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RESEARCH ARTICLE

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Explosive AR(1) process with independent but not identically distributed errors

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Abstract
Anderson (The Annals of Mathematical Statistics 30(3):676–687, 1959) studied the limiting distribution of the least square estimator for explosive AR(1) process under the independent and identically distributed (iid) condition on error i.e., $X_t = \rho X_{t-1} + e_t$ where $\rho > 1$ and e_t is iid error with $Ee = 0$ and $Ee^2 < \infty$. This paper is mainly concerned about the limiting distribution of the least square estimator of ρ, that is $\hat{\rho}$, when errors are not identically distributed. In addition, we provide an approximate description of the limiting distribution of $\sum_{j=0}^{n-1} \rho^{-j} e_{n-j}$ when $\rho > 1$ as $n \to \infty$.

Keywords Explosive AR(1) process · Non-identical distribution · Least square estimator

Mathematics Subject Classification 62M10

1 Introduction and main results

Consider the following non-stationary AR(1) process defined by

$$X_t = \rho X_{t-1} + e_t, t = 1, 2, \ldots \tag{1}$$

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where the coefficient $\rho \geq 1$, the initial value $X_0 = 0$ and the error process $\{e_t, t \geq 1\}$ is iid with mean zero and finite variance. Observe that

$$X_t = \sum_{j=0}^{t-1} \rho^j e_{t-j}.$$

The random walk (RW) refers to $\rho = 1$. For $\rho > 1$, $\{X_t, t \geq 0\}$ is referred to as an 'explosive' process. See, for instance, Hwang and Basawa (2005) and Hwang (2013). The two cases, viz., $\rho = 1$ (RW) and $\rho > 1$ (explosive), are well investigated and documented in the literatures for the case of iid errors. In this paper, we are mainly concerned about the explosive case ($\rho > 1$) which depends on the initial condition X_0 or e_0 critically. In the literatures it is employed to study economic bubble and often referred to as random chaotic system in the sense that it is highly sensitive to the initial condition. See, e.g., Kim and Hwang (2019) or Lee (2018).

When $X_1, X_2, \ldots, X_n$ denote a sample of size n, the least squares estimator of ρ,

$$\hat{\rho} = \sum_{t=1}^{n-1} X_t X_{t+1} / \sum_{t=1}^{n-1} X_t^2,$$

is known to be n-consistent and ρ^n-consistent according to $\rho = 1$ and $\rho > 1$, respectively (see Fuller 1996, Ch.10). Now the following expression is useful to derive limit distributions of $\hat{\rho}$,

$$\hat{\rho} - \rho = \frac{\sum_{t=1}^{n-1} X_t e_{t+1}}{\sum_{t=1}^{n-1} X_t^2} = \frac{\sum_{t=1}^{n-1} \sum_{j=0}^{t-1} \rho^j e_{t-j} e_{t+1}}{\sum_{t=1}^{n-1} X_t^2} = \frac{U_n}{V_n}$$

where

$$U_n = \sum_{i=2}^{n} \sum_{j=1}^{i-1} \rho^{i-j-1} e_i e_j$$

and

$$V_n = \sum_{t=1}^{n-1} X_t^2 = \sum_{t=1}^{n-1} \left(\sum_{j=0}^{t-1} \rho^j e_{t-j} \right)^2 = \sum_{t=1}^{n-1} \sum_{j=0}^{t-1} \sum_{k=0}^{t-1} \rho^{j+k} e_{t-j} e_{t-k}.$$

For $\rho > 1$, Anderson (1959) showed that under the iid condition on error process $\{e_t, t \geq 1\}$

$$plim_{n\to\infty}(U_n \rho^{-n+2} - Y_n Z_n) = 0 \quad (2)$$

and

$$plim_{n\to\infty}(V_n \rho^{-2n+4} - \rho^2(\rho^2 - 1)^{-1} Z_n^2) = 0 \quad (3)$$

where

$$Z_t = \rho^{-t+2} X_{t-1} = \sum_{j=0}^{t-2} \rho^{-(t-2-j)} e_{t-1-j} \text{ and } Y_t = \sum_{j=0}^{t-1} \rho^{-j} e_{t-j}$$

(refer to Theorem 2.1 and Theorem 2.2 there). Now observe that by (2) and (3)

$$\begin{aligned} U_n/V_n &= (\rho^{n-2} U_n \rho^{-n+2})/(\rho^{2n-4} V_n \rho^{-2n+4}) = (\rho^{n-2} Y_n Z_n)/(\rho^{2n-2}(\rho^2-1)^{-1} Z_n^2) \\ &= Y_n/(\rho^n(\rho^2-1)^{-1} Z_n) \end{aligned}$$

in probability limit or $plim_{n\to\infty}$. Thus

$$\rho^n(\rho^2-1)^{-1} U_n/V_n = Y_n/Z_n \tag{4}$$

in probability limit. By showing that the random variables in Y_n and Z_n are asymptotically disjoint (refer to Theorem 2.3 and 2.4 there), he derived the related limiting distribution of $\hat{\rho}$ for $\rho > 1$. Indeed

$$\rho^n(\rho^2-1)^{-1}(\hat{\rho}-\rho) \Rightarrow \frac{Y}{Z}$$

where Y is limiting distribution of Y_n and Z is limiting distribution of Z_n. Refer to Theorem 2.5 there.

Now, we are mainly concerned about limiting distribution of $\rho^n(\rho^2-1)^{-1}(\hat{\rho}-\rho)$ when errors are not identically distributed. This is a practical issue because economic bubble might be caused by non-identical shocks and severely affected by the initial shock. From this point of view, it would be quite essential to check how sensitively the initial condition give effects to $\hat{\rho}$ under non-identical shocks. Our main results address these issues which are given below.

Theorem 1 *Let* $Y_n^{(1)} = \sum_{j=0}^{[c_n^{(2)}]-1} \rho^{-j} e_{n-j}$ *and* $Z_n^{(1)} = \sum_{j=1}^{[c_n^{(1)}]} \rho^{-(j-1)} e_j$ *where* $1 \le c_n^{(i)} = o(n)$ *for* $i = 1, 2$ *are sequences going to infinity slowly. Assuming* $sup_t E e_t^2 < \infty$ *and* $\rho > 1$, *distributions of* Y_n/Z_n *and* $Y_n^{(1)}/Z_n^{(1)}$ *are asymptotically equivalent in the sense that*

$$\lim_{n\to\infty} \frac{P(Y_n/Z_n \le x)}{P(Y_n^{(1)}/Z_n^{(1)} \le x)} = 1.$$

Remark 1 Theorem translates Theorem 2.5 of Anderson (1959) under relaxed condition $sup_t E e_t^2 < \infty$. We also introduce $c_n^{(1)}$ and $c_n^{(2)}$ for better concise descriptions of Y_n and Z_n. These mainly follow from

$$plim_{n\to\infty}(Z_n - Z_n^{(1)}) = 0 \text{ and } plim_{n\to\infty}(Y_n - Y_n^{(1)}) = 0. \tag{5}$$

