Communications on Applied Mathematics and Computation (2019) 1:545–563
https://doi.org/10.1007/s42967-019-00025-w

ORIGINAL PAPER

A Compact Difference Scheme for Multi-point Boundary Value Problems of Heat Equations

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Received: 15 August 2018 / Revised: 14 December 2018 / Accepted: 5 January 2019 / Published online: 7 May 2019

Abstract

In this paper, a compact difference scheme is established for the heat equations with multi-point boundary value conditions. The truncation error of the difference scheme is $O(\tau^2 + h^4)$, where τ and h are the temporal step size and the spatial step size. A prior estimate of the difference solution in a weighted norm is obtained. The unique solvability, stability and convergence of the difference scheme are proved by the energy method. The theoretical statements for the solution of the difference scheme are supported by numerical examples.

Keywords Heat equation · Multi-point boundary value condition · Compact difference scheme · Energy method

Mathematics Subject Classification 65M06 · 65M12 · 65M15

1 Introduction

The classical type of conditions is referred to as local conditions when the values of the unknown function or its derivative are specified only at the boundary points of the problem domain, while the non-local boundary conditions are proposed where the values of the unknown function at all or some points inside the problem domain take part in the condition formulation. The development of numerical methods for the solution of non-local boundary value problems has been a very important research area. There are many models and works about non-local boundary conditions, such as elliptic equations [10, 15], elliptic–parabolic equations [3, 4], hyperbolic equations [2, 7, 16, 17], hyperbolic–parabolic equations [5] and parabolic equations [1, 8, 9, 11–13].

We recall two cases of non-local boundary conditions. The first one is the integral boundary conditions

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$$u(0,t)=\int_0^L \alpha(s)u(s,t)\mathrm{d}s+g_1(t),$$
$$u(L,t)=\int_0^L \beta(s)u(s,t)\mathrm{d}s+g_2(t),\quad 0\le t\le T, \tag{1}$$

where α,β,g_1,g_2 are known functions. Another one is the multi-point boundary value conditions

$$u(0,t)=\sum_{r=1}^{M}\alpha_r(t)u(\xi_r,t)+g_1(t),$$
$$u(L,t)=\sum_{s=1}^{N}\beta_s(t)u(\eta_s,t)+g_2(t),\quad 0\le t\le T, \tag{2}$$

where α_r,β_s,g_1,g_2 are known functions.

There is a lot of research on the integral boundary conditions and multi-point boundary conditions under different models. Sun considered the heat equations with integral boundary conditions (1) and got unconditional solvability and L_∞ convergence for the difference scheme which was second order in time and fourth order in space [13]. Martin-Vaquero and Vigo-Aguiar provided a compact difference scheme for the same problem by the fourth-order Simpson's composite formula and Crandall's formula [11]. They improved the accuracy of this algorithm and studied the convergence later in [12]. They all used the composite Simpson rule to approximate the boundary conditions. Yildirim and Uzun established stable difference schemes with third and fourth order for the hyperbolic multi-point non-local boundary value problem [16]. They provided stability estimates and numerical analysis for the solutions of the difference schemes. Ashyralyev and Gercek considered a finite difference method for solving the multi-point elliptic–parabolic partial differential equation and obtained stability, and coercive stability for the solution of the difference scheme [4]. Alikhanov studied multi-point boundary conditions (2) for the heat equation with variable coefficients in the differential and finite-difference settings [1]. He established the difference scheme which is second order both in space and in time. Using the method of energy inequalities, prior estimates for the corresponding differential and finite-difference problems are obtained. Due to the characteristic of the multi-point boundary condition, he just proved the prior estimates in a weighted L^2 norm.

Our work is a good supplement to the previous researches. In this article, we construct a compact difference scheme for the multi-point boundary value problem of the heat equation taking the form of

$$\frac{\partial u}{\partial t}-a\frac{\partial^2 u}{\partial x^2}+bu=f(x,t),\quad 0<x<L,\quad 0<t\le T, \tag{3}$$

$$u(0,t)=\sum_{r=1}^{M}\alpha_r(t)u(\xi_r,t)+\mu_1(t),$$
$$u(L,t)=\sum_{s=1}^{N}\beta_s(t)u(\eta_s,t)+\mu_2(t),\quad 0<t\le T, \tag{4}$$

$$u(x,0)=\varphi(x),\quad 0\le x\le L, \tag{5}$$

where $a > 0$ and $b > 0$ are given constants, $\alpha_r(t)$, $\beta_s(t)$, $\mu_1(t)$, $\mu_2(t) \in C[0, T]$, $0 < \xi_1 < \xi_2 < \cdots < \xi_M < L$, $0 < \eta_1 < \eta_2 < \cdots < \eta_N < L$, f and φ are continuous functions. We establish a compact difference scheme with the truncation error $O(\tau^2 + h^4)$ and get a weighted $L2$ norm prior estimate. Then, we prove the unique solvability, convergence and stability using the energy method.

The rest of this paper is organized as follows: some notations are introduced and several important lemmas are given in Sect. 2. Then, a compact difference scheme is constructed in Sect. 3 and a prior estimate is provided in Sect. 4. Based on a prior estimate, the unique solvability, stability and convergence are proved in Sect. 5. Besides, a compact finite-difference scheme is also given for the multi-point boundary value problems of the heat equation with variable coefficients in Sect. 6. At last, two numerical examples are presented in Sect. 7 and a brief conclusion is given in Sect. 8, respectively.

2 Preliminary

In this section, some useful notations and lemmas will be prepared.

For finite-difference approximation, we discretize equally the interval [0, L] with $x_i = ih$ $(0 \le i \le m)$, $[0, T]$ with $t_k = k\tau$ $(0 \le k \le n)$, where $h = L/m$ and $\tau = T/n$ are the spatial and temporal step sizes, respectively. Denote $t_{k+\frac{1}{2}} = (t_k + t_{k+1})/2$, $\Omega_h = \{x_i \mid 0 \le i \le m\}$, $\Omega_\tau = \{t_k \mid 0 \le k \le n\}$, then the computational domain $[0, L] \times [0, T]$ is covered by $\Omega_h \times \Omega_\tau$. For any mesh function $v = \{v_i^k \mid 0 \le i \le m,\ 0 \le k \le n\}$ defined on $\Omega_h \times \Omega_\tau$, introduce the following notations:

$$v_i^{k+\frac{1}{2}} = \frac{1}{2}\left(v_i^k + v_i^{k+1}\right), \quad \delta_t v_i^{k+\frac{1}{2}} = \frac{1}{\tau}\left(v_i^{k+1} - v_i^k\right),$$
$$\delta_x v_{i+\frac{1}{2}}^k = \frac{1}{h}\left(v_{i+1}^k - v_i^k\right), \quad \delta_x^2 v_i^k = \frac{1}{h}\left(\delta_x v_{i+\frac{1}{2}}^k - \delta_x v_{i-\frac{1}{2}}^k\right).$$

Let $v^k = (v_0^k, v_1^k, \ldots, v_m^k)$, then v^k is a mesh function defined on Ω_h.

