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ORIGINAL ARTICLE

Separation Principle-Based Positive Output-Feedback l_∞–l_∞ Disturbance Attenuation

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Abstract
We propose a robust positive observer-based output-feedback (OBOF) controller design method for linear time-invariant systems subject to parametric uncertainties and l_∞ disturbances. The contributions of this paper include the following: (i) the OBOF controller design condition is formulated as a convex optimization problem although there exist uncertainties in the system; (ii) a separation principle is demonstrated in the l_∞–l_∞ disturbance attenuation.

Keywords Observer-based output feedback · Positive systems · l_∞–l_∞ disturbance attenuation · Separation principle

1 Introduction

Dynamic systems in which a nonnegative initial condition excites the nonnegativity of their state are called positive systems. Positive systems have gathered considerable interest because many physical processes involve nonnegative quantities such as the level of liquids [1], population levels [14], and epidemic dynamics [6]. It is worth noting that positive stabilization requires that the closed-loop state is confined within the positive orthant cone rather than the whole state space. However, the state is not entirely measurable in various control processes [7, 8].

It is known that the observer and controller in the observer-based output-feedback (OBOF) framework are very difficult to design separately when the system possesses uncertainties [11]. The uncertain closed-loop system matrix is no longer triangular, and the separation principle is not directly identified from a matrix analysis viewpoint. A non-separate design often provokes a nonconvex optimization problem subject to bilinear matrix inequality (BMI) constraints [15]. Kheloufi et al. [5] recovered the convexity by applying Young's inequality at the expense of a certain conservatism. Even for systems that do not include uncertainties, studies on the separate design of OBOF controllers aiming at disturbance attenuations are few. In one study, for example, the plant dynamics in an OBOF closed loop includes the estimation error, which hinders the separate design for disturbance attenuation problems [9].

Moreover, imposing closed-loop positivity via an output-feedback controller with a typical Luenberger observer is indeed challenging. This is because a single gain in a controller cannot force the upper-left and upper-right blocks in a closed-loop system matrix to be simultaneously positive, and similarly, a single gain in an observer cannot force the lower-left block in a closed-loop system matrix to be positive because the gain is not placed therein. The positivity of fuzzy observer-control systems was investigated in [3] by regarding the observer system matrix as a decision variable. However, the separation principle was not clearly discussed.

In this paper, we propose a separate OBOF controller design methodology for a discrete-time uncertain linear time-invariant (LTI) system to exhibit closed-loop positivity as well as attenuation performance against l_∞ disturbances and robustness against norm-bounded parametric uncertainties. The contributions of this study are as follows: (i) to ensure that the uncertain closed-loop plant is positive, a novel OBOF controller is used; (ii) to design the controller and observer separately, decoupled l_∞–l_∞ disturbance attenuation performances are introduced; (iii) the design condition for the controller and observer is formulated as two independent convex optimization problems in terms of linear matrix inequalities (LMI) and linear matrix equalities (LME); (iv) a separation principle is explicitly revealed in

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the concerned problem. A numerical example of a dynamic Leontief input–output model of a multisector economy [13] demonstrates the efficacy of the proposed method.

Notation: The index set is defined as $\mathcal{I}_N := \{1, \dots, n\} \subset \mathbb{N}$. For matrices A and $B \in \mathbb{R}^{n\times m}$, $(A)_{ij}$ denotes the entry of A located in the ith row and jth column. In addition, $A \gg B$ indicates that $(A-B)_{ij} \geqslant 0$, $(i,j) \in \mathcal{I}_N \times \mathcal{I}_M$. The relation $P \succ Q$ indicates that the matrix $P-Q$ is positive definite. The shorthands $\mathrm{He}\{X\} := X + X^{\mathrm{T}}$ and $XY* := XYX^{\mathrm{T}}$ are adopted, and the transposed element in the symmetric positions is denoted by $*$. $e_l \in \mathbb{R}^n$ denotes the lth standard unit vector. I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$.

2 Positive Model

Consider the following uncertain discrete-time LTI model:

$$\begin{cases} x_{j+1} = (A + \Delta A)x_j + Bu_j + B_w w_j \\ y_j = Cx_j \\ z_j = C_z x_j \end{cases} \tag{1}$$

where $x_j \in \mathbb{R}^n$ is the state, $u_j \in \mathbb{R}^m$ is the input, $w_j \in \mathbb{R}^l$ is the disturbance in l_∞, $y_j \in \mathbb{R}^p$ is the measurement output, and $z_j \in \mathbb{R}^q$ is the controlled output. ΔA is the time-varying uncertaint matrix.

Definition 1 (positivity) Suppose $u_j = 0$, $j \in \mathbb{Z}_{\geqslant 0}$ in (1). System (1) is positive for all $x_0 \gg 0$ and $w_j \gg 0$ if $x_j \gg 0$ and $y_j \gg 0$ for all $j \in \mathbb{Z}_{\geqslant 0}$.

