k-1}\int_{t_{\ell}-1}^{t_\ell}(s-t_{\ell-1})(t_\ell-s)(t_k-s)^{-\alpha-1}\mathrm{d}s + \int_{t_{k-1}}^{t_k}(s-t_{k-1})(t_k-s)^{-\alpha}\mathrm{d}s
\end{aligned}$$

with

$$\sum_{\ell=1}^{k-1}\int_{t_{\ell}-1}^{t_\ell}(s-t_{\ell-1})(t_\ell-s)(t_k-s)^{-\alpha-1}\mathrm{d}s \leqslant \frac{\tau^2}{4}\sum_{\ell=1}^{k-1}\int_{t_{\ell}-1}^{t_\ell}(t_k-s)^{-\alpha-1}\mathrm{d}s \leqslant \frac{1}{4\alpha}\tau^{2-\alpha}, \tag{C3}$$

and

$$\int_{t_{k-1}}^{t_k}(s-t_{k-1})(t_k-s)^{-\alpha}\mathrm{d}s=\int_0^{\tau}(\tau-y)y^{-\alpha}\mathrm{d}y=\frac{\tau^{2-\alpha}}{(1-\alpha)(2-\alpha)}. \tag{C4}$$

Recalling $v(t)=\mathrm{e}^{\lambda t}u(t)$, we have

$$\max_{t_0\leqslant t\leqslant t_k}\left|v''(t)\right|\leqslant \mathrm{e}^{\lambda t_k}\left[\lambda^2\max_{t_0\leqslant t\leqslant t_k}|u(t)|+2\lambda\max_{t_0\leqslant t\leqslant t_k}\left|u'(t)\right|+\max_{t_0\leqslant t\leqslant t_k}\left|u''(t)\right|\right]. \tag{C5}$$

Combining Eqs. (C3)–(C5) with Eq. (C2) we arrive at Eq. (10).

Appendix D: Proof of Lemma 4

(i) For $k=2$, $\overline{d}_{c,1}^{(k,\alpha)}=c_1^{(k,\alpha)}\mathrm{e}^{\lambda(t_0-t_2)}$, and

$$c_1^{(k,\alpha)}=a_1^{(\alpha)}-b_0^{(\alpha)}=2^{1-\alpha}-\frac{1}{2-\alpha}-\frac{1}{2},\quad \alpha\in(0,1),$$

which is strictly monotone decreasing, so $c_1^{(k,\alpha)}\in(-1/2,1)$. Moreover, $\alpha_1\approx 0.673\,6$ is the unique zero point for $\alpha\in(0,1)$, which means

$$c_1^{(k,\alpha)}\begin{cases}>0, & \text{if } \alpha\in(0,\alpha_1),\\ \leqslant 0, & \text{if } \alpha\in[\alpha_1,1).\end{cases}$$

Thus, $\overline{d}_{c,1}^{(k,\alpha)}\leqslant c_1^{(k,\alpha)}$ for $\alpha\in(0,\alpha_1)$.

(ii) For $k=3$, $\overline{d}_{c,2}^{(k,\alpha,\sigma)}-\widetilde{d}_{c,1}^{(k,\alpha)}=\left(c_1^{(k,\alpha)}-c_2^{(k,\alpha)}\right)\mathrm{e}^{\lambda(t_1-t_3)}$ and $c_1^{(k,\alpha)}-c_2^{(k,\alpha)}=a_1^{(\alpha)}+2b_1^{(\alpha)}-b_0^{(\alpha)}-a_2^{(\alpha)}$. Moreover, the zero point of $c_1^{(k,\alpha)}-c_2^{(k,\alpha)}$ showed in Fig. 6a is $\alpha_2\approx 0.390\,9$, i.e.,

$$c_1^{(k,\alpha)}-c_2^{(k,\alpha)}\begin{cases}>0, & \text{if } \alpha\in(0,\alpha_2),\\ \leqslant 0, & \text{if } \alpha\in[\alpha_2,1).\end{cases}$$

Thus $\overline{d}_{c,2}^{(k,\alpha)}-\widetilde{d}_{c,1}^{(k,\alpha)}\leqslant c_1^{(k,\alpha)}-c_2^{(k,\alpha)}$ for $\alpha\in(0,\alpha_2)$.

(iii) For $k>3$, $\overline{d}_{c,k-1}^{(k,\alpha)}-\widetilde{d}_{c,k-2}^{(k,\alpha)}=\left(c_1^{(k,\alpha)}-c_2^{(k,\alpha)}\right)\mathrm{e}^{\lambda(t_{k-2}-t_k)}\leqslant c_1^{(k,\alpha)}-c_2^{(k,\alpha)}$ for $\alpha\in(0,\alpha_3)$, where the zero point α_3 is about 0.373 9 in Fig. 6b and $c_1^{(k,\alpha)}-c_2^{(k,\alpha)}=a_1^{(\alpha)}+2b_1^{(\alpha)}-b_0^{(\alpha)}-a_2^{(\alpha)}-b_2^{(\alpha)}$.