Denote

$$V_h = \left\{v \mid v = (v_0, v_1, \ldots, v_m)\right\}$$

and

$$p(x) = \sqrt{x(L-x)}, \quad 0 \le x \le L.$$

For any $v \in V_h$, introduce the following norms or seminorms:

$$\|v\|_\infty = \max_{0\le i\le m} |v_i|, \quad \|v\| = \sqrt{h\left(\frac{1}{2}v_0^2 + \sum_{i=1}^{m-1} v_i^2 + \frac{1}{2}v_m^2\right)}, \quad |v|_1 = \sqrt{h\sum_{i=0}^{m-1}\left(\delta_x v_{i+\frac{1}{2}}\right)^2},$$
$$\|v\|_0 = \sqrt{h\sum_{i=1}^{m-1} v_i^2}, \quad \|pv\|_0 = \sqrt{h\sum_{i=1}^{m-1} p_i^2 v_i^2}, \quad |pv|_1 = \sqrt{h\sum_{i=0}^{m-1} \frac{p_i^2 + p_{i+1}^2}{2}\left(\delta_x v_{i+\frac{1}{2}}\right)^2}.$$

For any $0 < c < d < L$, define

$$x_{m_0} = \min_{x_i\in[c,d]} x_i, \quad x_{n_0} = \max_{x_i\in[c,d]} x_i.$$

With an assumption that $2h < d - c$, we give similar definitions in the interval $[c, d]$:

$$\|v\|_{\infty,[c,d]} = \max_{c \le x_i \le d} |v_i|, \quad \|v\|_{[c,d]} = \sqrt{h\left(\frac{1}{2}v_{m_0}^2 + \sum_{i=m_0+1}^{n_0-1} v_i^2 + \frac{1}{2}v_{n_0}^2\right)},$$

$$|v|_{1,[c,d]} = \sqrt{h\sum_{i=m_0}^{n_0-1}\left(\delta_x v_{i+\frac{1}{2}}\right)^2},$$

$$\|pv\|_{[c,d]} = \sqrt{h\left(\frac{1}{2}p_{m_0}^2 v_{m_0}^2 + \sum_{i=m_0+1}^{n_0-1} p_i^2 v_i^2 + \frac{1}{2}p_{n_0}^2 v_{n_0}^2\right)},$$

$$|pv|_{1,[c,d]} = \sqrt{h\sum_{i=m_0}^{n_0-1}\frac{p_i^2 + p_{i+1}^2}{2}(\delta_x v_{i+\frac{1}{2}})^2}.$$

For any grid function $w \in V_h$, define

$$(\mathcal{A}w)_i = \begin{cases} \frac{1}{12}(w_{i-1} + 10w_i + w_{i+1}), & 1 \le i \le m-1, \\ w_i, \quad i = 0,\ m. \end{cases}$$

We need some lemmas for establishing and analyzing the difference scheme for (3)–(5).

Lemma 2.1 [14] *Let $v \in V_h$. Then, for any $\varepsilon > 0$, we have*

$$\|v\|_\infty^2 \le \varepsilon|v|_1^2 + \left(\frac{1}{\varepsilon} + \frac{1}{L}\right)\|v\|^2,$$

$$\|v\|_{\infty,[c,d]}^2 \le \varepsilon|v|_{1,[c,d]}^2 + \left(\frac{1}{\varepsilon} + \frac{1}{x_{n_0} - x_{m_0}}\right)\|v\|_{[c,d]}^2.$$

Lemma 2.2 *Let $v \in V_h$ and $0 < c < d < L$. If $h < (d-c)/4$, then for any $\varepsilon > 0$, we have*

$$\|v\|_{\infty,[c,d]}^2 \le \varepsilon|pv|_1^2 + \left(\frac{1}{c_0^2\varepsilon} + \frac{2}{c_0(d-c)}\right)\|pv\|_0^2,$$

where $c_0 = \min\{c(L-c), d(L-d)\}$.

Proof According to Lemma 2.1, we have

$$\begin{aligned}\|v\|^2_{\infty,[c,d]} &\le c_0\varepsilon|v|^2_{1,[c,d]} + \left(\frac{1}{c_0\varepsilon} + \frac{1}{x_{n_0} - x_{m_0}}\right)\|v\|^2_{[c,d]}\\ &\le c_0\varepsilon|v|^2_{1,[c,d]} + \left(\frac{1}{c_0\varepsilon} + \frac{1}{d-c-2h}\right)\|v\|^2_{[c,d]}\\ &\le \varepsilon|pv|^2_{1,[c,d]} + \left(\frac{1}{c_0^2\varepsilon} + \frac{2}{c_0(d-c)}\right)\|pv\|^2_{[c,d]}\\ &\le \varepsilon|pv|^2_{1} + \left(\frac{1}{c_0^2\varepsilon} + \frac{2}{c_0(d-c)}\right)\|pv\|^2_{0}.\end{aligned}$$

This completes the proof.

Lemma 2.3 *For any grid function $v \in V_h$, we have*

$$\|pv\|_0^2 \le \frac{12}{5}\|p(\mathcal{A}v)\|_0^2 + \frac{h^2L}{6}(v_0^2 + v_m^2).$$

Proof From the definition, we can get

$$\begin{aligned}\|p(\mathcal{A}v)\|_0^2 &= h\sum_{i=1}^{m-1} p_i^2(\mathcal{A}v_i)^2 = \frac{h}{144}\sum_{i=1}^{m-1} p_i^2(v_{i-1} + 10v_i + v_{i+1})^2\\ &= \frac{h}{144}\sum_{i=1}^{m-1} p_i^2\left[(v_{i-1} + v_{i+1})^2 + 100v_i^2 + 20v_i(v_{i-1} + v_{i+1})\right]\\ &\ge \frac{h}{144}\sum_{i=1}^{m-1} p_i^2\left[100v_i^2 - 20v_i^2 - 10(v_{i-1}^2 + v_{i+1}^2)\right]\\ &= \frac{40h}{72}\sum_{i=1}^{m-1} p_i^2v_i^2 - \frac{5h}{72}\left(\sum_{i=0}^{m-2} p_{i+1}^2v_i^2 + \sum_{i=2}^{m} p_{i-1}^2v_i^2\right)\\ &\ge \frac{5}{12}\|pv\|^2 + \frac{5h}{72}\sum_{i=1}^{m-1}(2p_i^2 - p_{i-1}^2 - p_{i+1}^2)v_i^2 - \frac{5h^2(L-h)}{72}(v_0^2 + v_m^2)\\ &\ge \frac{5}{12}\|pv\|^2 - \frac{5h^2L}{72}(v_0^2 + v_m^2).\end{aligned}$$

In obtaining the last inequality, we have used $2p_i^2 - p_{i-1}^2 - p_{i+1}^2 > 0$ when $1 \le i \le m-1$.
This completes the proof.

Lemma 2.4 *Suppose $f(x) \in C[d, d+3h]$. Taking $d,\ d+h,\ d+2h,\ d+3h$ as the interpolation points, we obtain the third-order interpolation polynomial of $f(x)$:*

$$L_3(x) = \sum_{i=0}^{3} f(d+ih)\prod_{\substack{j=0\\ j\ne i}}^{3}\frac{x-(d+jh)}{(d+ih)-(d+jh)}.$$

Then, it satisfies

$$\max_{d+h\le x\le d+2h} |L_3(x)| \le \frac{5}{4} \max_{0\le i\le 3} |f(d+ih)|.$$

Proof Let

$$s = (x-d)/h, \qquad x \in [d+h, d+2h].$$

We have

$$\sum_{i=0}^{3} \prod_{\substack{j=0 \\ j\ne i}}^{3} \left| \frac{x-(d+jh)}{(d+ih)-(d+jh)} \right| = \sum_{i=0}^{3} \prod_{\substack{j=0 \\ j\ne i}}^{3} \left| \frac{s-j}{i-j} \right| = \frac{5}{4} - \left(s - \frac{3}{2}\right)^2, \quad 1 \le s \le 2.$$

Therefore, we obtain

$$\begin{aligned} \max_{d+h\le x\le d+2h} |L_3(x)| &\le \sum_{i=0}^{3} \prod_{\substack{j=0 \\ j\ne i}}^{3} \left| \frac{x-(d+jh)}{(d+ih)-(d+jh)} \right| \max_{0\le i\le 3} |f(d+ih)| \\ &= \left[\frac{5}{4} - \left(s-\frac{3}{2}\right)^2 \right] \max_{0\le i\le 3} |f(d+ih)| \le \frac{5}{4} \max_{0\le i\le 3} |f(d+ih)|. \end{aligned}$$

This completes the proof.