Lemma 1 ([1]) *System (1) is positive if and only if $A + \Delta A \gg 0$, $B_w \gg 0$, and $C \gg 0$.*

Lemma 2 *Matrix $A \in \mathbb{R}^{n\times n}$ is positive if and only if*

$$\sum_{l=1}^{n} \mathrm{diag}\{e_l\} A \begin{bmatrix} 0 & 1 \\ I_{n-1} & 0 \end{bmatrix}^k \mathrm{diag}\{e_l\} \geqslant 0, \quad k \in \mathcal{I}_N. \tag{2}$$

Proof We define the operator that permutes the first column vector of A and the remaining $n-1$ vectors as

$$T(A) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n} := A \begin{bmatrix} 0 & 1 \\ I_{n-1} & 0 \end{bmatrix}.$$

The operator that eliminates the off-diagonal entries of A is defined as

$$U(A) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n} := \sum_{l=1}^{n} \mathrm{diag}\{e_l\} A (\mathrm{diag}\{e_l\}).$$

Thus, (2) is equivalent to

$$\begin{aligned} &U \circ \underbrace{T \circ \cdots \circ T}_{k}(A) \\ &= \mathrm{diag}\left\{(A)_{1(k+1)}, \dots, (A)_{(n-k)n}, (A)_{(n-k+1)1}, \dots, (A)_{nk}\right\} \\ &\geqslant 0 \end{aligned}$$

implying $(A)_{ij} \geqslant 0$ for all $(i,j) \in \mathcal{I}_N \times \mathcal{I}_N$. □

Assumption 1 *Matrices B_w and C are positive.*

Assumption 2 *Matrix B is full column rank.*

Assumption 3 *There exist known compatible constant matrices D and E and an unknown time-varying matrix Δ satisfying $\Delta^{\mathrm{T}}\Delta \leqslant I$ for all $j \in \mathbb{Z}_{\geqslant 0}$ such that $\Delta A = D\Delta E$.*

Remark 1 Through the appropriate column-row expansion of ΔA, one can construct Δ as a diagonal matrix with $\left|(\Delta)_{jj}\right| \leqslant 1$ and D and E as positive matrices without loss of generality. Then it follows that

$$-DE \ll D\Delta E.$$

Assumption 4 *Only y_j is available for feedback.*

Lemma 3 ([12]) *Given compatible matrices D, E, $S = S^{\mathrm{T}}$, with $\Delta \ni \Delta^{\mathrm{T}}\Delta \leqslant I$, there exists $\epsilon \in \mathbb{R}_{>0}$ such that*

$$S + \mathrm{He}\{D\Delta E\} \prec 0 \iff S + \epsilon^{-1}D^{\mathrm{T}}D + \epsilon EE^{\mathrm{T}} \prec 0.$$

3 Positive l_∞–l_∞ Disturbance Attenuation and Separation Principle

Considering Assumption 4, the following OBOF controller is adopted:

$$\begin{cases} \hat{x}_{j+1} = A\hat{x}_j + Bu_j + L(y_j - \hat{y}_j) + Hy_j \\ \hat{y}_j = C\hat{x}_j \\ u_j := K\hat{x}_j + Fy_j. \end{cases} \tag{3}$$

Define $e_j := x_j - \hat{x}_j$ and $\xi_j := (x_j, e_j)$. The closed-loop system is then constructed as

$$\xi_{j+1} = \begin{bmatrix} A + \Delta A + BK + BFC & -BK \\ \Delta A - HC & A - LC \end{bmatrix} \xi_j + \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_j. \tag{4}$$

Remark 2 The $\hat{x}_j$ dynamics in (3) has an additional correction through Hy_j. By using this term for leverage, one can effectively preserve the positivity of the lower-left block of the closed-loop system matrix in (4). Otherwise, there is no remedy to guarantee the positivity of the e_j dynamics in the presence of uncertainties. This aspect has not been addressed in existing studies, for example, [10]. Similarly, the controller equation in (3) has an additional compensation through Fy_j, which increases the possibility that the upper-left and upper-right blocks of the closed-loop system matrix in (4) are simultaneously positive.

We are interested in the following problem.

Problem 1 (l_∞–l_∞) Determine K, F, L, and H such that the uncertain LTI system (1) closed by the OBOF controller (3) is positive and robustly asymptotically stable against the norm-bounded parametric uncertainties when $w_j = 0$, $j \in \mathbb{Z}_{\geqslant 0}$, and for some attenuation level $\gamma \in \mathbb{R}_{>0}$, it exhibits the l_∞–l_∞-disturbance attenuation performance defined as

$$\|z_j\|_\infty \leqslant \gamma \|w_j\|_\infty \tag{5}$$

when $w_j \geqslant\!\!\geqslant 0$ and $w_j \in l_\infty$.

