

Effect of instantaneous change of surface temperature and density on an unsteady liquid–vapour front in a porous medium

Zafar Hayat Khan^{1,2} (✉), Rashid Ahmad^{3,4}, Licheng Sun¹

1. State Key Laboratory of Hydraulics and Mountain River Engineering, College of Water Resource & Hydropower, Sichuan University, Chengdu 610065, China

2. Key Laboratory of Advanced Reactor Engineering and Safety, Ministry of Education, Tsinghua University, Beijing 100084, China

3. School of Mathematics and Physics, University of Queensland, St Lucia, Brisbane 4072, Queensland, Australia

4. Faculty of Engineering Sciences, GIK Institute of Engineering Sciences and Technology, Topi, Swabi, KPK, Pakistan

Abstract

This article presents a comprehensive analysis of time dependent condensation model embedded in a porous medium with variations in liquid–vapour densities. Both similarity and asymptotic solutions for the unsteady liquid–vapour phase change front are obtained with the manifestation of various pertinent parameters. The obtained results are compared which congregate well as depicted clearly in graphs. Results indicate that with different diffusivity and contrast ratios, the similarity front parameter is found to be gradually declining with variation in a density ratio. We have shown for the condensation process, the ratio of sensible to latent heat is independent of time and is equal to the half of the Stefan number of the liquid phase.

Keywords

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1 Introduction

Liquid–vapour phase change process is a well-known phenomenon which exemplifies the basic thermophysics and transport principles that motivate the mechanisms of condensation and vaporization processes (Carey, 2007). Phase change procedures in porous materials exhibit a wide range of thermodynamic configurations. These procedures are commonly influenced by three factors (i) by means of the orientation of heating and cooling the surfaces, (ii) via both the micro- and macroscale geometry of a porous material, and (iii) by interactions with convective and conductive procedures in nearby regions. Moving boundary value problems in porous media have practical significance in thermal energy storage, freezing of biological tissues, cooling of electronic equipment for food processing (Masur et al., 1989; Mortensen et al., 1989). The analysis of heat transfer is based on the two-phase model at the boundary between a homogeneous fluid and a porous medium as

portrayed by Ochoa-Tapia and Whitaker (1997). Torrance (1986) established that more or less of the thermodynamic structures are linked with the boiling and freezing of water in horizontal porous layers. Phase transition of a material is described by a particular kind of boundary value problem for partial differential equation, where phase boundary can move with time (Dutil et al., 2011). The existence of the solution of the boundary value problem was verified by Evans (1951). Afterwards, the uniqueness was proved by Douglas (1957). The aim of the Stefan problem was to pronounce the temperature dissemination in a homogeneous medium via a phase change, for instance ice passing to water is profound by solving the heat equation and imposing the initial temperature distribution on the entire medium as depicted by several authors: Bear and Buchlin (1981) briefly discussed the modelling and applications of transport phenomena in porous media. Beckett et al. (2001) formed the two-dimensional Stefan problem solution with the help of a moving mesh finite element technique for the

✉ zhkhan@scu.edu.cn

two-dimensional heat conduction problems with a phase change. Bodvarsson et al. (1986) deeply explained the modelling of geothermal systems. Bonacina et al. (1973) examined the approximate solution for the phase change problems and found that the approximate results were matching well with the analytical solutions. Date (1991) studied the enthalpy formulation for the Stefan problem with the help of finite difference scheme. Gupta (2003) analyzed the classical Stefan problem in its basic concepts and modelling and further portrayed that the Stefan problem in its simplest form is a macroscopic model for phase transition in a pure material that is carried out purely by heat conduction. Hager and Whitaker (2000) investigated the vapour–liquid jump conditions inside a porous medium. In their study, they established the mass and energy jump conditions for a vapour–liquid boundary inside a porous medium. Their analysis was restricted to a single fluid component which resulted the appropriate jump conditions for an evaporation front and they further established that local thermal equilibrium condition was expected to exist in the homogeneous liquid and vapour regions. Harris et al. (2001) examined the phase change phenomena in a porous medium for the non-local thermal equilibrium. An approximate two-temperature model was studied analytically and the conditions that guaranteed the existence of local thermal equilibrium were presented. Rubin and Schweitzer (1972) performed the heat transfer analysis in a porous media with phase change. The objective of their study was to investigate the relative significance of convection against conduction, and the main parameters influencing the temperature distribution and the interface position. They found the exact solutions for the steady state problem wherein the properties were constant. Khan and Pritchard (2013) investigated the stability and instability of the liquid–vapour front in a porous medium along a bifurcation curve under both the isothermal and isoflux boundary conditions. Khan (2014) studied the instability of the liquid–vapour front in a geothermal system with a cooling flux at the liquid boundary with small perturbation at the front. The mechanisms contributing to the stability and instability of such systems were investigated by incorporating a separate-phase model with a sharp interface between liquid and vapour. He further showed that advection was not vital for instability. More recently, Khan and Pritchard (2015) examined the instability of the liquid–vapour front in a geothermal system with isothermal boundaries. A two-dimensional linear stability analysis of the isothermal basic state established that the Rayleigh–Taylor mechanism was a dominant contributor to the instability.