Lemma 2.5 [14] *Let $h > 0$ and c be two constants. Suppose $g(x) \in C^6[c-h, c+h]$. Then,*

$$\begin{aligned} &\frac{1}{12}[g''(c-h) + 10g''(c) + g''(c+h)] \\ &= \frac{1}{h^2}\left[g(c+h) - 2g(c) + g(c-h)\right] + \frac{h^4}{240} g^{(6)}(\xi),\ c-h < \xi < c+h. \end{aligned}$$

Lemma 2.6 [6] *Let $\{F^k \mid k \ge 0\}$ and $\{G^k \mid k \ge 0\}$ be two nonnegative sequences and satisfy*

$$F^{k+1} \le (1+c\tau)F^k + \tau G^k, \quad k = 0, 1, 2, \dots,$$

where c is a nonnegative constant. Then, we have

$$F^{k+1} \le e^{c(k+1)\tau}\left(F^0 + \tau \sum_{l=0}^{k} G^l \right), \quad k = 0, 1, 2, \dots.$$

3 Derivation of the Difference Scheme

Define a grid function

$$U = \left\{ U_i^k \mid 0 \le i \le m,\ 0 \le k \le n \right\}$$

on $\Omega_h \times \Omega_\tau$, where

$$U_i^k = u(x_i, t_k), \quad 0 \le i \le m,\ 0 \le k \le n.$$

Suppose h is small enough satisfying $\xi_1 > 2h$, $\xi_M < L - 2h$, $\eta_1 > 2h$, and $\eta_N < L - 2h$, that is, $h < \frac{1}{2}\min\{\xi_1, L - \xi_M, \eta_1, L - \eta_N\}$.

Considering the differential equation (3) at point $(x_i, t_{k+\frac{1}{2}})$, we have

$$\frac{\partial u}{\partial t}\left(x_i, t_{k+\frac{1}{2}}\right) - a\frac{\partial^2 u}{\partial x^2}\left(x_i, t_{k+\frac{1}{2}}\right) + bu\left(x_i, t_{k+\frac{1}{2}}\right) = f\left(x_i, t_{k+\frac{1}{2}}\right), \quad 0 \le i \le m,\ 0 \le k \le n-1. \tag{6}$$

By the Taylor expansion, we get

$$\begin{aligned} &\delta_t U_i^{k+\frac{1}{2}} - \frac{a}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1})\right] + bU_i^{k+\frac{1}{2}} \\ &= f_i^{k+\frac{1}{2}} + O(\tau^2), \quad 0 \le i \le m,\ 0 \le k \le n-1, \end{aligned}$$

where $f_i^{k+\frac{1}{2}} = f(x_i, t_{k+\frac{1}{2}})$. Acting the operator $\mathcal{A}$ on the above equation, we obtain

$$\begin{aligned} &\mathcal{A}\delta_t U_i^{k+\frac{1}{2}} - \frac{a}{2}\left[\mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1})\right] + b\mathcal{A}U_i^{k+\frac{1}{2}} \\ &= \mathcal{A}f_i^{k+\frac{1}{2}} + O(\tau^2), \quad 1 \le i \le m-1,\ 0 \le k \le n-1. \end{aligned} \tag{7}$$

Using Lemma 2.5, we have

$$\frac{1}{2}\left[\mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1})\right] = \delta_x^2 U_i^{k+\frac{1}{2}} + O(h^4). \tag{8}$$

Substituting (8) into (7), we obtain

$$\mathcal{A}\delta_t U_i^{k+\frac{1}{2}} - a\delta_x^2 U_i^{k+\frac{1}{2}} + b\mathcal{A}U_i^{k+\frac{1}{2}} = \mathcal{A}f_i^{k+\frac{1}{2}} + R_i^{k+\frac{1}{2}}, \quad 1 \le i \le m-1,\ 0 \le k \le n-1. \tag{9}$$

There exists a constant $\hat{c}_1$ such that

$$|R_i^{k+\frac{1}{2}}| \le \hat{c}_1(\tau^2 + h^4), \quad 1 \le i \le m-1,\ 0 \le k \le n-1. \tag{10}$$

Considering boundary value conditions (4) at $t_{k+\frac{1}{2}}$, we have

$$U_0^{k+\frac{1}{2}} = \sum_{r=1}^{M} \alpha_r\left(t_{k+\frac{1}{2}}\right)\frac{u\left(\xi_r, t_k\right) + u\left(\xi_r, t_{k+1}\right)}{2} + \mu_1\left(t_{k+\frac{1}{2}}\right) + O\left(\tau^2\right), \quad 0 \le k \le n-1, \tag{11}$$

$$U_m^{k+\frac{1}{2}} = \sum_{s=1}^{N} \beta_s\left(t_{k+\frac{1}{2}}\right)\frac{u\left(\eta_s, t_k\right) + u\left(\eta_s, t_{k+1}\right)}{2} + \mu_2\left(t_{k+\frac{1}{2}}\right) + O\left(\tau^2\right), \quad 0 \le k \le n-1. \tag{12}$$

For any ξ_r, there exists a unique i_r such that $\xi_r \in [x_{i_r+1}, x_{i_r+2})$. Taking $x_{i_r}, x_{i_r+1}, x_{i_r+2}, x_{i_r+3}$ as the interpolation points, we obtain the third-order interpolation polynomial of $u(x, t)$:

$$L_r^{(0)}(x,t) = \sum_{p=i_r}^{i_r+3} u(x_p,t) \prod_{\substack{q=i_r \\ q\neq p}}^{i_r+3} \frac{x-x_q}{x_p-x_q}$$

and we have

$$u(\xi_r,t) = L_r^{(0)}(\xi_r,t) + O(h^4), \quad 1 \le r \le M. \tag{13}$$

Similarly, for any η_s, there is a unique j_s satisfying $\eta_s \in [x_{j_s+1}, x_{j_s+2})$. Taking $x_{j_s}, x_{j_s+1}, x_{j_s+2}$, x_{j_s+3} as interpolation points, we obtain the third-order interpolation polynomial of $u(x, t)$:

$$L_s^{(1)}(x,t) = \sum_{p=j_s}^{j_s+3} u(x_p,t) \prod_{\substack{q=j_s \\ q\neq p}}^{j_s+3} \frac{x-x_q}{x_p-x_q}$$