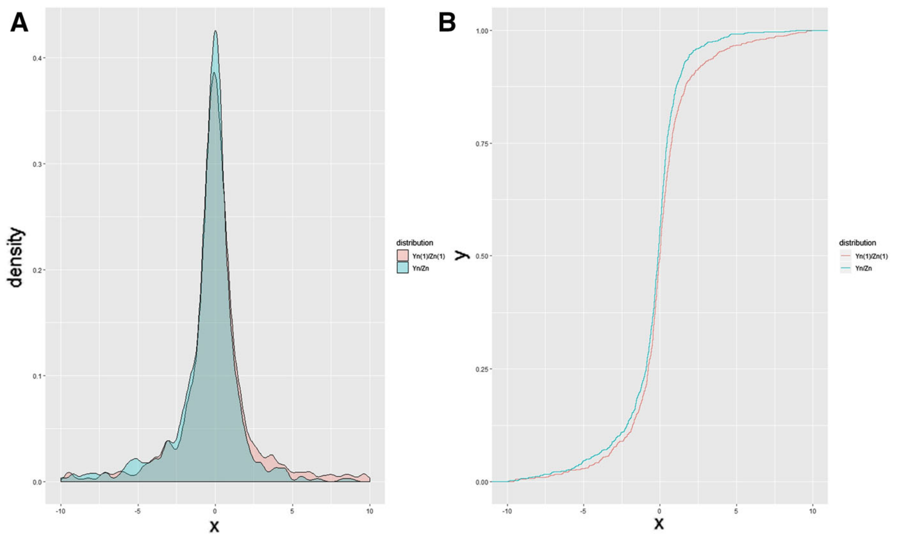

Fig. 1 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (i) with $\beta = 0.5$ and $n = 200$

It is worth mentioning that these sequences of $c_n^{(1)}$ and $c_n^{(2)}$ precisely attributes the independence of $Y_n^{(1)}$ and $Z_n^{(1)}$. For possible $c_n^{(1)}$ and $c_n^{(2)}$, one may consider slowly varying function which is defined as $L(x) : (0, \infty) \to (0, \infty)$ such that

$$\lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1 \text{ for all } a > 0.$$

For example, the function $L(x) = (\log x)^\beta$ for any $\beta \in R$ is slowly varying. In such case one can introduce $(\log n)^{\beta_i}$, $\beta_i > 0, i = 1, 2$ for $c_n^{(i)}$. This means that most errors are negligible except for very slowly increasing number of errors at both ends and hence error distribution might be allowed to vary on most occasions. In other words, any type error distribution is allowed between indices $c_n^{(1)} + 1$ and $n - c_n^{(2)}$. Refer to Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 and 18 for related simulation results.

Remark 2 Since $c_n^{(1)}$ and $c_n^{(2)}$ slowly go to infinity, one may reasonably approximate the limiting distribution of $Y/Z \sim e_n/e_1$ in practice because e_n and e_1 respectively influence the limiting distributions of Y and Z most. For instance, if both e_1 and e_n are $N(0, \sigma_0^2)$ and $\rho > 1$ is large, then one may reasonably approximate

$$\rho^n(\rho^2 - 1)^{-1}(\hat{\rho} - \rho) \Rightarrow \frac{N(0, \sigma_0^2)}{N(0, \sigma_0^2)} \sim Cauchy(0, 1).$$

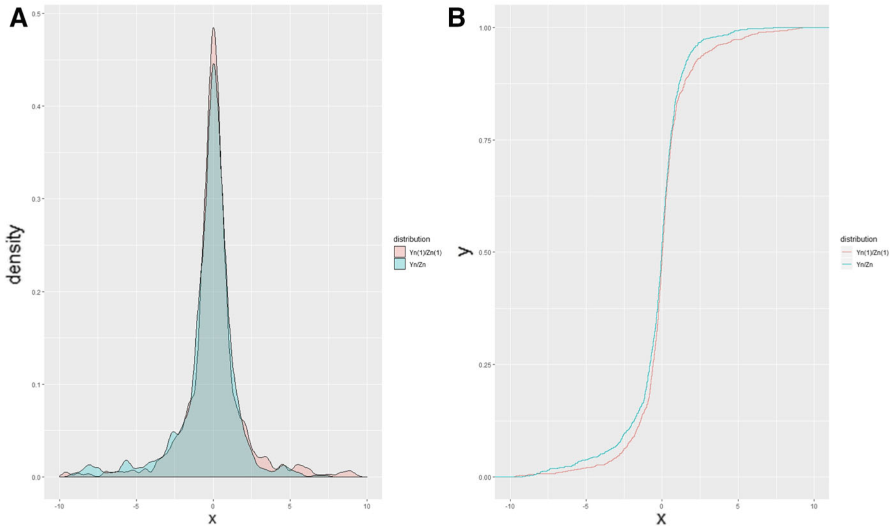

Fig. 2 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (i) with $\beta = 1.5$ and $n = 200$

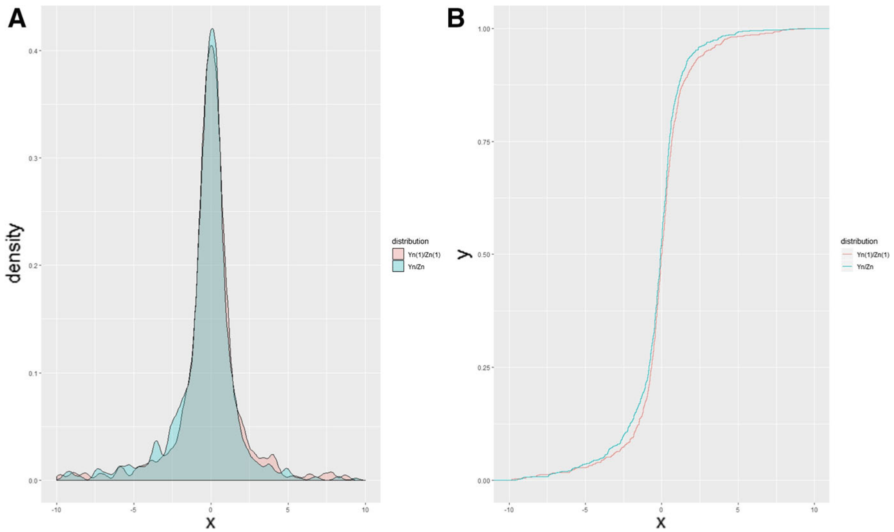

Fig. 3 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (i) with $\beta = 2.5$ and $n = 200$

If both e_1 and e_n are uniform$(-1, 1)$ and ρ is large,

$$\rho^n(\rho^2 - 1)^{-1}(\hat{\rho} - \rho) \Rightarrow \frac{U(-1, 1)}{U(-1, 1)}$$

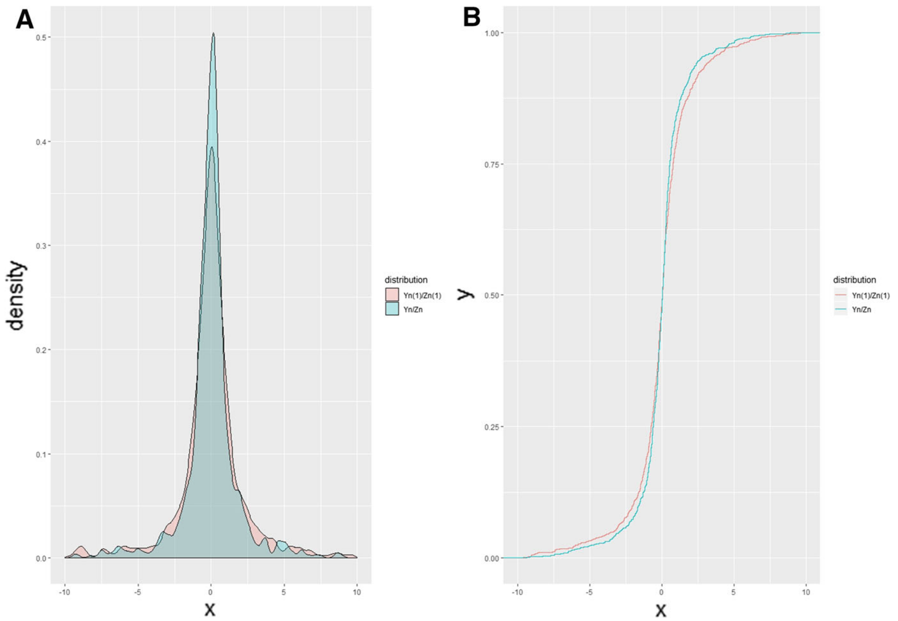

Fig. 4 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (i) with $\beta = 0.5$ and $n = 500$