All the aforementioned studies were based on the steady state flow in two-phase model without variations in density and so far, there is no such study that could be established on the density change between liquid (ρ_{liq}) and vapour

(ρ_{vap}) regions embedded in a porous medium with the unsteady flow. Two cases have been discussed: (i) when the densities of liquid and vapour phases remain the same, i.e., $\rho_{\text{liq}} = \rho_{\text{vap}}$ and (ii) when the densities of liquid and vapour regions are not the same, i.e., $\rho_{\text{liq}} \neq \rho_{\text{vap}}$, which further elaborate that although the liquid is static, there is flow in the vapour region. Typically, we assume that the density of liquid is greater than the vapour, i.e., $\rho_{\text{liq}} > \rho_{\text{vap}}$. Therefore, to cover the gap, the analysis has been accomplished for the unsteady form of two-phase model embedded in a porous layer with variations in densities of vapour and liquid regions. The system of partial differential equations (PDEs) of the two-dimensional heat conduction model has been converted into a system of ordinary differential equations (ODEs) with the help of similarity transformation. The collection of similarity solutions have been briefly specified by Bernoff and Witelski (2010), Chiarelli et al. (1994), Carslaw and Jaeger (1959), and Lunardini (1981). The layout of the rest of the paper has been demonstrated in the forthcoming sections.

2 Problem formulation

2.1 Instantaneous change of surface temperature

We consider an unsteady condensation problem. A material which exists in two phases (liquid and vapour) fills the half space $x \geq 0$ (see Fig. 1). For time $t \leq 0$, the material is in the vapour phase at a constant temperature $T_v > T_s$, where T_s is the phase change temperature. At time $t = 0$, the temperature of the surface $x = 0$ is instantaneously lowered and maintained at $T_l < T_s$. This will cause a layer of liquid to be formed adjacent to the surface $x = 0$ and as time increases this layer will expand into the vapour. We first assume that the densities are the same in the liquid and vapour regions, which will ensure no flow. In the forthcoming Section 2.2 we will assume that the densities no longer remain the same, i.e., $\rho_{\text{liq}} \neq \rho_{\text{vap}}$. This is more realistic, but also more complicated mathematically. The phase change temperature T_s is assumed to be constant.

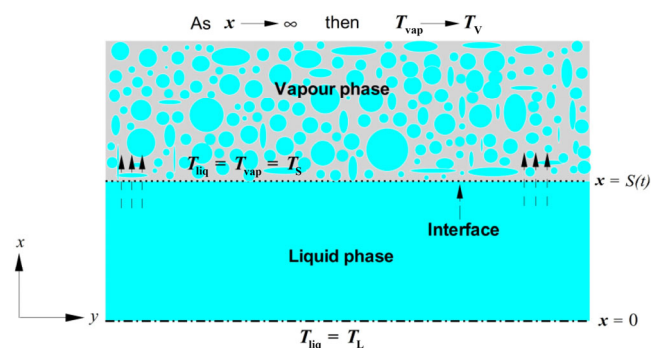


Fig. 1 Schematic diagram of the unsteady condensation problem.