and we have

$$u(\eta_s,t) = L_s^{(1)}(\eta_s,t) + O(h^4), \quad 1 \le s \le N. \tag{14}$$

Substituting (13) into (11) and (14) into (12), we get

$$U_0^{k+\frac{1}{2}} = \sum_{r=1}^{M} \alpha_r\left(t_{k+\frac{1}{2}}\right)\frac{L_r^{(0)}\left(\xi_r, t_k\right) + L_r^{(0)}\left(\xi_r, t_{k+1}\right)}{2} + \mu_1\left(t_{k+\frac{1}{2}}\right) + R_0^{k+\frac{1}{2}}, \quad 0 \le k \le n-1,$$
$$U_m^{k+\frac{1}{2}} = \sum_{s=1}^{N} \beta_s\left(t_{k+\frac{1}{2}}\right)\frac{L_s^{(1)}\left(\eta_s, t_k\right) + L_s^{(1)}\left(\eta_s, t_{k+1}\right)}{2} + \mu_2\left(t_{k+\frac{1}{2}}\right) + R_m^{k+\frac{1}{2}}, \quad 0 \le k \le n-1,$$

or

$$U_0^{k+\frac{1}{2}} = \sum_{r=1}^{M} \alpha_r\left(t_{k+\frac{1}{2}}\right)\left(\sum_{p=i_r}^{i_r+3} U_p^{k+\frac{1}{2}} \prod_{\substack{q=i_r \\ q\neq p}}^{i_r+3} \frac{\xi_r-x_q}{x_p-x_q}\right) + \mu_1\left(t_{k+\frac{1}{2}}\right) + R_0^{k+\frac{1}{2}}, \quad 0 \le k \le n-1, \tag{15}$$

$$U_m^{k+\frac{1}{2}} = \sum_{s=1}^{N} \beta_s\left(t_{k+\frac{1}{2}}\right)\left(\sum_{p=j_s}^{j_s+3} U_p^{k+\frac{1}{2}} \prod_{\substack{q=j_s \\ q\neq p}}^{j_s+3} \frac{\eta_s-x_q}{x_p-x_q}\right) + \mu_2\left(t_{k+\frac{1}{2}}\right) + R_m^{k+\frac{1}{2}}, \quad 0 \le k \le n-1, \tag{16}$$

and there exists a constant $\hat{c}_2$ such that

$$|R_0^{k+\frac{1}{2}}| \le \hat{c}_2\left(\tau^2 + h^4\right), \quad |R_m^{k+\frac{1}{2}}| \le \hat{c}_2\left(\tau^2 + h^4\right), \quad 0 \le k \le n-1.$$

Noticing the initial condition (5),

$$U_i^0 = \varphi(x_i), \quad 0 \le i \le m, \tag{17}$$

omitting the small items $R_i^{k+\frac{1}{2}}$ $(0 \le i \le m)$ in the formula (9), (15) and (16), and replacing U_i^k by u_i^k, we obtain the following difference scheme:

$$\mathcal{A}\delta_t u_i^{k+\frac{1}{2}} - a\delta_x^2 u_i^{k+\frac{1}{2}} + b\mathcal{A}u_i^{k+\frac{1}{2}} = \mathcal{A}f_i^{k+\frac{1}{2}}, \quad 1 \le i \le m-1,\ 0 \le k \le n-1, \tag{18}$$

$$u_0^{k+\frac{1}{2}} = \sum_{r=1}^{M} \alpha_r\left(t_{k+\frac{1}{2}}\right)\left(\sum_{p=i_r}^{i_r+3} u_p^{k+\frac{1}{2}} \prod_{\substack{q=i_r\\ q\neq p}}^{i_r+3} \frac{\xi_r - x_q}{x_p - x_q}\right) + \mu_1\left(t_{k+\frac{1}{2}}\right), \quad 0 \le k \le n-1, \tag{19}$$

$$u_m^{k+\frac{1}{2}} = \sum_{s=1}^{N} \beta_s\left(t_{k+\frac{1}{2}}\right)\left(\sum_{p=j_s}^{j_s+3} u_p^{k+\frac{1}{2}} \prod_{\substack{q=j_s\\ q\neq p}}^{j_s+3} \frac{\eta_s - x_q}{x_p - x_q}\right) + \mu_2\left(t_{k+\frac{1}{2}}\right), \quad 0 \le k \le n-1, \tag{20}$$

$$u_i^0 = \varphi\left(x_i\right), \quad 0 \le i \le m. \tag{21}$$

4 Prior Estimates

Lemma 4.1 *The solution of* (18)–(21) *satisfies the following equality*:

$$\begin{aligned} &\frac{1}{2\tau}\left(\left\|p\left(\mathcal{A}u^{k+1}\right)\right\|_0^2 - \|p\left(\mathcal{A}u^k\right)\|_0^2\right) + a\left|pu^{k+\frac{1}{2}}\right|_1^2 + a\left\|u^{k+\frac{1}{2}}\right\|^2 \\ &- \frac{ah^2}{12}\left\|p\left(\delta_x^2 u^{k+\frac{1}{2}}\right)\right\|_0^2 + b\left\|p\left(\mathcal{A}u^{k+\frac{1}{2}}\right)\right\|^2 = h\sum_{i=1}^{m-1} p_i^2\left(\mathcal{A}f_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right) \\ &+ \frac{aL}{2}\left[\left(u_m^{k+\frac{1}{2}}\right)^2 + \left(u_0^{k+\frac{1}{2}}\right)^2\right], \quad 0 \le k \le n-1. \end{aligned} \tag{22}$$

Proof Multiplying equality (18) by $h\mathcal{A}u_i^{k+\frac{1}{2}}$ and summing the result with respect to i from η to ξ, we obtain

$$
\begin{aligned}
&h\sum_{i=\eta}^{\xi}\left(\mathcal{A}\delta_t u_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right)-ah\sum_{i=\eta}^{\xi}\left(\delta_x^2 u_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right)+bh\sum_{i=\eta}^{\xi}\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right)^2\\
&=h\sum_{i=\eta}^{\xi}\left(\mathcal{A}f_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right),\quad 1\le\eta\le\xi\le m-1,\ 0\le k\le n-1.
\end{aligned}
\tag{23}
$$

We multiply the above equality by h and sum up for ξ from η to $m-1$, and then multiply the result by h and sum up for η from 1 to $m-1$ to get

$$
\begin{aligned}
&h\sum_{i=1}^{m-1}p_i^2\left(\mathcal{A}\delta_t u_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right)-ah\sum_{i=1}^{m-1}p_i^2\left(\delta_x^2 u_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right)\\
&+bh\sum_{i=1}^{m-1}p_i^2\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right)^2=h\sum_{i=1}^{m-1}p_i^2\left(\mathcal{A}f_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right),\quad 0\le k\le n-1.
\end{aligned}
\tag{24}
$$

Due to

$$
\left(\mathcal{A}\delta_t u_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right)=\frac{1}{2\tau}\left[\left(\mathcal{A}u_i^{k+1}\right)^2-\left(\mathcal{A}u_i^{k}\right)^2\right]
$$

and

$$
-ah\sum_{i=1}^{m-1}p_i^2\left(\delta_x^2 u_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right)=-ah\sum_{i=1}^{m-1}p_i^2\left(\delta_x^2 u_i^{k+\frac{1}{2}}\right)u_i^{k+\frac{1}{2}}-\frac{ah^3}{12}\sum_{i=1}^{m-1}p_i^2\left(\delta_x^2 u_i^{k+\frac{1}{2}}\right)^2,
$$

we obtain

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\left\|p\left(\mathcal{A}u^{k+1}\right)\right\|_0^2-\left\|p\left(\mathcal{A}u^{k}\right)\right\|_0^2\right)-ah\sum_{i=1}^{m-1}p_i^2\left(\delta_x^2 u_i^{k+\frac{1}{2}}\right)u_i^{k+\frac{1}{2}}\\
&-\frac{ah^2}{12}\left\|p\left(\delta_x^2 u^{k+\frac{1}{2}}\right)\right\|_0^2+b\left\|p\left(\mathcal{A}u^{k+\frac{1}{2}}\right)\right\|^2=h\sum_{i=1}^{m-1}p_i^2\left(\mathcal{A}f_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right),\quad 0\le k\le n-1.
\end{aligned}
\tag{25}
$$