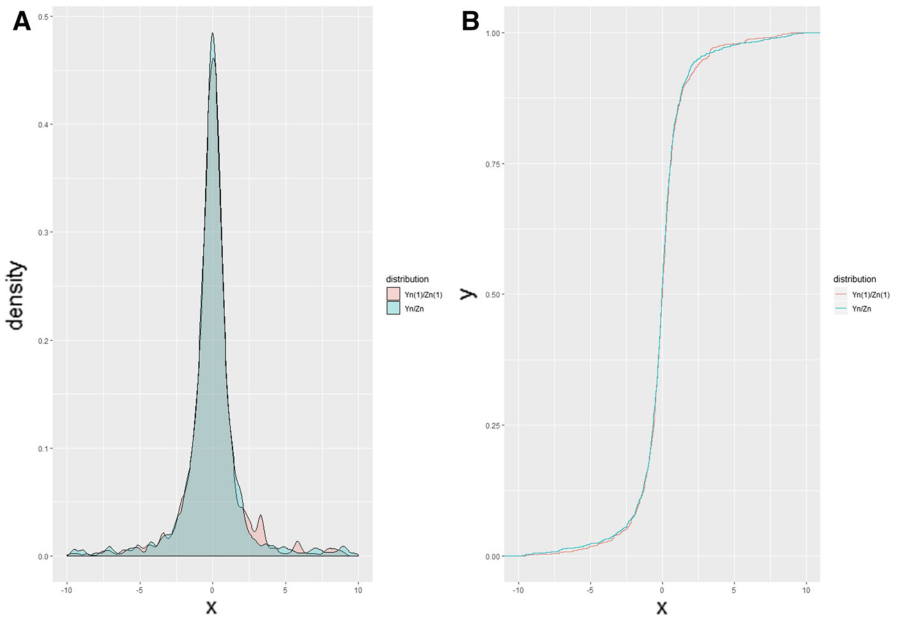

Fig. 5 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (i) with $\beta = 1.5$ and $n = 500$

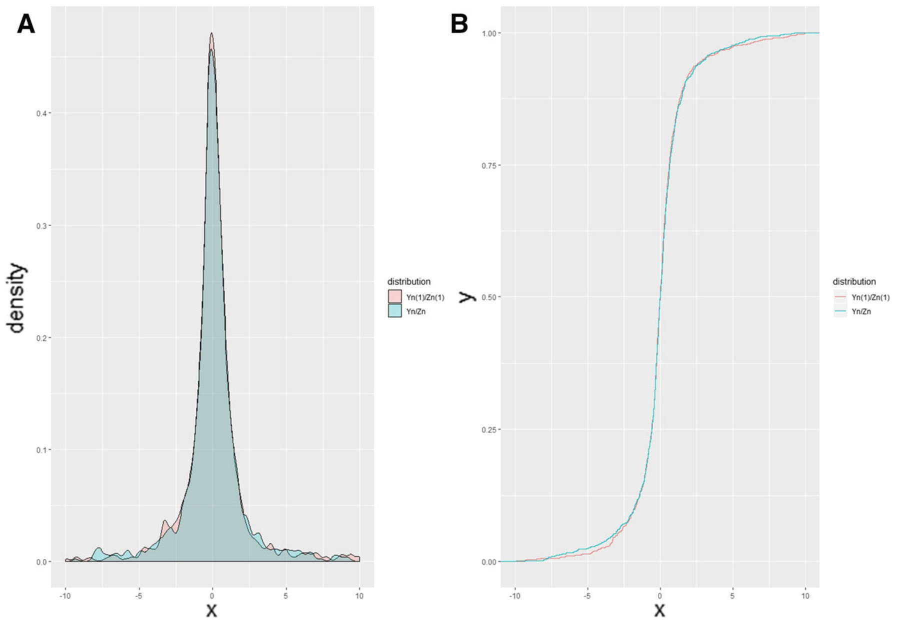

Fig. 6 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (i) with $\beta = 2.5$ and $n = 500$

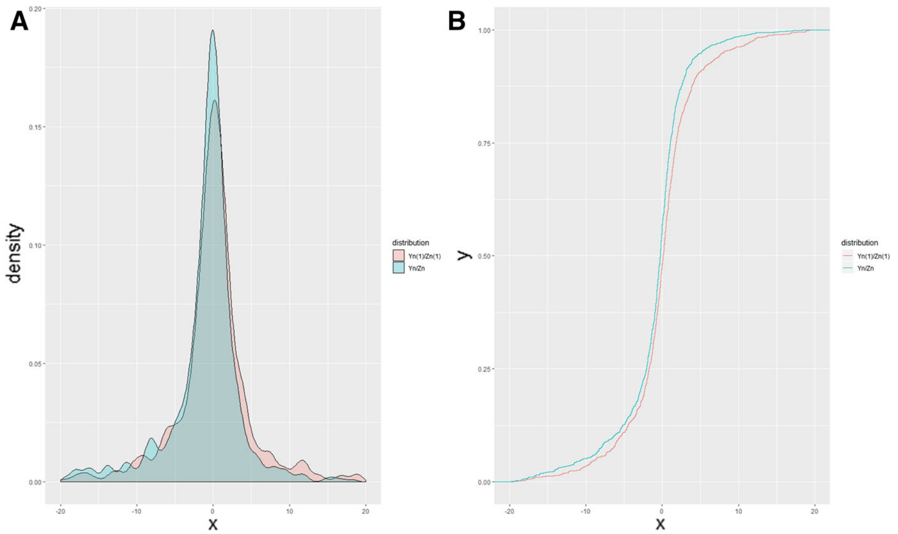

Fig. 7 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (ii) with $\beta = 0.5$ and $n = 200$

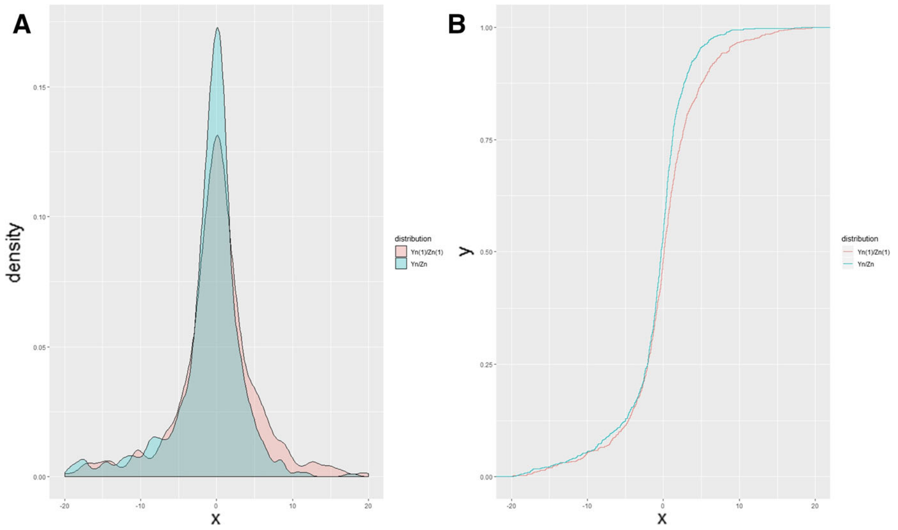

Fig. 8 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (ii) with $\beta = 1.5$ and $n = 200$

