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ORIGINAL PAPER

A High-Order Scheme for Fractional Ordinary Differential Equations with the Caputo–Fabrizio Derivative

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Abstract

In this paper, we consider numerical solutions of fractional ordinary differential equations with the Caputo–Fabrizio derivative, and construct and analyze a high-order time-stepping scheme for this equation. The proposed method makes use of quadratic interpolation function in sub-intervals, which allows to produce fourth-order convergence. A rigorous stability and convergence analysis of the proposed scheme is given. A series of numerical examples are presented to validate the theoretical claims. Traditionally a scheme having fourth-order convergence could only be obtained by using block-by-block technique. The advantage of our scheme is that the solution can be obtained step by step, which is cheaper than a block-by-block-based approach.

Keywords Caputo–Fabrizio derivative · Fractional differential equations · High-order numerical scheme

Mathematics Subject Classification 26A33 · 34A08 · 65M12 · 65M06

1 Introduction

In the last decades the fractional calculus had a remarkable development as shown by many mathematical volumes dedicated to it. We can see for instance the monograph [26] and the references therein. For a general right-hand side function *f*, it is usually difficult to obtain the analytical solution to a fractional differential equation. Thus there is a need to develop

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numerical methods for that equations. Cao and Xu [9] considered the fractional ordinary differential equations, and gave a modified block-by-block method for approximation to the fractional-order time derivative. Yang et al. [33] devoted to applications of fractional multistep methods for the fractional diffusion-wave equation. Sun and Wu [30] and Lin and Xu [21] analyzed a finite difference schema for the time discretization of the time-fractional diffusion equation, and proved that the convergence in time is of $2-\alpha$ order. Gao et al. [14] gave a $3-\alpha$ order L1-2 formula to approximate the Caputo fractional derivative of order α. Huang et al. [18] proved that the convergence of this method is of order at least three. Some other related work includes fast solvers for time-fractional diffusion equations [2, 7, 8, 11, 20, 22, 25, 31, 34, 35] and special care for treating the starting time singularity of the solutions [17, 19, 29]. We would like also to mention some relevant work for similar problems such as numerical methods for the time-fractional coupled mKdV equation, fractional Fisher's type equations [15, 27].

However, some issues have been raised for the somewhat cumbersome mathematical expression of fractional operators and the consequent complications in the solutions of the associated equations. Caputo and Fabrizio [10] introduced in 2015 a new definition of fractional derivative with a smooth kernel, which is called the Caputo–Fabrizio fractional derivative. The Caputo–Fabrizio fractional derivative can be useful to understand some phenomena, such as the thermal analysis of a second grade fluid [28], the electrochemical phenomena [16], and the anomalous diffusion [32]. Numerically, Firoozjaee et al. [13] used the Ritz approximation for the Caputo–Fabrizio fractional derivative. In [4–6, 12, 23, 24], the authors proposed some second-order finite difference schemes for the Caputo–Fabrizio fractional derivative. Very recently, Akman et al. [1] constructed a third-order finite difference scheme for this derivative. However, to the best of our knowledge, the convergence order of the existing schemes is no more than three.

Inspired by the idea in [9], the current paper aims at constructing and analyzing a higher order numerical method for fractional ordinary differential equations with the Caputo–Fabrizio derivative. The outline of this paper is as follows: in Sect. 2, we present some basic properties of the fractional ordinary differential equation under consideration. In Sect. 3, we describe the detailed construction of the high-order scheme for the Caputo–Fabrizio derivative. The error estimation and stability analysis are given in Sect. 4. We provide some numerical examples in Sect. 5 to support the theoretical results. Finally, some concluding remarks are given in the final section.

2 Problem and Basic Properties

We consider the following initial value problem: $\alpha \in (0,1)$,

$$D_t^\alpha u(t) = f(t), \quad 0 < t \le T, \tag{2.1}$$

subject to the initial condition $u(0)=u_0$. In (2.1), the operator D_t^α is the Caputo–Fabrizio derivative, defined by

$$D_t^\alpha u(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t u'(\tau)\exp\left(-\alpha\frac{t-\tau}{1-\alpha}\right)\mathrm{d}\tau, \tag{2.2}$$

where $M(\alpha)$ is a positive normalization function satisfying $M(0)=M(1)=1$.

This fractional derivative was first introduced by Caputo–Fabrizio in [10] with the aim of replacing the singular kernel in the traditional Caputo derivative by a regular kernel. It was claimed that the new definition can better describe some class of material heterogeneities, which cannot be well described by classical local theories or by fractional models with a singular kernel. Precisely, the Caputo–Fabrizio derivative is obtained by changing the kernel $(t-\tau)^{-\alpha}$ in the Caputo derivative by the exponential function $\exp(-\alpha(t-\tau)/(1-\alpha))$ and $1/\Gamma(1-\alpha)$ by $M(\alpha)/(1-\alpha)$. According to the new definition, it is readily seen that if u is a constant function, then $D_t^\alpha u \equiv 0$ as in the Caputo derivative. Contrary to the traditional definition, the main difference of the new definition is that the new kernel has no singularity for $t=\tau$.

It can be directly verified; see also [10], that the Laplace transform of the Caputo–Fabrizio derivative has the following expression:

$$\mathcal{L}[D_t^\alpha u(t)](s) = \frac{M(\alpha)(s\mathcal{L}[u(t)](s) - u(0))}{s+\alpha(1-s)}. \tag{2.3}$$

This is one of the interesting properties of the Caputo–Fabrizio derivative since the Laplace transform of $D_t^\alpha u(t)$ is linked to the Laplace transform of $u(t)$ in a very simple way. Thus if u is a solution of (2.1), then it follows from the relationship (2.3) that

$$\mathcal{L}[u(t)](s) = \frac{1}{s}u(0) + \frac{\alpha}{sM(\alpha)}\mathcal{L}[f(t)](s) + \frac{1-\alpha}{M(\alpha)}\mathcal{L}[f(t)](s).$$

Using known properties of the inverse Laplace transform, we deduce that

$$u(t) = u(0) + \frac{\alpha}{M(\alpha)}\int_0^t f(\tau)\mathrm{d}\tau + \frac{1-\alpha}{M(\alpha)}f(t). \tag{2.4}$$

It is readily seen from (2.4) that $u(t)$ satisfies the initial condition $u(0)=u_0$ if and only if $f(0)=0$. In fact $f(0)=0$ is a necessary condition for (2.1) to admit a C^1 solution. This can be directly observed by taking limit of the both sides of (2.1) as $t\to 0$.

Let us consider the case $f\equiv 0$, i.e., $D_t^\alpha u(t)=0$. It follows from (2.4) that $u(t)=u(0)$ for all $t\geq 0$. This, together with the definition (2.2), proves that $D_t^\alpha u(t)=0$ if and only if u is a constant. This is a good property of this new derivative, which is shared by the fractional Caputo and classical first-order derivatives.

Let us now consider the case $f=\lambda u$, where λ is a constant. Then (2.1) becomes an eigenvalue problem associated to the Caputo–Fabrizio differential operator D_t^α. For integer-order differential equations such a problem has often been served as a standard test case for investigating the stability of numerical methods. We will do the same for the Caputo–Fabrizio differential operator under consideration. Therefore it is interesting to see how the eigenfunctions behave. First we notice that the solution of (2.1) with $f(t)=\lambda u(t)$ has the following expression:

$$u(t) = \frac{M(\alpha)u_0}{M(\alpha)-\lambda(1-\alpha)}\exp\left(\frac{\lambda\alpha t}{M(\alpha)-\lambda(1-\alpha)}\right). \tag{2.5}$$

In fact, by applying the Laplace transform to both sides of (2.1) and using (2.3), we obtain

$$\frac{M(\alpha)(s\mathcal{L}[u(t)](s)-u(0))}{s+\alpha(1-s)} = \lambda\mathcal{L}[u(t)](s).$$

Rearranging it, we arrive at the following equality:

$$\mathcal{L}[u(t)](s) = \frac{M(\alpha)u_0}{[M(\alpha)-\lambda(1-\alpha)]s-\lambda\alpha}. \tag{2.6}$$

Finally, taking the inverse Laplace transform on both sides of (2.6) gives (2.5). However, we immediately realize that the solution (2.5) does not satisfy the initial condition unless $\lambda = 0$ or $u_0 = 0$. In the latter case, we have $u \equiv 0$. This means that the eigenvalue problem

$$\begin{cases} D_t^\alpha u(t) = \lambda u(t), \ t > 0, \\ u(0) = u_0 \end{cases}$$

is not well defined since either eigenvalue or eigenfunction has to be zero.

