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ORIGINAL PAPER

Numerical Algorithm for the Time-Caputo and Space-Riesz Fractional Diffusion Equation

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Abstract

In this paper, we develop a novel finite-difference scheme for the time-Caputo and space-Riesz fractional diffusion equation with convergence order $\mathcal{O}(\tau^{2-\alpha} + h^2)$. The stability and convergence of the scheme are analyzed by mathematical induction. Moreover, some numerical results are provided to verify the effectiveness of the developed difference scheme.

Keywords Caputo derivative · Riesz derivative · Fractional diffusion equation

Mathematics Subject Classification 65M06 · 65M12

1 Introduction

With the development of science, it has been found that the fractional derivatives and fractional differential equations provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Therefore, they are widely used in the field of science and technology, such as the fields of control theory, biology, electrochemical processes, porous media, viscoelastic materials [7, 14, 15].

However, unfortunately, for most fractional differential equations, it is not an easy task to seek for their analytical solutions. For some simple linear equations, even if the analytic solutions are obtained, it is not convenient to calculate, because the analytic solutions contain some special functions. Therefore, it is essential to develop the effective numerical solutions of the fractional differential equations.

Since the numerical approximation of fractional derivatives is the most important step in numerical solutions of fractional differential equations, we first review the progress made in numerical approximation of fractional derivatives. As for the Caputo derivative, Gao et al. proposed a so-called $L1-2$ formula with order $(3-\alpha)$ [8]. Using the different methods, Li et al. also got a numerical differential formula with convergence order $(3-\alpha)$ [11]. Later, Alikhanov proposed another $(3-\alpha)$th order numerical differential formula at the superconvergence point $t = t_{j+\sigma}$, and named it as the $L2-1$

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formula [1]. Furthermore, Li et al. developed a series of high-order formulas using the rth ($r \geq 4$ is a positive integer) degree interpolation function [9]. For approximating the Riesz fractional derivative, the first-order accurate normal/shifted Grünwald formula [13], second-order accurate weighted and shifted Grünwald by choosing the appropriate weight coefficients [16], second-order accurate fractional centered difference formula [3], and some other higher-order formulas [5, 6, 20] have been constructed. Based on the above mentioned and other approximation formulas, a tremendous amount of finite-difference methods for solving the fractional differential equations have been developed. For example, Cui constructed a compact finite-difference scheme with the temporal accuracy of first order and spatial accuracy of fourth order for the one-dimensional fractional diffusion equation in [2]. Wang and Vong [17] developed two high-order finite-difference schemes for the fractional modified anomalous subdiffusion equation and the diffusion-wave equation, respectively. Based on the fractional multistep methods in time and central difference formula in space, Zeng [19] proposed several finite-difference schemes for solving the time-fractional diffusion-wave equation.

In this paper, we propose a novel finite-difference scheme for the following time-Caputo and space-Riesz fractional diffusion equation:

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha} u(x,t) = \dfrac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + f(x,t),\ 0 < x < L,\ 0 < t \leq T, \\ u(x,0) = \varphi(x),\ 0 \leq x \leq L, \\ u(0,t) = u(L,t) = 0,\ 0 < t \leq T. \end{cases} \tag{1}$$

Here, ${}_C\mathrm{D}_{0,t}^{\alpha} u(x,t)$ denotes the Caputo derivative of order $\alpha \in (0,1)$ and defined by [15]

$${}_C\mathrm{D}_{0,t}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^{\alpha}} \mathrm{d}s,\ 0 < \alpha < 1,$$

and $\dfrac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}}$ is the Riesz derivative of order $\beta \in (1,2)$ which is defined below [15],

$$\frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} = -\frac{1}{2\cos\left(\frac{\pi\beta}{2}\right)} \left({}_{RL}\mathrm{D}_{a,x}^{\beta} + {}_{RL}\mathrm{D}_{x,b}^{\beta} \right) u(x,t),\ 1 < \beta \leq 2,$$

where ${}_{RL}\mathrm{D}_{a,x}^{\beta}$ denotes the left Riemann–Liouville derivative

$${}_{RL}\mathrm{D}_{a,x}^{\beta} u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{u(r,t)}{(x-r)^{\beta-1}} \mathrm{d}r, \quad 1 < \beta < 2,$$

and ${}_{RL}\mathrm{D}_{x,b}^{\beta}$ is the right Riemann–Liouville derivative

$${}_{RL}\mathrm{D}_{x,b}^{\beta} u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_x^b \frac{u(r,t)}{(r-x)^{\beta-1}} \mathrm{d}r, \quad 1 < \beta < 2.$$

This paper is organized as follows. In Sect. 2, based on a second-order accuracy approximation operator for the Riesz fractional derivative, we develop a finite-difference scheme for the time-Caputo and space-Riesz fractional diffusion equation. The stability and convergence analysis of the constructed scheme is studied in Sect. 3. Numerical results are provided in Sect. 4 to demonstrate the effectiveness of the numerical algorithm.