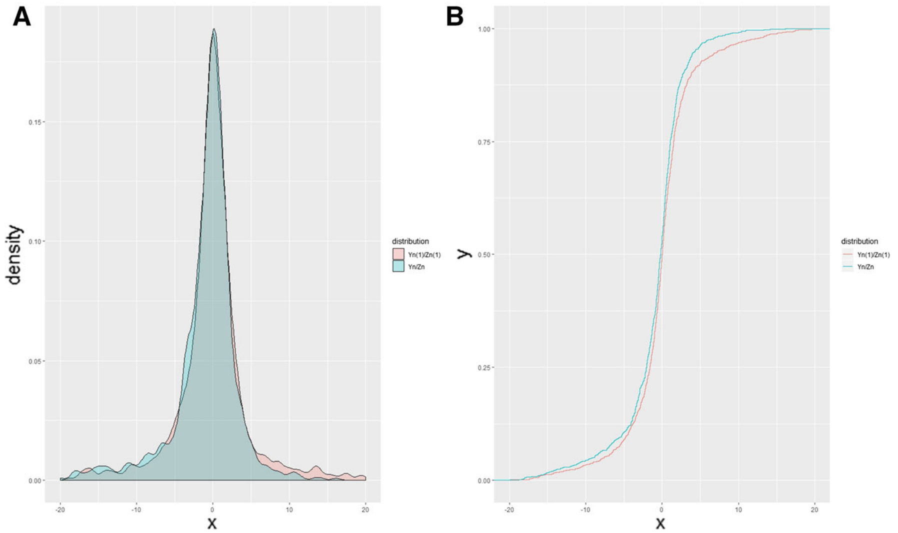

Fig. 9 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (ii) with $\beta = 2.5$ and $n = 200$

whose density is given by

$$g(z) = \begin{cases} \frac{1}{4z^2}, & |z| > 1, \\ \frac{1}{4}, & |z| \leq 1. \end{cases}$$

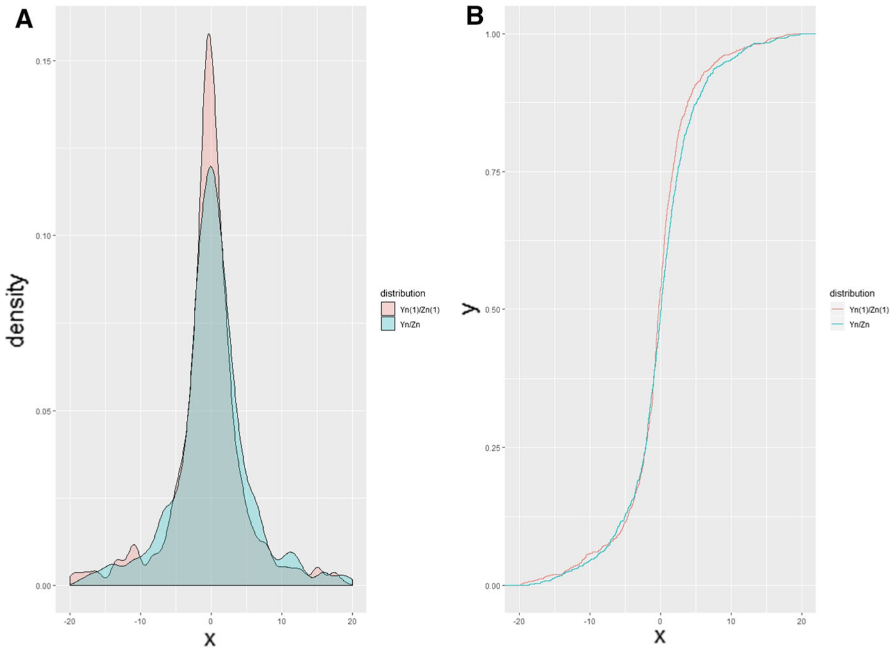

Fig. 10 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (ii) with $\beta = 0.5$ and $n = 500$

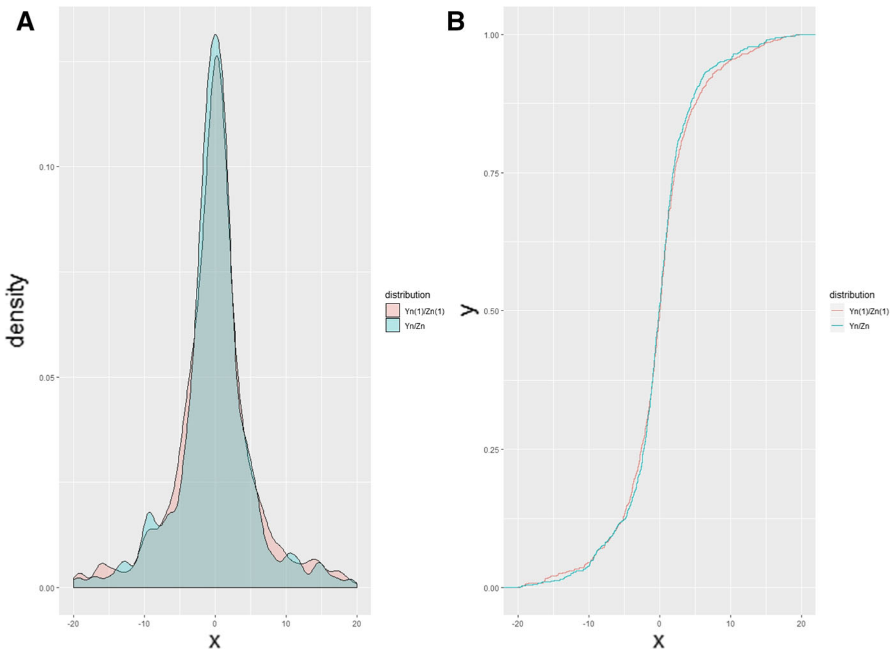

Fig. 11 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (ii) with $\beta = 1.5$ and $n = 500$

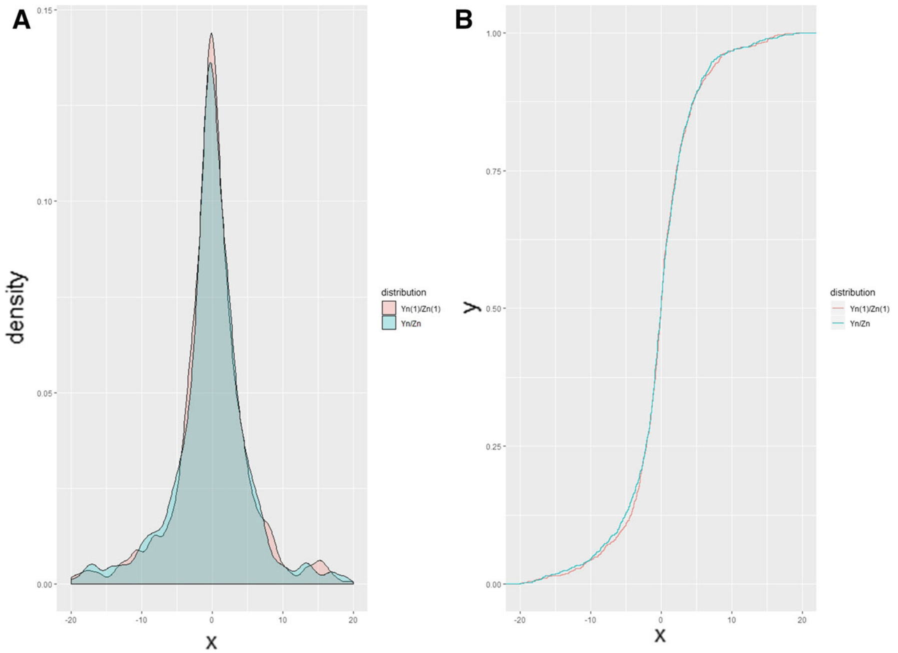

Fig. 12 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (ii) with $\beta = 2.5$ and $n = 500$