The above discussion motivates us to consider an alternative eigenvalue problem as follows:

$$\begin{cases} D_t^\alpha u(t) = f(t), \ t > 0, \\ u(0) = u_0, \end{cases} \tag{2.7}$$

where

$$f(t) = \lambda \Big[u(t) - u_0 \exp\Big(-\frac{\alpha}{1-\alpha} t \Big) \Big]. \tag{2.8}$$

We first notice that the function f takes value 0 at $t = 0$, which is a necessary condition for (2.7) to admit a non-trivial solution. In fact, it is not difficult to prove that the problem (2.7) admits the unique solution

$$u(t) = u_0 \exp\left(\frac{\lambda \alpha t}{M(\alpha) - \lambda(1-\alpha)} \right). \tag{2.9}$$

A direct calculation with (2.9) shows that the solution is decreasing when $u_0 \lambda < 0$. This property will be used to analyze the stability of the scheme to be constructed in the next section.

3 A Finite Difference Approximation to the Caputo–Fabrizio Derivative

In this section, we will construct and analyze an efficient numerical scheme for the problem (2.1). Particularly the scheme to be proposed will be tested for the problem (2.7) to see if the numerical solutions share same monotonic property as the exact solution (2.9).

First we propose a high-order approximation to the Caputo–Fabrizio derivative, and analyze its approximation property. Let us consider the following gird in [0, T]: $t_j = jh, j = 0, 1, 2, \cdots, 2N$, where N is a positive integer, and $h = \frac{T}{2N}$ is the grid size. We use u_i to denote $u(t_i), i = 0, 1, 2, \cdots, 2N$.

The question is how to efficiently approximate $D_t^\alpha u(t)$ at a given point t. Note that $u(t)$ can be approximated in $[t_0, t_1]$ by using the quadratic interpolation as

$$u(t) \approx \varphi_{0,0}(t)u_0 + \varphi_{1,0}(t)u_{1/2} + \varphi_{2,0}(t)u_1 \doteq I_{[t_0,t_1]}u(t) \tag{3.1}$$

with $u_{1/2} = u(t_{1/2}), t_{1/2} = t_0 + \frac{1}{2}h$, and

$$\varphi_{0,0}(t) = \frac{2(t-t_{1/2})(t-t_1)}{h^2}, \varphi_{1,0}(t) = \frac{-4(t-t_0)(t-t_1)}{h^2}, \varphi_{2,0}(t) = \frac{2(t-t_0)(t-t_{1/2})}{h^2}.$$

Plugging (3.1) into (2.2), we obtain an approximation

$$\begin{aligned} D_t^{\alpha}u(t_1) =&\frac{M(\alpha)}{1-\alpha}\int_0^{t_1} u'(\tau)\exp\Big(-\alpha\frac{t_1-\tau}{1-\alpha}\Big)\mathrm{d}\tau\\ \approx&\frac{M(\alpha)}{1-\alpha}\int_0^{t_1} [\varphi_{0,0}(\tau)u_0+\varphi_{1,0}(\tau)u_{1/2}+\varphi_{2,0}(\tau)u_1]'\exp\Big(-\alpha\frac{t_1-\tau}{1-\alpha}\Big)\mathrm{d}\tau\\ =&w_1^{0,0}u_0+w_1^{1,0}u_{1/2}+w_1^{2,0}u_1, \end{aligned} \tag{3.2}$$

where

$$w_1^{i,0}=\frac{M(\alpha)}{1-\alpha}\int_0^{t_1}\varphi_{i,0}'(\tau)\exp\Big(-\frac{\alpha}{1-\alpha}(t_1-\tau)\Big)\mathrm{d}\tau,\ \ i=0,1,2,$$

which can be exactly computed. The value of $u_{1/2}$ will be obtained by the following interpolation:

$$u_{1/2}\approx\frac{3}{8}u_0+\frac{3}{4}u_1-\frac{1}{8}u_2.$$

Inserting the above approximation into (3.2), we get

$$D_t^{\alpha}u(t_1)\approx a_1^{0,0}u_0+a_1^{1,0}u_1+a_1^{2,0}u_2\doteq D_h^{\alpha}u(t_1), \tag{3.3}$$

where

$$a_1^{0,0}=w_1^{0,0}+\frac{3}{8}w_1^{1,0},\quad a_1^{1,0}=\frac{3}{4}w_1^{1,0}+w_1^{2,0},\quad a_1^{2,0}=-\frac{1}{8}w_1^{1,0}.$$

Similarly, $u(t)$ in $[t_0,t_2]$ can be approximated by

$$u(t)\approx\psi_{0,0}(t)u_0+\psi_{1,0}(t)u_1+\psi_{2,0}(t)u_2\doteq I_{[t_0,t_2]}u(t),$$

where $\psi_{i,0}(t), i=0,1,2$, are defined as follows:

$$\psi_{0,0}(t)=\frac{(t-t_1)(t-t_2)}{2h^2},\psi_{1,0}(t)=\frac{(t-t_0)(t-t_2)}{-h^2},\psi_{2,0}(t)=\frac{(t-t_0)(t-t_1)}{2h^2}.$$

Using this approximation yields

$$\begin{aligned} D_t^{\alpha}u(t_2) =&\frac{M(\alpha)}{1-\alpha}\int_0^{t_2} u'(\tau)\exp\Big(-\alpha\frac{t_2-\tau}{1-\alpha}\Big)\mathrm{d}\tau\\ \approx&\frac{M(\alpha)}{1-\alpha}\int_0^{t_2} [\psi_{0,0}(\tau)u_0+\psi_{1,0}(\tau)u_1+\psi_{2,0}(\tau)u_2]'\exp\Big(-\alpha\frac{t_2-\tau}{1-\alpha}\Big)\mathrm{d}\tau\\ =&a_2^{0,0}u_0+a_2^{1,0}u_1+a_2^{2,0}u_2\doteq D_h^{\alpha}u(t_2), \end{aligned} \tag{3.4}$$

where

$$a_2^{i,0}=\frac{M(\alpha)}{1-\alpha}\int_0^{t_2}\psi_{i,0}'(\tau)\exp\Big(-\frac{\alpha}{1-\alpha}(t_2-\tau)\Big)\mathrm{d}\tau,\ \ i=0,1,2.$$

Now assuming that the values of u at the grid points $t_j, j=0,1,\cdots,2m$, are already known, we want to derive an approximation to $D_t^{\alpha}u(t_{2m+1})$ and $D_t^{\alpha}u(t_{2m+2})$. Similar to the previous sub-intervals, we use the following quadratic interpolation functions to approximate u in $[t_{2k-1}, t_{2k+1}], k = 1, 2, \cdots, m$:

$$u(t) \approx \varphi_{0,k}(t)u_{2k-1} + \varphi_{1,k}(t)u_{2k} + \varphi_{2,k}(t)u_{2k+1} \doteq I_{[t_{2k-1},t_{2k+1}]}u(t),$$

where $\varphi_{0,k}(t) = \frac{(t-t_{2k})(t-t_{2k+1})}{2h^2}, \varphi_{1,k}(t) = \frac{(t-t_{2k-1})(t-t_{2k+1})}{-h^2}, \varphi_{2,k}(t) = \frac{(t-t_{2k-1})(t-t_{2k})}{2h^2}$. This suggests the following approach:

