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RESEARCH ARTICLE

Markov-modulated fluid flow model with server maintenance period

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Abstract
We consider a Markov-modulated fluid flow model with server maintenance period. As soon as the fluid level reaches zero, the server begins a maintenance period of a random length. During the maintenance period, fluid arrives from outside depending on the state of the background Markov process and the level increases either vertically or linearly. This model can be applied to various real-world systems such as inventory systems and production systems. We first derive the distribution of the fluid level and the mean performance measures. Then, we present some numerical examples to show the effect of the maintenance time.

Keywords Markov-modulated fluid flow · MMFF · Server maintenance period

1 Introduction

We consider a Markov-modulated fluid (MMFF) model with a server maintenance period. The MMFF model is a famous stochastic fluid models in which the fluid level linearly increases or decreases depending on the phase (state) of an underlying Markov chain (UMC). It has been widely applied to several real-world systems, such as communication systems, risk analysis, insurance models, production systems, and inventory systems. Related analyses and applications, have been presented by Aggarwal et al. (2005), Ahn (2009), Ahn et al. (2007, 2005), Ahn and Ramaswami (2003,

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2004, 2005), Anick et al. (1982), Asmussen (1995), Badescu et al. (2005); Badescu and Landriault (2009), Kulkarni and Yan (2007), Mitra (1988), Miyazawa and Takada (2002), Takada (2001), Yan and Kulkarni (2008), and Yeralan et al. (1986).

In the conventional MMFF model, the idle server reactivates and process the fluid as soon as the zero fluid level becomes positive. This simple behavior of the MMFF model has restricted wider applications to real-world systems due to the lack of a controlling server. To overcome this drawback, we consider a new modification of the conventional MMFF model such that the system has a vacation period (later, we call this vacation period as a maintenance period) whenever the fluid level reaches zero. During the vacation period, no service is provided, and the fluid level only increases by the influx of the fluid. This modification is meaningful in engineering and economic point of views.

For example, let us suppose a chemical production system in which a very high back-up cost occurs when the server breaks down during its processing period. To avoid this cost, it is beneficial to schedule server maintenance whenever the system becomes empty. During the maintenance period, the processing rate becomes zero, and the fluid level only increases from the inflow of the arriving fluid. In a real-world setting, maintenance takes several weeks or even months, and overall performance of the system is affected by the maintenance period. However, modeling the system with the conventional MMFF is not possible. Therefore, for the analysis of the system with a maintenance period, the proposed modification is meaningful from an engineering point of view.

Furthermore, taking a vacation period is beneficial from an economic point of view, especially when the reactivation (startup) cost of the server is very high. In the conventional model, to process the fluid, an idle server should be turned on as soon as the level of the fluid in the buffer becomes positive. This model has a high server reactivation cost due to frequent server reactivations. However, if the vacation period is employed, the fluid level at a server reactivation point gets higher, and the mean cycle length (the length between two consecutive points at which the workload level becomes 0) becomes longer. Therefore, the server of the modified model reactivates less frequently than the conventional one so that the cost per unit time can be reduced. In a later section, we will provide several numerical examples to show the effect of vacation (maintenance) period from engineering and economic point of views.

For wider applications of our model, we consider two patterns of fluid increase during the maintenance period: Type-V and Type-L denoting vertical and linear increases, respectively. Our model can be applied to various real-world systems, including inventory, production, and stochastic clearing systems. More examples of MMFF models with server control policies can be found in Baek et al. (2011, 2013a, b).

In the queueing context, our maintenance model is analogous to vacation queueing systems. Vacation systems have been discussed, for example, by Doshi (1986), Lee and Srinivasan (1989), and Levy and Yechiali (1975). Studies on the MAP(BMAP)/G/1 queue with the server vacation model can be found in Baek et al. (2008), Chang et al. (2002), Lee et al. (2001), and Lee and Baek (2005).

Mao et al. (2010a, b) studied a coupled queueing-fluid model, in which an independent M/M/1 vacation queue determines the stochastic behaviors of a fluid process; so their model is called a fluid model driven by background server vacation queues.

In some sense, their model is similar to the proposed model. However, the difference is evident because we concentrate on the evaluation of the fluid process when the vacation is employed to the fluid process itself. Therefore, our model fits the classical sense of the server vacation queueing model that can be widely applied to more diverse real-world systems.

2 Model and preliminaries

In this section, we describe our model in more detail and review some known results of the conventional MMFF model for later use.

2.1 Model

The MMFF model has the following specifications:

(1) The fluid level process is controlled by a continuous time Markov chain with an infinitesimal generator $\boldsymbol{Q}$. We will call this background process the UMC.
(2) During the busy period (processing period), the fluid level changes at a rate r_i when the phase of UMC is i. Therefore, if $r_i > 0$, the fluid level increases linearly, and decreases if $r_i < 0$. For analytic convenience, we assume that $r_i \neq 0$. We define $\Im_1$ and $\Im_2$ as the sets of UMC phases with increasing and decreasing rates, respectively. Then, the set $\Im$ of the UMC phases can be given by $\Im = \Im_1 \cup \Im_2$.
(3) As soon as the fluid level becomes zero, the server takes a maintenance period of random length, V, with distribution function (DF), $V(x)$.
(4) During the maintenance period, the outflow rate becomes 0. Thus, the fluid level either increases or remains constant during the period. The following two different types of systems occur, depending on the increase pattern of the fluid level during the maintenance period:

(4-a) Type-V system: During maintenance, customers arrive in the system with a random amount, S, of fluid [(with DF $S(x)$] according to the Markovian arrival process (MAP, Lucantoni (1991)) with parameter matrices $\boldsymbol{C}$ and $\boldsymbol{D}$. Therefore, the fluid level jumps up vertically at the arrival instance of the customer. From the definition of MAP, we note that $\boldsymbol{Q} = \boldsymbol{C} + \boldsymbol{D}$. At the end of the maintenance period, if there is any fluid, the server starts to process the fluid immediately. If there is no fluid at the end of the maintenance period, the server waits in the system (dormant period) until the first arrival occurs.
(4-b) Type-L system: During the maintenance period, if the UMC phase is i, the fluid level increases linearly at a rate ν_i, where ν_i is either zero or positive depending on the UMC phase. We note that ν_i may be different from r_i; the rate during the busy period. There are no vertical increases. Let $\Im_1^{idle}$ be the set of UMC phases with increasing rates and $\Im_2^{idle}$ is the set with zero rates. There is then a second partition, $\Im = \Im_1^{idle} \cup \Im_2^{idle}$, of the UMC phases during the maintenance period. At the end of the maintenance period, if the fluid level is not zero, the busy period starts immediately. If the fluid level is zero and the

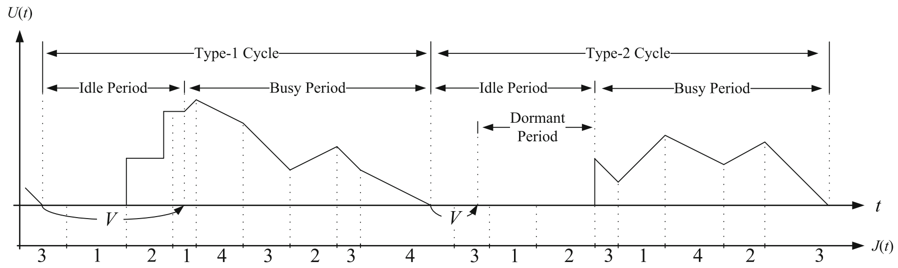


Fig. 1 Type-V system

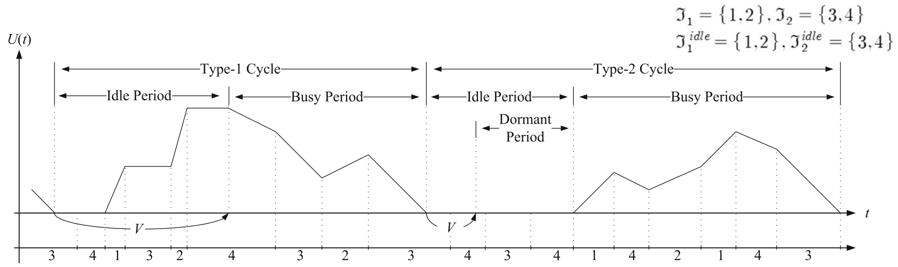


Fig. 2 Type-L system

UMC phase belongs to $\Im_2^{idle}$, the server waits in the system (dormant period) until the first ($\Im_2^{idle} \to \Im_1^{idle}$) transition occurs.

(5) The S, V, and UMC process are independent.

Figures 1 and 2 show Type-V and Type-L systems. For both systems, the system is idle if it is either in the maintenance or dormant periods. A cycle is defined as an interval between two successive busy period termination points. A cycle consists of an idle period and a busy period.

We note that the stochastic behavior of the fluid level processes during the busy period of our model is identical to that of the conventional MMFF process, except that the busy period starts with a random amount of initial fluid, which is accumulated during the idle period.

2.2 Preliminaries

To analyze our model, we used the theoretical results of Ahn and Ramaswami (2004, 2005). In this section, we review the results.

The conventional MMFF process is a stochastic fluid process whose rates of changes are all linear and are governed by UMCs with an infinitesimal generator $\boldsymbol{Q}$. Specifically, the fluid level changes at a rate r_i during the sojourn time in the UMC phase i. Let us partition the set $\Im$ of UMC phases into two sets $\{\Im_1, \Im_2\}$ where $\Im_1$ and $\Im_2$ are the sets of UMC phases with increasing and decreasing rates, respectively. Let us define the diagonal matrices $\boldsymbol{R}$ of r_i and $\boldsymbol{\Gamma}$ of $\gamma_i = |r_i|$, $(i \in \Im = \Im_1 \cup \Im_2)$. Then, as for $\Im_1$ and $\Im_2$, we partition $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{\Gamma}$ as

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix}, \quad \boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_2 \end{pmatrix}, \quad \boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{\Gamma}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Gamma}_2 \end{pmatrix}.$$

If we define $U(t)$ as the fluid level and $J(t)$ as the UMC phase at time t, the two dimensional stochastic process $\{U(t), J(t), t \geq 0\}$ is the MMFF process.