Remark 3 The separate design of the OBOF controller subject to (5) is challenging. Even if we design the OBOF controller in an integrated way, the synthesis conditions often result in nonconvex BMIs, rather than convex LMIs. The main reason for this is that x_j appears in the e_j dynamics coupled with ΔA.

To resolve the difficulty mentioned in Remark 3, we regard x_j as a disturbance in the e_j dynamics and define the following l_∞–l_∞ performance:

$$\|e_j\|_\infty \leqslant \mu_2 \|x_j\|_\infty + \gamma_2 \|w_j\|_\infty \tag{6}$$

where μ_2 and $\gamma_2 \in \mathbb{R}_{>0}$. Similarly, we consider e_j in the x_j dynamics as a disturbance and introduce the following l_∞–l_∞ performance:

$$\|z_j\|_\infty \leqslant \mu_1 \|e_j\|_\infty + \gamma_1 \|w_j\|_\infty \tag{7}$$

where μ_1 and $\gamma_1 \in \mathbb{R}_{>0}$.

Now, we present the separate design condition for Problem 1 based on (7) and (6).

Theorem 1 *Given $\gamma_1, \gamma_2, \epsilon_1, \epsilon_2, \lambda_1, \lambda_2, \mu_1$, and $\mu_2 \in \mathbb{R}_{>0}$, the uncertain LTI system (1) closed by the OBOF controller (3) exhibits the γ-disturbance attenuation performance (5) with asymptotic stability and is positive if there exist M, diagonal $P_1 = P_1^{\mathrm{T}} > 0$ and $P_2 = P_2^{\mathrm{T}} > 0$, and W, X, Y, and Z, in addition to ρ_1, ρ_2, ϕ_1, and $\phi_2 \in \mathbb{R}_{>0}$ such that*

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ (2,1) & -P_1 + \lambda_1 P_1 & * & * & * & * \\ -X^{\mathrm{T}}B^{\mathrm{T}} & 0 & -\phi_1 I & * & * & * \\ B_w^{\mathrm{T}}P_1 & 0 & 0 & -\rho_1 I & * & * \\ D^{\mathrm{T}}P_1 & 0 & 0 & 0 & -\epsilon_1 I & * \\ E & 0 & 0 & 0 & 0 & -\epsilon_1^{-1} I \end{bmatrix} < 0 \tag{8}$$

$$P_1 B - BM = 0 \tag{9}$$

$$\lambda_1 P_1 - \lambda_{\max}(C_z^{\mathrm{T}} C_z) I > 0 \tag{10}$$

$$\mu_1^2 - \phi_1 > 0 \tag{11}$$

$$\gamma_1^2 - \rho_1 > 0 \tag{12}$$

$$\sum_{l=1}^{n} \mathrm{diag}\{e_l\}(P_1 A - P_1 DE + BX + BYC) \begin{bmatrix} 0 & 1 \\ I_{n-1} & 0 \end{bmatrix}^k \times \mathrm{diag}\{e_l\} > 0 \tag{13}$$

$$\sum_{l=1}^{n} \mathrm{diag}\{e_l\} BX \begin{bmatrix} 0 & 1 \\ I_{n-1} & 0 \end{bmatrix}^k \mathrm{diag}\{e_l\} < 0, \quad k \in \mathcal{I}_N \tag{14}$$

$$\begin{bmatrix} \lambda_{\max}(C_z^{\mathrm{T}} C_z) & * \\ \phi_1 & \frac{1}{4} \end{bmatrix} > 0 \tag{15}$$

$$\begin{bmatrix} -P_2 & * & * & * & * & * \\ -C^{\mathrm{T}}W^{\mathrm{T}} & -\phi_2 I & * & * & * & * \\ A^{\mathrm{T}}P_2 - C^{\mathrm{T}}Z^{\mathrm{T}} & 0 & -P_2 + \lambda_2 P_2 & * & * & * \\ B_w^{\mathrm{T}}P_2 & 0 & 0 & -\rho_2 I & * & * \\ D^{\mathrm{T}}P_2 & 0 & 0 & 0 & -\epsilon_2 I & * \\ E & 0 & 0 & 0 & 0 & -\epsilon_2^{-1} I \end{bmatrix} < 0 \tag{16}$$

$$\lambda_2 P_2 - I > 0 \tag{17}$$

$$\mu_2^2 - \phi_2 > 0 \tag{18}$$

$$\gamma_2^2 - \rho_2 > 0 \tag{19}$$

$$\sum_{l=1}^{n} \mathrm{diag}\{e_l\}(P_2 DE + WC) \begin{bmatrix} 0 & 1 \\ I_{n-1} & 0 \end{bmatrix}^k \mathrm{diag}\{e_l\} < 0 \tag{20}$$