Some calculation yields

$$
\begin{aligned}
-ah\sum_{i=1}^{m-1}p_i^2\left(\delta_x^2 u_i^{k+\frac{1}{2}}\right)u_i^{k+\frac{1}{2}}&=\frac{ah}{2}\sum_{i=0}^{m-1}\left(p_i^2+p_{i+1}^2\right)\left(\delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}}\right)^2\\
&\quad+\frac{a}{2}\sum_{i=0}^{m-1}\left(L-2x_i-h\right)\left[\left(u_{i+1}^{k+\frac{1}{2}}\right)^2-\left(u_i^{k+\frac{1}{2}}\right)^2\right]\\
&=a\left|pu^{k+\frac{1}{2}}\right|_1^2+a\left\|u^{k+\frac{1}{2}}\right\|^2-\frac{aL}{2}\left[\left(u_m^{k+\frac{1}{2}}\right)^2+\left(u_0^{k+\frac{1}{2}}\right)^2\right].
\end{aligned}
$$

Substituting the above equality into (25), we can get (22).

This completes the proof.

Theorem 4.1 *Let* $\{u_i^k \mid 0 \le i \le m, 0 \le k \le n\}$ *be the solution of* (18)–(21). *Denote*

$$\begin{aligned}
&\alpha_0 = \max_{0\le k\le n-1}\left(\frac{5}{4}\sum_{r=1}^{M}\left|\alpha_r\left(t_{k+\frac{1}{2}}\right)\right|\right)^2, \quad \beta_0 = \max_{0\le k\le n-1}\left(\frac{5}{4}\sum_{s=1}^{N}\left|\beta_s\left(t_{k+\frac{1}{2}}\right)\right|\right)^2,\\
&c_0 = \min\left\{\frac{\xi_1}{2}\left(L-\frac{\xi_1}{2}\right), \frac{\xi_M}{2}\left(L-\frac{\xi_M}{2}\right), \frac{\eta_1}{2}\left(L-\frac{\eta_1}{2}\right), \frac{\eta_N}{2}\left(L-\frac{\eta_N}{2}\right)\right\},\\
&c_1 = \min\left\{\xi_M-\xi_1, \eta_N-\eta_1\right\},\\
&c_2 = \frac{1}{2}+\frac{24(\alpha_0+\beta_0)aL}{5}\left(\frac{6(\alpha_0+\beta_0)L}{c_0^2}+\frac{2}{c_0c_1}\right), \qquad c_3 = \max\{1, 4aL\}.
\end{aligned} \tag{26}$$

When h and τ *are small enough, we have*

$$\begin{aligned}
\left\|p\left(\mathcal{A}u^{k+1}\right)\right\|_0^2 \le e^{3c_2(k+1)\tau}\Bigg[\left\|p\left(\mathcal{A}u^0\right)\right\|_0^2 + \frac{3c_3}{2}\tau\sum_{l=0}^{k}\Bigg(\left\|p\left(\mathcal{A}f^{l+\frac{1}{2}}\right)\right\|_0^2\\
+\mu_1^2\left(t_{l+\frac{1}{2}}\right)+\mu_2^2\left(t_{l+\frac{1}{2}}\right)\Bigg)\Bigg], \; 0\le k\le n-1.
\end{aligned} \tag{27}$$

Proof By Lemma 4.1 and inequalities

$$\begin{aligned}
&h\sum_{i=1}^{m-1}p_i^2\left(\mathcal{A}f_i^{k+\frac{1}{2}}\right)\left(\mathcal{A}u_i^{k+\frac{1}{2}}\right) \le \frac{1}{2}\left\|p\left(\mathcal{A}u^{k+\frac{1}{2}}\right)\right\|_0^2+\frac{1}{2}\left\|p\left(\mathcal{A}f^{k+\frac{1}{2}}\right)\right\|_0^2,\\
&\frac{ah^2}{12}\left\|p\left(\delta_x^2u^{k+\frac{1}{2}}\right)\right\|_0^2 \le \frac{ah}{6}\sum_{i=1}^{m-1}p_i^2\left[\left(\delta_xu_{i+\frac{1}{2}}^{k+\frac{1}{2}}\right)^2+\left(\delta_xu_{i-\frac{1}{2}}^{k+\frac{1}{2}}\right)^2\right] \le \frac{a}{3}|pu^{k+\frac{1}{2}}|_1^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{2\tau}\left(\left\|p\left(\mathcal{A}u^{k+1}\right)\right\|_0^2-\left\|p\left(\mathcal{A}u^{k}\right)\right\|_0^2\right)+\frac{2a}{3}|pu^{k+\frac{1}{2}}|_1^2\\
&\le \frac{1}{2}\left\|p\left(\mathcal{A}u^{k+\frac{1}{2}}\right)\right\|_0^2+\frac{1}{2}\left\|p\left(\mathcal{A}f^{k+\frac{1}{2}}\right)\right\|_0^2+\frac{aL}{2}\left[\left(u_0^{k+\frac{1}{2}}\right)^2+\left(u_m^{k+\frac{1}{2}}\right)^2\right].
\end{aligned} \tag{28}$$

According to Lemmas 2.2 and 2.3, when $4h \le \min\{\xi_1, L-\xi_M, \eta_1, L-\eta_n, \xi_M-\xi_1, \eta_N-\eta_1\}$, we have

$$
\begin{aligned}
\frac{1}{2}\left(u_0^{k+\frac{1}{2}}\right)^2 &\le \alpha_0\left\|u^{k+\frac{1}{2}}\right\|^2_{\infty,[x_{i_1},x_{i_M+3}]} + \mu_1^2\left(t_{k+\frac{1}{2}}\right)\\
&\le \alpha_0\varepsilon|pu^{k+\frac{1}{2}}|_1^2 + \alpha_0\left(\frac{1}{c_0^2\varepsilon} + \frac{2}{c_0\left(x_{i_M+3}-x_{i_1}\right)}\right)\left\|pu^{k+\frac{1}{2}}\right\|_0^2 + \mu_1^2\left(t_{k+\frac{1}{2}}\right)\\
&\le \alpha_0\varepsilon\left|pu^{k+\frac{1}{2}}\right|_1^2 + \frac{12\alpha_0}{5}\left(\frac{1}{c_0^2\varepsilon} + \frac{2}{c_0\left(\xi_M-\xi_1\right)}\right)\left\|p\left(\mathcal{A}u^{k+\frac{1}{2}}\right)\right\|_0^2\\
&\quad + \frac{\alpha_0 h^2 L}{6}\left(\frac{1}{c_0^2\varepsilon} + \frac{2}{c_0\left(\xi_M-\xi_1\right)}\right)\left[\left(u_0^{k+\frac{1}{2}}\right)^2 + \left(u_m^{k+\frac{1}{2}}\right)^2\right] + \mu_1^2\left(t_{k+\frac{1}{2}}\right),\\
\frac{1}{2}\left(u_m^{k+\frac{1}{2}}\right)^2 &\le \beta_0\left\|u^{k+\frac{1}{2}}\right\|^2_{\infty,[x_{j_1},x_{j_N+3}]} + \mu_2^2\left(t_{k+\frac{1}{2}}\right)\\
&\le \beta_0\varepsilon\left|pu^{k+\frac{1}{2}}\right|_1^2 + \beta_0\left(\frac{1}{c_0^2\varepsilon} + \frac{2}{c_0\left(x_{j_N+3}-x_{j_1}\right)}\right)\left\|pu^{k+\frac{1}{2}}\right\|_0^2 + \mu_2^2\left(t_{k+\frac{1}{2}}\right)\\
&\le \beta_0\varepsilon\left|pu^{k+\frac{1}{2}}\right|_1^2 + \frac{12\beta_0}{5}\left(\frac{1}{c_0^2\varepsilon} + \frac{2}{c_0\left(\eta_N-\eta_1\right)}\right)\left\|p\left(\mathcal{A}u^{k+\frac{1}{2}}\right)\right\|_0^2\\
&\quad + \frac{\beta_0 h^2 L}{6}\left(\frac{1}{c_0^2\varepsilon} + \frac{2}{c_0\left(\eta_N-\eta_1\right)}\right)\left[\left(u_0^{k+\frac{1}{2}}\right)^2 + \left(u_m^{k+\frac{1}{2}}\right)^2\right] + \mu_2^2\left(t_{k+\frac{1}{2}}\right).
\end{aligned}
$$