We define a probability matrix $\boldsymbol{P}$ as

$$\boldsymbol{P} = \frac{\boldsymbol{\Gamma}^{-1}\boldsymbol{Q}}{\lambda} + \boldsymbol{I}, \tag{1}$$

where λ is a positive number with $\lambda \geq \max_{i\in\Im}[-\boldsymbol{\Gamma}^{-1}\boldsymbol{Q}]_{ii}$.

$\boldsymbol{P}$ is used to compute the performance measures of the MMFF model by using the uniformization technique. It can be partitioned as $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{pmatrix}$, correspondingly to the sets $\Im_1$ and $\Im_2$.

First passage times play important roles in the analysis of MMFF-related systems, and we will restate the results here. Let $\tau = inf\{t > 0, U(t) = 0\}$ be the first passage time to level 0. We define the following Laplace-Stieltjes Transforms (LST),

$$\begin{aligned}
[\boldsymbol{\Psi}^*(\theta)]_{ij} &= E[e^{-\theta\tau}, J(\tau) = j|U(0) = 0, J(0) = i], (i \in \Im_1, j \in \Im_2),\\
[\boldsymbol{G}_{12}^*(\theta|x)]_{ij} &= E[e^{-\theta\tau}, J(\tau) = j|U(0) = x, J(0) = i], (i \in \Im_1, j \in \Im_2),\\
[\boldsymbol{G}_{22}^*(\theta|x)]_{ij} &= E[e^{-\theta\tau}, J(\tau) = j|U(0) = x, J(0) = i], (i \in \Im_2, j \in \Im_2).
\end{aligned}$$

$\boldsymbol{\Psi}^*(\theta)$ is the matrix LST of the first returning time to level 0 of the simple MMFF system. $\boldsymbol{G}_{12}^*(\theta|x)$ and $\boldsymbol{G}_{22}^*(\theta|x)$ are the matrix LSTs of the first depletion times under the condition that it starts with x amount of initial fluid at increasing and decreasing slopes, respectively. From Ahn and Ramaswami (2005), it is known that

$$\boldsymbol{\Psi}^*(\theta) = \left[\left(\boldsymbol{P}_{11} - \frac{\theta}{\lambda}\boldsymbol{\Gamma}_1{}^{-1}\right)\boldsymbol{\Psi}^*(\theta) + \boldsymbol{P}_{12}\right]\left[\boldsymbol{I} - \frac{\boldsymbol{H}^*(\theta)}{\lambda}\right]^{-1}, \tag{2}$$

$$\boldsymbol{G}_{12}^*(\theta|x) = \boldsymbol{\Psi}^*(\theta)\boldsymbol{G}_{22}^*(\theta|x), \ (x > 0), \tag{3}$$

$$\boldsymbol{G}_{22}^*(\theta|x) = e^{\boldsymbol{H}^*(\theta)x}, \ (x > 0), \tag{4}$$

where

$$\boldsymbol{H}^*(\theta) = \boldsymbol{\Gamma}_2^{-1}[\boldsymbol{Q}_{22} - \theta\boldsymbol{I} + \boldsymbol{Q}_{21}\boldsymbol{\Psi}^*(\theta)]. \tag{5}$$

In our model, we assume that there is no zero rate UMC states during the busy period for analytic convenience. For the case in which zero rate states are allowed, we need to generalize Eq. (2)–(5). More details of the generalized results are provided in Ahn and Ramaswami (2004) and Baek et al. (2011).

3 Analysis of Type-V system

In this section, we analyze the Type-V system. Let us define the following notation:

r_i: rate of change (slope) of fluid process during UMC phase i,

$\gamma_i = |r_i|$,

$\Im_1$: set of UMC phases with increasing slopes,

$\Im_2$: set of UMC phases with decreasing slopes,

$\Im = \Im_1 \cup \Im_2$,

m_i: number of UMC phases in $\Im_i$, $(i = 1, 2)$,

m: number of UMC phases $(= m_1 + m_2)$,

V: length of a maintenance period (random variable),

$V(x)$, $v(x)$: distribution function (DF) and probability density function (pdf) of V,

S: amount of fluid brought in by an arrival during the maintenance period (random variable),

$S(x)$, $s(x)$: DF and pdf of S,

$S^*(\theta)$: LST of $S(x)$,

$\boldsymbol{R}_i$: diagonal matrix of rates in $\Im_i$, $(i = 1, 2)$,

$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}_2 \end{pmatrix}$, $\quad \boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{\Gamma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_2 \end{pmatrix}$

$\pi_i = \lim_{t\to\infty} Pr[J(t) = i]$, $(1 \le i \le m)$,

$\boldsymbol{\pi} = \{\pi_1, \pi_2, \ldots, \pi_m\}$ (i.e., steady-state phase probability of UMC process),

$\xi(t) = \begin{cases} 0, & \text{(system is under maintenance at } t), \\ 1, & \text{(system is dormant at } t), \\ 2, & \text{(system is busy at } t), \end{cases}$

$\mathbf{e}$: column vector of 1's.

Let us define the following probabilities,

$$\begin{aligned} U_i^{(V)}(x,t) &= Pr[U(t) \le x, J(t) = i, \xi(t) = 0],\ (x \ge 0), \\ U_i^{(D)}(x,t) &= Pr[U(t) \le x, J(t) = i, \xi(t) = 1],\ (x \ge 0), \\ U_{idle,i}(x,t) &= U_i^{(V)}(x,t) + U_i^{(D)}(x,t), \\ U_{busy,i}(x,t) &= Pr[U(t) \le x, J(t) = i, \xi(t) = 2],\ (x \ge 0), \end{aligned}$$

and steady-state quantities as

$$U_{idle,i}(x) = \lim_{t\to\infty} U_{idle,i}(x,t), \quad U_{busy,i}(x) = \lim_{t\to\infty} U_{busy,i}(x,t).$$

Let us define the vectors and vector LSTs as follows:

$$\begin{aligned} \boldsymbol{U}_{idle}(x) &= \left(U_{idle,1}(x), U_{idle,2}(x), \ldots, U_{idle,m}(x)\right), \\ \boldsymbol{U}_{busy}(x) &= \left(U_{busy,1}(x), U_{busy,2}(x), \ldots, U_{busy,m}(x)\right), \\ \boldsymbol{u}^*_{idle}(\theta) &= \int_0^\infty e^{-\theta x} d\boldsymbol{U}_{idle}(x), \quad \boldsymbol{u}^*_{busy}(\theta) = \int_0^\infty e^{-\theta x} d\boldsymbol{U}_{busy}(x). \end{aligned}$$

Then, the vector LST $\boldsymbol{u}^*(\theta)$ of a fluid level at an arbitrary time point can be obtained from the following equation;

$$\boldsymbol{u}^*(\theta) = \boldsymbol{u}^*_{idle}(\theta) + \boldsymbol{u}^*_{busy}(\theta). \tag{6}$$

3.1 Analysis of the idle period

In this section, we derive the vector LST $\boldsymbol{u}^*_{idle}(\theta)$ of the fluid level at an arbitrary time point during an idle period. We note that the fluid level process during an idle period of the Type-V system is identical to that of the MAP/G/1 queueing system with a single vacation. Thus, we can use the results of Lee et al. (2001). Let us define the following probability,

$[\tilde{\boldsymbol{V}}_n(x)]_{ij} = Pr(n$ jumps (arrivals) occur during a maintenance period, the length of the maintenance period is less than or equal to x, and the UMC phase is j at the end of the maintenance period, under the condition that the maintenance period starts with UMC phase i, $(n \geq 0, x \geq 0)$.

We define the following matrix transforms,

$$\boldsymbol{V}^*_n(\theta) = \int_0^\infty e^{-\theta x} d\tilde{\boldsymbol{V}}_n(x), \tag{7}$$

$$\boldsymbol{V}^*(z,\theta) = \sum_{n=0}^\infty \boldsymbol{V}^*_n(\theta) z^n = \int_0^\infty e^{-[\theta \boldsymbol{I} - (\boldsymbol{C}+\boldsymbol{D}z)]x} dV(x), \tag{8}$$

$$\boldsymbol{V}(z) = \boldsymbol{V}^*(z,\theta)|_{\theta=0} = \sum_{n=0}^\infty \boldsymbol{V}_n z^n = \int_0^\infty e^{(\boldsymbol{C}+\boldsymbol{D}z)x} dV(x), \tag{9}$$

where $\tilde{\boldsymbol{V}}_n(x)$ is the matrix of $[\tilde{\boldsymbol{V}}_n(x)]_{ij}$.

Let $E(V)$ be the mean length of a maintenance period, and $\boldsymbol{\kappa}$ be the stationary probability vector of the UMC phase at the start of a cycle (i.e., at the start of an arbitrary idle period). Then, from Lee et al. (2001), the mean length $E(I)$ of an idle period is given by

$$E(I) = E(V) + \boldsymbol{\kappa} \boldsymbol{V}_0 (-\boldsymbol{C})^{-1} \mathbf{e}, \tag{10}$$

where $\boldsymbol{V}_0 = \boldsymbol{V}(z)|_{z=0}$.

Equation (10) is obvious because $\boldsymbol{\kappa}\boldsymbol{V}_0$ represents the probability that there are no arrivals during a maintenance time, and $(-\boldsymbol{C})^{-1}$ represents the mean length of an inter-arrival time of the MAP.