2 The Development of the Numerical Algorithm

Let $h=\frac{L}{M}$ and $\tau=\frac{T}{N}$ be the spatial and temporal stepsizes, respectively. Set $x_j=jh\ (0\le j\le M)$, $t_k=n\tau\ (0\le k\le N)$. Denote

$$u_j^j=u(x_j,t_k),\ f_j^k=f(x_j,t_k).$$

First, we introduce the following $L1$ formula [10, 14] to numerical treatment of the Caputo fractional derivative ${}_C\mathrm{D}_{0,t}^{\alpha}u(t)$ at $t=t_n$ $(n=0,1,\dots,N)$:

$$\begin{aligned}{}_C\mathrm{D}_{0,t}^{\alpha}u(t)|_{t=t_n}&=\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n-s)^{-\alpha}u'(s)\mathrm{d}s\\&=\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n-s)^{-\alpha}\left[\frac{u(t_{k+1})-u(t_k)}{\tau}+\mathcal{O}(\tau)\right]\mathrm{d}s\\&=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}b_{n-k-1}\big(u(t_{k+1})-u(t_k)\big)+\mathcal{O}\big(\tau^{2-\alpha}\big),\end{aligned}\tag{2}$$

where the weights are defined by

$$b_k=(k+1)^{1-\alpha}-k^{1-\alpha},\ k=0,1,\dots,n-1,$$

and which have the following properties.

Lemma 1 [12] *Let* $b_k=(k+1)^{1-\alpha}-k^{1-\alpha},\ k=0,1,2,\dots$ *and* $0<\alpha<1$. *Then, one has*

$$\begin{cases}1=b_0>b_1>b_2>\cdots>b_k\to 0, \textit{as } k\to+\infty,\\ c_1k^{\alpha}\le(b_k)^{-1}\le c_2k^{\alpha}, \textit{where}\ \ c_1\ \ \textit{and}\ \ c_2\ \ \textit{are}\ \ \textit{constants},\\ \displaystyle\sum_{k=0}^{n}(b_k-b_{k+1})+b_{n+1}=(1-b_1)+\sum_{k=1}^{n-1}(b_k-b_{k+1})+b_n=1.\end{cases}$$

In [4], the authors constructed the following second-order numerical differential formula:

$$\frac{\partial^{\beta}u(x)}{\partial|x|^{\beta}}=-\frac{1}{2\cos\left(\frac{\pi}{2}\beta\right)}\left({}^{L}\mathcal{B}_2^{\beta}u(x)+{}^{R}\mathcal{B}_2^{\beta}u(x)\right)+\mathcal{O}(h^2)\tag{3}$$

for the Riesz space fractional derivative, where the operators

$${}^{L}\mathcal{B}_2^{\beta}u(x)=\frac{1}{h^{\beta}}\sum_{\ell=0}^{\infty}\kappa_{2,\ell}^{(\beta)}u(x-(\ell-1)h),$$

and

$$ {}^{R}\mathcal{B}_2^{\beta}u(x) = \frac{1}{h^{\beta}}\sum_{\ell=0}^{\infty}\kappa_{2,\ell}^{(\beta)}u(x+(\ell-1)h). $$

Here, the coefficients

$$ \kappa_{2,\ell}^{(\beta)} = (-1)^{\ell}\left(\frac{3\beta-2}{2\beta}\right)^{\beta}\sum_{m=0}^{\ell}\left(\frac{\beta-2}{3\beta-2}\right)^{m}\binom{\beta}{m}\binom{\beta}{\ell-m}, \quad \ell = 0, 1, \dots $$

can be obtain by the novel generating function

$$ \widetilde{W}_2(z) = \left(\frac{3\beta-2}{2\beta} - \frac{2(\beta-1)}{\beta}z + \frac{\beta-2}{2\beta}z^2\right)^{\beta}. $$

Besides, we can also calculate the coefficients $\kappa_{2,\ell}^{(\beta)}$ by the following recursive relations:

$$ \begin{cases} \kappa_{2,0}^{(\beta)} = \left(\dfrac{3\beta-2}{2\beta}\right)^{\beta}, \\ \kappa_{2,1}^{(\beta)} = \dfrac{4\beta(1-\beta)}{3\beta-2}\kappa_{2,0}^{(\beta)}, \\ \kappa_{2,\ell}^{(\beta)} = \dfrac{1}{\ell(3\beta-2)}\Big[4(1-\beta)(\beta-\ell+1)\kappa_{2,\ell-1}^{(\beta)} \\ \qquad +(\beta-2)(2\beta-\ell+2)\kappa_{2,\ell-2}^{(\beta)}\Big], \quad \ell \geq 2. \end{cases} $$

Next, we list the properties of the coefficients $\kappa_{2,\ell}^{(\beta)}$ $(\ell = 0, 1, \dots)$.

Lemma 2 [4] *The coefficients* $\kappa_{2,\ell}^{(\beta)}$ $(\ell = 0, 1, \dots)$ *have the following properties for* $1 < \beta < 2$:

i) $\kappa_{2,0}^{(\beta)} = \left(\dfrac{3\beta-2}{2\beta}\right)^{\beta} > 0, \kappa_{2,1}^{(\beta)} = \dfrac{4\beta(1-\beta)}{3\beta-2}\kappa_{2,0}^{(\beta)} < 0;$

ii) $\kappa_{2,2}^{(\beta)} = \dfrac{\beta(8\beta^3-21\beta^2+16\beta-4)}{(3\beta-2)^2}\kappa_{2,0}^{(\beta)}. \ \kappa_{2,2}^{(\beta)} < 0 \ \mathit{if}\ \beta \in (1, \beta^*),\ \mathit{while}\ \kappa_{2,2}^{(\beta)} \geq 0 \ \mathit{if}\ \beta \in [\beta^*, 2),$