Taking $\varepsilon = 1/(6L(\alpha_0+\beta_0))$, we have

$$
\begin{aligned}
&\left[\frac{1}{2} - \frac{\left(\alpha_0+\beta_0\right)h^2L}{6}\left(\frac{6\left(\alpha_0+\beta_0\right)L}{c_0^2} + \frac{2}{c_0c_1}\right)\right]\left[\left(u_0^{k+\frac{1}{2}}\right)^2 + \left(u_m^{k+\frac{1}{2}}\right)^2\right]\\
&\le \frac{1}{6L}\left|pu^{k+\frac{1}{2}}\right|_1^2 + \frac{12\left(\alpha_0+\beta_0\right)}{5}\left(\frac{6\left(\alpha_0+\beta_0\right)L}{c_0^2} + \frac{2}{c_0c_1}\right)\left\|p\left(\mathcal{A}u^{k+\frac{1}{2}}\right)\right\|_0^2\\
&\quad + \mu_1^2\left(t_{k+\frac{1}{2}}\right) + \mu_2^2\left(t_{k+\frac{1}{2}}\right).
\end{aligned}
$$

When $h < \frac{c_0}{2}\sqrt{\frac{3c_1}{(\alpha_0+\beta_0)L[3c_1(\alpha_0+\beta_0)L+c_0]}}$, we get

$$
\begin{aligned}
\left(u_0^{k+\frac{1}{2}}\right)^2 + \left(u_m^{k+\frac{1}{2}}\right)^2 &\le \frac{2}{3L}\left|pu^{k+\frac{1}{2}}\right|_1^2 + \frac{48\left(\alpha_0+\beta_0\right)}{5}\left(\frac{6\left(\alpha_0+\beta_0\right)L}{c_0^2} + \frac{2}{c_0c_1}\right)\left\|p\left(\mathcal{A}u^{k+\frac{1}{2}}\right)\right\|_0^2\\
&\quad + 4\mu_1^2\left(t_{k+\frac{1}{2}}\right) + 4\mu_2^2\left(t_{k+\frac{1}{2}}\right).
\end{aligned}
$$

Substituting the above inequality into (28), we obtain

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\left\|p\left(\mathcal{A}u^{k+1}\right)\right\|_0^2-\left\|p\left(\mathcal{A}u^{k}\right)\right\|_0^2\right)\\
&\le \frac{c_2}{2}\left(\left\|p\left(\mathcal{A}u^{k}\right)\right\|_0^2+\left\|p\left(\mathcal{A}u^{k+1}\right)\right\|_0^2\right)+\frac{c_3}{2}\left[\left\|p\left(\mathcal{A}f^{k+\frac{1}{2}}\right)\right\|_0^2+\mu_1^2\left(t_{k+\frac{1}{2}}\right)+\mu_2^2\left(t_{k+\frac{1}{2}}\right)\right].
\end{aligned}
$$

That is

$$
\begin{aligned}
\left(1-c_2\tau\right)\left\|p\left(\mathcal{A}u^{k+1}\right)\right\|_0^2 \le &\left(1+c_2\tau\right)\left\|p\left(\mathcal{A}u^{k}\right)\right\|_0^2\\
&+c_3\tau\left[\left\|p\left(\mathcal{A}f^{k+\frac{1}{2}}\right)\right\|_0^2+\mu_1^2\left(t_{k+\frac{1}{2}}\right)+\mu_2^2\left(t_{k+\frac{1}{2}}\right)\right].
\end{aligned}
\tag{29}
$$

When $c_2\tau \le 1/3$, we have

$$
\begin{aligned}
\left\|p\left(\mathcal{A}u^{k+1}\right)\right\|_0^2 \le &\left(1+3c_2\tau\right)\left\|p\left(\mathcal{A}u^{k}\right)\right\|_0^2+\frac{3}{2}c_3\tau\left[\left\|p\left(\mathcal{A}f^{k+\frac{1}{2}}\right)\right\|_0^2\right.\\
&\left.+\mu_1^2\left(t_{k+\frac{1}{2}}\right)+\mu_2^2\left(t_{k+\frac{1}{2}}\right)\right], \quad 0\le k\le n-1.
\end{aligned}
$$

By Lemma 2.6, we obtain (27).

This completes the proof.

5 The Unique Solvability, Stability and Convergence

5.1 Unique Solvability

Theorem 5.1 *Difference scheme* (18)–(21) *has a unique solution.*

Proof The difference scheme (18)–(21) is a linear system of algebraic equations. Let $u^k=(u_0^k,u_1^k,\dots,u_m^k)$. According to (21), we obtain the value of u^0. If the value u^k of the k-th time level is obtained, then we can obtain the value of u^{k+1} through (18)–(20). Consider the homogeneous system about u^{k+1}:

$$
\frac{2}{\tau}\mathcal{A}u_i^{k+1}-a\delta_x^2u_i^{k+1}+b\mathcal{A}u_i^{k+1}=0, \quad 1\le i\le m-1, \tag{30}
$$

$$
u_0^{k+1}=\sum_{r=1}^{M}\alpha_r\left(t_{k+\frac{1}{2}}\right)\left(\sum_{p=i_r}^{i_r+3}u_p^{k+1}\prod_{\substack{q=i_r\\ q\ne p}}^{i_r+3}\frac{\xi_r-x_q}{x_p-x_q}\right), \tag{31}
$$

$$u_m^{k+1} = \sum_{s=1}^{N} \beta_s\left(t_{k+\frac{1}{2}}\right)\left(\sum_{p=j_s}^{j_s+3} u_p^{k+1} \prod_{\substack{q=j_s \\ q\neq p}}^{j_s+3} \frac{\eta_s - x_q}{x_p - x_q}\right). \tag{32}$$

According to Lemma 4.1 and similar to the derivation of (29), we get the result

$$(1 - c_2\tau)\left\|p(\mathcal{A}u^{k+1})\right\|_0^2 \leq 0.$$

When $\tau < 1/c_2$, we have $\|p(\mathcal{A}u^{k+1})\|_0^2 = 0$, that implies that

$$\mathcal{A}u_i^{k+1} = 0, \quad 1 \leq i \leq m-1.$$

Then, it follows from (30) that

$$\delta_x^2 u_i^{k+1} = 0, \quad 1 \leq i \leq m-1.$$

Both above equalities yield that

$$u_i^{k+1} = \mathcal{A}u_i^{k+1} - \frac{h^2}{12}\delta_x^2 u_i^{k+1} = 0, \quad 1 \leq i \leq m-1.$$

Combining with (31) and (32), we know $u_i^{k+1} = 0,\ 0 \leq i \leq m$. Thus, the homogeneous system only has a trivial solution.