Assuming that the heat transfer is only due to conduction, the problem can be described by a pair of Fourier heat conduction equations. According to the above assumptions, the energy equation and the corresponding boundary conditions for this model will take the following form:

$$\left\{ \begin{array}{l} (\rho c_p)_{m,\text{liq}} \frac{\partial T_{\text{liq}}}{\partial t} = k_{m,\text{liq}} \frac{\partial^2 T_{\text{liq}}}{\partial x^2}, \text{ for } t > 0, 0 < x < S(t), \\ T_{\text{liq}}(0) = T_L, T_{\text{liq}}(S) = T_S \\ (\rho c_p)_{m,\text{vap}} \frac{\partial T_{\text{vap}}}{\partial t} = k_{m,\text{vap}} \frac{\partial^2 T_{\text{vap}}}{\partial x^2}, \text{ for } t > 0, S(t) < x, \\ T_{\text{vap}}(S) = T_S \text{ as } x \rightarrow \infty \text{ then } T_{\text{vap}} = T_V \end{array} \right. \quad (2.1)$$

The heat flux condition is to be applied at the unknown position of the interface $S(t)$. Conservation of heat requires the latent heat of condensation be diffused away from the interface so that

$$\phi \lambda \rho_{\text{liq}} \frac{dS(t)}{dt} = k_{m,\text{liq}} \frac{\partial T_{\text{liq}}}{\partial x} \Big|_{x=S} - k_{m,\text{vap}} \frac{\partial T_{\text{vap}}}{\partial x} \Big|_{x=S} \quad (2.2)$$

The above heat flux condition is a special form of Stefan condition under the assumption that there is no flow of liquid, i.e., $u_{\text{liq}} = 0$.

2.1.1 Similarity solution

In a similarity solution, a similarity variable combining the space and time variables, is sought that transforms the governing partial differential equations into a set of ordinary differential equations with the similarity variable as the independent variable (Matteij et al., 2005). Let us introduce the dilation transformation:

$$m = \varepsilon^a x, n = \varepsilon^b t, Y(m, n) = \varepsilon^c T(\varepsilon^{-a} m, \varepsilon^{-b} n) \quad (2.3)$$

Using the transformation in Eq. (2.3), the heat equation becomes

$$\varepsilon^{b-c} \frac{\partial Y(m, n)}{\partial n} = \varepsilon^{2a-c} \alpha_{m,\text{liq,vap}} \frac{\partial^2 Y(m, n)}{\partial m^2} \quad (2.4)$$

Now if $b - c = 2a - c$ (i.e., $b = 2a$), then the heat equation Eq. (2.1) in both phases (liquid and vapour) is invariant under the dilation transformation Eq. (2.3), i.e., if $T(x, t)$ is the solution of the heat equation in the variables x and t , then for $m, n, Y(m, n)$ given by Eq. (2.3), $Y(m, n)$ solves the heat equation in the variables m and n . Note that

$$Y n^{-\frac{c}{b}} = (\varepsilon^c T)(\varepsilon^b t)^{-\frac{c}{b}} = T t^{-\frac{c}{b}}$$

and

$$\frac{m}{n^{\frac{a}{b}}} = \frac{\varepsilon^a x}{(\varepsilon^b t)^{\frac{a}{b}}} = \frac{x}{t^{\frac{a}{b}}}$$

So both groupings of variables are invariant under the transformation Eq. (2.3) for all choices of a, b, c . This suggests that we look for a solution for Eq. (2.1) that is of the form

$$T(x, t) = t^{\frac{c}{2a}} F(\eta) \text{ for } \eta = \frac{x}{t^{a/b}} = \frac{x}{\sqrt{t}} \text{ since } b = 2a \quad (2.5)$$

Now using the transformation Eq. (2.5), we have

$$\frac{\partial T_{\text{liq,vap}}}{\partial t} = t^{\frac{c}{2a}-1} \left[\frac{c}{2a} F_{\text{liq,vap}}(\eta) - \frac{\eta}{2} \frac{dF_{\text{liq,vap}}}{d\eta} \right] \quad (2.6)$$

and

$$\frac{\partial^2 T_{\text{liq,vap}}}{\partial x^2} = t^{\frac{c}{2a}-1} \frac{d^2 F_{\text{liq,vap}}(\eta)}{d\eta^2} \quad (2.7)$$