Let ρ be the probability that the system is busy at an arbitrary time point. Then, using the result in Lee et al. (2001), we have

$$\boldsymbol{u}^*_{idle}(\theta) = (1-\rho) \frac{\boldsymbol{\kappa}\left\{\boldsymbol{V}_0(-\boldsymbol{C})^{-1} + [\boldsymbol{V}[S^*(\theta)] - \boldsymbol{I}](\boldsymbol{C} + \boldsymbol{D}S^*(\theta))^{-1}\right\}}{E(I)}, \tag{11}$$

where $\boldsymbol{V}[S^*(\theta)] = \boldsymbol{V}(z)|_{z=S^*(\theta)}$.

$\boldsymbol{\kappa}$ and ρ will be derived later.

3.2 Analysis of the busy period

To derive $\boldsymbol{u}^*_{busy}(\theta)$, we first set up the system equation that represents the level change during an infinitesimal time, Δt. Let q_{ij} be the (i, j)-element of the infinitesimal generator $\boldsymbol{Q}$. Let us define $u_{busy,i}(x, t) = \frac{d}{dx} U_{busy,i}(x, t)$, $(x > 0)$, and $\phi_{B,i}(x, t)$ as the rate (number of occurrences per unit time) at which the system becomes busy with fluid level x and UMC phase i at time t, then;

$$u_{busy,i}(x, t + \Delta t) = u_{busy,i}(x - r_i \Delta t, t)(1 + q_{ii}\Delta t) + \sum_{\substack{j=1 \\ (j \neq i)}}^{m} u_{busy,j}(x - r_j \Delta t, t) q_{ji} \Delta t + \phi_{B,i}(x, t)\Delta t. \tag{12}$$

Let us define $\phi_{B,i}(x) = \lim_{t\to\infty} \phi_{B,i}(x, t)$ and a vector $\boldsymbol{\phi}_B(x) = \{\phi_{B,1}(x), \cdots, \phi_{B,m}(x)\}$. Then, Eq. (12) can be written in vector form as

$$\frac{d}{dx}\boldsymbol{u}_{busy}(x)\boldsymbol{R} = \boldsymbol{u}_{busy}(x)\boldsymbol{Q} + \boldsymbol{\phi}_B(x), \ (x > 0). \tag{13}$$

Let us define the following probability,

$U_{B,ij}(x) = Pr$(A busy period starts with fluid level x and UMC phase j under the condition that the UMC phase is i at the start of the previous idle period), $(x > 0)$.

Let $E(C)$ be the mean length of an arbitrary cycle. Then we have

$$\phi_{B,i}(x) = \sum_{j \in \Im} \frac{\kappa_j}{E(C)} dU_{B,ji}(x), \ (x > 0). \tag{14}$$

$E(C)$ will be obtained later.

Defining $\boldsymbol{U}_B(x)$ as the matrix of $U_{B,ij}(x)$, we can express Eq. (14) in vector form as

$$\boldsymbol{\phi}_B(x) = \frac{\boldsymbol{\kappa}}{E(C)} d\boldsymbol{U}_B(x). \tag{15}$$

To solve Eq. (13), we need to obtain $u_{busy,i}(0)$, which is the rate at which the busy period ends with the UMC phase i. We note that the fluid level decreases at a rate r_i when the UMC phase is $i \in \Im_2$. Because $\frac{\kappa_i}{E(C)}$ is the mean number of events per unit time that an idle period starts with phase i, then;

$$u_{busy,i}(0)(-r_i) = \frac{\kappa_i}{E(C)}, \ (i \in \Im_2). \tag{16}$$

We note that no busy period ends with a UMC phase $i \in \Im_1$, and

$$u_{busy,i}(0) = 0, \ (i \in \Im_1). \tag{17}$$

Using Eqs. (16) and (17), we have

$$\boldsymbol{u}_{busy}(0)(-\boldsymbol{R}) = \frac{\boldsymbol{\kappa}}{E(C)}. \tag{18}$$

Let $\boldsymbol{U}_B^*(\theta) = \int_0^\infty e^{-\theta x} d\boldsymbol{U}_B(x)$. Taking the Laplace transforms of both sides of Eq. (13), then;

$$\theta \boldsymbol{u}_{busy}^*(\theta)\boldsymbol{R} - \boldsymbol{u}_{busy}(0)\boldsymbol{R} = \boldsymbol{u}_{busy}^*(\theta)\boldsymbol{Q} + \frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}_B^*(\theta). \tag{19}$$

Using Eqs. (18) in (19) yields

$$\boldsymbol{u}_{busy}^*(\theta) = \frac{\boldsymbol{\kappa}}{E(C)}[-\boldsymbol{I} + \boldsymbol{U}_B^*(\theta)][\theta\boldsymbol{R} - \boldsymbol{Q}]^{-1}. \tag{20}$$

Next, $\boldsymbol{U}_B^*(\theta)$ needs to be derived. As the fluid level process during an idle period of the Type-V system is identical to that of the MAP/G/1 queueing system with a single vacation, from Lee et al. (2001), we have

$$\boldsymbol{U}_B(x) = \boldsymbol{V}_0(-\boldsymbol{C})^{-1}\boldsymbol{D}S(x) + \left[\sum_{n=1}^{\infty} \boldsymbol{V}_n S^{(n)}(x)\right], \ (x > 0), \tag{21}$$

and

$$\boldsymbol{U}_B^*(\theta) = \boldsymbol{V}_0(-\boldsymbol{C})^{-1}\boldsymbol{D}S^*(\theta) + [\boldsymbol{V}[S^*(\theta)] - \boldsymbol{V}_0], \tag{22}$$

where $\boldsymbol{V}[S^*(\theta)] = \boldsymbol{V}(z)|_{z=S^*(\theta)}$, $\boldsymbol{V}_n = \boldsymbol{V}_n^*(\theta)|_{\theta=0}$, and $S^{(n)}(x)$ is the DF of the n-fold convolution of S itself.

Using Eqs. (22) in (20), we have

$$\boldsymbol{u}_{busy}^*(\theta) = \frac{\boldsymbol{\kappa}}{E(C)}\left[-\boldsymbol{I} + \left[\boldsymbol{V}_0(-\boldsymbol{C})^{-1}\boldsymbol{D}S^*(\theta) + \left\{\boldsymbol{V}[S^*(\theta)] - \boldsymbol{V}_0\right\}\right]\right](\theta\boldsymbol{R} - \boldsymbol{Q})^{-1}. \tag{23}$$

From Eq. (23), $\boldsymbol{u}_{busy}^*(\theta)$ can be completely determined once $\boldsymbol{\kappa}$ and $E(C)$ are known.

3.3 Determining $\boldsymbol{\kappa}$ and $E(C)$

Let $\boldsymbol{K}^*(\theta)$ be the matrix LST of a cycle length. We then have the following theorem.

Theorem 3.1

$$\boldsymbol{K}^*(\theta) = \int_0^\infty \left\{ \boldsymbol{V}_0^*(\theta)(\theta \boldsymbol{I} - \boldsymbol{C})^{-1}\boldsymbol{D} dS(x) + \sum_{n=1}^\infty \left[\boldsymbol{V}_n^*(\theta) dS^{(n)}(x) \right] \right\} \boldsymbol{G}^*(\theta|x), \tag{24}$$

where $\boldsymbol{G}^*(\theta|x) = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{G}_{12}^*(\theta|x) \\ \boldsymbol{0} & \boldsymbol{G}_{22}^*(\theta|x) \end{pmatrix}$.

Proof Let $\boldsymbol{W}_I^*(\theta, x)$ be the matrix LST of the length of an idle period including the probability that the fluid level is less than or equal to x at the end of the idle period. The idle period of the Type-V system is stochastically equivalent to that of the MAP/G/1 queue with a single vacation. Therefore, using the result in Lee et al. (2001), we have

$$\boldsymbol{W}_I^*(\theta, x) = \boldsymbol{V}_0^*(\theta)(\theta \boldsymbol{I} - \boldsymbol{C})^{-1}\boldsymbol{D} S(x) + \sum_{n=1}^\infty \left[\boldsymbol{V}_n^*(\theta) S^{(n)}(x) \right]. \tag{25}$$

If a busy period of our model starts with x amount of initial fluid, the busy period is stochastically equivalent to that of the conventional MMFF model with the same amount of fluid. We note that $\boldsymbol{G}^*(\theta|x)$ represents the length of the first depletion time of the conventional model. Thus, postmultiplying Eq. (25) by $\boldsymbol{G}^*(\theta|x)$ finishes the proof. □

We note that $\boldsymbol{K} = \boldsymbol{K}^*(\theta)|_{\theta=0}$ represents the phase shift probability during a cycle. Then;

$$\begin{aligned} \boldsymbol{K} &= \int_0^\infty \left[\boldsymbol{V}_0(-\boldsymbol{C})^{-1}\boldsymbol{D} dS(x) + \sum_{n=1}^\infty \left[\boldsymbol{V}_n dS^{(n)}(x) \right] \right] [\boldsymbol{G}^*(\theta|x)]_{\theta=0} \\ &= \int_0^\infty \left[\boldsymbol{V}_0(-\boldsymbol{C})^{-1}\boldsymbol{D} dS(x) + \sum_{n=1}^\infty \left[\boldsymbol{V}_n dS^{(n)}(x) \right] \right] \begin{pmatrix} \boldsymbol{0} & \boldsymbol{\Psi} e^{\boldsymbol{H}x} \\ \boldsymbol{0} & e^{\boldsymbol{H}x} \end{pmatrix}, \end{aligned} \tag{26}$$

where $\boldsymbol{H} = \boldsymbol{H}^*(\theta)|_{\theta=0} = \boldsymbol{\Gamma}_2^{-1}[\boldsymbol{Q}_{22} + \boldsymbol{Q}_{21}\boldsymbol{\Psi}]$, and $\boldsymbol{\Psi} = \boldsymbol{\Psi}^*(\theta)|_{\theta=0}$ can be obtained from the algorithm of Ahn and Ramaswami (2005). The algorithm for $\boldsymbol{V}_n$ can be obtained from Lucantoni (1991).