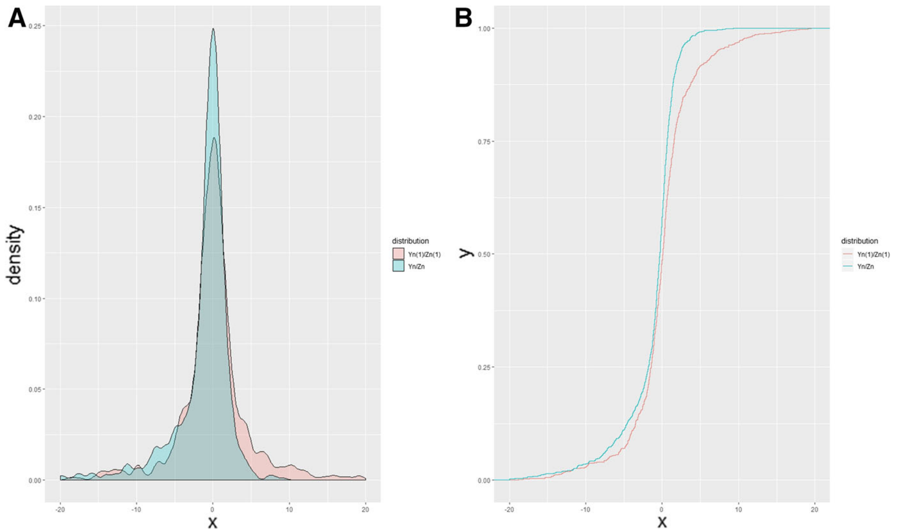

Fig. 13 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (iii) with $\beta = 0.5$ and $n = 200$

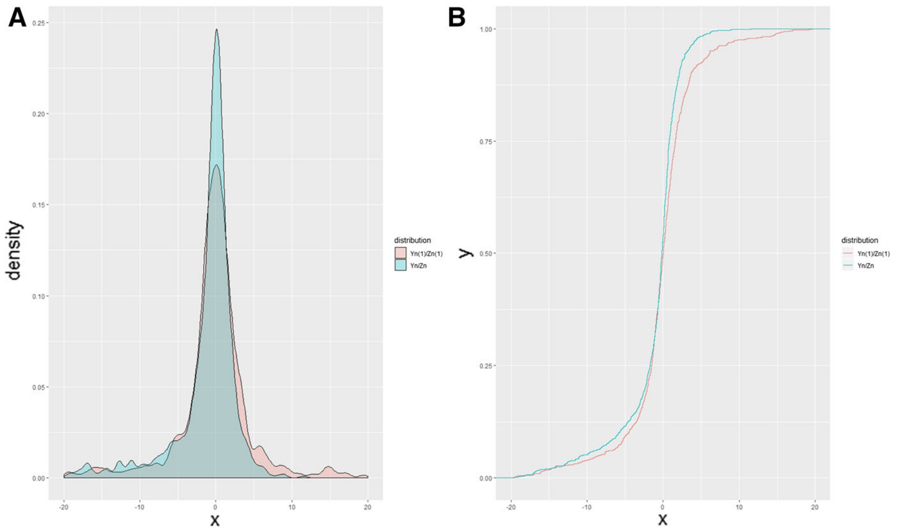

Fig. 14 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (iii) with $\beta = 1.5$ and $n = 200$

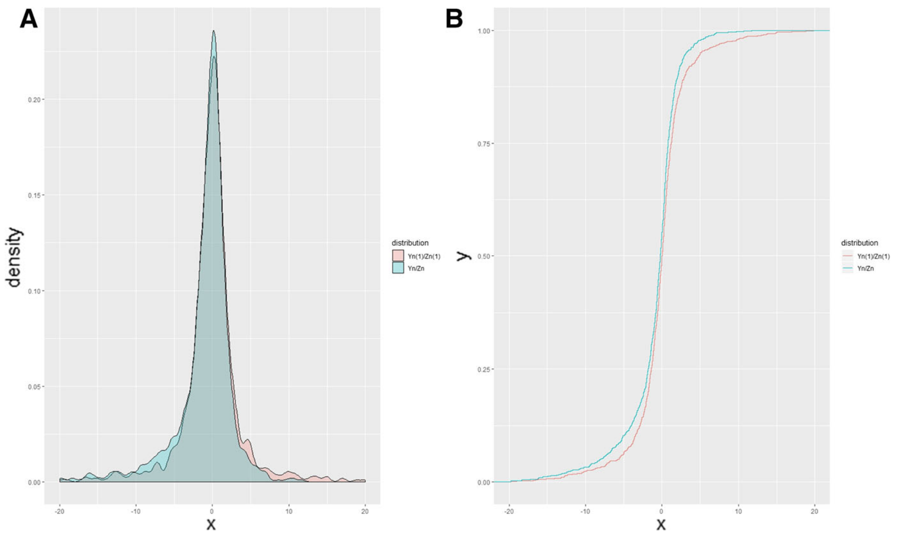

Fig. 15 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (iii) with $\beta = 2.5$ and $n = 200$

From Theorem 1, one might establish limiting distribution for $\hat{\rho}$ by imposing additional condition.

Theorem 2 *Assume that* $\sup_t Ee_t^2 < \infty$ *and* $\rho > 1$. *Let* $\{e_{t,n}\}$ *be triangular array of errors for* $t = 1, \ldots, n$ *and* $n = 1, \ldots$ *If*

$$e_{n,n} \to e_0 \text{ in distribution} \tag{6}$$

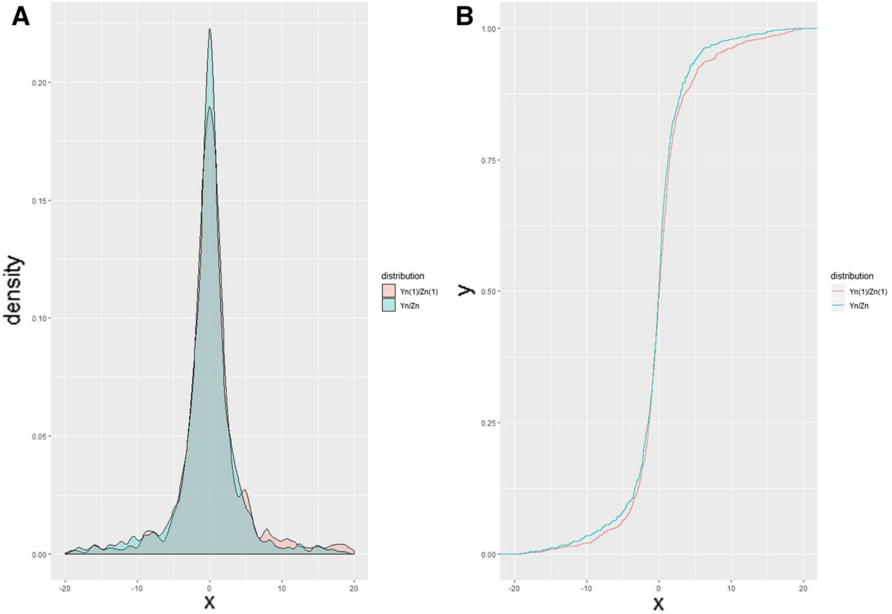

Fig. 16 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (iii) with $\beta = 0.5$ and $n = 500$