$\quad \mathit{where}\ \beta^* = \dfrac{7}{8} + \dfrac{\sqrt[3]{621+48\sqrt{87}}}{24} + \dfrac{19}{\sqrt[3]{621+48\sqrt{87}}} \approx 1.533\,3;$

iii) $\kappa_{2,\ell}^{(\beta)} \geq 0 \ \mathit{if}\ \ell \geq 3;$

iv) $\kappa_{2,\ell}^{(\beta)} \sim -\dfrac{\sin(\pi\beta)\Gamma(\beta+1)}{\pi}\ell^{-\beta-1} \ \mathit{as}\ \ell \to \infty;$

v) $\kappa_{2,\ell}^{(\beta)} \to 0 \ \mathit{as}\ \ell \to \infty;$

vi) $\displaystyle\sum_{\ell=0}^{\infty}\kappa_{2,\ell}^{(\beta)} = 0, \quad \sum_{\ell=0}^{m}\kappa_{2,\ell}^{(\beta)} < 0, \ m \geq 2.$

Now, we consider Eq. (1) at point (x_j, t_k). For the space fractional derivative, we apply the second-order formula (3) to approximate the Riesz derivative for $x \in (0, L)$, that is,

$$\frac{\partial^\beta u(x_j, t_n)}{\partial |x|^\beta} = -\frac{1}{2\cos\left(\frac{\pi}{2}\beta\right)}\left({}^L\mathcal{A}_2^\beta + {}^R\mathcal{A}_2^\beta\right)u(x_j, t_n) + \mathcal{O}\left(h^2\right), \tag{4}$$

where

$${}^L\mathcal{A}_2^\beta u(x_j, t_n) = \frac{1}{h^\beta}\sum_{\ell=0}^{j+1}\kappa_{2,\ell}^{(\beta)} u\left(x_j - (\ell-1)h, t_n\right),$$

and

$${}^R\mathcal{A}_2^\beta u(x_j, t_n) = \frac{1}{h^\beta}\sum_{\ell=0}^{M-j+1}\kappa_{2,\ell}^{(\beta)} u\left(x_j + (\ell-1)h, t_n\right).$$

Finally, substituting (2) and (4) into (1), and omitting the high-order terms $\mathcal{O}\left(\tau^{2-\alpha} + h^2\right)$. Replacing the function $u(x_j, t_n)$ with its numerical approximation value U_j^n, we can obtain the following finite-difference scheme:

$$\begin{cases} U_j^n + q\left[\sum\limits_{\ell=0}^{j+1}\kappa_{2,\ell}^{(\beta)} U_{j-\ell+1}^n + \sum\limits_{\ell=0}^{M-j+1}\kappa_{2,\ell}^{(\beta)} U_{j+\ell-1}^n\right] = U_j^{n-1} \\ -\sum\limits_{k=1}^{n-1} b_k\left(U_j^{n-k} - U_j^{n-k-1}\right) + \tau^\alpha \Gamma(2-\alpha) f_j^n, \\ U_j^0 = \varphi\left(x_j\right),\ 0 \le j \le M, \\ U_0^n = U_M^n = 0,\ 1 \le n \le N, \end{cases} \tag{5}$$

where $q = \dfrac{\tau^\alpha \Gamma(2-\alpha)}{2h^\beta \cos\left(\frac{\pi}{2}\beta\right)}$.

3 Stability and Convergence Analysis

In this section, the stability and convergence analysis of the above difference scheme are studied in detail.

3.1 Stability Analysis

From Lemma 2, we easily know that

Lemma 3 *Under the condition*

$$\frac{7}{8} + \frac{\sqrt[3]{621 + 48\sqrt{87}}}{24} + \frac{19}{\sqrt[3]{621 + 48\sqrt{87}}} \le \beta < 2, \tag{6}$$

the coefficient $\kappa_{2,2}^{(\beta)}$ *satisfy*

$$\kappa_{2,2}^{(\beta)} \geq 0.$$

Theorem 1 *Under the condition* (6) *and* $0 < \alpha < 1$, *the finite-difference scheme for the time-Caputo and space-Riesz fractional diffusion equation* (1) *is unconditionally stable.*

Proof Let V_j^n be the exact solution of the finite-difference scheme (5). Denote $\xi_j^n = V_j^n - U_j^n$, then we can obtain the following perturbation equation:

$$\begin{cases} \left(1 + 2q\kappa_{2,1}^{(\beta)}\right)\xi_j^1 + q\left[\sum\limits_{\ell=0,\ell\neq 1}^{j+1} \kappa_{2,\ell}^{(\beta)}\xi_{j-\ell+1}^1 + \sum\limits_{\ell=0,\ell\neq 1}^{M-j+1} \kappa_{2,\ell}^{(\beta)}\xi_{j+\ell-1}^1\right] = \xi_j^0, \\ \left(1 + 2q\kappa_{2,1}^{(\beta)}\right)\xi_j^n + q\left[\sum\limits_{\ell=0,\ell\neq 1}^{j+1} \kappa_{2,\ell}^{(\beta)}\xi_{j-\ell+1}^n + \sum\limits_{\ell=0,\ell\neq 1}^{M-j+1} \kappa_{2,\ell}^{(\beta)}\xi_{j+\ell-1}^n\right] \\ = \left(1 - b_1\right)\xi_j^{n-1} + \sum\limits_{k=2}^{n-1} \left(b_{k-1} - b_k\right)\xi_j^{n-k} + b_{n-1}\xi_j^0. \end{cases}$$