By using Eq. (26), $\boldsymbol{\kappa}$ can be obtained from

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}\boldsymbol{K}, \quad \boldsymbol{\kappa}\mathbf{e} = 1. \tag{27}$$

The mean length $E(C)$ of a cycle is in order.

Theorem 3.2 *The mean length of a cycle is*

$$\begin{aligned} E(C) &= E(V) + \boldsymbol{\kappa}\boldsymbol{V}_0(-\boldsymbol{C})^{-1}\boldsymbol{e} \\ &\quad + \int_0^\infty \left[\boldsymbol{V}_0(-\boldsymbol{C})^{-1}\boldsymbol{D}\boldsymbol{g}(x) dS(x) + \sum_{n=1}^\infty \boldsymbol{V}_n \boldsymbol{g}(x) dS^{(n)}(x) \right], \end{aligned} \tag{28}$$

where

$$\boldsymbol{g}(x) = \begin{pmatrix} \boldsymbol{g}_1(x) \\ \boldsymbol{g}_2(x) \end{pmatrix} = \begin{pmatrix} -\frac{d}{d\theta}\boldsymbol{G}_{12}^*(\theta|x)\big|_{\theta=0}\mathbf{e} \\ -\frac{d}{d\theta}\boldsymbol{G}_{22}^*(\theta|x)\big|_{\theta=0}\mathbf{e} \end{pmatrix}.$$

Proof We note that $E(C) = \boldsymbol{\kappa}\left[-\frac{d}{d\theta}\boldsymbol{K}^*(\theta)\right]_{\theta=0}\mathbf{e}$. Therefore, taking the derivative of Eq. (24), then

$$\begin{aligned} E(C) &= \boldsymbol{\kappa}\boldsymbol{V}_0(-\boldsymbol{C})^{-1}\mathbf{e} + \boldsymbol{\kappa}\left[-\frac{d}{d\theta}\sum_{n=0}^{\infty}\boldsymbol{V}_n^*(\theta)\right]_{\theta=0}\mathbf{e} \\ &\quad + \boldsymbol{\kappa}\boldsymbol{V}_0(-\boldsymbol{C})^{-1}\boldsymbol{D}\int_0^{\infty}\boldsymbol{g}(x)dS(x) + \boldsymbol{\kappa}\sum_{n=1}^{\infty}\boldsymbol{V}_n\int_0^{\infty}\boldsymbol{g}(x)dS^{(n)}(x). \end{aligned} \tag{29}$$

Using $E(V) = \boldsymbol{\kappa}\left[-\frac{d}{d\theta}\sum_{n=0}^{\infty}\boldsymbol{V}_n^*(\theta)\right]_{\theta=0}\mathbf{e}$ in Eq. (29) finishes the proof. □

Now, we need to obtain $\boldsymbol{g}(x)$ used in Eq. (28). Let $\boldsymbol{Q}_{cen} = \boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{22} + \boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{21}\boldsymbol{\Psi}$, and $\boldsymbol{\pi}_{cen}$ be the stationary vector of a Markov chain with infinitesimal generator $\boldsymbol{Q}_{cen}$. Using the results in Baek et al. (2011, 2013b), we have

$$\begin{aligned} \boldsymbol{g}_2(x) = (\boldsymbol{Q}_{cen} + \mathbf{e}\boldsymbol{\pi}_{cen})^{-1}(e^{\boldsymbol{Q}_{cen}x} + x\mathbf{e}\boldsymbol{\pi}_{cen} - \boldsymbol{I}) \\ \left[\boldsymbol{\Gamma}_2^{-1}\mathbf{e} + \boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{21}\left(-\boldsymbol{\Psi}^{(1)}\mathbf{e}\right)\right], (x > 0), \end{aligned} \tag{30}$$

and

$$\boldsymbol{g}_1(x) = -\boldsymbol{\Psi}^{(1)}\mathbf{e} + \boldsymbol{\Psi}\boldsymbol{g}_2(x), (x > 0), \tag{31}$$

where

$$\boldsymbol{\Psi}^{(1)}\mathbf{e} = \frac{d}{d\theta}\boldsymbol{\Psi}^*(\theta)\bigg|_{\theta=0}\mathbf{e} = [\boldsymbol{\Psi}\boldsymbol{P}_{21} + \boldsymbol{P}_{11} - \boldsymbol{I}]^{-1}\left(\frac{\boldsymbol{\Gamma}_1^{-1}\boldsymbol{\Psi}}{\lambda} + \frac{\boldsymbol{\Psi}\boldsymbol{\Gamma}_2^{-1}}{\lambda}\right)\mathbf{e}. \tag{32}$$

Using Eqs. (10) and (28), we have

$$\rho = \frac{E(C) - E(I)}{E(C)} = 1 - \frac{E(I)}{E(C)}. \tag{33}$$

3.4 Fluid level at an arbitrary time point

In this section, we derive the vector LST $\boldsymbol{u}^*(\theta)$ of the fluid level at an arbitrary time and confirm the result in Baek et al. (2013a). Using Eqs. (11), (20), and (6), we have the following theorem.

Theorem 3.3

$$\begin{aligned}\boldsymbol{u}^*(\theta) &= \boldsymbol{u}^*_{idle}(\theta)\left[\theta\boldsymbol{R} - \boldsymbol{D} + \boldsymbol{D}S^*(\theta)\right](\theta\boldsymbol{R} - \boldsymbol{Q})^{-1} \\ &= (1-\rho)\frac{\boldsymbol{\kappa}\left[\boldsymbol{V}_0(-\boldsymbol{C})^{-1} + [\boldsymbol{V}[S^*(\theta)] - \boldsymbol{I}][\boldsymbol{C} + \boldsymbol{D}S^*(\theta)]^{-1}\right]}{E(I)} \\ &\quad \left[\theta\boldsymbol{R} - \boldsymbol{D} + \boldsymbol{D}S^*(\theta)\right](\theta\boldsymbol{R} - \boldsymbol{Q})^{-1}. \end{aligned} \tag{34}$$

Proof Let us rewrite Eq. (11) in the following way

$$\boldsymbol{u}^*_{idle}(\theta)\left[\boldsymbol{C} + \boldsymbol{D}S^*(\theta)\right] = (1-\rho)\frac{\boldsymbol{\kappa}[-\boldsymbol{V}_0 + \boldsymbol{V}_0(-\boldsymbol{C})^{-1}\boldsymbol{D}S^*(\theta) + \boldsymbol{V}[S^*(\theta)] - \boldsymbol{I}]}{E(I)}. \tag{35}$$

Using Eqs. (22) and (33) in (35), we then have

$$\begin{aligned}\boldsymbol{u}^*_{idle}(\theta)[\boldsymbol{C} + \boldsymbol{D}S^*(\theta)] &= \frac{\boldsymbol{\kappa}[\boldsymbol{V}_0(-\boldsymbol{C})^{-1}\boldsymbol{D}S^*(\theta) + \left[\boldsymbol{V}[S^*(\theta)] - \boldsymbol{V}_0\right]}{E(C)} - \frac{\boldsymbol{\kappa}}{E(C)} \\ &= \frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}^*_B(\theta) - \frac{\boldsymbol{\kappa}}{E(C)}. \end{aligned} \tag{36}$$

Using Eq. (20) and $\boldsymbol{Q} = \boldsymbol{C} + \boldsymbol{D}$, then;

$$\theta\boldsymbol{u}^*_{busy}(\theta)\boldsymbol{R} - \boldsymbol{u}^*_{busy}(\theta)\boldsymbol{C} - \boldsymbol{u}^*_{busy}(\theta)\boldsymbol{D} = \frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}^*_B(\theta) - \frac{\boldsymbol{\kappa}}{E(C)}. \tag{37}$$

Directly subtracting Eqs. (37) from (36) finishes the proof. □

We note that Eq. (34) shows that the vector LST $\boldsymbol{u}^*(\theta)$ of the fluid level at an arbitrary time point is factored into two parts, one of which is the vector LST $\boldsymbol{u}^*_{idle}(\theta)$ of the fluid level at an arbitrary time point in idle period. This confirms the result in Baek et al. (2013a).

3.5 Mean performance measures

In this section, we derive the mean $E(U)$ of the fluid level at an arbitrary time point in the steady state. Let $E(U_{idle})$ and $E(U_{busy})$ be the mean fluid levels at an arbitrary time point during the idle and busy periods, respectively. Then, we have

$$E(U) = E(U_{idle}) + E(U_{busy}). \tag{38}$$

Theorem 3.4 *To derive $E(U_{idle})$, we use Eq. (11) and get*

$$E(U_{idle}) = \left[-\frac{d}{d\theta}\boldsymbol{u}^*_{idle}(\theta)\right]_{\theta=0}\mathbf{e} = \frac{(1-\rho)\boldsymbol{\kappa}(\mathbf{e}\boldsymbol{\pi}+\boldsymbol{Q})^{-2}}{E(I)}$$
$$\left[\frac{1}{2}E(V^2)\mathbf{e}\boldsymbol{\pi} + \boldsymbol{V} - E(V)\boldsymbol{Q} - \boldsymbol{I}\right]\boldsymbol{D}\mathbf{e}E(S), \tag{39}$$

where $\boldsymbol{V} = \boldsymbol{V}(z)|_{z=1} = \int_0^\infty e^{(\boldsymbol{C}+\boldsymbol{D})x}dV(x)$.