(iv) The computed numerically as in the Fig. 6.

Appendix E: Proof of Lemma 5

For $k=1$, we find that Eq. (12) actually is the tempered L1 formula of the Caputo-tempered fractional derivative and its numerical accuracy is $\mathcal{O}(\tau^{2-\alpha})$, i.e.,

$$\begin{aligned}\left|T^1\right|&=\frac{\mathrm{e}^{-\lambda t_1}}{\Gamma(1-\alpha)}\left|\int_{t_0}^{t_1}\left[v(s)-P_{1,1}v(s)\right]'(t_1-s)^{-\alpha}\mathrm{d}s\right|\\&=\frac{\alpha\mathrm{e}^{-\lambda t_1}}{2\Gamma(3-\alpha)}\max_{t_0\leqslant t\leqslant t_1}\left|v''(t)\right|\tau^{2-\alpha},\end{aligned}$$

thus Eq. (17) is established.

For $k \geqslant 2$, we get

$$T^k = \frac{e^{-\lambda t_k}}{\Gamma(1-\alpha)}\left\{ \int_{t_0}^{t_1} \left[v(s) - P_{1,1}v(s)\right]'(t_k - s)^{-\alpha}ds + \sum_{\ell=2}^{k}\int_{t_{\ell-1}}^{t_\ell} \left[v(s) - P_{2,\ell}v(s)\right]'(t_k - s)^{-\alpha}ds \right\}$$
$$= \frac{-\alpha e^{-\lambda t_k}}{\Gamma(1-\alpha)}\left\{ \int_{t_0}^{t_1} \left[v(s) - P_{1,1}v(s)\right](t_k - s)^{-\alpha-1}ds + \sum_{\ell=2}^{k}\int_{t_{\ell-1}}^{t_\ell} \left[v(s) - P_{2,\ell}v(s)\right](t_k - s)^{-\alpha-1}ds \right\}. \tag{E1}$$

In [14] there exist

$$\left| \int_{t_0}^{t_1} \left[v(s) - P_{1,1}v(s)\right](t_k - s)^{-\alpha-1}ds \right| \leqslant \frac{1}{12}\max_{t_0 \leqslant t \leqslant t_1} |v''(t)|(t_k - t_1)^{-\alpha-1}\tau^3, \tag{E2}$$

$$\left| \sum_{\ell=2}^{k-1}\int_{t_{\ell-1}}^{t_\ell} \left[v(s) - P_{2,\ell}v(s)\right](t_k - s)^{-\alpha-1}ds \right| \leqslant \frac{1}{12\alpha}\max_{t_0 \leqslant t \leqslant t_{k-1}} |v'''(t)|\tau^{3-\alpha}, \tag{E3}$$

$$\left| \int_{t_{k-1}}^{t_k} \left[v(s) - P_{2,k}v(s)\right](t_k - s)^{-\alpha-1}ds \right| \leqslant \frac{1}{3}\frac{1}{(1-\alpha)(2-\alpha)}\left(\frac{1}{2} + \frac{1}{3-\alpha}\right)\max_{t_{k-2} \leqslant t \leqslant t_k} v'''(t)\tau^{3-\alpha}, \tag{E4}$$

and we have

$$\max_{t_0 \leqslant t \leqslant t_k} |v'''(t)| \leqslant e^{\lambda t_k}\left[\lambda^3 \max_{t_0 \leqslant t \leqslant t_k} |u(t)| + 3\lambda^2 \max_{t_0 \leqslant t \leqslant t_k} |u'(t)| + 3\lambda \max_{t_0 \leqslant t \leqslant t_k} |u''(t)| + \max_{t_0 \leqslant t \leqslant t_k} |u'''(t)|\right]. \tag{E5}$$

The substitution of Eqs. (E2)–(E5) into Eq. (E1) can lead to Eq. (18).