$$\begin{aligned}
&D_t^\alpha u(t_{2m+1}) \\
&= \frac{M(\alpha)}{1-\alpha}\int_0^{t_{2m+1}} u'(\tau)\exp\left(-\alpha\frac{t_{2m+1}-\tau}{1-\alpha}\right)\mathrm{d}\tau \\
&= \frac{M(\alpha)}{1-\alpha}\left[\int_0^{t_1} u'(\tau)\exp\left(-\alpha\frac{t_{2m+1}-\tau}{1-\alpha}\right)\mathrm{d}\tau + \sum_{k=1}^m\int_{t_{2k-1}}^{t_{2k+1}} u'(\tau)\exp\left(-\alpha\frac{t_{2m+1}-\tau}{1-\alpha}\right)\mathrm{d}\tau\right] \\
&\approx \frac{M(\alpha)}{1-\alpha}\int_0^{t_1}[I_{[t_0,t_1]}u(\tau)]'\exp\left(-\alpha\frac{t_{2m+1}-\tau}{1-\alpha}\right)\mathrm{d}\tau \\
&\quad + \sum_{k=1}^m\frac{M(\alpha)}{1-\alpha}\int_{t_{2k-1}}^{t_{2k+1}}[I_{[t_{2k-1},t_{2k+1}]}u(\tau)]'\exp\left(-\alpha\frac{t_{2m+1}-\tau}{1-\alpha}\right)\mathrm{d}\tau \\
&= a_{2m+1}^{0,0}u_0 + a_{2m+1}^{1,0}u_1 + a_{2m+1}^{2,0}u_2 + \sum_{k=1}^m[a_{2m+1}^{0,k}u_{2k-1} + a_{2m+1}^{1,k}u_{2k} + a_{2m+1}^{2,k}u_{2k+1}] \\
&\doteq D_h^\alpha u(t_{2m+1}),
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
&a_{2m+1}^{0,0} = w_{2m+1}^{0,0} + \frac{3}{8}w_{2m+1}^{1,0}, \quad a_{2m+1}^{1,0} = \frac{3}{4}w_{2m+1}^{1,0} + w_{2m+1}^{2,0}, \quad a_{2m+1}^{2,0} = -\frac{1}{8}w_{2m+1}^{1,0}, \\
&a_{2m+1}^{i,k} = \frac{M(\alpha)}{1-\alpha}\int_{t_{2k-1}}^{t_{2k+1}}\varphi_{i,k}'(\tau)\exp\left(-\frac{\alpha}{1-\alpha}(t_{2m+1}-\tau)\right)\mathrm{d}\tau, i = 0, 1, 2; k = 1, 2, \cdots, m
\end{aligned}$$

with

$$w_{2m+1}^{i,0} = \frac{M(\alpha)}{1-\alpha}\int_0^{t_1}\varphi_{i,0}'(\tau)\exp\left(-\frac{\alpha}{1-\alpha}(t_{2m+1}-\tau)\right)\mathrm{d}\tau, i = 0, 1, 2.$$

In the sub-intervals $[t_{2k}, t_{2k+2}], k = 0, 1, \cdots, m$, we approximate u by

$$u(t) \approx \psi_{0,k}(t)u_{2k} + \psi_{1,k}(t)u_{2k+1} + \psi_{2,k}(t)u_{2k+2} \doteq I_{[t_{2k},t_{2k+2}]}u(t),$$

where $\psi_{0,k}(t) = \frac{(t-t_{2k+1})(t-t_{2k+2})}{2h^2}, \psi_{1,k}(t) = \frac{(t-t_{2k})(t-t_{2k+2})}{-h^2}, \psi_{2,k}(t) = \frac{(t-t_{2k})(t-t_{2k+1})}{2h^2}$.

As a consequence

$$\begin{aligned}
&D_t^\alpha u(t_{2m+2}) \\
&= \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^m\int_{t_{2k}}^{t_{2k+2}} u'(\tau)\exp\left(-\alpha\frac{t_{2m+2}-\tau}{1-\alpha}\right)\mathrm{d}\tau \\
&\approx \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^m\int_{t_{2k}}^{t_{2k+2}}[I_{[t_{2k},t_{2k+2}]}u(\tau)]'\exp\left(-\alpha\frac{t_{2m+2}-\tau}{1-\alpha}\right)\mathrm{d}\tau \\
&= \sum_{k=0}^m\left(a_{2m+2}^{0,k}u_{2k} + a_{2m+2}^{1,k}u_{2k+1} + a_{2m+2}^{2,k}u_{2k+2}\right) \doteq D_h^\alpha u(t_{2m+2}),
\end{aligned} \tag{3.6}$$

where

$$a_{2m+2}^{i,k} = \begin{cases} \frac{M(\alpha)}{1-\alpha} \int_{t_{2k}}^{t_{2k+2}} \psi'_{i,k}(\tau) \exp\left(-\frac{\alpha}{1-\alpha}(t_{2m+2}-\tau)\right) \mathrm{d}\tau, \\ i = 0, 1, 2;\ k = 0, 1, \cdots, m. \end{cases} \tag{3.7}$$

We are in a position to construct our numerical scheme for the fractional differential equation (2.1) subject to the initial condition u_0. Based on the finite difference operator D_h^α defined in (3.3)–(3.6), we propose the following scheme:

$$D_h^\alpha u(t_k) = f(t_k), k = 1, 2, \cdots, 2N \tag{3.8}$$

to numerically solve the problem (2.1).

4 Error Estimate and Stability Analysis

This section is devoted to carry out the stability and convergence analysis, and derive error estimates for the numerical solutions. We start with deriving an error estimate for the finite difference operator D_h^α.

Theorem 4.1 *Assume $u(t) \in C^4[0, T]$. Let*

$$r_k := D_t^\alpha u(t_k) - D_h^\alpha u(t_k), \quad k = 1, 2, \cdots, 2N. \tag{4.1}$$

Then it holds

$$|r_k| \leq ch^4, \quad k = 1, 2, \cdots, 2N, \tag{4.2}$$

where c is a constant independent of h, but depends on u.

Proof Applying the Taylor theorem, we have

$$\left| u_{1/2} - \left(\frac{3}{8} u_0 + \frac{3}{4} u_1 - \frac{1}{8} u_2 \right) \right| \leq ch^3, \tag{4.3}$$

$$\begin{cases} u(t) - I_{[t_0,t_1]} u(t) = \frac{u^{(3)}(\xi_0(t))}{6}(t-t_0)(t-t_{1/2})(t-t_1), \xi_0(t) \in (t_0, t_1), \forall t \in [t_0, t_1], \\ u(t) - I_{[t_{2k-1},t_{2k+1}]} u(t) = \frac{u^{(3)}(\xi_k(t))}{6}(t-t_{2k-1})(t-t_{2k})(t-t_{2k+1}), \\ \xi_k(t) \in (t_{2k-1}, t_{2k+1}), \forall t \in [t_{2k-1}, t_{2k+1}], k = 1, 2, \cdots, m, \\ u(t) - I_{[t_{2k},t_{2k+2}]} u(t) = \frac{u^{(3)}(\eta_k(t))}{6}(t-t_{2k})(t-t_{2k+1})(t-t_{2k+2}), \\ \eta_k(t) \in (t_{2k}, t_{2k+2}), \forall t \in [t_{2k}, t_{2k+2}], k = 0, 1, 2, \cdots, m. \end{cases} \tag{4.4}$$

We begin with estimating r_1. It follows from (3.3), (4.3), and (4.4):