Below, we will discuss the stability of the numerical algorithm by mathematical induction. Denote

$$\left\|E^1\right\|_\infty = \left|\xi_\ell^1\right| = \max_{1\leq j\leq M-1} \left|\xi_j^1\right|.$$

Note that Lemma 2, that is, $\sum\limits_{\ell=0}^{m} \kappa_{2,\ell}^{(\beta)} < 0$ $(m \geq 2)$, then we have

$$\begin{aligned} \left\|E^1\right\|_\infty = \left|\xi_\ell^1\right| &\leq \left(1 + 2q\kappa_{2,1}^{(\beta)}\right)\left|\xi_\ell^1\right| + q\left[\sum_{\ell=0,\ell\neq 1}^{j+1} \kappa_{2,\ell}^{(\beta)}\left|\xi_\ell^1\right| + \sum_{\ell=0,\ell\neq 1}^{M-j+1} \kappa_{2,\ell}^{(\beta)}\left|\xi_\ell^1\right|\right] \\ &\leq \left(1 + 2q\kappa_{2,1}^{(\beta)}\right)\left|\xi_j^1\right| + q\left[\sum_{\ell=0,\ell\neq 1}^{j+1} \kappa_{2,\ell}^{(\beta)}\left|\xi_{j-\ell+1}^1\right| + \sum_{\ell=0,\ell\neq 1}^{M-j+1} \kappa_{2,\ell}^{(\beta)}\left|\xi_{j+\ell-1}^1\right|\right] \\ &\leq \left|\left(1 + 2q\kappa_{2,1}^{(\beta)}\right)\xi_j^1 + q\left[\sum_{\ell=0,\ell\neq 1}^{j+1} \kappa_{2,\ell}^{(\beta)}\xi_{j-\ell+1}^1 + \sum_{\ell=0,\ell\neq 1}^{M-j+1} \kappa_{2,\ell}^{(\beta)}\xi_{j+\ell-1}^1\right]\right| \\ &= \left|\xi_l^0\right| \leq \left\|E^0\right\|_\infty. \end{aligned}$$

Furthermore, let

$$\|E^n\|_\infty = \left|\xi_\ell^n\right| = \max_{1\leq j\leq M-1} \left|\xi_j^n\right|,$$

and assuming that we have proved that $\left\|E^k\right\|_\infty \le \left\|E^0\right\|_\infty$ for $1 \le k \le n-1$. Then, we also know that

$$\begin{aligned}
\|E^n\|_\infty &= \left|\xi_\ell^n\right| \le \left(1 + 2q\kappa_{2,1}^{(\beta)}\right)\left|\xi_\ell^n\right| + q\left[\sum_{\ell=0,\ell\ne1}^{j+1} \kappa_{2,\ell}^{(\beta)}\left|\xi_\ell^n\right| + \sum_{\ell=0,\ell\ne1}^{M-j+1} \kappa_{2,\ell}^{(\beta)}\left|\xi_\ell^n\right|\right] \\
&\le \left(1 + 2q\kappa_{2,1}^{(\beta)}\right)\left|\xi_j^n\right| + q\left[\sum_{\ell=0,\ell\ne1}^{j+1} \kappa_{2,\ell}^{(\beta)}\left|\xi_{j-\ell+1}^n\right| + \sum_{\ell=0,\ell\ne1}^{M-j+1} \kappa_{2,\ell}^{(\beta)}\left|\xi_{j+\ell-1}^n\right|\right] \\
&\le \left|\left(1 + 2q\kappa_{2,1}^{(\beta)}\right)\xi_j^n + q\left[\sum_{\ell=0,\ell\ne1}^{j+1} \kappa_{2,\ell}^{(\beta)}\xi_{j-\ell+1}^n + \sum_{\ell=0,\ell\ne1}^{M-j+1} \kappa_{2,\ell}^{(\beta)}\xi_{j+\ell-1}^n\right]\right| \\
&= \left|(1-b_1)\xi_j^n + \sum_{k=2}^{n-1}\left(b_{k-1}-b_k\right)\xi_j^{n-k} + b_{n-1}\xi_j^0\right| \\
&\le \left(1-b_1\right)\|E^n\|_\infty + b_{n-1}\left\|E^0\right\|_\infty + \sum_{k=2}^{n-1}\left(b_{k-1}-b_k\right)\left\|E^{n-k}\right\|_\infty \\
&\le \left(1-b_1\right)\left\|E^0\right\|_\infty + b_{n-1}\left\|E^0\right\|_\infty + \sum_{k=2}^{n-1}\left(b_{k-1}-b_k\right)\left\|E^0\right\|_\infty \\
&= \|E^0\|_\infty.
\end{aligned}$$

This ends the proof.

3.2 Convergence Analysis

Theorem 2 *Denote by* $u(x_j, t_n)$ $(j = 1, 2, \ldots, M-1; n = 1, 2, \ldots, N)$ *the exact solution of* (1) *at mesh point* (x_i, t_n), *and let* $\{U_j^n \mid 0 \le j \le M, 0 \le n \le N\}$ *be the solution of the finite-difference scheme* (5). *Define*

$$\varepsilon_j^n = u(x_j, t_n) - U_j^n, \;\; j = 1, 2, \ldots, M; \, n = 1, 2, \ldots, N,$$

then there exists a positive constant C, *such that*

$$||\varepsilon^n||_\infty \le C\left(\tau^{2-\alpha} + h^2\right), \;\; 0 \le n \le N,$$

under the condition (6) *and* $0 < \alpha < 1$.