$$\sum_{l=1}^{n} \mathrm{diag}\{e_l\}(P_2 A - ZC) \begin{bmatrix} 0 & 1 \\ I_{n-1} & 0 \end{bmatrix}^k \mathrm{diag}\{e_l\} > 0, \quad k \in \mathcal{I}_N \tag{21}$$

$$\begin{bmatrix} \lambda_{\max}(C_z^{\mathrm{T}}C_z) & * \\ \phi_2 & \frac{1}{4} \end{bmatrix} \succ 0 \tag{22}$$

where $(2,1) := A^{\mathrm{T}}P_1 + X^{\mathrm{T}}B^{\mathrm{T}} + C^{\mathrm{T}}Y^{\mathrm{T}}B^{\mathrm{T}}$. *In this case, the gains are given by* $K = M^{-1}X$, $F = M^{-1}Y$, $L = P_2^{-1}Z$, *and* $H = P_2^{-1}W$.

Proof Let $V_1 := x_j^{\mathrm{T}}P_1x_j$ and $\Delta V_1 := x_{j+1}^{\mathrm{T}}P_1x_{j+1} - x_j^{\mathrm{T}}P_1x_j$. We then calculate as

$$\Delta V_1 = \begin{bmatrix} \xi_j \\ w_j \end{bmatrix}^{\mathrm{T}} \bar{\mathcal{M}} * -\lambda_1 V_1 + \phi_1 e_j^{\mathrm{T}} e_j + \rho_1 w_j^{\mathrm{T}} w_j$$

where

$$\bar{\mathcal{M}} := \begin{bmatrix} A + \Delta A + BK + BFC & -BK & B_w \end{bmatrix}^{\mathrm{T}} P_1 * \\ - \mathrm{diag}\{(1-\lambda_1)P_1, \phi_1 I, \rho_1 I\}.$$

By using the Schur complement, a congruence transformation, Assumption 3, Remark 1, and Lemma3, and denoting $MK =: X$ and $MF =: Y$, the following implication holds:

$$(8) \text{ and } (9) \implies \bar{\mathcal{M}} \prec 0. \tag{23}$$

Then, the comparison lemma leads to

$$\Delta V_1 + \lambda_1 V_1 - \phi_1 e_j^{\mathrm{T}} e_j - \rho_1 w_j^{\mathrm{T}} w_j < 0 \\ \implies V_1 \leqslant \frac{1-(1-\lambda_1)^j}{\lambda_1}\left(\phi_1 \|e_j\|_\infty^2 + \rho_1 \|w_j\|_\infty^2\right).$$

In addition, from (10)–(12), we can obtain the following inequality:

$$\begin{bmatrix} \xi_j \\ w_j \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \begin{bmatrix} \lambda_1 P_1 & * \\ 0 & (\mu_1^2 - \phi_1)I \end{bmatrix} & * \\ 0 & (\gamma_1^2 - \rho_1)I \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} C_z^{\mathrm{T}} \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} I * \right) * > 0 \\ \iff \lambda_1 V_1 + (\mu_1^2 - \phi_1) e_j^{\mathrm{T}} e_j + (\gamma_1^2 - \rho_1) w_j^{\mathrm{T}} w_j - z_j^{\mathrm{T}} z_j > 0$$

and arrive at

$$\lambda_1\left(\frac{1-(1-\lambda_1)^j}{\lambda_1}\right)\left(\phi_1\|e_j\|_\infty^2 + \rho_1\|w_j\|_\infty^2\right) \\ + (\mu_1^2 - \phi_1)\|e_j\|_\infty^2 + (\gamma_1^2 - \rho_1)\|w_j\|_\infty^2 > \|z_j\|^2 \\ \implies \mu_1^2\|e_j\|_\infty^2 + \gamma_1^2\|w_j\|_\infty^2 > \|z_j\|^2 \\ \implies \sqrt{(\mu_1\|e_j\|_\infty + \gamma_1\|w_j\|_\infty)^2 - 2\mu_1\gamma_1\|e_j\|_\infty\|w_j\|_\infty} \\ > \|z_j\|_\infty \\ \implies (7).$$

With Lyapunov function candidate $V_2 := e_j^{\mathrm{T}} P_2 e_j$ and

$$\Delta V_2 = \begin{bmatrix} \xi_j \\ w_j \end{bmatrix}^{\mathrm{T}} \bar{\mathcal{N}} * -\lambda_2 V_2 + \phi_2 x_j^{\mathrm{T}} x_j + \rho_2 w_j^{\mathrm{T}} w_j$$

along the e_j dynamics in (4), we can derive that:

$$(16) \implies \bar{\mathcal{N}} \prec 0 \tag{24}$$

where

$$\bar{\mathcal{N}} := \begin{bmatrix} \Delta A - HC & A - LC & B_w \end{bmatrix}^{\mathrm{T}} P_2 * \\ - \mathrm{diag}\{\phi_2 I, (1-\lambda_2)P_2, -\rho_2 I\}.$$