$$\begin{aligned}
|r_1| &= |D_t^\alpha u(t_1) - D_h^\alpha u(t_1)| \\
&= \left|\frac{M(\alpha)}{1-\alpha}\int_0^{t_1} u'(\tau)\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\mathrm{d}\tau - (a_1^{0,0}u_0 + a_1^{1,0}u_1 + a_1^{2,0}u_2)\right| \\
&= \left|\frac{M(\alpha)}{1-\alpha}\int_0^{t_1} u'(\tau)\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\mathrm{d}\tau - \frac{M(\alpha)}{1-\alpha}\int_0^{t_1}[I_{[t_0,t_1]}u(\tau)]'\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\mathrm{d}\tau\right. \\
&\quad \left. + \frac{M(\alpha)}{1-\alpha}\int_0^{t_1}\left[u_{1/2} - \left(\frac{3}{8}u_0 + \frac{3}{4}u_1 - \frac{1}{8}u_2\right)\right]\varphi_{1,0}'(\tau)\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\mathrm{d}\tau\right| \\
&\le \frac{M(\alpha)}{1-\alpha}\left|\int_0^{t_1}[u(\tau) - I_{[t_0,t_1]}u(\tau)]'\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\mathrm{d}\tau\right| \\
&\quad + \frac{M(\alpha)}{1-\alpha}\left|\int_0^{t_1}\left[u_{1/2} - \left(\frac{3}{8}u_0 + \frac{3}{4}u_1 - \frac{1}{8}u_2\right)\right]\varphi_{1,0}'(\tau)\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\mathrm{d}\tau\right| \\
&\le \frac{M(\alpha)}{1-\alpha}\left|\int_0^{t_1}\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\mathrm{d}[u(\tau) - I_{[t_0,t_1]}u(\tau)]\right| \\
&\quad + \frac{M(\alpha)}{1-\alpha}ch^3\left|\int_0^{t_1}\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\mathrm{d}\varphi_{1,0}(\tau)\right| \\
&\le \frac{M(\alpha)}{1-\alpha}\left|\frac{u^{(3)}(\xi_0(\tau))}{6}(\tau-t_0)(\tau-t_{1/2})(\tau-t_1)\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\right|_{t_0}^{t_1} \\
&\quad \left. - \int_0^{t_1}\frac{u^{(3)}(\xi_0(\tau))}{6}(\tau-t_0)(\tau-t_{1/2})(\tau-t_1)\mathrm{d}\Big(\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\Big)\right| \\
&\quad + ch^3\frac{M(\alpha)}{1-\alpha}\left|\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)|\varphi_{1,0}(\tau)|_{t_0}^{t_1} - \int_0^{t_1}\varphi_{1,0}(\tau)\mathrm{d}\Big(\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\Big)\right| \\
&\le \frac{M(\alpha)}{1-\alpha}\left|\int_0^{t_1}\frac{u^{(3)}(\xi_0(t))}{6}(\tau-t_0)(\tau-t_{1/2})(\tau-t_1)\mathrm{d}\Big(\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\Big)\right| \\
&\quad + ch^3\frac{M(\alpha)}{1-\alpha}\left|\int_0^{t_1}\varphi_{1,0}(\tau)\mathrm{d}\Big(\exp\left(-\alpha\frac{t_1-\tau}{1-\alpha}\right)\Big)\right| \\
&\le \frac{cM(\alpha)}{(1-\alpha)^2}\alpha h^4\exp\left(-\alpha\frac{t_1-\eta}{1-\alpha}\right) + ch^4\frac{|\alpha M(\alpha)|}{(1-\alpha)^2}\exp\left(-\alpha\frac{t_1-\eta}{1-\alpha}\right) \\
&\le ch^4, \eta \in (t_0, t_1).
\end{aligned}$$

In the above derivation we have used the fact that $\exp(-\alpha\frac{t_1-\eta}{1-\alpha}) < 1$. In a similar way we can prove

$$|r_2| \le ch^4.$$

For r_{2m+1}, we have

$$
\begin{aligned}
|r_{2m+1}| =& \left| \frac{M(\alpha)}{1-\alpha} \int_0^{t_1} [u(\tau) - I_{[t_0,t_1]}u(\tau)]' \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \right. \\
&+ \frac{M(\alpha)}{1-\alpha} \int_0^{t_1} \left[u_{1/2} - \left(\frac{3}{8}u_0 + \frac{3}{4}u_1 - \frac{1}{8}u_2\right)\right] \varphi'_{1,0}(\tau) \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \\
&\left. + \sum_{k=1}^{m} \frac{M(\alpha)}{1-\alpha} \int_{t_{2k-1}}^{t_{2k+1}} [u(\tau) - I_{[t_{2k-1},t_{2k+1}]}u(\tau)]' \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \right| \\
\leq& \frac{M(\alpha)}{1-\alpha} \left| \int_0^{t_1} [u(\tau) - I_{[t_0,t_1]}u(\tau)]' \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \right| \\
&+ \frac{M(\alpha)}{1-\alpha} \left| \int_0^{t_1} \left[u_{1/2} - \left(\frac{3}{8}u_0 + \frac{3}{4}u_1 - \frac{1}{8}u_2\right)\right] \varphi'_{1,0}(\tau) \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \right| \\
&+ \sum_{k=1}^{m} \frac{M(\alpha)}{1-\alpha} \left| \int_{t_{2k-1}}^{t_{2k+1}} [u(\tau) - I_{[t_{2k-1},t_{2k+1}]}u(\tau)]' \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \right| \\
\leq& ch^4 + \frac{M(\alpha)}{1-\alpha} \sum_{k=1}^{m} \left| \int_{t_{2k-1}}^{t_{2k+1}} \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}[u(\tau) - I_{[t_{2k-1},t_{2k+1}]}u(\tau)] \right| \\
=& ch^4 + \frac{M(\alpha)}{1-\alpha} \sum_{k=1}^{m} \left| \frac{u^{(3)}(\xi_k(\tau))}{6} \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \right|_{t_{2k-1}}^{t_{2k+1}} \\
&\left. - \int_{t_{2k-1}}^{t_{2k+1}} \frac{u^{(3)}(\xi_k(\tau))}{6} \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \frac{\alpha}{1-\alpha} \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \right| \\
\leq& ch^4 + \frac{\alpha M(\alpha)}{(1-\alpha)^2} \sum_{k=1}^{m} \left| \int_{t_{2k-1}}^{t_{2k+1}} \frac{u^{(3)}(\xi_k(\tau))}{6} \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \right|.
\end{aligned} \tag{4.5}
$$

The second term in the right-hand side, which will be denoted by R hereafter, can be bounded by

$$
\begin{aligned}
R =& c_\alpha \sum_{k=1}^{m} \left| \int_{t_{2k-1}}^{t_{2k+1}} \frac{u^{(3)}(\xi_k(\tau))}{6} \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \right| \\
\leq& c_\alpha \sum_{k=1}^{m} \left(\left| \int_{t_{2k-1}}^{t_{2k+1}} \frac{u^{(3)}(\widetilde{\xi}_k)}{6} \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \right| \right. \\
&\left. + \left| \int_{t_{2k-1}}^{t_{2k+1}} \frac{u^{(3)}(\xi_k(\tau)) - u^{(3)}(\widetilde{\xi}_k)}{6} \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \exp\left(-\alpha \frac{t_{2m+1}-\tau}{1-\alpha}\right) \mathrm{d}\tau \right| \right) \\
\doteq& R_1 + R_2,
\end{aligned} \tag{4.6}
$$

where $c_\alpha = \frac{\alpha M(\alpha)}{(1-\alpha)^2}$, $\widetilde{\xi}_k = t_{2k}$, R_1 can be controlled as follows:

$$
\begin{aligned}
|R_1| \leq & c_\alpha B_1 \sum_{k=1}^{m} \left| \int_{t_{2k-1}}^{t_{2k}} \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \exp\left(-\alpha \frac{t_{2m+1} - \tau}{1-\alpha}\right) \mathrm{d}\tau \right. \\
& \left. + \int_{t_{2k}}^{t_{2k+1}} \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \exp\left(-\alpha \frac{t_{2m+1} - \tau}{1-\alpha}\right) \mathrm{d}\tau \right| \\
= & c_\alpha B_1 \sum_{k=1}^{m} \left| \exp\left(-\alpha \frac{t_{2m+1} - s_1^k}{1-\alpha}\right) \int_{t_{2k-1}}^{t_{2k}} \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \mathrm{d}\tau \right. \\
& \left. + \exp\left(-\alpha \frac{t_{2m+1} - s_2^k}{1-\alpha}\right) \int_{t_{2k}}^{t_{2k+1}} \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \mathrm{d}\tau \right| \\
= & \frac{1}{4} c_\alpha B_1 h^4 \sum_{k=1}^{m} \left| \exp\left(-\alpha \frac{t_{2m+1} - s_1^k}{1-\alpha}\right) - \exp\left(-\alpha \frac{t_{2m+1} - s_2^k}{1-\alpha}\right) \right| \\
= & \frac{1}{4} c_\alpha B_1 h^4 \frac{\alpha}{1-\alpha} \sum_{k=1}^{m} \left| \int_{s_1^k}^{s_2^k} \exp\left(-\alpha \frac{t_{2m+1} - \tau}{1-\alpha}\right) \mathrm{d}\tau) \right| \\
\leq & c_\alpha B_1 \frac{\alpha}{1-\alpha} h^4 \sum_{k=1}^{m} \left| \int_{t_{2k-1}}^{t_{2k+1}} \exp\left(-\alpha \frac{t_{2m+1} - \tau}{1-\alpha}\right) \mathrm{d}\tau \right| \leq ch^4,
\end{aligned} \tag{4.7}
$$

where B_1 is an upper bound of $u^{(3)}$, $t_{2k-1} \leq s_1^k \leq t_{2k} \leq s_2^k \leq t_{2k+1}$.