Proof Denote $\varepsilon^n = \left(\varepsilon_1^n, \varepsilon_2^n, \ldots, \varepsilon_{M-1}^n\right)^{\mathrm{T}}$. then it follows from (1) and (5) that

$$\begin{cases} \left(1+2q\kappa_{2,1}^{(\beta)}\right)\varepsilon_j^1 + q\left[\sum\limits_{\ell=0,\ell\neq 1}^{j+1} \kappa_{2,\ell}^{(\beta)}\varepsilon_{j-\ell+1}^1 + \sum\limits_{\ell=0,\ell\neq 1}^{M-j+1} \kappa_{2,\ell}^{(\beta)}\varepsilon_{j+\ell-1}^1\right] = \varepsilon_j^0 + R_j^1, \\ \left(1+2q\kappa_{2,1}^{(\beta)}\right)\varepsilon_j^n + q\left[\sum\limits_{\ell=0,\ell\neq 1}^{j+1} \kappa_{2,\ell}^{(\beta)}\varepsilon_{j-\ell+1}^n + \sum\limits_{\ell=0,\ell\neq 1}^{M-j+1} \kappa_{2,\ell}^{(\beta)}\varepsilon_{j+\ell-1}^n\right] \\ = \left(1-b_1\right)\varepsilon_j^n + \sum\limits_{k=2}^{n-1}\left(b_{k-1}-b_k\right)\varepsilon_j^{n-k} + b_{n-1}\varepsilon_j^0 + R_j^n. \end{cases}$$

Here, the truncation error R_j^n satisfies

$$\left|R_j^n\right| \le \widetilde{C}\left(\tau^2+\tau^\alpha h^2\right),\ j=1,2,\dots,M-1; n=1,2,\dots,N,$$

where $\widetilde{C}$ is a non-negative constant.

Below, we give the convergence result using mathematical induction. First, for the case of $n=1$, let

$$\left\|\varepsilon^1\right\|_\infty = \left|\varepsilon_\ell^1\right|/\tau^\alpha = \max_{1\le j\le M-1}\left|\varepsilon_j^1\right|/\tau^\alpha.$$

Then, one has

$$\begin{aligned} \tau^\alpha\left\|\varepsilon^1\right\|_\infty = \left|\varepsilon_\ell^1\right| &\le \left(1+2q\kappa_{2,1}^{(\beta)}\right)\left|\varepsilon_\ell^1\right| + q\left[\sum_{\ell=0,\ell\neq 1}^{j+1}\kappa_{2,\ell}^{(\beta)}\left|\varepsilon_\ell^1\right| + \sum_{\ell=0,\ell\neq 1}^{M-j+1}\kappa_{2,\ell}^{(\beta)}\left|\varepsilon_\ell^1\right|\right] \\ &\le \left(1+2q\kappa_{2,1}^{(\beta)}\right)\left|\varepsilon_j^1\right| + q\left[\sum_{\ell=0,\ell\neq 1}^{j+1}\kappa_{2,\ell}^{(\beta)}\left|\varepsilon_{j-\ell+1}^1\right| + \sum_{\ell=0,\ell\neq 1}^{M-j+1}\kappa_{2,\ell}^{(\beta)}\left|\varepsilon_{j+\ell-1}^1\right|\right] \\ &\le \left|\left(1+2q\kappa_{2,1}^{(\beta)}\right)\varepsilon_j^1 + q\left[\sum_{\ell=0,\ell\neq 1}^{j+1}\kappa_{2,\ell}^{(\beta)}\varepsilon_{j-\ell+1}^1 + \sum_{\ell=0,\ell\neq 1}^{M-j+1}\kappa_{2,\ell}^{(\beta)}\varepsilon_{j+\ell-1}^1\right]\right| \\ &= \left|\varepsilon_l^0 + R_\ell^1\right|. \end{aligned}$$

Using $\varepsilon^0=0$ and $\left|R_\ell^1\right| \le \widetilde{C}(\tau^2+\tau^\alpha h^2)$, then there holds that

$$\left\|\varepsilon^1\right\|_\infty \le \widetilde{C}\left(\tau^{2-\alpha}+h^2\right).$$

As the before, set

$$\|\varepsilon^n\|_\infty = \left|\varepsilon_\ell^n\right|/\tau^\alpha = \max_{1\le j\le M-1}\left|\varepsilon_j^n\right|/\tau^\alpha,$$

and

$$\left\|\varepsilon^k\right\|_\infty \le \widehat{C}(\tau^2+\tau^\alpha h^2), \quad k=1,2,\dots,n-1,$$