Proof Let us rewrite Eq. (11) in the following way

$$\boldsymbol{u}^*_{idle}(\theta) = \frac{(1-\rho)\boldsymbol{\kappa}}{E(I)}\left\{\boldsymbol{V}_0(-\boldsymbol{C})^{-1} + \int_0^\infty \sum_{k=0}^\infty \frac{x^{k+1}}{(k+1)!}\left[\boldsymbol{C}+\boldsymbol{D}S^*(\theta)\right]^k dV(x)\right\}. \tag{40}$$

Taking the derivative of Eq. (40) with respect to θ and $\theta = 0$, we have

$$E(U_{idle}) = -\left[\frac{d}{d\theta}\boldsymbol{u}^*_{idle}(\theta)\right]_{\theta=0}\mathbf{e}$$
$$= \frac{(1-\rho)\boldsymbol{\kappa}(\mathbf{e}\boldsymbol{\pi}+\boldsymbol{Q})^{-2}}{E(I)}\int_0^\infty \sum_{k=0}^\infty \frac{x^{k+2}}{(k+2)!}(\mathbf{e}\boldsymbol{\pi}+\boldsymbol{Q})^2\boldsymbol{Q}^k\boldsymbol{D}dV(x)\mathbf{e}E(S). \tag{41}$$

We note that

$$\sum_{k=0}^\infty \frac{x^{k+2}}{(k+2)!}(\mathbf{e}\boldsymbol{\pi}+\boldsymbol{Q})^2\boldsymbol{Q}^k = \frac{x^2}{2}\mathbf{e}\boldsymbol{\pi} + e^{\boldsymbol{Q}x} - x\boldsymbol{Q} - \boldsymbol{I}. \tag{42}$$

Then, using Eqs. (42) in (41) finishes the proof. □

The mean fluid level $E(U_{busy})$ at an arbitrary time point during the busy period can be obtained from Eq. (20) and becomes, after a laborious simplification effort;

$$E(U_{busy}) = \left[-\frac{d}{d\theta}\boldsymbol{u}^*_{busy}(\theta)\right]_{\theta=0}\boldsymbol{e}$$
$$= \frac{1}{-\boldsymbol{\pi}\boldsymbol{R}\boldsymbol{e}}\left\{\frac{1}{2}\frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}_B^{(2)}\boldsymbol{e} + (\boldsymbol{\pi}-\boldsymbol{h})\left[\boldsymbol{Q}(-\boldsymbol{R})^{-1} - \boldsymbol{e}\boldsymbol{\pi}\right]^{-1}\boldsymbol{R}\boldsymbol{e}\right.$$
$$\left.\frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}_B^{(1)}\boldsymbol{R}^{-1}\left[\boldsymbol{Q}(-\boldsymbol{R})^{-1} - \boldsymbol{e}\boldsymbol{\pi}\right]^{-1}\boldsymbol{R}\boldsymbol{e}\right\}, \tag{43}$$

where

$$\boldsymbol{h} = \boldsymbol{u}^*_{idle}(\theta)|_{\theta=0} = \frac{(1-\rho)\boldsymbol{\kappa}}{E(I)}\Big[\boldsymbol{V}_0(-\boldsymbol{C})^{-1} + \big\{E(V)\mathbf{e}\boldsymbol{\pi} + \boldsymbol{V} - \boldsymbol{I}\big\}(\mathbf{e}\boldsymbol{\pi} + \boldsymbol{Q})^{-1}\Big], \tag{44}$$

and $\boldsymbol{U}_B^{(k)} = (-1)^k \frac{d^k}{d\theta^k}\boldsymbol{U}^*_B(\theta)\Big|_{\theta=0}$ can be obtained from Eq. (22).

4 Analysis of the Type-L system

In this section, we analyze the Type-L system. During the idle period, the fluid level increases linearly with a rate ν_i, where ν_i is either zero or positive depending on the UMC phase. In addition to the notation defined in Section 3, let us define the following:

$\Im_1^{idle}$: set of UMC phases with $\nu_i > 0$,
$\Im_2^{idle}$: set of UMC phases with $\nu_i = 0$,
$\boldsymbol{R}_L$: diagonal matrix of ν_i, $(i \in \Im_1^{idle})$,

and the probability

$[\tilde{\boldsymbol{V}}(w,x)]_{ij} = Pr($ if the amount of fluid is less than or equal to w, the UMC phase is j at the end of a maintenance period, and the length of the maintenance period is less than or equal to x under the condition that the maintenance period starts with UMC phase i), $(w \geq 0, x \geq 0)$.

We also define

$$\boldsymbol{V}^*(w,\theta) = \int_{x=0}^{\infty} e^{-\theta x} d_x \tilde{\boldsymbol{V}}(w,x), \ (w > 0), \tag{45}$$

$$\boldsymbol{V}_F(w) = \boldsymbol{V}^*(w,\theta)|_{\theta=0} = \int_{x=0}^{\infty} d_x \tilde{\boldsymbol{V}}(w,x), \ (w > 0), \tag{46}$$

and

$$\boldsymbol{V}^*_F(\theta) = \int_{w=0}^{\infty} e^{-\theta w} d_w \boldsymbol{V}_F(w). \tag{47}$$

Then, we have the following theorem.

Theorem 4.1 $\boldsymbol{V}^*_F(\theta)$ *is the matrix LST of the amount of fluid which arrives in the system during a maintenance period, and becomes*

$$\boldsymbol{V}^*_F(\theta) = \int_0^{\infty} e^{\boldsymbol{Q}^*_{\Lambda}(\theta)x} d V(x), \tag{48}$$

where

$$\boldsymbol{Q}^*_{\Lambda}(\theta) = \begin{pmatrix} -\theta \boldsymbol{R}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \boldsymbol{Q} \tag{49}$$

Proof Let $U_I(t)$ be the amount of fluid, which arrives in the system during a time interval $(0, t]$ in the idle period ($U_I(0) = 0$). Let us define the following joint probability;

$$[\boldsymbol{F}(w,t)]_{ij} = [\boldsymbol{F}(w,t|0)]_{ij} = Pr[U_I(t) \leq w, J(t) = j | U_I(0) = 0, J(0) = i].$$

Then, we have

$$\begin{aligned} &[\boldsymbol{F}(w,t+\Delta t)]_{ij}\mathbf{1}_{[j\in\Im_1^{idle}]} \\ &\quad = \Bigg[[\boldsymbol{F}(w-\nu_i\Delta t,t)]_{ij}(1+q_{jj}\Delta t) + \sum_{\substack{k\in\Im_1^{idle}\\(k\neq j)}} [\boldsymbol{F}(w-\nu_k\Delta t,t)]_{ik}q_{kj}\Delta t \\ &\qquad + \sum_{k\in\Im_2^{idle}} [\boldsymbol{F}(w,t)]_{ik}q_{kj}\Delta t \Bigg] \mathbf{1}_{[j\in\Im_1^{idle}]}, \end{aligned} \tag{50}$$

and

$$\begin{aligned} &[\boldsymbol{F}(w,t+\Delta t)]_{ij}\mathbf{1}_{[j\in\Im_2^{idle}]} \\ &\quad = \Bigg[[\boldsymbol{F}(w,t)]_{ij}(1+q_{jj}\Delta t) + \sum_{\substack{k\in\Im_1^{idle}\\(k\neq j)}} [\boldsymbol{F}(w-\nu_k\Delta t,t)]_{ik}q_{kj}\Delta t \\ &\qquad + \sum_{k\in\Im_2^{idle}} [\boldsymbol{F}(w,t)]_{ik}q_{kj}\Delta t \Bigg] \mathbf{1}_{[j\in\Im_2^{idle}]}, \end{aligned} \tag{51}$$

where the indicator function $\mathbf{1}_{[A]}$ is defined as 1 if A is true and 0 otherwise. Expressing Eqs. (50) and (51) in matrix form, we get the following matrix partial differential equation;

$$\frac{\partial}{\partial t}\boldsymbol{F}(w,t) + \frac{\partial}{\partial w}\boldsymbol{F}(w,t)\begin{pmatrix} \boldsymbol{R}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \boldsymbol{F}(w,t)\boldsymbol{Q}. \tag{52}$$

Let us define the following matrix LST,

$$\boldsymbol{F}^*(\theta,t) = \int_{w=0}^{\infty} e^{-\theta w} d_w \boldsymbol{F}(w,t). \tag{53}$$

Taking the LSTs of both sides of Eq. (52), we get

$$\frac{d}{dt}\boldsymbol{F}^*(\theta,t) = \boldsymbol{F}^*(\theta,t)\left[\begin{pmatrix} -\theta\boldsymbol{R}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \boldsymbol{Q}\right]. \tag{54}$$

The solution of Eq. (54) with initial condition $\boldsymbol{F}(0,0) = \boldsymbol{I}$ becomes $\boldsymbol{F}^*(\theta,t) = e^{\boldsymbol{Q}^*_\Lambda(\theta)t}$. Then $\boldsymbol{V}^*_F(\theta) = \int_0^\infty \boldsymbol{F}^*(\theta,x)dV(x)$ finishes the proof. □

4.1 Analysis of the idle period

To derive the vector LST $\boldsymbol{u}^*_{idle}(\theta)$ of the fluid level at an arbitrary time during an idle period, we first find the mean length $E(I)$ of an idle period. AS for $\Im_1^{idle}$ and $\Im_2^{idle}$, let us consider another partition of $\boldsymbol{Q}$ as

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{Q}^I_{11} & \boldsymbol{Q}^I_{12} \\ \boldsymbol{Q}^I_{21} & \boldsymbol{Q}^I_{22} \end{pmatrix}.$$

We then have the following theorem.