Noticing that

$$
|u^{(3)}(\xi_k(\tau)) - u^{(3)}(\widetilde{\xi}_k)| \leq B_2 h, \forall \tau \in [t_{2k-1}, t_{2k+1}],
$$

where B_2 depends on the upper bound of $u^{(4)}$, we have

$$
\begin{aligned}
|R_2| \leq & \frac{c_\alpha B_2 h}{6} \sum_{k=1}^{m} \int_{t_{2k-1}}^{t_{2k+1}} \exp\left(-\alpha \frac{t_{2m+1} - \tau}{1-\alpha}\right) \left| \prod_{i=0}^{2} (\tau - t_{2k-1+i}) \right| \mathrm{d}\tau \\
\leq & c_\alpha B_2 h^4 \sum_{k=1}^{m} \int_{t_{2k-1}}^{t_{2k+1}} \exp\left(-\alpha \frac{t_{2m+1} - \tau}{1-\alpha}\right) \mathrm{d}\tau \\
\leq & c_\alpha B_2 h^4 \int_{t_0}^{t_{2m+1}} \exp\left(-\alpha \frac{t_{2m+1} - \tau}{1-\alpha}\right) \mathrm{d}\tau \leq ch^4.
\end{aligned} \tag{4.8}
$$

Bringing (4.5)–(4.8) together, we obtain

$$
|r_{2m+1}| < ch^4.
$$

Similarly, we can prove

$$
|r_{2m+2}| \leq ch^4.
$$

This completes the proof of Theorem 4.1.

In order to simply the stability analysis, we rewrite the scheme in the following form:

$$\widehat{b}_0 u_0 + \widehat{b}_1 u_1 + \widehat{b}_2 u_2 = f(t_1), \tag{4.9}$$

$$\widetilde{b}_0 u_0 + \widetilde{b}_1 u_1 + \widetilde{b}_2 u_2 = f(t_2), \tag{4.10}$$

$$\sum_{k=0}^{2m+1} b_k^{(m)} u_k = f(t_{2m+1}), m = 1, 2, \cdots, N-1, \tag{4.11}$$

$$\sum_{k=0}^{2m+2} \overline{b}_k^{(m)} u_k = f(t_{2m+2}), m = 1, 2, \cdots, N-1, \tag{4.12}$$

where

$$\begin{cases} \widehat{b}_i = a_1^{i,0}, \widetilde{b}_i = a_2^{i,0}, i = 0, 1, 2; \\ b_0^{(m)} = a_{2m+1}^{0,0}, b_1^{(m)} = a_{2m+1}^{1,0} + a_{2m+1}^{0,1}, b_2^{(m)} = a_{2m+1}^{2,0} + a_{2m+1}^{1,1}, \\ b_{2k}^{(m)} = a_{2m+1}^{1,k}, b_{2k-1}^{(m)} = a_{2m+1}^{2,k-1} + a_{2m+1}^{0,k}, k = 2, 3, \cdots, m;\ b_{2m+1}^{(m)} = a_{2m+1}^{2,m}; \\ \overline{b}_0^{(m)} = a_{2m+2}^{0,0}, \overline{b}_{2k}^{(m)} = a_{2m+2}^{2,k-1} + a_{2m+2}^{0,k}, k = 1, 2, \cdots, m; \\ \overline{b}_{2k+1}^{(m)} = a_{2m+2}^{1,k}, k = 0, 1, 2, \cdots, m;\ \overline{b}_{2m+2}^{(m)} = a_{2m+2}^{2,m}. \end{cases} \tag{4.13}$$

In the following lemma, we study the sign of the coefficients given in (4.13), which plays an important role in analyzing the stability of the scheme.

Lemma 4.1 *Some of the coefficients defined in* (4.13) *have different signs under some conditions as listed in Table 1.*

Proof First, we investigate the sign of $b_k^{(m)}$. A direct computation gives

Table 1 Coefficients defined in (4.13)

Coefficients in (4.13)	Sign	Condition
$b_0^{(m)}$	< 0	$h < \frac{4}{3}\frac{1-\alpha}{\alpha}$
$b_1^{(m)}$	< 0	$h < \frac{1}{3}\frac{1-\alpha}{\alpha}$
$b_2^{(m)}$	< 0	-
$b_{2k}^{(m)}, k = 2, \cdots, m$	< 0	-
$b_{2k-1}^{(m)}, k = 2, \cdots, m$	< 0	$h < \frac{1}{2}\frac{1-\alpha}{\alpha}$
$b_{2m+1}^{(m)}$	> 0	-
$\overline{b}_0^{(m)}$	< 0	$h < \frac{1}{3}\frac{1-\alpha}{\alpha}$
$\overline{b}_{2k}^{(m)}, k = 1, \cdots, m$	< 0	$h < \frac{1}{2}\frac{1-\alpha}{\alpha}$
$\overline{b}_{2k+1}^{(m)}, k = 0, 1, \cdots, m$	< 0	-
$\overline{b}_{2m+2}^{(m)}$	> 0	-

$$b_0^{(m)} = \frac{M(\alpha)}{\alpha h^2} F_1(h) \exp\Big(\frac{\alpha}{1-\alpha}(-2mh)\Big),$$

where $F_1(h) := -\frac{h}{2} - \frac{1-\alpha}{\alpha} + (\frac{3}{2}h + \frac{1-\alpha}{\alpha})\exp(\frac{\alpha}{1-\alpha}(-h))$. It can be shown that $F_1(h) < 0$ if $h < \frac{4}{3}\frac{1-\alpha}{\alpha}$. In fact, we have

$$\begin{aligned} F_1'(h) &= -\frac{1}{2} + \Big(\frac{1}{2} - \frac{3}{2}\frac{h\alpha}{1-\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big), \\ F_1''(h) &= \frac{\alpha}{1-\alpha}\Big(-2 + \frac{3}{2}\frac{h\alpha}{1-\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big). \end{aligned}$$

Thus, $F_1''(h) < 0$ if $h < \frac{4}{3}\frac{1-\alpha}{\alpha}$. Consequently, $F_1'(h) < F_1'(0) = 0$, and furthermore $F_1(h) < F_1(0) = 0$. This gives $b_0^{(m)} < 0$.

For $b_1^{(m)}$, we have

$$b_1^{(m)} = \frac{M(\alpha)}{\alpha h^2} F_2(h) \exp\Big(\frac{\alpha}{1-\alpha}(-2m+2)h\Big),$$

where $F_2(h) := \frac{h}{2} - \frac{1-\alpha}{\alpha} + (\frac{3}{2}h + \frac{3(1-\alpha)}{\alpha})\exp(\frac{\alpha}{1-\alpha}(-2h)) - 2(h + \frac{1-\alpha}{\alpha})\exp(\frac{\alpha}{1-\alpha}(-3h))$, and we calculate the first and second derivatives of $F_2(h)$ as follows:

$$\begin{aligned} F_2'(h) =& \frac{1}{2} - \Big(\frac{9}{2} + 3\frac{h\alpha}{1-\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big) + 2\Big(2 + 3\frac{h\alpha}{1-\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-3h)\Big), \\ F_2''(h) =& \frac{-6\alpha}{1-\alpha}\Big[-1 - \frac{h\alpha}{1-\alpha} + \Big(1 + 3\frac{h\alpha}{1-\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big)\Big]\exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big) \\ \doteq& \frac{-6\alpha}{1-\alpha} G_1(h) \exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big). \end{aligned}$$

It can be checked that

$$G_1(h) := -1 - \frac{h\alpha}{1-\alpha} + \Big(1 + 3\frac{h\alpha}{1-\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big) > 0$$

if $h < \frac{1}{3}\frac{1-\alpha}{\alpha}$. Therefore, $F_2''(h) < 0$, $F_2'(h) < F_2'(0) = 0$, $F_2(h) < F_2(0) = 0$. This gives $b_1^{(m)} < 0$.

For $b_2^{(m)}$, it holds

$$b_2^{(m)} = -\frac{M(\alpha)}{\alpha h^2} F_3(h) \exp\Big(\frac{\alpha}{1-\alpha}(-2m+2)h\Big),$$

where $F_3(h) := 2(h - \frac{1-\alpha}{\alpha}) + (\frac{3}{2}h + \frac{3(1-\alpha)}{\alpha})\exp(\frac{\alpha}{1-\alpha}(-2h)) - (\frac{h}{2} + \frac{1-\alpha}{\alpha})\exp(\frac{\alpha}{1-\alpha}(-3h))$.