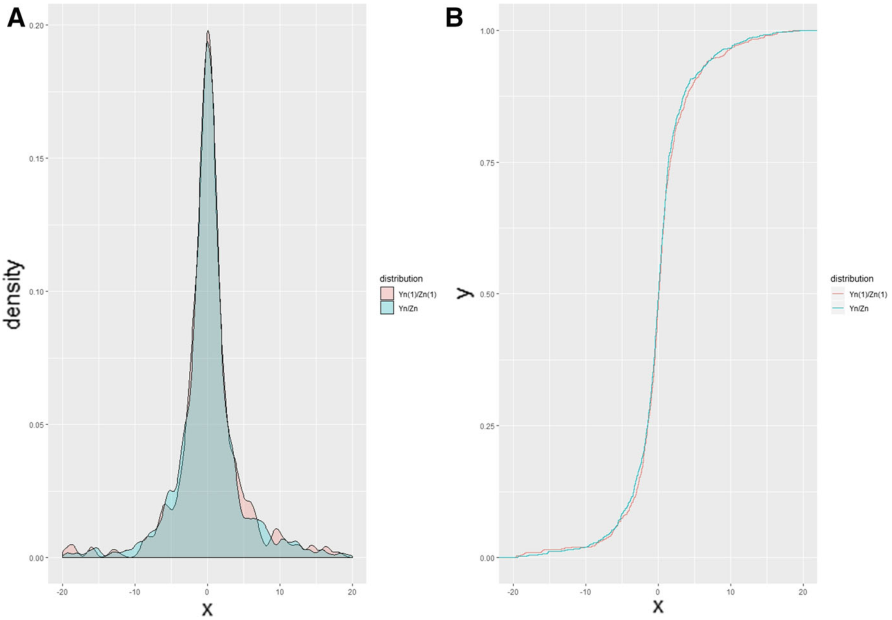

Fig. 17 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (iii) with $\beta = 1.5$ and $n = 500$

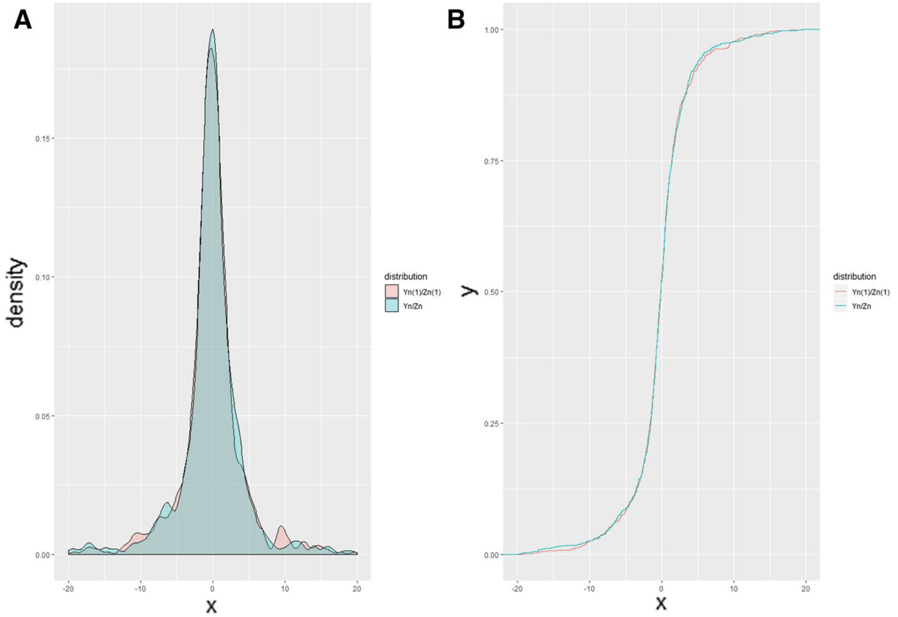

Fig. 18 Kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ for error sequence (iii) with $\beta = 2.5$ and $n = 500$

as $n \to \infty$, *then we have*

$$\rho^n(\rho^2 - 1)^{-1}(\hat{\rho} - \rho) \Rightarrow \frac{Y}{Z}$$

where Y *and* Z *are independent.*

Remark 3 Theorem 2 corresponds to Theorem 2.5 of Anderson (1959). Major differences between the two Theorems are to replace identically distributed e's and $Ee^2 < \infty$ by (6) and $sup_t Ee_t^2 < \infty$. Note that (6) allows any types of distributions for $e_{t,n}$ for $t \leq n - 1$. Refer to Figs. 19, 20, 21 and 22 for related simulation results. One may apply Theorem 2 to test economic bubble under non-identical shocks. Note that it is usually recommended to fix $\rho > 1$ close to 1 for testing economic bubble because it is unrealistic to imagine ρ far greater than 1 under economic bubble.

Remark 4 Notice that if $E\epsilon_t^2 = \sigma_0^2$ for all t, $Var(Y) = Var(Z) = \sigma_0^2/(1 - \rho^{-2})$ and central limit theorem could result in $Y \sim N(0, 1)$ or $Z \sim N(0, 1)$ as ρ decreases to one from above. In fact (Phillips and Magdalinos 2007) verify this by taking $\rho_n = 1+1/n^k$ for some $0 < k < 1$ (see Theorem 4.3 there). This in turn suggests that Y and Z tend to have central tendency to zero at the cost of increased variance as ρ decreases to one.

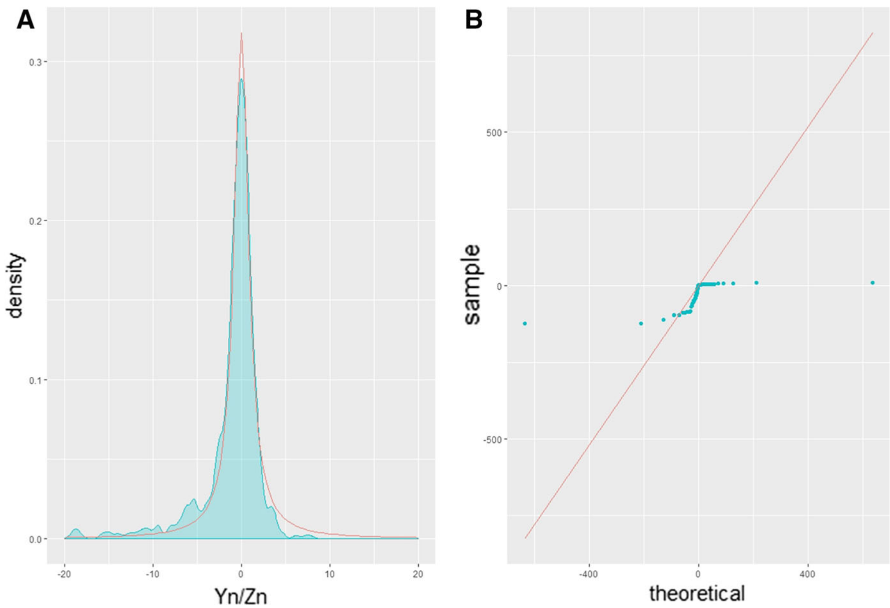

Fig. 19 Kernel density estimate $\hat{f}_n$ and qq plot Q_n for error sequence (iv) with $n = 200$

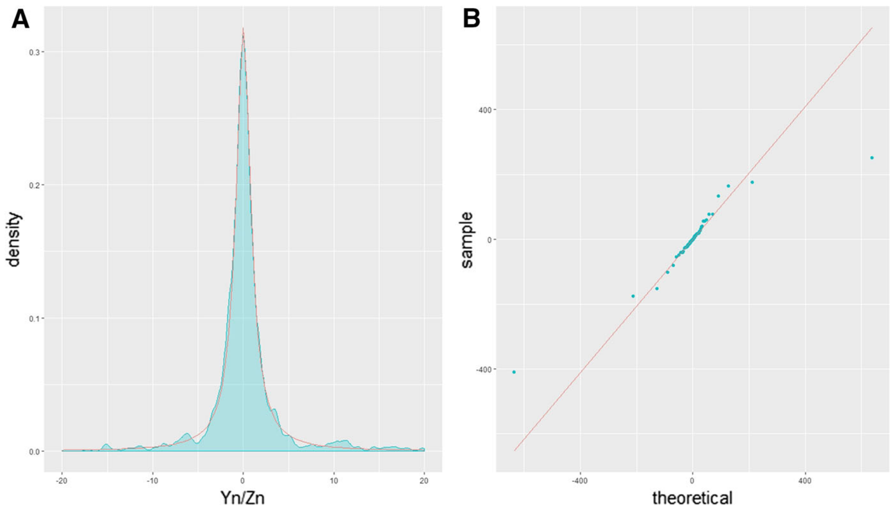

Fig. 20 Kernel density estimate $\hat{f}_n$ and qq plot Q_n for error sequence (iv) with $n = 500$