This implies

$$\Delta V_2 + \lambda_2 V_2 - \phi_2 x_j^{\mathrm{T}} x_j - \rho_2 w_j^{\mathrm{T}} w_j < 0 \\ \implies \lambda_2\left(\frac{1-(1-\lambda_2)^j}{\lambda_2}\right)\left(\phi_2\|x_j\|_\infty^2 + \rho_2\|w_j\|_\infty^2\right) \\ + (\mu_2^2 - \phi_2)\|x_j\|_\infty^2 + (\gamma_2^2 - \rho_2)\|w_j\|_\infty^2 > \|e_j\|^2 \\ \implies (6)$$

under (17)–(18), equivalently,

$$\lambda_2 V_2 + (\mu_2^2 - \phi_2) x_j^{\mathrm{T}} x_j + (\gamma_2^2 - \rho_2) w_j^{\mathrm{T}} w_j - e_j^{\mathrm{T}} e_j > 0.$$

Next, we demonstrate the closed-loop positivity. By Assumption 3, Remark 1, and Lemma 2,

$$(13) \implies P_1(A - DE + BK + BFC) \geqslant\!\!\geqslant 0 \\ \implies A + \Delta A + BK + BFC \geqslant\!\!\geqslant 0$$

because $P_1 > 0$ is diagonal. Similarly, (20) $\implies A - LC \geqslant\!\!\geqslant 0$, (14) $\implies -BK \geqslant\!\!\geqslant 0$, and (21) $\Delta A - HC \geqslant\!\!\geqslant 0$. According to Assumption 1, and Lemma 1, this guarantees that x_j in (4) is positive.

Finally, we prove the separation principle. To this end, we show that there exist γ, λ, and $\rho \in \mathbb{R}_{>0}$ for some ψ_1 and $\psi_2 \in \mathbb{R}_{>0}$ such that the following two inequalities for $V := \psi_1 V_1 + \psi_2 V_2$ along the closed-loop trajectory of (4) with K, F, L, and H designed above hold:

$$\Delta V + \lambda V - \rho w_j^{\mathrm{T}} w_j = \begin{bmatrix} \xi_j \\ w_j \end{bmatrix}^{\mathrm{T}} \bar{\mathcal{O}} \begin{bmatrix} \xi_j \\ w_j \end{bmatrix} < 0 \tag{25}$$

$$\lambda V + (\gamma^2 - \rho) w_j^{\mathrm{T}} w_j - z_j^{\mathrm{T}} z_j \\ = \begin{bmatrix} \xi_j \\ w_j \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{diag}\{\lambda\psi_1 P_1 - C_z^{\mathrm{T}} C_z, \lambda\psi_2 P_2\} & * \\ 0 & (\gamma^2 - \rho)I \end{bmatrix} * \\ > 0$$

where

$$\bar{\mathcal{O}} := \psi_1\bar{\mathcal{M}} + \psi_2\bar{\mathcal{N}} + \mathrm{diag}\{\psi_2\phi_2 I - \psi_1\lambda_1 P_1 + \lambda\psi_1 P_1, \\ \psi_1\phi_1 I - \psi_2\lambda_2 P_2 + \lambda\psi_2 P_2, (\psi_1\rho_1 + \psi_2\rho_2 - \rho)I\}.$$

Considering (23) and (24), it suffices to show that

$$\psi_2\phi_2 - \psi_1(\lambda_1 - \lambda)\lambda_{\min}(P_1) < 0$$
$$\psi_1\phi_1 - \psi_2(\lambda_2 - \lambda)\lambda_{\min}(P_2) < 0, \quad \psi_1\rho_1 + \psi_2\rho_2 - \rho < 0$$
$$\lambda\psi_1\lambda_{\min}(P_1) - \lambda_{\max}(C_z^{\mathrm{T}}C_z) > 0, \quad \gamma^2 - \rho > 0. \tag{26}$$

The first two inequalities in (26) hold if and only if there exist ψ_1, ψ_2, and $\lambda \in \mathbb{R}_{>0}$ such that

$$0 < \frac{\phi_1}{\lambda_{\min}(P_2)(\lambda_2 - \lambda)} < \frac{\psi_2}{\psi_1} < \frac{\lambda_{\min}(P_1)(\lambda_1 - \lambda)}{\phi_2} \tag{27}$$

and equivalently, there exists $\lambda \in \mathbb{R}_{[0,\lambda_1]}$ such that

$$f(\lambda) := \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 - \frac{\phi_1\phi_2}{\lambda_{\min}(P_1)\lambda_{\min}(P_2)} > 0$$

assuming $\lambda_1 < \lambda_2$ without loss of generality. It follows that

$$f(\lambda_1) = -\frac{\phi_1\phi_2}{\lambda_{\min}(P_1)\lambda_{\min}(P_2)} < 0.$$