Theorem 4.2 *Let $\boldsymbol{\kappa}$ be the UMC phase at the starting point of an arbitrary idle period. We then have*

$$E(I) = E(V) + \boldsymbol{\kappa}\boldsymbol{V}_F(0)\boldsymbol{T}_D\mathbf{e}, \tag{55}$$

where $\boldsymbol{T}_D = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (-\boldsymbol{Q}^I_{22})^{-1} \end{pmatrix}$, *and* $\boldsymbol{V}_F(0) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \int_0^\infty e^{\boldsymbol{Q}^I_{22}x} dV(x) \end{pmatrix}$ *denotes the probability that no fluid arrives during a maintenance period.*

Proof As soon as the system becomes empty, the server undergoes a maintenance period with mean length $E(V)$. At the end of the maintenance time, if there is no fluid and the UMC phases are in $\Im_2^{idle}$ (with probability $\boldsymbol{\kappa}\boldsymbol{V}_F(0)$), the server stays dormant to find the first ($\Im_2^{idle} - > \Im_1^{idle}$) transition of UMC. Let $(\boldsymbol{T}_D)_{ij}$ be the mean sojourn time of the dormant process in the UMC phase j under the condition that it starts with phase i. Then, we have

$$\boldsymbol{T}_D = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (-\boldsymbol{Q}^I_{22})^{-1} \end{pmatrix}. \tag{56}$$

Thus, we can denote the mean length of a dormant period by $\boldsymbol{\kappa}\boldsymbol{V}_F(0)\boldsymbol{T}_D\mathbf{e}$, which finishes the proof. □

Theorem 4.3 *Let ρ be the probability that the system is idle at an arbitrary time point. We then have*

$$\boldsymbol{u}^*_{idle}(\theta) = (1-\rho)\frac{\boldsymbol{\kappa}\left\{\boldsymbol{V}_F(0)\boldsymbol{T}_D + [\boldsymbol{V}^*_F(\theta) - \boldsymbol{I}][\boldsymbol{Q}^*_\Lambda(\theta)]^{-1}\right\}}{E(I)}. \tag{57}$$

Proof An arbitrary time point during an idle period is included either in the maintenance period (with probability $E(V)/E(I)$) or in the dormant period (with probability $\boldsymbol{\kappa}\boldsymbol{V}_F(0)\boldsymbol{T}_D\mathbf{e}/E(I)$). The phase probability vector at an arbitrary time point during a dormant period becomes

$$\frac{\boldsymbol{\kappa}\boldsymbol{V}_F(0)\boldsymbol{T}_D}{\boldsymbol{\kappa}\boldsymbol{V}_F(0)\boldsymbol{T}_D\mathbf{e}}. \tag{58}$$

We note that the fluid level at an arbitrary time point during a maintenance period is equal to the total amount of fluid that arrives into the system during an elapsed maintenance period. Then, the vector LST is given by

$$\int_0^\infty \boldsymbol{F}^*(\theta, t)\frac{1-V(t)}{E(V)}dt = \int_0^\infty e^{\boldsymbol{Q}^*_\Lambda(\theta)t}\frac{1-V(t)}{E(V)}dt = \frac{[\boldsymbol{V}^*_F(\theta) - \boldsymbol{I}][\boldsymbol{Q}^*_\Lambda(\theta)]^{-1}}{E(V)}. \tag{59}$$

Now, using Eqs. (58) and (59), we finish the proof. □

4.2 Analysis of the busy period

To derive $\boldsymbol{u}^*_{busy}(\theta)$, we need to find the distribution of the fluid level at the start of a busy period.

Theorem 4.4 *We have*

$$\boldsymbol{U}_B(w) = \begin{cases} \boldsymbol{V}_F(0)\boldsymbol{T}_D\boldsymbol{Q}_B, & (w = 0), \\ \boldsymbol{V}_F(w), & (w > 0), \end{cases} \tag{60}$$

and

$$\boldsymbol{U}^*_B(\theta) = \boldsymbol{V}_F(0)\boldsymbol{T}_D\boldsymbol{Q}_B + [\boldsymbol{V}^*_F(\theta) - \boldsymbol{V}_F(0)], \tag{61}$$

where $\boldsymbol{Q}_B = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{Q}^I_{21} & \boldsymbol{0} \end{pmatrix}$.

Proof If there is any amount of fluid at the end of a maintenance period, the busy period starts immediately. Thus, we obtain the second equality of Eq. (60). Because the dormant period ends as soon as a $(\Im^{idle}_2 \rightarrow \Im^{idle}_1)$ transition occurs, the change of the UMC phase during the dormant period is represented by $\boldsymbol{V}_F(0)\boldsymbol{T}_D\boldsymbol{Q}_B$ with a zero fluid level at the end. Thus, we have the first equality of Eq. (60). Taking LSTs of both sides of Eq. (60) yields Eq. (61), and finishes the proof. □

It is noteworthy that the stochastic behavior of the busy period of our model with x amount of initial fluid is identical to the busy period of the conventional MMFF model with the same amount of initial fluid. Thus, using Eqs. (61) in (20), we get

$$\begin{aligned} \boldsymbol{u}^*_{busy}(\theta) &= \frac{\boldsymbol{\kappa}}{E(C)}[-\boldsymbol{I} + \boldsymbol{U}^*_B(\theta)][\theta\boldsymbol{R} - \boldsymbol{Q}]^{-1} \\ &= \frac{\boldsymbol{\kappa}}{E(C)}\Big[-\boldsymbol{I} + \big\{\boldsymbol{V}_F(0)\boldsymbol{T}_D\boldsymbol{Q}_B + [\boldsymbol{V}^*_F(\theta) - \boldsymbol{V}_F(0)]\big\}\Big](\theta\boldsymbol{R} - \boldsymbol{Q})^{-1}. \end{aligned} \tag{62}$$

4.3 Determining $\boldsymbol{\kappa}$ and $E(C)$

In this section, we derive the mean length $E(C)$ of a cycle and the stationary probability vector $\boldsymbol{\kappa}$ of the UMC phase at an arbitrary idle period starting point. Let $\boldsymbol{V}^*(0,\theta)$ be the joint matrix transform that represents the length of a maintenance period, including the probability that no fluid arrives into the system during the same period. We then have

$$\boldsymbol{V}^*(0,\theta) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \int_0^\infty e^{(-\theta \boldsymbol{I} + \boldsymbol{Q}_{22}^I)x} dV(x) \end{pmatrix}. \tag{63}$$

Then, $\boldsymbol{K}^*(\theta)$ is given in the following theorem.

Theorem 4.5 *We have*

$$\boldsymbol{K}^*(\theta) = \boldsymbol{V}^*(0,\theta)\boldsymbol{T}_D^*(\theta)\boldsymbol{Q}_B\boldsymbol{B}^*(\theta) + \int_{w=0+}^\infty d_w \boldsymbol{V}^*(w,\theta)\boldsymbol{G}^*(\theta|w), \tag{64}$$

where

$$\boldsymbol{T}_D^*(\theta) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\theta \boldsymbol{I} - \boldsymbol{Q}_{22}^I)^{-1} \end{pmatrix}, \tag{65}$$

and

$$\boldsymbol{B}^*(\theta) = \begin{pmatrix} \mathbf{0} & \boldsymbol{\Psi}^*(\theta) \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{66}$$

Proof Because a dormant period ends with ($\Im_2^{idle} \rightarrow \Im_1^{idle}$) UMC transition, $\boldsymbol{T}_D^*(\theta)\boldsymbol{Q}_B$ represents the length of the dormant period. The behavior during the busy period is stochastically identical to the conventional MMFF. Thus, we have the first part of Eq. (64). If there is any amount of fluid at the end of the maintenance period, the busy period immediately starts. Under the condition that busy period starts with a fluid level w, the length of the busy period is represented by $\boldsymbol{G}^*(\theta|w)$, which explains the second part of Eq. (64). □

The phase transition matrix $\boldsymbol{K}$ during a cycle becomes

$$\boldsymbol{K} = \boldsymbol{K}^*(\theta)|_{\theta=0} = \boldsymbol{V}_F(0)\boldsymbol{T}_D\boldsymbol{Q}_B\boldsymbol{B} + \int_{w=0+}^\infty d_w \boldsymbol{V}_F(w) \begin{pmatrix} \mathbf{0} & \boldsymbol{\Psi} e^{\boldsymbol{H}w} \\ \mathbf{0} & e^{\boldsymbol{H}w} \end{pmatrix}, \tag{67}$$

in which $\boldsymbol{H} = \boldsymbol{H}^*(\theta)|_{\theta=0} = \boldsymbol{\Gamma}_2^{-1}[\boldsymbol{Q}_{22} + \boldsymbol{Q}_{21}\boldsymbol{\Psi}]$, $\boldsymbol{B} = \boldsymbol{B}^*(\theta)|_{\theta=0} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\Psi} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, and $\boldsymbol{\Psi}$ can be computed from the algorithm of Ahn and Ramaswami (2005).

Using Eq. (67), the stationary vector $\boldsymbol{\kappa}$ of UMC phases at an arbitrary cycle starting point can be obtained from

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}\boldsymbol{K}, \quad \boldsymbol{\kappa}\boldsymbol{e} = 1. \tag{68}$$

The mean length of a cycle is given in the following theorem.