It can be proved $F_3(h) > 0$ by following the same lines. Thus $b_2^{(m)} < 0$.

For $k = 2, \cdots, m$, we have

$$\begin{aligned} b_{2k-1}^{(m)} &= \frac{M(\alpha)}{\alpha h^2} F_4(h) \exp\Big(\frac{\alpha}{1-\alpha}(2k-2-2m)h\Big), \\ b_{2k}^{(m)} &= -\frac{2M(\alpha)}{\alpha h^2} F_5(h) \exp\Big(\frac{\alpha}{1-\alpha}(2k-2m)h\Big), \end{aligned}$$

where

$$F_4(h) = \Big(\frac{h}{2} - \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(2h)\Big) + 3h + \Big(\frac{h}{2} + \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big), \tag{4.14}$$

$$F_5(h) = h - \frac{1-\alpha}{\alpha} + \Big(h + \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big). \tag{4.15}$$

It can be directly verified that $F_4(h) < 0$ if $h < \frac{1}{2}\frac{1-\alpha}{\alpha}$ and $F_5(h) > 0$ for all $h > 0$. This proves that for all $k = 2, \cdots, m$, $b_{2k-1}^{(m)} < 0$ if $h < \frac{1}{2}\frac{1-\alpha}{\alpha}$, and $b_{2k}^{(m)} < 0$ for all $h > 0$.

Some more calculation gives

$$b_{2m+1}^{(m)} = \frac{M(\alpha)}{\alpha h^2}\Big(\frac{3}{2}h - \frac{1-\alpha}{\alpha} + \Big(\frac{h}{2} + \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big)\Big) > 0.$$

Now we turn to check the sign of the coefficients $\overline{b}_k^{(m)}$.

For $\overline{b}_0^{(m)}$, we have

$$\overline{b}_0^{(m)} = \frac{M(\alpha)}{\alpha h^2}F_6(h)\exp\Big(\frac{\alpha}{1-\alpha}(-2mh)\Big),$$

where $F_6(h) := \frac{1}{2}h - \frac{1-\alpha}{\alpha} + (\frac{3}{2}h + \frac{1-\alpha}{\alpha})\exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big)$, which is negative if $h < \frac{1}{3}\frac{1-\alpha}{\alpha}$. Therefore, $\overline{b}_0^{(m)} < 0$ under the same condition.

Furthermore, it holds

$$\begin{aligned}
\overline{b}_{2k}^{(m)} &= \frac{M(\alpha)}{\alpha h^2}F_4(h)\exp\Big(\frac{\alpha}{1-\alpha}(2k-2-2m)h\Big) < 0, \quad \text{if } h < \frac{1}{2}\frac{1-\alpha}{\alpha}, \quad k = 1, \cdots, m;\\
\overline{b}_{2k+1}^{(m)} &= -\frac{2M(\alpha)}{\alpha h^2}F_5(h)\exp\Big(\frac{\alpha}{1-\alpha}(2k-2m)h\Big) < 0, \quad k = 0, 1, \cdots, m;\\
\overline{b}_{2m+2}^{(m)} &= \frac{M(\alpha)}{\alpha h^2}\Big(\frac{3}{2}h - \frac{1-\alpha}{\alpha} + \Big(\frac{h}{2} + \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big)\Big) > 0.
\end{aligned}$$

The proof is completed.

Some more properties of the coefficients in (4.13) are given in Lemma 4.2.

Lemma 4.2 *It holds*

$$\beta_1 := \widehat{b}_1 + \widetilde{b}_2 > 0, \tag{4.16}$$

$$\beta_2 := \exp\Big(\frac{\alpha}{1-\alpha}(-t_1)\Big)\widetilde{b}_2 - \widehat{b}_0 - \exp\Big(\frac{\alpha}{1-\alpha}(-t_2)\Big)\widehat{b}_2 > 0. \tag{4.17}$$

Furthermore, if $h < \frac{2}{3}\frac{1-\alpha}{\alpha}$, it holds

$$\beta_3 := \exp\Big(\frac{\alpha}{1-\alpha}(-t_2)\Big)\widehat{b}_1 - \widetilde{b}_0 - \exp\Big(\frac{\alpha}{1-\alpha}(-t_1)\Big)\widetilde{b}_1 > 0. \tag{4.18}$$

Proof A direct computation gives

$$
\begin{aligned}
\widehat{b}_1 =& \frac{2M(\alpha)}{\alpha h^2}\Big[\frac{1-\alpha}{\alpha} - \Big(h + \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big)\Big],\\
\widetilde{b}_2 =& \frac{M(\alpha)}{\alpha h^2}\Big[\frac{3}{2}h - \frac{1-\alpha}{\alpha} + \Big(\frac{h}{2} + \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big)\Big].
\end{aligned}
$$

Summing these two equalities, we obtain

$$
\beta_1 = \frac{M(\alpha)}{\alpha h^2}\Big[\Big(\frac{h}{2} + \frac{1-\alpha}{\alpha}\Big)\Big(\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big) - 1\Big)^2 - h\Big(\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big) - 1\Big)\Big] > 0.
$$

This proves (4.16). To prove (4.17), noticing

$$
\begin{aligned}
\widehat{b}_0 =& \frac{M(\alpha)}{\alpha h^2}\Big[-\frac{1}{2}h - \frac{1-\alpha}{\alpha} + \Big(\frac{3}{2}h + \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big)\Big],\\
\widehat{b}_2 =& \frac{M(\alpha)}{\alpha h^2}\Big[\frac{1}{2}h - \frac{1-\alpha}{\alpha} + \Big(\frac{1}{2}h + \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big)\Big].
\end{aligned}
$$

We have

$$
\begin{aligned}
\beta_2 =& \frac{M(\alpha)}{\alpha h^2}\Big[\Big(\frac{h}{2} + \frac{1-\alpha}{\alpha}\Big)\Big(\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big) - 1\Big)^2\\
& - h\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big)\Big(\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big) - 1\Big)\Big] > 0.
\end{aligned}
$$

Finally, using the equalities as follows:

$$
\begin{aligned}
\widetilde{b}_0 =& \frac{M(\alpha)}{\alpha h^2}\Big[\frac{1}{2}h - \frac{1-\alpha}{\alpha} + \Big(\frac{3}{2}h + \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big)\Big],\\
\widetilde{b}_1 =& -\frac{2M(\alpha)}{\alpha h^2}\Big[h - \frac{1-\alpha}{\alpha} + \Big(h + \frac{1-\alpha}{\alpha}\Big)\exp\Big(\frac{\alpha}{1-\alpha}(-2h)\Big)\Big],
\end{aligned}
$$

we get

$$
\begin{aligned}
\beta_3 =& \frac{M(\alpha)}{\alpha h^2}\Big[\Big(-\frac{3}{2}h + \frac{1-\alpha}{\alpha}\Big)\Big(\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big) - 1\Big)^2\\
& - h\Big(\exp\Big(\frac{\alpha}{1-\alpha}(-h)\Big) - 1\Big)\Big],
\end{aligned}
$$

which is positive if $h < \frac{2}{3}\frac{1-\alpha}{\alpha}$. This proves the lemma.

A byproduct of the above lemmas is the well-posedness of the discrete problem (3.8), which is given in the following theorem.

Theorem 4.2 *For any given function f(t) and initial condition u_0, the linear system (3.8) admits a unique solution $(u_1, u_2, \cdots, u_{2N})^{\mathrm{T}}$.*

Proof First, it follows from (3.3) and (3.4):

$$S_2 \boldsymbol{u}_2 = \boldsymbol{f}_2, \tag{4.19}$$

where $S_2 = (a_i^{j,0})_{i,j=1}^2, \boldsymbol{u}_2 = (u_1, u_2)^{\mathrm{T}}, \boldsymbol{f}_2 = (f(t_1) - a_1^{0,0}u_0, f(t_2) - a_2^{0,0}u_0)^{\mathrm{T}}$.