In addition, according to (10) and (17)

$$\begin{aligned}
(15) \text{ and } (22) &\implies \lambda_{\max}(C_z^{\mathrm{T}}C_z) - 4\phi_1\phi_2 > 0 \\
&\iff \frac{\lambda_1\lambda_2\lambda_{\max}(C_z^{\mathrm{T}}C_z)}{\lambda_1\lambda_2} - 4\phi_1\phi_2 > 0 \\
&\implies \lambda_1\lambda_2\lambda_{\min}(P_1)\lambda_{\min}(P_2) - 4\phi_1\phi_2 > 0 \\
&\iff f(0) = \lambda_1\lambda_2 - \frac{4\phi_1\phi_2}{\lambda_{\min}(P_1)\lambda_{\min}(P_2)} > 0.
\end{aligned}$$

This implies that there exist a $\lambda \in (0, \lambda_1)$ and a set of all pairs of (ψ_1, ψ_2) that meet (27). Within this set, the pair $(\psi_1, \psi_2) \ni \psi_1 := \frac{\lambda_1}{\lambda}$ satisfies the fourth inequality in (26) through (10). It is simple to determine ρ and γ such that the third and fifth inequalities in (26) are true. When $w_j = 0$, $j \in \mathbb{Z}_{\geqslant 0}$, (25) can be expressed as $\Delta V + \lambda V < 0 \implies \Delta V < 0$. Therefore, (4) exhibits the γ -disturbance attenuation performance (5) and is robustly asymptotically stable. In (9), Assumption 2 implies M is full rank, and thus invertible. □

Remark 4 The compensation through Fy_j in the controller equation of (3) to relax the positivity constraint is designed using the convexification technique in [2]. This scheme also linearizes the bilinear term BKP_1 without Young's inequality [5], serving two ends.

Remark 5 The proposed design condition in Theorem 1 can be solved by using the semidefinite programming or by simply changing (9) into the following LMI

$$\begin{bmatrix} -\alpha I & * \\ P_1B - BM & -\alpha I \end{bmatrix} \prec 0$$

with a very small $\alpha \in \mathbb{R}_{>0}$, to use the LMI Solvers in MATLAB.

Remark 6 The proposed technique can be applied to real-world research areas such as economic cybernetics. Although the Leontief input–output model is quite efficient to this end, its dynamic behavior needs to imitate the real economic nature like the positivity [4].

4 An Example

A dynamic Leontief input–output model of a multisector economy is described as

$$\bar{x}_j = L\bar{x}_j + V(\bar{x}_{j+1} - \bar{x}_j) + \bar{u}_j \tag{28}$$

where $\bar{x}_j$ is the vector of the output levels and $\bar{u}_j$ is the final demand excluding investments. By slightly abusing notations, L and V denote the Leontief input–output matrix satisfying $(I - L)^{-1} \gg 0$ [4] and the capital coefficient matrix, respectively. Their numerical data

$$L = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.4 \end{bmatrix}, \quad V = \begin{bmatrix} 0.1 & 0.3 & 0.8 \\ 0.5 & 0.1 & 0.9 \\ 0.8 & 0.5 & 0.7 \end{bmatrix}$$

are borrowed from [13]. Let $A := I + V^{-1}(I - L)$ and $B := V^{-1}$. The minimum demand vector $r \gg 0$ is given. The corresponding minimum vector of the output levels is calculated by $\bar{x} = (I - A)^{-1}Br \gg 0$. We define $x_j := \bar{x}_j - \bar{x}$. Then, the Leontief input–output model in (28) is equivalently cast into the form of (1), which is absent from uncertainties and disturbances, which is required to be positive because $\bar{x}_j$ should be not less than $\bar{x}$.

Validation is performed through a comparison with the following conventional Luenberger OBOF controller:

$$\begin{cases} \hat{x}_{j+1} = A\hat{x}_j + Bu_j + L(y_j - \hat{y}_j) \\ \hat{y}_j = C\hat{x}_j \\ u_j := K\hat{x}_j \end{cases}$$

where

$$C := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The gain matrices, $K := XP_1^{-1}$ and $L := P_2^{-1}Z$ that asymptotically stabilize the equivalent Leontief input–output model are obtained as

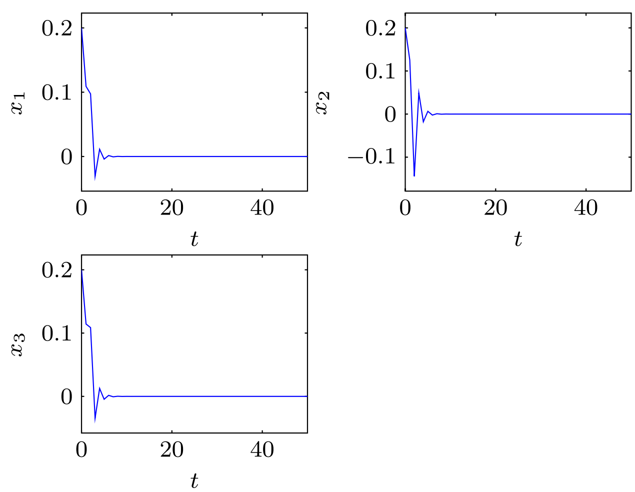

Fig. 1 Time response of a Leontief input–output model without uncertainties nor disturbances by the conventional Luenberger OBOF controller