then we further have

$$\begin{aligned}
\tau^\alpha \|\varepsilon^n\|_\infty = \left|\varepsilon_\ell^n\right| &\le \left(1+2q\kappa_{2,1}^{(\beta)}\right)\left|\varepsilon_\ell^n\right| + q\left[\sum_{\ell=0,\ell\neq 1}^{j+1}\kappa_{2,\ell}^{(\beta)}\left|\varepsilon_\ell^n\right| + \sum_{\ell=0,\ell\neq 1}^{M-j+1}\kappa_{2,\ell}^{(\beta)}\left|\varepsilon_\ell^n\right|\right]\\
&\le \left(1+2q\kappa_{2,1}^{(\beta)}\right)\left|\varepsilon_j^n\right| + q\left[\sum_{\ell=0,\ell\neq 1}^{j+1}\kappa_{2,\ell}^{(\beta)}\left|\varepsilon_{j-\ell+1}^n\right| + \sum_{\ell=0,\ell\neq 1}^{M-j+1}\kappa_{2,\ell}^{(\beta)}\left|\varepsilon_{j+\ell-1}^n\right|\right]\\
&\le \left|\left(1+2q\kappa_{2,1}^{(\beta)}\right)\varepsilon_j^n + q\left[\sum_{\ell=0,\ell\neq 1}^{j+1}\kappa_{2,\ell}^{(\beta)}\varepsilon_{j-\ell+1}^n + \sum_{\ell=0,\ell\neq 1}^{M-j+1}\kappa_{2,\ell}^{(\beta)}\varepsilon_{j+\ell-1}^n\right]\right|\\
&= \left|\left(1-b_1\right)\varepsilon_j^n + \sum_{k=2}^{n-1}\left(b_{k-1}-b_k\right)\varepsilon_j^{n-k} + b_{n-1}\varepsilon_j^0 + R_j^n\right|\\
&\le \left(1-b_1\right)\|\varepsilon^n\|_\infty + b_{n-1}\left\|\varepsilon^0\right\|_\infty + \sum_{k=2}^{n-1}\left(b_{k-1}-b_k\right)\left\|\varepsilon^{n-k}\right\|_\infty + \left|R_j^n\right|\\
&\le \left\{\left(1-b_1\right) + \sum_{k=2}^{n-1}\left(b_{k-1}-b_k\right)\right\}\widehat{C}(\tau^2+\tau^\alpha h^2) + \widetilde{C}\left(\tau^2+\tau^\alpha h^2\right).
\end{aligned}$$

Therefore, there exists a positive constant C, such that

$$\|\varepsilon^n\|_\infty \le C(\tau^{2-\alpha}+h^2).$$

This finishes the proof.

4 Numerical Examples

In this section, we apply the method proposed in this paper to solve the fractional partial differential equation. We obtain the numerical results and plot graphs for these problems with the help of MATLAB routines.

Example 1 Let us consider the following equation:

$${}_C\mathrm{D}_{0,t}^\alpha u(x,t) = \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t), \ \alpha \in (0,1), \ \beta \in (1,2)$$

on a finite domain $0 \le x \le 1, 0 \le t \le 1$ with a given force term

$$f(x,t) = \frac{\Gamma(3+\alpha)}{2} t^2 x^2 (1-x)^2 + \frac{t^{2+\alpha}}{\cos(\pi\beta/2)} \sum_{\ell=0}^{2} (-1)^{\ell} \frac{(2+\ell)!}{\ell!(2-\ell)!\Gamma(3+\ell-\beta)} \left[x^{2+\ell-\beta} + (1-x)^{2+\ell-\beta}\right].$$

Its analytical solution is

$$u(x,t) = t^{2+\alpha} x^2 (1-x)^2.$$

Tables 1 and 2 list the maximum errors and convergence orders using the finite-difference scheme (5) at time $t = 1$ with different stepsizes. It is observed that the numerical convergence orders are consistent with our theoretical analysis. In addition, Figs. 1, 2, 3 and 4 compare the graphs of the exact and approximate solutions with different values of α, β, τ, and h. The graphs show excellent agrement between the solutions.

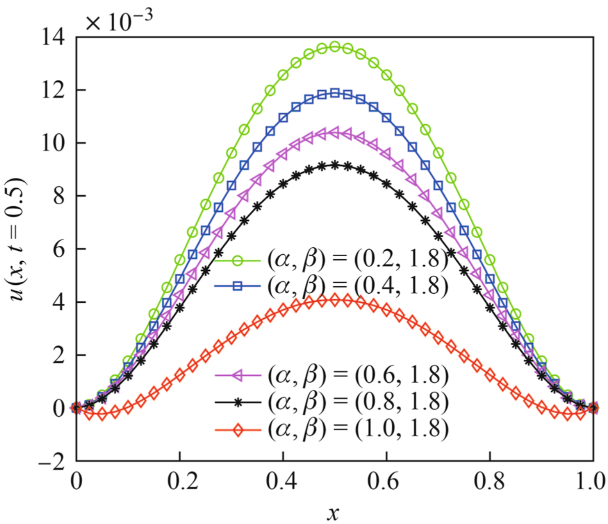
Table 1 Temporal convergence orders of Example 1 with $\beta = 1.6$ and $h = \frac{1}{1000}$

α	τ	The maximum errors	The temporal convergence orders
0.2	$\frac{1}{10}$	3.100 519E−005	–
	$\frac{1}{15}$	1.573 542E−005	1.672 7
	$\frac{1}{20}$	9.722 013E−006	1.673 8
	$\frac{1}{25}$	6.700 211E−006	1.668 2
	$\frac{1}{30}$	4.951 918E−006	1.658 4
0.4	$\frac{1}{10}$	1.245 707E−004	–
	$\frac{1}{15}$	6.713 991E−005	1.524 4
	$\frac{1}{20}$	4.318 225E−005	1.534 2
	$\frac{1}{25}$	3.063 135E−005	1.538 9
	$\frac{1}{30}$	2.312 765E−005	1.541 2
0.6	$\frac{1}{10}$	3.782 137E−004	–
	$\frac{1}{15}$	2.189 031E−004	1.348 6
	$\frac{1}{20}$	1.480 504E−004	1.359 4
	$\frac{1}{25}$	1.091 674E−004	1.365 4
	$\frac{1}{30}$	8.505 257E−005	1.369 1
0.8	$\frac{1}{10}$	1.010 018E−003	–
	$\frac{1}{15}$	6.309 231E−004	1.160 5
	$\frac{1}{20}$	4.504 827E−004	1.171 0
	$\frac{1}{25}$	3.464 536E−004	1.176 7
	$\frac{1}{30}$	2.793 741E−004	1.180 3