2 Simulation study

In this section, we carried out some Monte Carlo simulations to check validity of Theorems 1 and 2 in reality or at reasonable finite sample size. For this, we compare kernel density estimate $\hat{f}_n$ and empirical distribution function $\hat{F}_n$ of Y_n/Z_n with those of $Y_n^{(1)}/Z_n^{(1)}$ in Theorem 1 under various non-identical error situations. In particular,

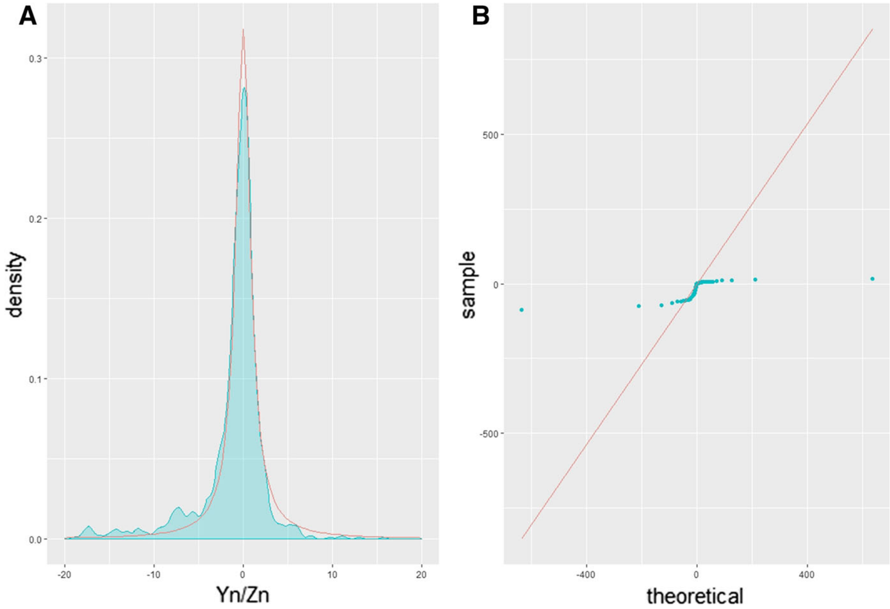

Fig. 21 Kernel density estimate $\hat{f}_n$ and qq plot Q_n for error sequence (v) with $n = 200$

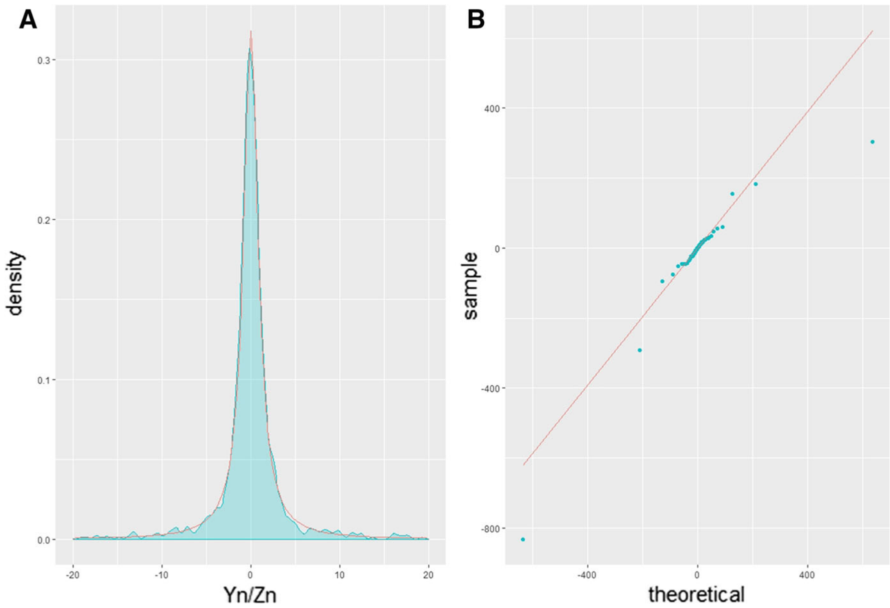

Fig. 22 Kernel density estimate $\hat{f}_n$ and qq plot Q_n for error sequence (v) with $n = 500$

we fix $\rho = 1.02$ as recommended in Remark 3 and then generate model (1) with non-identical error sequences $\{e_t\}$ as follows:

(i) $e_1, \ldots, e_{[n/2]} \sim U(-1, 1), e_{[n/2]+1} \ldots, e_n \sim N(0, 1)$,
(ii) $e_1, \ldots, e_{[n/2]} \sim t(3), e_{[n/2]+1}, \ldots, e_n \sim N(0, 1)$,
(iii) $e_1, \ldots, e_{[n/10]} \sim U(-1, 1), e_{[n/10]+1}, \ldots, e_{[9n/10]} \sim N(0, 1)$, $e_{[9n/10]+1}, \ldots, e_n \sim t(3)$,

Recall $[x]$ denotes the greatest integer not exceeding x. Also we set

$$c_n^{(1)} = c_n^{(2)} = (\log n)^{\beta}, \quad \beta = 0.5, 1.5, 2.5$$

at $n = 200$ and 500. For each simulation setting, 1, 000 repetitions are made. Simulation results are reported in Figs. 1, 2, 3, 4, 5 and 6 (error sequence (i)), Figs. 7, 8, 9, 10, 11 and 12 (error sequence (ii)) and Figs. 13, 14, 15, 16, 17 and 18 (error sequence (iii)). It follows from these simulations that the distributions of Y_n/Z_n (blue line) or $Y_n^{(1)}/Z_n^{(1)}$ (red line) are getting close to each other as n or β increase, which is as expected.

Regarding Theorem 2, the following triangular array of errors $\{e_{t,n}\}$ are considered:

(iv) $e_{t,n} \sim N(0, 1 - 1/n)), t = 1, \ldots, n$,
(v) $e_{1,n} \sim N(0, 1 - 1/n), e_{2,n}, \ldots, e_{n-1,n} \sim U(-1, 1), e_{n,n} \sim N(0, 1 - 1/n)$,

and then compare kernel density estimate $\hat{f}_n$ and qq plot $\hat{Q}_n$ of $\rho^n(\rho^2 - 1)^{-1}(\hat{\rho} - \rho)$ with those of the standard Cauchy distribution. Note that the Cauchy(0,1) is the limiting distribution for the error sequences of (iv) and (v) according to Theorem 2. Simulation results are reported in Figs. 19 and 20 (error sequence (iv)) and Figs. 21 and 22 (error sequence (v)). In Figs. 19, 20, 21 and 22, the distributions of $\rho^n(\rho^2 - 1)^{-1}(\hat{\rho} - \rho)$ (blue line) and the standard Cauchy (red line) are getting close to each other as the sample size increases. Our simulation results confirms that Theorem 2 is regardless of the types of error distributions in between the first and the last standard normal errors.