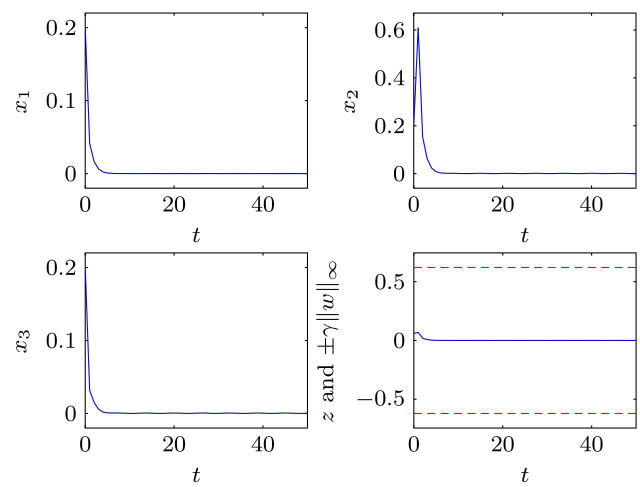

Fig. 2 Time response of a Leontief input–output model with uncertainties and disturbances

$$K = \begin{bmatrix} 0.5 & 0.2 & 0.7 \\ 0.4 & 0.6 & 0.7 \\ 0.7 & 0.3 & 1.3 \end{bmatrix}, \quad L = \begin{bmatrix} 0.011 & 0.465 \\ -0.512 & -0.107 \\ 0.245 & 0.278 \end{bmatrix}$$

by solving

$$\begin{bmatrix} -P_1 & * \\ AP + BX & -P_1 \end{bmatrix} \prec 0, \quad \begin{bmatrix} -P_2 & * \\ A^{\mathrm{T}}P_2 - C^{\mathrm{T}}Z^{\mathrm{T}} & -P_2 \end{bmatrix} \prec 0$$

where $P_1 \succ 0$ and $P_2 \succ 0$. Fig. 1 represents the simulation results with $x_0 = (0.2, 0.2, 0.2)$ and $\hat{x}_0 = (0.1, 0.1, 0.1)$. For $j \in [0, 5]$, the closed-loop state is not positive, even if neither uncertainties nor disturbances are implemented. This phenomenon occurs because the foregoing design condition does not guarantee the positivity of the closed-loop Leontief input–output model. In particular, the computed K and L fails in positifying $A + BK$, $-BK$, and $A - LC$, although they are designed separately.

To highlight the advantage of our method, we introduce

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.09\delta \\ 0 & 0 & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ 0.0015 \end{bmatrix}, \quad C_z = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}^{\mathrm{T}}$$

where $\delta \ni |\delta| \leqslant 1$ randomly varies in time. The l_∞ disturbance is defined as $w_j := 0.2\cos j + 0.2, j \in \mathbb{Z}_{\geqslant 0}$. To resolve the difficulty in the validation example, the additional feedback gain, F is introduced to increase the possibility that $A + \Delta A + BK + BFC$ and $-BK$ are positive. Moreover, the additional correction gain, H is introduced to ensure that $\Delta A - HC$ is positive. Let $\gamma_1 = \mu_1 = \gamma_2 = \mu_2 = 0.6$. According to Assumption 3, ΔA is factorized as

$$D = \begin{bmatrix} 0 & 0.3 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \Delta = \delta, \quad E = \begin{bmatrix} 0 & 0 & 0.3 \end{bmatrix}.$$

By solving Theorem 1, the controller gain matrices

$$K = \begin{bmatrix} 0.018 & 0.016 & 0.018 \\ 0.014 & 0.013 & 0.014 \\ 0.030 & 0.026 & 0.030 \end{bmatrix}, \quad F = \begin{bmatrix} 0.010 & 0.148 \\ 0.148 & 0.422 \\ -0.133 & 0.363 \end{bmatrix}$$

and the observer gain matrices

$$L = \begin{bmatrix} 0.035 & 0.114 \\ 0.820 & -1.698 \\ 0.300 & 0.209 \end{bmatrix}, \quad H = \begin{bmatrix} -0.096 & -0.010 \\ -0.002 & -0.003 \\ -0.000 & -0.000 \end{bmatrix}$$

are independently searched. The parameters $\gamma = 1.552$, $\lambda = 0.378$, $\rho = 2.410$, $\psi_1 = 2.118$, and $\psi_2 = 7.793$ with $\lambda_{\min}(P_1) = 1.369$ and $\lambda_{\min}(P_2) = 2.078$ satisfy (26), proving the closed-loop stability. This result indicates that the separation principle is established for Problem 1. The closed-loop time responses are shown in Fig. 2. Unlike in the results of the compared method, our results demonstrate that the state is positive and well guided to zero in the presence of parametric uncertainties. As shown in the lower-right subfigure, the proposed controller satisfies the l_∞–l_∞ disturbance attenuation performance in (5). The specific evaluated value is $\frac{\|z_j\|_\infty}{\|w_j\|_\infty} = 0.171 < \gamma\ (= 1.552)$.