In virtue of the proof of Lemma 4.2, we have

$$a_1^{1,0}a_2^{2,0} - a_1^{2,0}a_2^{1,0} = \widehat{b}_1\widetilde{b}_2 - \widehat{b}_2\widetilde{b}_1 = \left(\frac{M(\alpha)}{h\alpha}\right)^2 \left(\exp\left(-\frac{\alpha}{1-\alpha}h\right) - 1\right)^2 \neq 0.$$

Thus S_2 is invertible, and consequently the system (4.19) admits a unique solution $(u_1, u_2)^{\mathrm{T}}$.

Furthermore, we deduce from (3.5) and (3.6),

$$S_N \boldsymbol{u}_N = \boldsymbol{f}_N, \tag{4.20}$$

where

$$S_N = \begin{pmatrix} a_3^{2,1} & 0 & \cdots & 0 & 0 \\ a_4^{1,1} & a_4^{2,1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{2N-1}^{2,1} + a_{2N-1}^{0,2} & a_{2N-1}^{1,2} & \cdots & a_{2N-1}^{2,N-1} & 0 \\ a_{2N}^{1,1} & a_{2N}^{2,1} + a_{2N}^{0,2} & \cdots & a_{2N}^{1,N-1} & a_{2N}^{2,N-1} \end{pmatrix},$$

$\boldsymbol{u}_N = (u_3, u_4, \cdots, u_{2N})^{\mathrm{T}}, \boldsymbol{f}_N = (\widehat{f}_3, \widehat{f}_4, \cdots, \widehat{f}_{2N})^{\mathrm{T}}$ with $\widehat{f}_{2m+1} = f(t_{2m+1}) - a_{2m+1}^{0,0}u_0 - (a_{2m+1}^{1,0} + a_{2m+1}^{0,1})u_1 - (a_{2m+1}^{2,0} + a_{2m+1}^{1,1})u_2, \widehat{f}_{2m+2} = f(t_{2m+2}) - a_{2m+2}^{0,0}u_0 - a_{2m+2}^{1,0}u_1 - (a_{2m+2}^{2,0} + a_{2m+2}^{0,1})u_2$, $\widehat{f}_{2m+2} = f(t_{2m+2}) - a_{2m+2}^{0,0}u_0 - a_{2m+2}^{1,0}u_1 - (a_{2m+2}^{2,0} + a_{2m+2}^{0,1})u_2, m = 1, \cdots, N-1.$

We see that S_N is a lower triangular matrix. According to Lemma 4.1, all diagonal entries of S_N are positive. Therefore, the linear system (4.20) admits a unique solution $(u_3, u_4, \cdots, u_{2N})^{\mathrm{T}}$. This proves the theorem.

Now we turn to analyze the stability property of the scheme (4.9)–(4.12).

Theorem 4.3 *If $u_0 > 0, \lambda < 0$, the scheme (4.9)–(4.12) applied to (2.1) with f given in (2.8) is stable with respect to the initial value under the condition*

$$h < \frac{1-\alpha}{3\alpha}.$$

That is, the numerical solution $\{u_j\}_{j=0}^{2N}$ satisfies

$$0 < u_j < u_0, \quad j = 1, 2, \cdots, 2N. \tag{4.21}$$

Proof First, inserting (2.8) into (4.9) and (4.10) gives

$$
\begin{aligned}
\widehat{b}_0 u_0 + \widehat{b}_1 u_1 + \widehat{b}_2 u_2 =& \lambda \Big[u_1 - u_0 \exp\Big(-\frac{\alpha}{1-\alpha} t_1 \Big) \Big], \\
\widetilde{b}_0 u_0 + \widetilde{b}_1 u_1 + \widetilde{b}_2 u_2 =& \lambda \Big[u_2 - u_0 \exp\Big(-\frac{\alpha}{1-\alpha} t_2 \Big) \Big].
\end{aligned}
$$

The solution of this equation can be expressed by

$$
u_1 = \frac{G_1}{G_2} u_0, \quad u_2 = \frac{G_3}{G_2} u_0,
$$

where

$$
\begin{aligned}
G_1 =& G - \lambda \beta_2 + \lambda^2 \exp\Big(-\frac{\alpha}{1-\alpha} t_1 \Big), \\
G_2 =& G - \lambda \beta_1 + \lambda^2, \\
G_3 =& G - \lambda \beta_3 + \lambda^2 \exp\Big(-\frac{\alpha}{1-\alpha} t_2 \Big)
\end{aligned}
$$

with $G = \Big(\frac{M(\alpha)}{h\alpha} \Big)^2 \Big(\exp(-\frac{\alpha}{1-\alpha} h) - 1 \Big)^2 > 0$ and β_1, β_2, and β_3 defined in Lemma 4.2.

In virtue of Lemma 4.2, we have $\beta_1 > 0, \beta_2 > 0, \beta_3 > 0$ under the condition $h < \frac{2}{3} \frac{1-\alpha}{\alpha}$. This means $G_1 > 0, G_2 > 0, G_3 > 0$. Furthermore, a direct calculation shows

$$
\begin{aligned}
G_1 - G_2 =& \lambda \frac{M(\alpha)}{\alpha h} \Big(\exp\Big(-\frac{\alpha}{1-\alpha} t_1 \Big) - 1 \Big)^2 + \lambda^2 \Big(\exp\Big(-\frac{\alpha}{1-\alpha} t_1 \Big) - 1 \Big) < 0, \\
G_3 - G_2 =& \lambda \frac{2M(\alpha)}{\alpha h} \Big(\exp\Big(-\frac{\alpha}{1-\alpha} t_1 \Big) - 1 \Big)^2 + \lambda^2 \Big(\exp(-\frac{\alpha}{1-\alpha} t_2) - 1 \Big) < 0.
\end{aligned}
$$

As a result, we get

$$
0 < u_1 < u_0, \quad 0 < u_2 < u_0. \tag{4.22}
$$

It follows from (4.11),

$$
u_3 = \Big[\Big(-b_0^{(1)} - \lambda \exp\Big(-\frac{\alpha}{1-\alpha} t_3 \Big) \Big) u_0 - b_1^{(1)} u_1 - b_2^{(1)} u_2 \Big] \Big/ (b_3^{(1)} - \lambda).
$$

According to Lemma 4.1, it holds

$$
-b_0^{(1)} > 0, -b_1^{(1)} > 0, -b_2^{(1)} > 0, b_3^{(1)} > 0.
$$

This, together with (4.22), leads to

$$
0 < u_3 < \Big[-b_0^{(1)} - \lambda \exp\Big(-\frac{\alpha}{1-\alpha} t_3 \Big) - b_1^{(1)} - b_2^{(1)} \Big] u_0 \Big/ (b_3^{(1)} - \lambda).
$$

Furthermore, it can be directly verified that

$$0 < -b_0^{(1)} - \lambda \exp\left(-\frac{\alpha}{1-\alpha}t_3\right) - b_1^{(1)} - b_2^{(1)} < b_3^{(1)} - \lambda.$$

Therefore we get

$$0 < u_3 < u_0. \tag{4.23}$$

Next we deduce from (4.12),

$$u_4 = \left[\left(-\overline{b}_0^{(1)} - \lambda \exp\left(-\frac{\alpha}{1-\alpha}t_4\right)\right)u_0 - \overline{b}_1^{(1)}u_1 - \overline{b}_2^{(1)}u_2 - \overline{b}_3^{(1)}u_3\right] \Big/ (\overline{b}_4^{(1)} - \lambda).$$

Once again, it follows from Lemma 4.1,

$$-\overline{b}_0^{(1)} > 0,\ -\overline{b}_1^{(1)} > 0,\ -\overline{b}_2^{(1)} > 0,\ -\overline{b}_3^{(1)} > 0,\ \overline{b}_4^{(1)} > 0.$$

Then using some relationships between these coefficients and (4.22)–(4.23), we obtain

$$0 < u_4 < u_0. \tag{4.24}$$

Now we prove the remaining results by using mathematical induction. Assume (4.21) holds for all $j = 1, 2, \cdots, 2m; m = 1, 2, \cdots, N-1$, we want to prove that it also holds for $j = 2m+1$ and $j = 2m+2$.