This completes the proof.

5.2 Stability and Convergence

According to Theorem 4.1, we can obtain the following result easily.

Theorem 5.2 *The difference scheme* (18)–(21) *is stable to the initial value and the right term in the sense that*: *let* $\{u_i^k \mid 0 \leq i \leq m, 0 \leq k \leq n\}$ *be the solution of difference scheme* (18)–(21), *then we have*

$$\left\|p\left(\mathcal{A}u^{k+1}\right)\right\|_0^2 \leq e^{3c_2(k+1)\tau}\left[\left\|p\left(\mathcal{A}u^0\right)\right\|_0^2 + \frac{3c_3}{2}\tau\sum_{l=0}^{k}\left(\left\|p\left(\mathcal{A}f^{l+\frac{1}{2}}\right)\right\|_0^2 + \mu_1^2\left(t_{l+\frac{1}{2}}\right) + \mu_2^2\left(t_{l+\frac{1}{2}}\right)\right)\right],\ 0 \leq k \leq n-1,$$

where c_2 *and* c_3 *are defined in* (26).

Theorem 5.3 *The finite-difference scheme* (18)–(21) *is convergent with the convergence order of* $O(\tau^2 + h^4)$ *in the weighted norm.*

Proof Let

$$e_i^k = U_i^k - u_i^k, \quad 0 \le i \le m,\ 0 \le k \le n.$$

Then, subtracting (18)–(21) from (9), (15), (16) and (17) yields the error equations

$$\mathcal{A}\delta_t e_i^{k+\frac{1}{2}} - a\delta_x^2 e_i^{k+\frac{1}{2}} + b\mathcal{A}e_i^{k+\frac{1}{2}} = R_i^{k+\frac{1}{2}}, \quad 1 \le i \le m-1,\ 0 \le k \le n-1, \tag{33}$$

$$e_0^{k+\frac{1}{2}} = \sum_{r=1}^{M} \alpha_r\left(t_{k+\frac{1}{2}}\right)\left(\sum_{p=i_r}^{i_r+3} e_p^{k+\frac{1}{2}} \prod_{\substack{q=i_r \\ q\neq p}}^{i_r+3} \frac{\xi_r - x_q}{x_p - x_q}\right) + R_0^{k+\frac{1}{2}}, \quad 0 \le k \le n-1, \tag{34}$$

$$;e_m^{k+\frac{1}{2}} = \sum_{s=1}^{N} \beta_s\left(t_{k+\frac{1}{2}}\right)\left(\sum_{p=j_s}^{j_s+3} e_p^{k+\frac{1}{2}} \prod_{\substack{q=j_s \\ q\neq p}}^{j_s+3} \frac{\eta_s - x_q}{x_p - x_q}\right) + R_m^{k+\frac{1}{2}}, \quad 0 \le k \le n-1, \tag{35}$$

$$e_i^0 = 0, \quad 0 \le i \le m. \tag{36}$$

From Theorem 4.1, we obtain

$$\left\|p\left(\mathcal{A}e^{k+1}\right)\right\|_0^2 \le \frac{3c_3}{2} e^{3c_2(k+1)\tau}\tau \sum_{j=0}^{k}\left(\left\|p\left(\mathcal{A}R^{j+\frac{1}{2}}\right)\right\|_0^2 + \left(R_0^{k+\frac{1}{2}}\right)^2 + \left(R_m^{k+\frac{1}{2}}\right)^2\right) = O\left(\left(\tau^2 + h^4\right)^2\right).$$

This completes the proof of the theorem.

6 A Compact Difference Scheme for Heat Equations with Variable Coefficients

We have discussed a compact difference scheme for the heat equation with the constant coefficients in above sections. In this section, we will consider the heat equation with variable coefficients in [1]:

$$w_t = a(x)w_{xx} + b(x)w_x + \bar{c}(x,t)w + g(x,t), \quad 0 < x < L,\ 0 < t \le T.$$

Let

$$w(x,t) = e^{k(x)}u(x,t), \quad k(x) = -\frac{1}{2}\int_0^x \frac{b(s)}{a(s)}\mathrm{d}s.$$

Then, the function $u(x, t)$ satisfies that

$$\frac{1}{a(x)}u_t = u_{xx} + c(x,t)u + f(x,t),$$

where $c(x,t) = k''(x) + (k'(x))^2 + b(x)k'(x)/a(x) + \bar{c}(x,t)/a(x)$, $f(x,t) = e^{-k(x)}g(x,t)$. The convection term disappears now. Thus, we consider the multi-point boundary value problem of the heat equation with variable coefficients:

$$q(x)u_t = u_{xx} + c(x,t)u + f(x,t), \quad 0 < x < L,\ 0 < t \le T, \tag{37}$$

$$\begin{aligned} u(0,t) &= \sum_{r=1}^{M} \alpha_r(t)u\big(\xi_r,t\big) + \mu_1(t), \\ u(L,t) &= \sum_{s=1}^{N} \beta_s(t)u\big(\eta_s,t\big) + \mu_2(t), \quad 0 < t \le T, \end{aligned} \tag{38}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le L. \tag{39}$$

Similar to the establishment of (18)–(21) for problem (3)–(5), we present a compact difference scheme for (37)–(39) as follows:

$$\mathcal{A}\left(q\big(x_i\big)\delta_t u_i^{k+\frac{1}{2}}\right) - \delta_x^2 u_i^{k+\frac{1}{2}} + \mathcal{A}(cu)_i^{k+\frac{1}{2}} = \mathcal{A}f_i^{k+\frac{1}{2}}, \quad 1 \le i \le m-1,\ 0 \le k \le n-1, \tag{40}$$

$$u_0^{k+\frac{1}{2}} = \sum_{r=1}^{M} \alpha_r\left(t_{k+\frac{1}{2}}\right)\left(\sum_{p=i_r}^{i_r+3} u_p^{k+\frac{1}{2}} \prod_{\substack{q=i_r \\ q\ne p}}^{i_r+3} \frac{\xi_r - x_q}{x_p - x_q}\right) + \mu_1\left(t_{k+\frac{1}{2}}\right), \quad 0 \le k \le n-1, \tag{41}$$

$$u_m^{k+\frac{1}{2}} = \sum_{s=1}^{N} \beta_s\left(t_{k+\frac{1}{2}}\right)\left(\sum_{p=j_s}^{j_s+3} u_p^{k+\frac{1}{2}} \prod_{\substack{q=j_s \\ q\ne p}}^{j_s+3} \frac{\eta_s - x_q}{x_p - x_q}\right) + \mu_2\left(t_{k+\frac{1}{2}}\right), \quad 0 \le k \le n-1, \tag{42}$$

$$u_i^0 = \varphi\big(x_i\big), \quad 0 \le i \le m. \tag{43}$$

The truncation errors of (40)–(42) are all $O(\tau^2 + h^4)$. The proof of solvability, stability and convergence of (40)–(43) is similar to that of (18)–(21), so we do not repeat it here and just show a numerical example in Sect. 7.