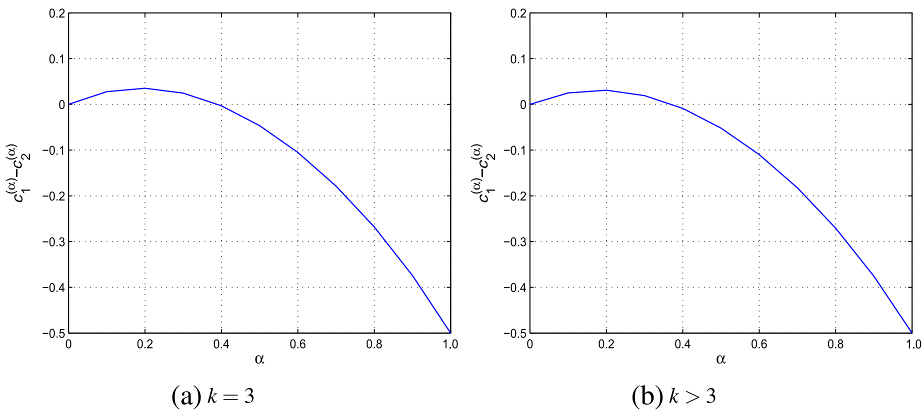

Fig. 6 The curves with $c_1^{(k,\alpha)} - c_2^{(k,\alpha)}$ with different k

Appendix F: Proof of Lemma 7

The error estimate is

$$T^{k+\sigma} = T^k + T_k^{k+\sigma}. \tag{F1}$$

Similarly to [1], we have

$$\begin{aligned} |T^k| &= \left| \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}}{\Gamma(1-\alpha)} \sum_{\ell=1}^{k} \int_{t_{\ell-1}}^{t_\ell} [v(s) - \Pi_{2,\ell} v(s)]'(t_{k+\sigma} - s)^{-\alpha} \mathrm{d}s \right| \\ &\leqslant \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}}{3\sigma^\alpha \Gamma(1-\alpha)} \max_{t_0 \leqslant t \leqslant t_{k+1}} |v'''(t)| \tau^{3-\alpha}, \end{aligned} \tag{F2}$$

and

$$\begin{aligned} |T_k^{k+\sigma}| &= \left| \frac{\mathrm{e}^{-\lambda t_{k+\sigma}}}{\Gamma(1-\alpha)} \int_{t_k}^{t_{k+\sigma}} (v'(s) - \delta_t v_{k+\frac{1}{2}})(t_{k+\sigma} - s)^{-\alpha} \mathrm{d}s \right| \\ &= \frac{\mathrm{e}^{-\lambda t_{k+\sigma}} v''(t_{k+\frac{1}{2}})}{\Gamma(1-\alpha)} \int_{t_k}^{t_{k+\sigma}} \frac{(s - t_{k+\frac{1}{2}})}{(t_{k+\sigma} - s)^\alpha} \mathrm{d}s + \mathcal{O}(\tau^{3-\alpha}), \end{aligned} \tag{F3}$$

where another term in (F3) besides $\mathcal{O}(\tau^{3-\alpha})$ is zero due to the special choice of $\sigma = 1 - \alpha/2$, so the order of truncation error $T^{k+\sigma}$ is $3-\alpha$. At the same time, there exists

$$\begin{aligned} \max_{t_0 \leqslant t \leqslant t_{k+1}} \left| v'''(t) \right| \leqslant \mathrm{e}^{\lambda t_{k+1}} \Big[& \lambda^3 \max_{t_0 \leqslant t \leqslant t_{k+1}} |u(t)| + 3\lambda^2 \max_{t_0 \leqslant t \leqslant t_{k+1}} \left| u'(t) \right| \\ & + 3\lambda \max_{t_0 \leqslant t \leqslant t_{k+1}} \left| u''(t) \right| + \max_{t_0 \leqslant t \leqslant t_{k+1}} \left| u'''(t) \right| \Big]. \end{aligned} \tag{F4}$$

Then substituting Eqs. (F2)–(F4) into Eq. (F1), we have

$$\begin{aligned} \left| T^{k+\sigma} \right| \leqslant \frac{\mathrm{e}^{\lambda(1-\sigma)\tau}}{3\sigma^\alpha \Gamma(1-\alpha)} \Big[& \lambda^3 \max_{t_0 \leqslant t \leqslant t_{k+1}} |u(t)| + 3\lambda^2 \max_{t_0 \leqslant t \leqslant t_{k+1}} \left| u'(t) \right| \\ & + 3\lambda \max_{t_0 \leqslant t \leqslant t_{k+1}} \left| u''(t) \right| + \max_{t_0 \leqslant t \leqslant t_{k+1}} \left| u'''(t) \right| \Big] \tau^{3-\alpha} + \mathcal{O}(\tau^{3-\alpha}), \end{aligned}$$

and using the Taylor expansion

$$\mathrm{e}^{\lambda(1-\sigma)\tau} = 1 + \lambda(1-\sigma)\tau + \frac{(\mathrm{e}^{\lambda(1-\sigma)\tau})''(\xi)}{2!} \tau^2,$$

thus Eq. (24) is proved.

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