Table 2 Spatial convergence orders of Example 1 with $\alpha = 0.4$ and $\tau = \frac{1}{1000}$

β	h	The maximum errors	The spatial convergence orders
1.6	$\frac{1}{10}$	2.155 992E−003	–
	$\frac{1}{15}$	9.409 768E−004	2.044 8
	$\frac{1}{20}$	5.276 218E−004	2.011 0
	$\frac{1}{25}$	3.335 257E−004	2.055 4
	$\frac{1}{30}$	2.303 227E−004	2.030 7
1.7	$\frac{1}{10}$	2.208 103E−003	–
	$\frac{1}{15}$	9.633 981E−004	2.045 6
	$\frac{1}{20}$	5.397 315E−004	2.014 0
	$\frac{1}{25}$	3.413 537E−004	2.053 2
	$\frac{1}{30}$	2.357 280E−004	2.030 7
1.8	$\frac{1}{10}$	2.230 580E−003	–
	$\frac{1}{15}$	9.751 993E−004	2.040 6
	$\frac{1}{20}$	5.467 731E−004	2.011 3
	$\frac{1}{25}$	3.464 373E−004	2.045 0
	$\frac{1}{30}$	2.394 909E−004	2.024 9
1.9	$\frac{1}{10}$	2.217 223E−003	–
	$\frac{1}{15}$	9.740 071E−004	2.028 8
	$\frac{1}{20}$	5.476 063E−004	2.001 7
	$\frac{1}{25}$	3.481 438E−004	2.029 8
	$\frac{1}{30}$	2.412 347E−004	2.012 1

Fig. 1 Comparison of exact and numerical solutions for Example 1 with $\tau = \frac{1}{20}, h = \frac{1}{40}$ at time $t = 0.5$



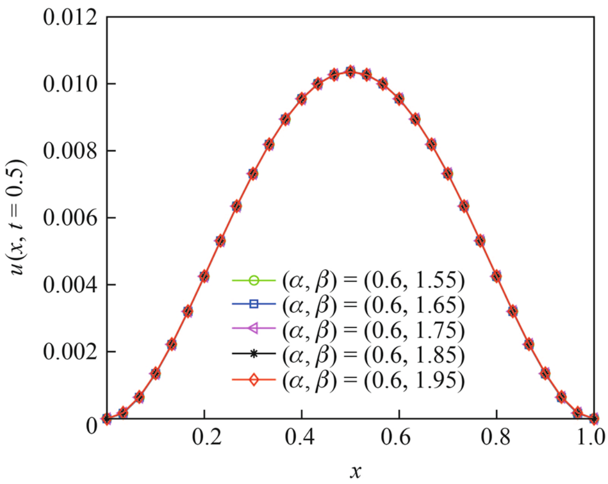


Fig. 2 Comparison of exact and numerical solutions for Example 1 with $\tau = \frac{1}{50}, h = \frac{1}{30}$ at time $t = 0.5$

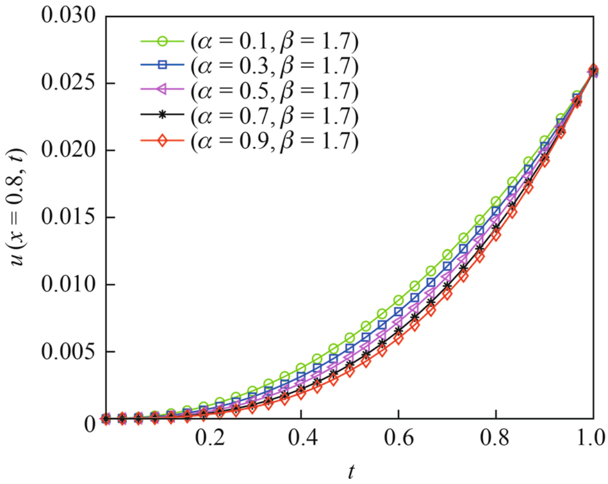


Fig. 3 Comparison of exact and numerical solutions for Example 1 with $\tau = \frac{1}{30}, h = \frac{1}{20}$ at space $x = 0.8$