It follows from (4.11),

$$\begin{aligned} u_{2m+1} =& \left[\left(-b_0^{(m)} - \lambda \exp\left(-\frac{\alpha}{1-\alpha}t_{2m+1}\right)\right)u_0 - b_1^{(m)}u_1 - b_2^{(m)}u_2 \right.\\ & \left. - \sum_{k=2}^{m} b_{2k}^{(m)}u_{2k} - \sum_{k=2}^{m} b_{2k-1}^{(m)}u_{2k-1}\right] \Big/ (b_{2m+1}^{(m)} - \lambda). \end{aligned} \tag{4.25}$$

It is not difficult to see, by using Lemma 4.1, that all coefficients in the right-hand side of (4.25) are positive. Thus we deduce from the induction assumption

$$u_{2m+1} < \frac{G_4}{G_5}u_0,$$

where

$$\begin{aligned} G_4 =& - b_0^{(m)} - \lambda \exp\left(-\frac{\alpha}{1-\alpha}t_{2m+1}\right) - b_1^{(m)} - b_2^{(m)} - \sum_{k=2}^{m} b_{2k}^{(m)} - \sum_{k=2}^{m} b_{2k-1}^{(m)} > 0,\\ G_5 =& b_{2m+1}^{(m)} - \lambda > 0. \end{aligned}$$

Using the fact that the scheme (4.9)–(4.12) is accurate for the constant solution, we have

$$b_0^{(m)} + b_1^{(m)} + b_2^{(m)} + \sum_{k=2}^{m} b_{2k}^{(m)} + \sum_{k=2}^{m} b_{2k-1}^{(m)} + b_{2m+1}^{(m)} = 0.$$

Thus

$$G_4 - G_5 = -\lambda\left[\exp\left(-\frac{\alpha}{1-\alpha}t_{2m+1}\right) - 1\right] < 0.$$

This proves

$$0 < u_{2m+1} < u_0.$$

In an exactly same way, we can deduce from (4.12) and Lemma 4.1:

$$0 < u_{2m+2} < u_0.$$

The proof is completed.

5 Numerical Results

We present several numerical examples to verify the theoretical results obtained in the previous sections. Precisely, our main purpose is to check the convergence order of the numerical solution with respect to the step size h.

We consider the initial value problem (2.1) with several right-hand side $f(t, u(t))$ as follows:

i) $f(t, u(t)) = G(t)$;
ii) $f(t, u(t)) = G(t) - t^3 + u(t)$;
iii) $f(t, u(t)) = G(t) + t^6 - u^2(t)$;

where

Table 2 Maximum errors and decay rates with $\alpha = 0.3$ and 0.7 for $f(t, u(t)) = G(t)$

h	$\alpha = 0.3$	Rate	$\alpha = 0.7$	Rate
$\frac{1}{4}$	2.863 25E-004	-	1.498 51E-003	-
$\frac{1}{8}$	2.181 21E-005	3.714 45	1.131 70E-004	3.726 95
$\frac{1}{16}$	1.495 51E-006	3.866 41	7.205 32E-006	3.973 29
$\frac{1}{32}$	9.776 73E-008	3.935 14	4.431 21E-007	4.023 28
$\frac{1}{64}$	6.247 42E-009	3.968 01	3.077 29E-008	3.847 96
$\frac{1}{128}$	3.947 84E-010	3.984 12	2.045 96E-009	3.910 80

$$G(t) = M(\alpha)\left[\frac{3}{\alpha}t^2 - \frac{6(1-\alpha)}{\alpha^2}t + \frac{6(1-\alpha)^2}{\alpha^3} - \frac{6(1-\alpha)^2}{\alpha^3}\exp\left(-\frac{\alpha}{1-\alpha}t\right)\right].$$

In the first case, f is independent of u, while in the second and third cases f is function of u. In particular, f is nonlinear with respect to u. However, it can be verified that the exact solution is $u(t) = t^3$ for all three cases. Note the main difference between these examples is that the first example considers a right-hand side function which is independent of the solution, while the second example addresses a right-hand side function linearly dependent of the solution, and the third one is a nonlinear function of the solution.

All the results presented in this example correspond to the numerical solution captured at $T = 1$. As in [1, 3], we choose a special normalization function $M(\alpha) = 1$ such that $M(0) = M(1) = 1$ in the numerical tests. In Tables 2, 3, 4, we list the maximum errors, i.e., $\max_i |u(t_i) - u_i|$ as a function of h for several α. Also shown are the corresponding error decay rates. It is observed from these tables that for all tested right-hand side functions and values of α, the convergence rate is close to four. This is in a good agreement with the theoretical prediction. In particular, it is worthy to emphasize that the non-linearity of f seems to have no impact on the accuracy of the scheme.

The next test concerns the stability investigation. To this end, the scheme is applied to the problem (2.1) with the fabricated right-hand side function $f(t,u(t)) = \frac{M(\alpha)}{\alpha^2+(1-\alpha)^2}[(1-\alpha)\sin t + \alpha\cos t - \alpha\exp(-\frac{\alpha}{1-\alpha}t)]$, so that the exact solution is $u(t) = \sin t$. The calculation is run up to $T = 1\,000$, long enough to study the stability of the scheme. Table 5 shows the error behavior as a function of t, computed with fixed $h = 0.01$. It is observed that the numerical solution remains to be good approximation to the exact solution even after long time computation. This test demonstrates good stability property of the proposed scheme.

Table 3 Maximum errors and decay rates with $\alpha = 0.3$ and 0.7 for $f(t,u(t)) = G(t) - t^3 + u(t)$

h	$\alpha = 0.3$	Rate	$\alpha = 0.7$	Rate
$\frac{1}{4}$	1.044 48E-003	-	2.550 63E-003	-
$\frac{1}{8}$	7.592 63E-005	3.782 04	2.102 50E-004	3.600 67
$\frac{1}{16}$	5.354 74E-006	3.825 70	1.295 48E-005	4.020 54
$\frac{1}{32}$	3.633 26E-007	3.881 47	7.477 94E-007	4.114 70
$\frac{1}{64}$	2.381 44E-008	3.931 35	4.419 42E-008	4.080 70
$\frac{1}{128}$	1.319 38E-009	4.173 90	2.930 49E-009	3.914 64

Table 4 Maximum errors and decay rates with $\alpha = 0.3$ and 0.7 for $f(t,u(t)) = G(t) - t^6 + u^2(t)$

h	$\alpha = 0.3$	Rate	$\alpha = 0.7$	Rate
$\frac{1}{4}$	1.775 85E-004	-	8.363 73E-004	-
$\frac{1}{8}$	2.028 73E-005	3.129 85	6.067 24E-005	3.785 03
$\frac{1}{16}$	1.481 71E-006	3.775 24	5.186 95E-006	3.548 08
$\frac{1}{32}$	9.765 41E-008	3.923 44	4.325 88E-007	3.583 81
$\frac{1}{64}$	6.246 52E-009	3.966 55	3.077 10E-008	3.813 35
$\frac{1}{128}$	3.947 76E-010	3.983 94	2.045 95E-009	3.910 72

Table 5 Errors $|u(t_i) - u_i|$ for $\alpha = 0.3, 0.5$ and 0.7

t_i	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$
1	2.174 438 407 109 847E-11	2.299 160 861 696 237E-11	5.039 940 687 012 745E-10
150	7.386 136 147 147 226E-11	3.120 245 173 349 190E-10	1.374 845 570 722 982E-09
300	2.050 879 466 253 264E-10	5.759 095 422 774 863E-10	1.879 522 759 651 309E-09
450	1.969 819 862 779 332E-10	5.625 367 949 235 738E-10	1.591 615 395 035 717E-09
600	2.666 145 498 819 716E-10	4.107 411 286 091 711E-10	7.394 196 227 528 127E-10
750	2.545 806 898 623 937E-10	2.454 503 267 301 789E-10	1.508 889 6798 687 30E-10
900	2.445 257 329 952 710E-10	1.306 380 559 285 003E-10	5.660 025 781 395 461E-10
1 000	2.393 513 165 444 006E-10	1.446 973 652 008 410E-10	4.171 569 7562 949 57E-10

6 Concluding Remarks

We have proposed an efficient high-order scheme for fractional ordinary differential equations with the Caputo–Fabrizio derivative. The stability and convergence analysis was carried out to prove that the proposed scheme is stable under a slight restriction on the step size, which only depends on the fractional order. The obtained error estimate shows that the proposed scheme is of order 4. The carried out numerical tests confirmed the theoretical prediction.

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