Theorem 4.6 *We have*

$$E(C) = E(V) + \boldsymbol{\kappa} \boldsymbol{V}_F(0)\boldsymbol{T}_D\mathbf{e} + \boldsymbol{\kappa} \boldsymbol{V}_F(0)\boldsymbol{T}_D\boldsymbol{Q}_B\left[-\frac{d}{d\theta}\boldsymbol{B}^*(\theta)\right]_{\theta=0}\mathbf{e} + \boldsymbol{\kappa}\int_{w=0+}^{\infty} d\boldsymbol{V}_F(w)\boldsymbol{g}(w), \tag{69}$$

where

$$-\frac{d}{d\theta}\boldsymbol{B}^*(\theta)\bigg|_{\theta=0} = \begin{pmatrix} \mathbf{0} & -\frac{d}{d\theta}\boldsymbol{\Psi}^*(\theta)\Big|_{\theta=0}\mathbf{e} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\Psi}^{(1)}\mathbf{e} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

$\boldsymbol{\Psi}^{(1)}\mathbf{e}$ *and* $\boldsymbol{g}(w)$ *were given in Eqs. (30)–(32).*

Proof Using Eq. (64), we get

$$\begin{aligned} E(C) &= \boldsymbol{\kappa}\left[-\frac{d}{d\theta}\boldsymbol{K}^*(\theta)\right]_{\theta=0}\boldsymbol{e} \\ &= \boldsymbol{\kappa}\left[-\frac{d}{d\theta}\boldsymbol{V}^*(0,\theta)\right]_{\theta=0}\boldsymbol{T}_D\boldsymbol{Q}_B\boldsymbol{e} + \boldsymbol{\kappa}\boldsymbol{V}_F(0)\left[-\frac{d}{d\theta}\boldsymbol{T}_D^*(\theta)\right]_{\theta=0}\boldsymbol{Q}_B\boldsymbol{e} \\ &\quad + \boldsymbol{\kappa}\boldsymbol{V}_F(0)\boldsymbol{T}_D\boldsymbol{Q}_B\left[-\frac{d}{d\theta}\boldsymbol{B}^*(\theta)\right]_{\theta=0}\boldsymbol{e} + \boldsymbol{\kappa}\int_{w=0+}^{\infty} d_w\left[-\frac{d}{d\theta}\boldsymbol{V}^*(w,\theta)\right]_{\theta=0}\boldsymbol{e} \\ &\quad + \boldsymbol{\kappa}\int_{w=0+}^{\infty} d\boldsymbol{V}_F(w)\boldsymbol{g}(w) \end{aligned} \tag{70}$$

where

$$-\frac{d}{d\theta}\boldsymbol{T}_D^*(\theta)\bigg|_{\theta=0} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{d}{d\theta}(\theta\boldsymbol{I} - \boldsymbol{Q}_{22})^{-1}\Big|_{\theta=0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (-\boldsymbol{Q}_{22})^{-2} \end{pmatrix} = (\boldsymbol{T}_D)^2. \tag{71}$$

Then, after some simplifications, we have

$$\begin{aligned} E(C) &= \boldsymbol{\kappa}\left[-\frac{d}{d\theta}\boldsymbol{V}^*(0,\theta)\right]_{\theta=0}\mathbf{e} + \boldsymbol{\kappa}\int_{w=0+}^{\infty} d_w\left[-\frac{d}{d\theta}\boldsymbol{V}^*(w,\theta)\right]_{\theta=0}\mathbf{e} \\ &\quad + \boldsymbol{\kappa}\boldsymbol{V}_F(0)\boldsymbol{T}_D\mathbf{e} + \boldsymbol{\kappa}\boldsymbol{V}_F(0)\boldsymbol{T}_D\boldsymbol{Q}_B\left[-\frac{d}{d\theta}\boldsymbol{B}^*(\theta)\right]_{\theta=0}\mathbf{e} \\ &\quad + \boldsymbol{\kappa}\int_{w=0+}^{\infty} d\boldsymbol{V}_F(w)\boldsymbol{g}(w). \end{aligned} \tag{72}$$

Using the following identity in Eq. (72), we get Eq. (69)

$$\boldsymbol{\kappa}\left[-\frac{d}{d\theta}\boldsymbol{V}^*(0,\theta)\right]_{\theta=0}\mathbf{e}+\boldsymbol{\kappa}\int_{w=0+}^{\infty}d_w\left[-\frac{d}{d\theta}\boldsymbol{V}^*(w,\theta)\right]_{\theta=0}\mathbf{e}=E(V), \tag{73}$$

and the proof is complete. □

Using Eqs. (55) and (69), we have

$$\rho=\frac{E(C)-E(I)}{E(C)}=1-\frac{E(I)}{E(C)}. \tag{74}$$

4.4 Fluid level at an arbitrary time point

In this section, we derive the vector LST $\boldsymbol{u}^*(\theta)$ of the fluid level at an arbitrary time point to confirm the result in Baek et al. (2013a). Using Eqs. (57) and (62), we have the following theorem.

Theorem 4.7

$$\begin{aligned}\boldsymbol{u}^*(\theta)&=\boldsymbol{u}^*_{idle}(\theta)\theta\left[\boldsymbol{R}-\begin{pmatrix}\boldsymbol{R}_L & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\right]\cdot(\theta\boldsymbol{R}-\boldsymbol{Q})^{-1}\\ &=\frac{(1-\rho)\boldsymbol{\kappa}\left\{\boldsymbol{V}_F(0)\boldsymbol{T}_D+[\boldsymbol{V}^*_F(\theta)-\boldsymbol{I}][\boldsymbol{Q}^*_\Lambda(\theta)]^{-1}\right\}}{E(I)}\left[\boldsymbol{R}-\begin{pmatrix}\boldsymbol{R}_L & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\right]\cdot(\theta\boldsymbol{R}-\boldsymbol{Q})^{-1},\end{aligned} \tag{75}$$

Proof Let us rewrite Eq. (57) in the following way

$$\boldsymbol{u}^*_{idle}(\theta)\boldsymbol{Q}^*_\Lambda(\theta)=\frac{(1-\rho)\boldsymbol{\kappa}}{E(I)}\left\{\boldsymbol{V}_F(0)\boldsymbol{T}_D\boldsymbol{Q}^*_\Lambda(\theta)+[\boldsymbol{V}^*_F(\theta)-\boldsymbol{I}]\right\}. \tag{76}$$

Using Eqs. (61) and (74) in Eq. (76), we have

$$\begin{aligned}\boldsymbol{u}^*_{idle}(\theta)\boldsymbol{Q}^*_\Lambda(\theta)&=\boldsymbol{u}^*_{idle}(\theta)\left[\begin{pmatrix}-\theta\boldsymbol{R}_L & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}+\boldsymbol{Q}\right]\\ &=\frac{\boldsymbol{\kappa}\boldsymbol{V}_F(0)\boldsymbol{T}_D\boldsymbol{Q}_B+[\boldsymbol{V}^*_F(\theta)-\boldsymbol{V}_F(0)]}{E(C)}-\frac{\boldsymbol{\kappa}}{E(C)}\\ &=\frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}^*_B(\theta)-\frac{\boldsymbol{\kappa}}{E(C)}.\end{aligned} \tag{77}$$

Using Eq. (62), then;

$$\theta\boldsymbol{u}^*_{busy}(\theta)\boldsymbol{R}-\boldsymbol{u}^*_{busy}(\theta)\boldsymbol{Q}=\frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}^*_B(\theta)-\frac{\boldsymbol{\kappa}}{E(C)}. \tag{78}$$

By subtracting Eqs. (78) from (77) finishes the proof.

□

Equation (75) shows that the vector LST $u^*(\theta)$ of the fluid level at an arbitrary time point is factored into two parts, one of which is the vector LST $u^*_{idle}(\theta)$ of the fluid level at an arbitrary idle time point. This confirms the result in Baek et al. (2013a).

4.5 Mean performance measures

In this section, we derive the mean fluid level $E(U)$ at an arbitrary time in the steady state. Let $E(U_{idle})$ and $E(U_{busy})$ be the mean fluid levels at an arbitrary time point during the idle and busy periods, respectively. Using the same technique in Theorem 3.4, then it is straightforward to show

$$\begin{aligned} E(U_{idle}) &= \left[-\frac{d}{d\theta}\boldsymbol{u}^*_{idle}(\theta)\right]_{\theta=0}\mathbf{e} \\ &= \frac{(1-\rho)\boldsymbol{\kappa}(\mathbf{e}\boldsymbol{\pi}+\boldsymbol{Q})^{-2}}{E(I)}\left[\frac{1}{2}E(V^2)\mathbf{e}\boldsymbol{\pi}+\boldsymbol{V}-E(V)\boldsymbol{Q}-\boldsymbol{I}\right]\begin{pmatrix}\boldsymbol{R}_L & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{e}, \end{aligned} \tag{79}$$

where $\boldsymbol{V} = \boldsymbol{V}^*_F(\theta)|_{\theta=0} = \int_0^\infty e^{\boldsymbol{Q}x}dV(x)$.

$E(U_{busy})$ can be obtained from Eq. (62) and becomes

$$\begin{aligned} E(U_{busy}) &= \left[-\frac{d}{d\theta}\boldsymbol{u}^*_{busy}(\theta)\right]_{\theta=0}\mathbf{e} \\ &= \frac{1}{-\boldsymbol{\pi}\boldsymbol{R}\mathbf{e}}\left\{\frac{1}{2}\frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}_B^{(2)}\mathbf{e}+(\boldsymbol{\pi}-\boldsymbol{h})\left[\boldsymbol{Q}(-\boldsymbol{R})^{-1}-\mathbf{e}\boldsymbol{\pi}\right]^{-1}\boldsymbol{R}\mathbf{e}\right. \\ &\quad \left.+\frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}_B^{(1)}\boldsymbol{R}^{-1}\left[\boldsymbol{Q}(-\boldsymbol{R})^{-1}-\mathbf{e}\boldsymbol{\pi}\right]^{-1}\boldsymbol{R}\mathbf{e}\right\}, \end{aligned} \tag{80}$$

where

$$\boldsymbol{h} = \boldsymbol{u}^*_{idle}(\theta)|_{\theta=0} = \frac{(1-\rho)\boldsymbol{\kappa}}{E(I)}\left[\boldsymbol{V}_F(0)\boldsymbol{T}_D+\left\{E(V)\mathbf{e}\boldsymbol{\pi}+\boldsymbol{V}-\boldsymbol{I}\right\}(\mathbf{e}\boldsymbol{\pi}+\boldsymbol{Q})^{-1}\right], \tag{81}$$

and $\boldsymbol{U}_B^{(k)} = (-1)^k\frac{d^k}{d\theta^k}\boldsymbol{U}^*_B(\theta)\Big|_{\theta=0}$ can be obtained from Eq. (61).