7 Numerical Tests

Example 1 Use the compact difference scheme (18)–(21) to solve the following problem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = -e^{x-t}, \qquad 0 < x < 1,\ 0 < t \le 1, \\ u(0,t) = 0.2 \times [u(0.215,t) + u(0.345,t) + u(0.455,t) + u(0.5,t) + u(0.785,t)] \\ \quad + \left[1 - 0.2 \times \left(e^{0.215} + e^{0.345} + e^{0.455} + e^{0.5} + e^{0.785}\right)\right] e^{-t}, \qquad 0 < t \le 1, \\ u(1,t) = 0.4 \times [u(0.375,t) + u(0.435,t) + u(0.575,t) + u(0.695,t)] \\ \quad + \left[e - 0.4 \times \left(e^{0.375} + e^{0.435} + e^{0.575} + e^{0.695}\right)\right] e^{-t}, \qquad 0 < t \le 1, \\ u(x,0) = e^x, \quad 0 \le x \le 1. \end{cases}$$

The exact solution is $u(x,t) = e^{x-t}$. Define

$$E(h,\tau) = \max_{1 \le k \le n} \left\| p(\mathcal{A}(U^k - u^k)) \right\|_0.$$

Table 1 presents the errors in a weighted L^2 norm when we take different space step sizes. When the time step size is fixed to 1/16 000 and the space step size shrinks, the spatial convergence order is 4. Table 2 lists the errors in a weighted L^2 norm when we take different time step sizes. When the space step is fixed to 1/1 600 and the time step shrinks, the temporal convergence order is 2. Table 3 provides the errors in a weighted L^2 norm when

Table 1 The spatial convergence order of difference scheme (18)–(21) ($\tau = 1/16\,000$)

i	h_i	u_m^n	U_m^n	$E(h,\tau)$	$\frac{\log(E(h_{i-1},\tau)/E(h_i,\tau))}{\log(h_{i-1}/h_i)}$
1	1/10	0.999 931 43	1.000 000 00	1.884E−5	
2	1/20	0.999 995 28	1.000 000 00	1.310E−6	3.846
3	1/30	0.999 999 11	1.000 000 00	2.433E−7	4.152
4	1/40	0.999 999 78	1.000 000 00	6.059E−8	4.832

Table 2 The temporal convergence order of difference scheme (18)–(21) ($h = 1/1\,600$)

i	τ_i	u_m^n	U_m^n	$E(h,\tau)$	$\frac{\log(E(h,\tau_{i-1})/E(h,\tau_i))}{\log(\tau_{i-1}/\tau_i)}$
1	1/10	1.010 799 55	1.000 000 00	2.881E−3	
2	1/20	1.002 695 21	1.000 000 00	7.189E−4	2.003
3	1/30	1.001 197 31	1.000 000 00	3.194E−4	2.001
4	1/40	1.000 673 45	1.000 000 00	1.796E−4	2.000

Table 3 The spatial and temporal convergence order of difference scheme (18)–(21)

(h,τ)	u_m^n	U_m^n	$E(h,\tau)$	$E(2h,4\tau)/E(h,\tau)$
(1/10, 1 /10)	1.010 728 98	1.000 000 00	2.845E−3	
(1/20, 1/40)	1.000 668 69	1.000 000 00	1.781E−4	15.977
(1/40, 1/160)	1.000 041 86	1.000 000 00	1.116E−5	15.955
(1/80, 1/640)	1.000 002 62	1.000 000 00	6.976E−7	15.997

we take different space and time step sizes. The numerical results are consistent with the theoretical analysis of convergence and stability.

Example 2 Use the compact difference scheme (40)–(43) to solve the following problem:

$$\begin{cases} e^{-x}\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + e^t u = e^x - e^{-t} - e^{x-t}, \quad 0 < x < 1,\ 0 < t \leq 1, \\ u(0,t) = 0.2 \times [u(0.215,t) + u(0.345,t) + u(0.455,t) + u(0.5,t) + u(0.785,t)] \\ \quad + \left[1 - 0.2 \times \left(e^{0.215} + e^{0.345} + e^{0.455} + e^{0.5} + e^{0.785}\right)\right] e^{-t}, \quad 0 < t \leq 1, \\ u(1,t) = 0.4 \times [u(0.375,t) + u(0.435,t) + u(0.575,t) + u(0.695,t)] \\ \quad + \left[e - 0.4 \times \left(e^{0.375} + e^{0.435} + e^{0.575} + e^{0.695}\right)\right] e^{-t}, \quad 0 < t \leq 1, \\ u(x,0) = e^x, \quad 0 \leq x \leq 1. \end{cases}$$

The exact solution is $u(x,t) = e^{x-t}$.

Table 4 presents the errors in a weighted L^2 norm when we take different space step sizes with fixing time step size, which shows the spatial convergence order is 4. Table 5 lists the errors in a weighted L^2 norm when we take different time step sizes with fixing space step, which presents the temporal convergence order is 2. Table 6 provides the errors in a weighted L^2 norm when we take different space and time step sizes. The numerical results are consistent with the truncation errors.

Table 4 The spatial convergence order of difference scheme (40)–(43) ($\tau = 1/16\,000$)

i	h_i	u_m^n	U_m^n	$E(h,\tau)$	$\frac{\log(E(h_{i-1},\tau)/E(h_i,\tau))}{\log(h_{i-1}/h_i)}$
1	1/10	0.999 916 61	1.000 000 00	2.276E−5	
2	1/20	0.999 994 24	1.000 000 00	1.587E−6	3.842
3	1/30	0.999 998 93	1.000 000 00	2.936E−7	4.162
4	1/40	0.999 999 73	1.000 000 00	7.335E−8	4.821

Table 5 The temporal convergence order of difference scheme (40)–(43) ($h = 1/1\,600$)

i	τ_i	u_m^n	U_m^n	$E(h,\tau)$	$\frac{\log(E(h,\tau_{i-1})/E(h,\tau_i))}{\log(\tau_{i-1}/\tau_i)}$
1	1/10	1.009 213 81	1.000 000 00	2.460E−3	
2	1/20	1.002 295 61	1.000 000 00	6.159E−4	1.998
3	1/30	1.001 019 48	1.000 000 00	2.734E−4	2.003
4	1/40	1.000 573 31	1.000 000 00	1.538E−4	1.999

Table 6 The spatial and temporal convergence order of difference scheme (40)–(43)

(h,τ)	u_m^n	U_m^n	$E(h,\tau)$	$E(2h,4\tau)/E(h,\tau)$
(1/10, 1/10)	1.009 127 10	1.000 000 00	2.421E−3	
(1/20, 1/40)	1.000 567 53	1.000 000 00	1.520E−4	15.927
(1/40, 1/160)	1.000 035 55	1.000 000 00	9.534E−6	15.945
(1/80, 1/640)	1.000 002 22	1.000 000 00	5.960E−7	15.997

8 Conclusion

In this article, a compact difference scheme is constructed to solve the multi-point boundary value problem of the heat conduction equation with constant coefficients. Using the energy method, the prior estimate is obtained and the unique solvability, stability and convergence are proved rigorously. Because of the complexity of the multi-point boundary conditions, the convergence order $O(\tau^2 + h^4)$ is obtained only in a weighted L^2 norm. Besides, a compact difference scheme is also constructed for the problem with variable coefficients. Numerical examples are provided to confirm the accuracy of the difference scheme, which are consistent with the theoretical analysis. In the future, efforts will be taken to perform an analysis on the difference scheme in L^2 norm and in L_∞ norm.

Acknowledgements The research is supported by the National Natural Science Foundation of China (No. 11671081) and the Fundamental Research Funds for the Central Universities (No. 242017K41044).

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