Example 2 Consider the following equation:

$$ {}_{C}\mathrm{D}_{0,t}^{\alpha} u(x,t) = \frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + f(x,t), \quad 0 < x < 1, \ 0 < t < 1, $$

where

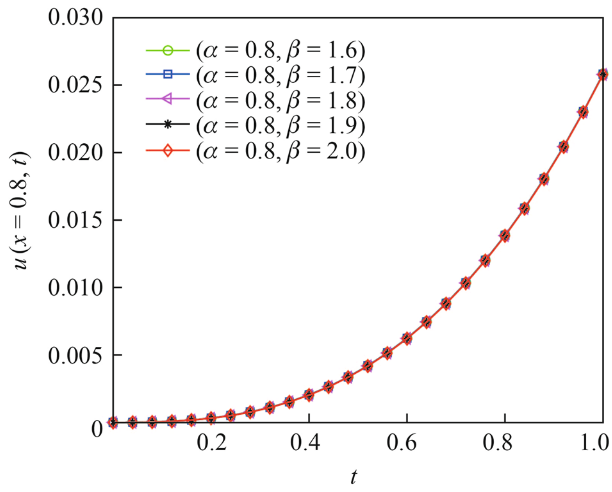


Fig. 4 Comparison of exact and numerical solutions for Example 1 with $\tau=\frac{1}{25}, h=\frac{1}{50}$ at space $x=0.8$

$$
\begin{aligned}
f(x,t)=&\frac{\Gamma(5)}{\Gamma(5-\alpha)}t^{4-\alpha}x^4(1-x)^4\\
&+\frac{t^4}{2\cos\left(\frac{\pi}{2}\beta\right)}\sum_{\ell=0}^{4}\frac{(-1)^{\ell}4!\,(4+\ell)!}{\ell!\,(4-\ell)!\,\Gamma(5+\ell-\beta)}\left[x^{4+\ell-\beta}+(1-x)^{4+\ell-\beta}\right].
\end{aligned}
$$

The exact solution is

$$u(x,t)=t^4x^4(1-x)^4.$$

In Table 3, we list the maximum error for $\beta=1.2,\ h=1/500$ and different values of α. In Table 4, we list the maximum error for $\alpha=0.7,\ \tau=1/400$ and different values of β. From these tables, we can conclude that the developed numerical solutions are in excellent agreement with the exact solution.

Table 3 Temporal convergence orders of Example 2 with $\beta = 1.2$ and $h = \frac{1}{500}$

α	τ	The maximum errors	The temporal convergence orders
0.15	$\frac{1}{10}$	5.992 925E−006	–
	$\frac{1}{15}$	3.119 961E−006	1.609 9
	$\frac{1}{20}$	1.961 602E−006	1.613 1
	$\frac{1}{25}$	1.372 343E−006	1.601 0
	$\frac{1}{30}$	1.028 883E−006	1.579 9
0.35	$\frac{1}{10}$	2.032 757E−005	–
	$\frac{1}{15}$	1.102 994E−005	1.507 8
	$\frac{1}{20}$	7.106 417E−006	1.528 1
	$\frac{1}{25}$	5.043 203E−006	1.536 9
	$\frac{1}{30}$	3.808 692E−006	1.539 9
0.55	$\frac{1}{10}$	5.827 195E−005	–
	$\frac{1}{15}$	3.382 003E−005	1.341 8
	$\frac{1}{20}$	2.284 980E−005	1.363 0
	$\frac{1}{25}$	1.681 387E−005	1.374 6
	$\frac{1}{30}$	1.306 956E−005	1.381 7
0.75	$\frac{1}{10}$	1.300 527E−004	–
	$\frac{1}{15}$	8.070 317E−005	1.176 8
	$\frac{1}{20}$	5.724 104E−005	1.194 1
	$\frac{1}{25}$	4.375 770E−005	1.203 7
	$\frac{1}{30}$	3.509 557E−005	1.209 9
0.95	$\frac{1}{10}$	2.682 839E−004	–
	$\frac{1}{15}$	1.781 985E−004	1.009 1
	$\frac{1}{20}$	1.328 735E−004	1.020 2
	$\frac{1}{25}$	1.056 765E−004	1.026 3
	$\frac{1}{30}$	8.758 086E−005	1.030 2

Table 4 Spatial convergence orders of Example 2 with $\alpha = 0.7$ and $\tau = \frac{1}{400}$

β	h	The maximum errors	The spatial convergence orders
1.55	$\frac{1}{10}$	1.625 279E−004	–
	$\frac{1}{12}$	1.142 747E−004	1.932 0
	$\frac{1}{14}$	8.480 268E−005	1.935 0
	$\frac{1}{16}$	6.548 636E−005	1.935 8
	$\frac{1}{18}$	5.214 072E−005	1.934 9
1.65	$\frac{1}{10}$	1.548 145E−004	–
	$\frac{1}{12}$	1.084 935E−004	1.950 1
	$\frac{1}{14}$	8.032 008E−005	1.950 5
	$\frac{1}{16}$	6.190 909E−005	1.949 7
	$\frac{1}{18}$	4.921 736E−005	1.947 8
1.75	$\frac{1}{10}$	1.426 306E−004	–
	$\frac{1}{12}$	9.974 747E−005	1.961 5
	$\frac{1}{14}$	7.372 765E−005	1.960 8
	$\frac{1}{16}$	5.675 414E−005	1.959 5
	$\frac{1}{18}$	4.507 008E−005	1.957 1
1.85	$\frac{1}{10}$	1.261 905E−004	–
	$\frac{1}{12}$	8.820 379E−005	1.964 3
	$\frac{1}{14}$	6.515 927E−005	1.964 4
	$\frac{1}{16}$	5.013 117E−005	1.963 5
	$\frac{1}{18}$	3.979 032E−005	1.961 4
1.95	$\frac{1}{10}$	1.056 483E−004	–
	$\frac{1}{12}$	7.398 705E−005	1.953 8
	$\frac{1}{14}$	5.471 822E−005	1.957 1
	$\frac{1}{16}$	4.212 694E−005	1.958 4
	$\frac{1}{18}$	3.345 178E−005	1.957 7

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