5 Numerical examples

In this section, we present numerical examples. First, we evaluate the effect of the maintenance period from an economic point of view. Specifically, we show that employing a maintenance period is beneficial when the server reactivation cost is high. We then present a cost optimization model under various cost options for the maintenance period.

Another issue in the production system with a maintenance period is managing the variability of the maintenance period because a high variance of the period might cause a high level of work-in-process. From this point of view, we show a numerical example that shows the effect of the variance of the maintenance period on the expected level of fluid.

5.1 Cost effect of the manitenance period

In many production systems, the server breaks down during its processing period causes a very high back-up cost. Therefore, to avoid the back-up cost, a manager of the system usually takes server maintenance whenever the system becomes empty. Furthermore, employing a maintenance period gives an economic advantage when the server reactivation cost is very high. In this section, we present a numerical example that shows the cost saving by applying a maintenance period. To this end, we compare the mean costs per unit time of the proposed model and the conventional MMFF model.

Let us define the following cost function:

$$EC = \frac{C_s}{E(C)} + C_h \cdot E(U), \tag{82}$$

where C_s is a cost incurred to turn the idle server on (server reactivation cost) and C_h is a cost incurred to keep a unit amount of fluid in the system per unit time (holding cost).

The cost function (82) is a traditional one to evaluate the total cost per unit time of a particular system. Balachandran and Tijms (1975) and Boxma (1976) used it first to find the optimal threshold for an M/G/1 queueing system under D-policy. Next, we use the following parameter matrices for the numerical example:

$$\boldsymbol{C}=\begin{pmatrix} -10 & 1 & 3 & 1 \\ 1 & -7 & 1 & 1 \\ 3 & 1 & -13 & 2 \\ 2 & 1 & 1 & -12 \end{pmatrix}, \quad \boldsymbol{D}=\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & 3 & 2 \end{pmatrix}, \quad \boldsymbol{R}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad \boldsymbol{R}_L=\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

and C_h is set to be 2.

Since the conventional model works as the same of the Type-L model except for the maintenance period, we only use the type for the clear comparison. We define $E_{Proposed}$ and E_{Normal} as the costs of the proposed and the conventional models with above parameter setting, respectively. We obtain $E_{Proposed}$ using Eqs. (69), (79), and (80) in Eq. (82). E_{Normal} is obtained by using the results in Ahn et al. (2005). Then, Fig. 3 shows the relative ratio $\frac{E_{Proposed}}{E_{Normal}}$ under various $E(V)$ and C_s settings.

From Fig. 3, we can observe that the ratio $\frac{E_{Proposed}}{E_{Normal}}$ is greater than 1, if the server reactivation cost (C_s) is small. It means that employing a maintenance period is not beneficial in cost when the reactivation cost is small. This is especially so when the expected length $E(V)$ of the maintenance period is long. It is because a longer

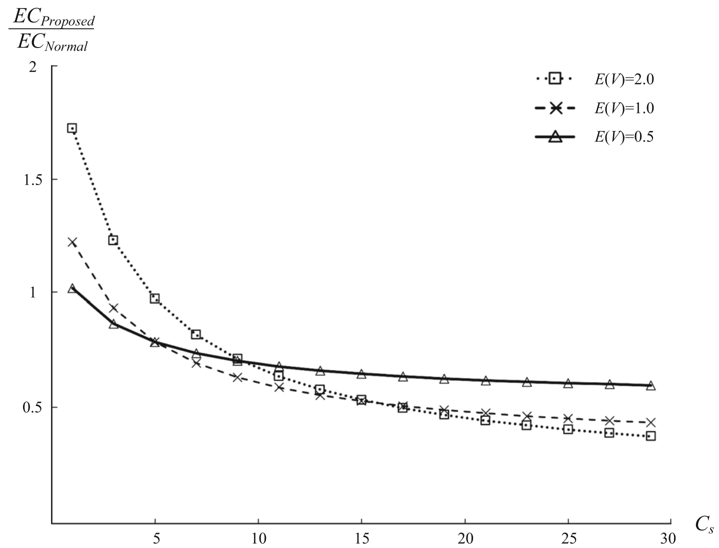


Fig. 3 Comparison between the proposed and the conventional models

maintenance period means the higher expected fluid level at the starting of a server reactivation time point and at an arbitrary time point. However, under all $E(V)$ settings, it is observed that the relative ratio between costs of the proposed model and conventional MMFF model gets smaller as C_s increases. This fits the conventional sense that employing a maintenance period may be more beneficial for cost saving as the server reactivation cost increases.

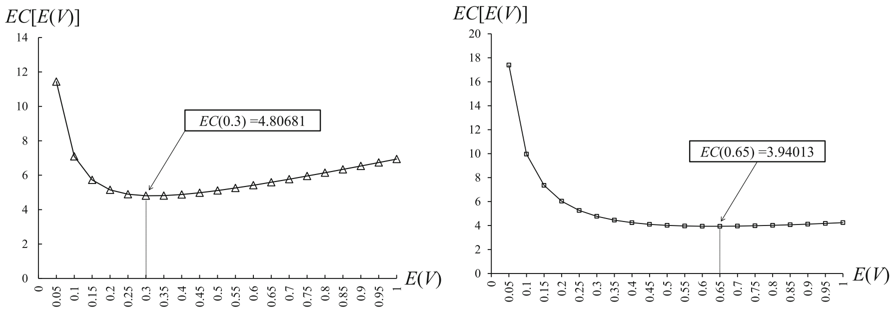
5.2 Cost optimization

In a typical production system, the length of a maintenance period can be reduced by spending more cost, e.g., hiring an artisan by paying more instead of hiring an average technician. From such point of view, we present a cost optimization model under various cost options for the maintenance period. We first consider a linear cost function as follows:

$$EC[E(V)] = \frac{C_s}{E(C)} + C_h \cdot E(U) + \frac{c_m/E(V)}{E(C)}. \tag{83}$$

The above cost function is a simple modification of Eq. (82). This modification is for modeling of the situation that one can choose maintenance time options given by the pair of the expected length $(E(V))$ and the cost $(\frac{c_m}{E(V)})$.

Next, we use the same parameter matrices in the previous section and assume that the jump size of the Type-V system follows the exponential distribution with mean $E(S) = 0.5$. We assume that $C_s = 3,\ C_h = 1.5,\ c_m = 1$. Then, using Eqs. (39), (43), (79), and (80) in Eq. (83), we can compute the mean length of the maintenance time to minimize the average operating cost for each type (see Fig. 4).



(a) Optimal maintenance time for Type-V system **(b)** Optimal maintenance time for Type-L system

Fig. 4 Optimal maintenance times

5.3 Effect of the variance of the maintenance period

As mentioned above, the variance of the length of the maintenance period severely affects the fluid level. Practically this means that the high variance of maintenance period might cause high work-in-process in the system. In this section, we show the effect of the variance on the system performance. To this end, we compare the mean fluid levels of the following two systems:

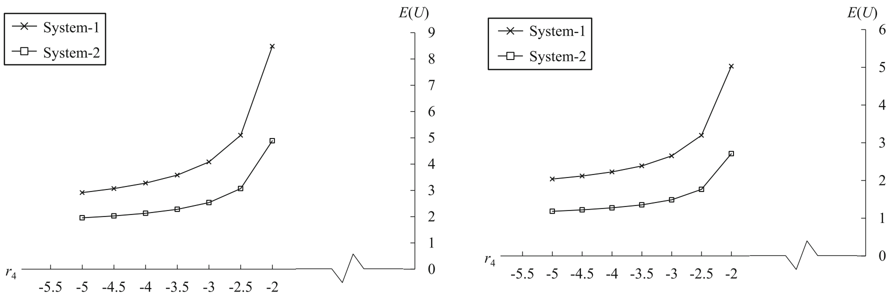
(System-1) MMFF system with exponential maintenance time
(System-2) MMFF system with Erlang (of order 3) maintenance time

From the above definition, it is easy to see that the variance of the maintenance time of the System-1 is three times higher than System-2's if the expected lengths are the same. Next, we assume that $E(V) = 0.5$ and $E(S) = 0.5$ for both systems. The parameter matrices are assumed to be

$$\boldsymbol{C} = \begin{pmatrix} -8 & 1 & 2 & 2 \\ 1 & -9.5 & 1 & 3 \\ 1 & 2 & -10 & 2 \\ 1 & 1 & 1 & -10 \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} 1 & 0.5 & 0.5 & 1 \\ 2 & 0.5 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{pmatrix},$$

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}, \quad \boldsymbol{R}_L = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}.$$

To vary the traffic intensity of the systems, we change $r_4 (< 0)$ with all other parameters fixed. It is obvious that the traffic intensity will increase as r_4 increases. Then, Fig. 5 shows that under the same parameter settings, the system with a higher variance has a higher mean fluid level. Furthermore, as the traffic intensity increases, the difference in the mean fluid levels between the two systems increases. This implies that the control of the variability of the maintenance time is important for managing system performances, especially for the system with heavy traffic intensity.



(a) Comparison for Type-V system **(b)** Comparison for Type-L system

Fig. 5 Effect of the variance of the maintenance time

6 Conclusions and future Work

In this study, we analyzed the MMFF model by using the server maintenance time. We considered two cases distinguished by their input patterns during the maintenance period: Type-V and Type-L for vertical and linear inputs, respectively. For both systems, we derived the vector LSTs of the fluid levels at an arbitrary time in the steady state, and expressed the LSTs in factorized form. The mean performance measures were shown. Using numerical examples, we showed the effect of the maintenance time on the system performances.

Future work could include various generalizations to make the system more realistic. One direct extension would be to generalize the pattern of the fluid input. For example, we can consider the input pattern that Type-V and Type-L are combined. Furthermore, a non-exhaustive system could be a possible extension, in which the server can take a maintenance period during its busy period. This generalization would be valuable to model a system in which the server should take urgent maintenance even though the system is not empty.

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