5 Conclusions

We present a robust positive OBOF l_∞–l_∞ disturbance attenuation technique for uncertain LTI systems. The design condition is formulated as a convex optimization problem in terms of LMIs and an LME. Another contribution of this paper is that the separation principle for the concerned design problem is established. The numerical simulation demonstrates that the proposed methodology is successfully applied to the Leontief input–output model.

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Declarations

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Benzaouia A, Hmamed A, Hajjaji AE (2010) Stabilization of controlled positive discrete-time T-S fuzzy systems by state feedback control. Int J Adapt Control Signal Process 24(12):1091–1106. https://doi.org/10.1002/acs.1185
2. Crusius CAR, Trofino A (1999) Sufficient LMI conditions for output feedback control problems. IEEE Trans Autom Control 44(5):1053–1057. https://doi.org/10.1109/9.763227
3. Han M, Lam H, Li Y, Liu F, Zhang C (2019) Observer-based control of positive polynomial fuzzy systems with unknown time delay. Neuorocomputing 349:77–90. https://doi.org/10.1016/j.neucom.2019.04.016
4. Jódar L, Merello P (2010) Positive solutions of discrete dynamic Leontief input–output model with possibly singular capital matrix. Math Comput Modell 52(7–8):1081–1087. https://doi.org/10.1016/j.mcm.2010.02.043
5. Kheloufi H, Zemouche A, Bedouhene F, Boutayeb M (2013) On LMI conditions to design observer-based controllers for linear systems with parameter uncertainties. Automatica 49(12):3700–3704. https://doi.org/10.1016/j.automatica.2013.09.046
6. Lee HJ (2022) Robust static output-feedback vaccination policy design for an uncertain SIR epidemic model with disturbances: Positive Takagi–Sugeno model approach. Biomed Signal Process Control 72(103):273. https://doi.org/10.1016/j.bspc.2021.103273
7. Lee J, Moon JH, Jee SC, Lee HJ (2021) Robust $\mathcal{L}_\infty$–l_∞ sampled-data dynamic output-feedback control for uncertain linear time-invariant systems through descriptor redundancy. J Electr Eng & Technol 16(2):1051–1058. https://doi.org/10.1007/s42835-020-00603-6
8. Moon JH, Kang HB, Lee HJ (2020) Robust $\mathcal{H}_\infty$ and $\mathcal{L}_\infty$–$\mathcal{L}_\infty$ sampled-data fuzzy static output-feedback controllers in Takagi–Sugeno form for singularly perturbed nonlinear systems with parametric uncertainty. J Franklin Inst 357(13):8508–8528. https://doi.org/10.1016/j.jfranklin.2020.05.005
9. Nguyen CM, Pathirana PN, Trinh H (2018) Robust observer-based control designs for discrete nonlinear systems with disturbances. Eur J Control 44:65–72. https://doi.org/10.1016/j.ejcon.2018.09.002
10. Pang B, Zhang Q (2018) Stability analysis and observer-based controllers design for T–S fuzzy positive systems. Neurocomputing 275:1468–1477. https://doi.org/10.1016/j.neucom.2017.09.087
11. Peaucelle D, Ebihara Y (2014) LMI results for robust control design of observer-based controllers, the discrete-time case with polytopic uncertainties. In: IFAC Proceedings Volumes, pp. 6527–6532. Elsevier BV (2014). https://doi.org/10.3182/20140824-6-za-1003.00218
12. Xie L (1996) Output feedback H_∞ control of systems with parameter uncertainties. Int J Control 63(4):741–750. https://doi.org/10.1080/00207179608921866
13. Xue W, Li K (2014) Positive finite-time stabilization for discrete-time linear systems. J Dyn Syst Meas Contr 137(1):014–502. https://doi.org/10.1115/1.4028141
14. Yang W, Heng L, Hongbin Z (2013) Stability analysis of discrete-time fuzzy positive systems with time delays. J Intell Fuzzy Syst 25(4):893–905. https://doi.org/10.3233/ifs-120692
15. Zemouche A, Rajamani R, Kheloufi H, Bedouhene F (2016) Robust observer-based stabilization of Lipschitz nonlinear uncertain systems via LMIs - discussions and new design procedure. Int J Robust Nonlinear Control 27(11):1915–1939. https://doi.org/10.1002/rnc.3644

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