3 Proofs

Proof of Theorem 1 In the below, the results assuming $Ee^2 = \sigma_0^2 < \infty$ by Anderson (1959) are extended to the case of $sup_t Ee_t^2 < \infty$. First, the proofs of (2) and (3) by Anderson (1959) (see Theorem 2.1 and 2.2 there) are extended easily to our case of $sup_t Ee_t^2 < \infty$. Then using (4), the proof of Theorem 1 will be completed if (5) is established, i.e.,

$$plim_{n\to\infty}\left(Z_n - \sum_{j=1}^{[c_n^{(1)}]} \rho^{-(j-1)} e_j\right)$$
$$= 0 \text{ and } plim_{n\to\infty}\left(Y_n - \sum_{j=0}^{[c_n^{(2)}]-1} \rho^{-j} e_{n-j}\right) = 0.$$

Without loss of generality, it is sufficient to consider the case $\sup_t |e_t| \le M$. This is so because by (14) and (15) in the proof of Lemma 1,

$$E(Z_n I(\sup_t |e_t| > M))^2 = \frac{1-\rho^{-2n}}{1-\rho^{-2}} E(e^2 I(\sup_t |e_t| > M)) \to 0 \tag{7}$$

and

$$E(Y_n I(\sup_t |e_t| > M))^2 = \frac{1-\rho^{-2n}}{1-\rho^{-2}} E(e^2 I(\sup_t |e_t| > M)) \to 0 \tag{8}$$

as $M \to \infty$. Now (5) follows by (12) and (13) of Lemma 1. For Z_n, letting

$$Z_n^* = \sum_{j=0}^{\lceil n-1-c_n^{(1)} \rceil -1} \rho^{-(n-2-j)} e_{n-1-j}$$

where $\lceil x \rceil$ denote the least integer greater than or equal to x, we have

$$Z_n = Z_n^* + \sum_{j=\lceil n-1-c_n^{(1)} \rceil}^{n-2} \rho^{-(n-2-j)} e_{n-1-j} = Z_n^* + \sum_{j=1}^{[c_n^{(1)}]} \rho^{-(j-1)} e_j.$$

Owing to (12) of Lemma 1, we obtain for any $\epsilon > 0$,

$$\begin{aligned} P(|Z_n^*| > \epsilon) &= P\left(\left| \sum_{j=0}^{\lceil n-1-c_n^{(1)} \rceil -1} \rho^{-(n-2-j)} e_{n-1-j} \right| > \epsilon \right) \\ &\le E\left(\sum_{j=0}^{\lceil n-1-c_n^{(1)} \rceil -1} \rho^{-(n-2-j)} e_{n-1-j} \right)^{2r} \epsilon^{-2r} = O(\rho^{-2r c_n^{(1)}}). \end{aligned}$$

Since the above holds for any positive integer r (recall $\sup_t |e_t| \le M$ at this moment) and $c_n^{(1)} \to \infty$ as $n \to \infty$, we have

$$plim_{n\to\infty} \left(Z_n - \sum_{j=1}^{[c_n^{(1)}]} \rho^{-(j-1)} e_j \right) = plim_{n\to\infty} Z_n^* = 0. \tag{9}$$

For Y_n, letting

$$Y_n^* = \sum_{j=[c_n^{(2)}]}^{n-1} \rho^{-j} e_{n-j},$$

we have

$$Y_n = Y_n^* + \sum_{j=0}^{[c_n^{(2)}]-1} \rho^{-j} e_{n-j}.$$

For any $\epsilon > 0$, by using (13) of Lemma 1,

$$P\left(\left|\sum_{j=[c_n^{(2)}]}^{n-1} \rho^{-j} e_{n-j}\right| > \epsilon\right) \le E\left(\sum_{j=[c_n^{(2)}]}^{n-1} \rho^{-j} e_{n-j}\right)^{2r} \epsilon^{-2r} = O(\rho^{-2rc_n^{(2)}}).$$

Since the above holds for any positive integer r and $c_n^{(2)} \to \infty$ as $n \to \infty$, we have

$$plim_{n\to\infty}\left(Y_n - \sum_{j=0}^{[c_n^{(2)}]-1} \rho^{-j} e_{n-j}\right) = plim_{n\to\infty} Y_n^* = 0. \quad (10)$$

Recall that

$$\begin{aligned}\hat{\rho} - \rho = U_n/V_n &= (\rho^{n-2} U_n \rho^{-n+2})/(\rho^{2n-4} V_n \rho^{-2n+4}) \\ &= (\rho^{n-2} Y_n Z_n)/(\rho^{2n-2}(\rho^2-1)^{-1} Z_n^2) \\ &= Y_n/(\rho^n(\rho^2-1)^{-1} Z_n).\end{aligned}$$

By (9) and (10), we have

$$\rho^n(\rho^2-1)^{-1}(\hat{\rho}-\rho) = \rho^n(\rho^2-1)^{-1} U_n/V_n = Y_n/Z_n = Y_n^{(1)}/Z_n^{(1)}$$

in probability. □

Proof of Theorem 2 Choose $c_n^{(1)} = c_n^{(2)} = (\log n)^\beta$ for $\beta > 0$. Then by Theorem 1, we have $\sum_{j=0}^{[c_n^{(2)}]-1} \rho^{-j} e_{n-j} \Rightarrow Y$ and $\sum_{j=1}^{[c_n^{(1)}]} \rho^{-(j-1)} e_j \Rightarrow Z$, which leads to

$$\rho^n(\rho^2-1)^{-1}(\hat{\rho}-\rho) \Rightarrow \frac{Y}{Z}.$$

Since the above holds for any $\beta > 0$, the proof is completed. □

Lemma 1 *Assume that* $\sup_t E e_t^{2r} < \infty$ *for* $r > 2$. *Then we have*

$$E(Z_n^2) = O(1) \text{ and } E(Y_n^2) = O(1). \quad (11)$$

Furthermore

$$E(Z_n^*)^{2r} = E\left(\sum_{j=0}^{\lceil n-1-c_n^{(1)}\rceil -1} \rho^{-(n-2-j)} e_{n-1-j}\right)^{2r} = O(\rho^{-2rc_n^{(1)}}) \quad (12)$$

where $c_n^{(1)} \to \infty$ *as* $n \to \infty$ *and*

$$E(Y_n^*)^{2r} = E\left(\sum_{j=[c_n^{(2)}]}^{n-1} \rho^{-j} e_{n-j}\right)^{2r} = O(\rho^{-2rc_n^{(2)}}) \tag{13}$$

where $c_n^{(2)} \to \infty$ *as* $n \to \infty$.

Proof Assume $Ee^2 = \sigma_0^2$ for the moment. Using independence and $Ee = 0$, one may obtain

$$E(Z_n^2)=E\left(\sum_{j=0}^{n-2} \rho^{-(n-2-j)} e_{n-1-j}\right)^2 =E\left(\sum_{t=1}^{n-1} \rho^{-(t-1)} e_t\right)^2 =\sigma_0^2\left(\frac{1-\rho^{-2(n-1)}}{1-\rho^{-2}}\right) \tag{14}$$

and

$$E(Y_n^2) = E\left(\sum_{t=0}^{n-1} \rho^{-t} e_{n-t}\right)^2 = \sigma_0^2 \sum_{t=0}^{n-1} \rho^{-2t} = \sigma_0^2\left(\frac{1-\rho^{-2n}}{1-\rho^{-2}}\right). \tag{15}$$

This establishes (11) under $Ee^2 = \sigma_0^2 < \infty$ and its extension to $\sup_t Ee_t^2 < \infty$ is trivial. To verify (12), let r be a positive integer. Observe that

$$\begin{aligned} E(Z_n^*)^{2r} &= E\left(\sum_{j=0}^{\lceil n-1-c_n^{(1)}\rceil -1} \rho^{-(n-2-j)} e_{n-1-j}\right)^{2r} = E\left(\sum_{j=[c_n^{(1)}]+1}^{n-1} \rho^{-(j-1)} e_j\right)^{2r} \\ &\leq \left(\sum_{j=[c_n^{(1)}]+1}^{n-1} \rho^{-(j-1)} (E(e_j^{2r}))^{1/2r}\right)^{2r} = O(\rho^{-2rc_n^{(1)}}). \end{aligned}$$

The above holds by the Minkowski inequality and $\sup_t Ee_t^{2r} < \infty$, which validates (12). Moreover, (13) can be yielded similarly to (12). Hence, the lemma is established. □

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