Substituting Eq. (2.6) and Eq. (2.7) into the heat equation Eq. (2.1), we have

$$t^{\frac{c}{2a}-1} \left[\alpha_{m,\text{liq,vap}} \frac{d^2 F_{\text{liq,vap}}(\eta)}{d\eta^2} + \frac{\eta}{2} \frac{dF_{\text{liq,vap}}(\eta)}{d\eta} - \frac{c}{2a} F_{\text{liq,vap}}(\eta) \right] = 0 \quad (2.8)$$

Now we will transform the boundary conditions using the transformation in Eq. (2.5), since $T_{\text{liq}}(0, t) = T_L$ then $T_{\text{liq}}(0, t) = t^{c/b} F_{\text{liq}}(0)$ and this can equal the constant T_L if and only if $c = 0$. The same is true for $T_{\text{liq,vap}}(S, t) = t^{c/b} F_{\text{liq,vap}}(S/\sqrt{t}) = T_S$ and finally, $T_{\text{vap}}(\infty, t) = t^{c/b} F_{\text{vap}}(\infty) = T_V$. In this case when $c = 0$ the problem for $T(x, t)$ reduces to

$$\left\{ \begin{array}{l} \frac{d^2 F_{\text{liq}}}{d\eta^2} + \frac{\eta}{2\alpha_{m,\text{liq}}} \frac{dF_{\text{liq}}}{d\eta} = 0 \\ F_{\text{liq}}(0) = T_L, F_{\text{liq,vap}}\left(\frac{S}{\sqrt{t}}\right) = T_S \\ \frac{d^2 F_{\text{vap}}}{d\eta^2} + \frac{\eta}{2\alpha_{m,\text{vap}}} \frac{dF_{\text{vap}}}{d\eta} = 0 \\ F_{\text{vap}} \rightarrow T_V \text{ as } \eta \rightarrow \infty \end{array} \right. \quad (2.9)$$

where α_m is the thermal diffusivity. The solution of Eq. (2.9) with the appropriate boundary conditions is

$$T_{\text{liq}}(x, t) = T_L - (T_L - T_S) \frac{\text{erf}\left(\frac{x}{2\sqrt{\alpha_{m,\text{liq}} t}}\right)}{\text{erf}(\beta)} \quad (2.10)$$

$$T_{\text{vap}}(x, t) = T_V + (T_S - T_V) \frac{\text{erfc}\left(\frac{x}{2\sqrt{\alpha_{m,\text{vap}} t}}\right)}{\text{erfc}(h_1 \beta)} \quad (2.11)$$

where $\beta = \frac{S(t)}{2\sqrt{\alpha_{m,\text{liq}} t}}$ and $h_1 = \sqrt{\frac{\alpha_{m,\text{liq}}}{\alpha_{m,\text{vap}}}}$. We seek a similarity solution in which the interface position is given by

$$S(t) = 2\beta\sqrt{\alpha_{m,liq}t} \tag{2.12}$$

Inserting Eq. (2.10), Eq. (2.11), and Eq. (2.12) into Eq. (2.2) gives

$$\varphi\sqrt{\pi}\beta H_{liq} = E_1 \left(\frac{\exp(-\beta^2)}{\text{erf}(\beta)} - \frac{\theta_0 h_1}{k} \frac{\exp(-\beta^2 h_1^2)}{\text{erfc}(\beta h_1)} \right) \tag{2.13}$$

where $E_1 = \frac{(\rho c_p)_{m,liq}}{(\rho c_p)_{liq}}$.

2.1.2 Interpretation of Stefan number

Solomon (1981) has shown that for a specific melting process the ratio of sensible to latent heat is independent of time and can be related to the Stefan number. We will now follow the same procedure while considering the condensation phase change problem. First we note that the latent heat stored at time t is

$$LH = \varphi\rho_{liq}\lambda S(t) = 2\varphi\beta\rho_{liq}\lambda\sqrt{\alpha_{m,liq}t} \tag{2.14}$$

The total heat (TH) removed from the system is the time integral of the surface heat flux which is

$$TH = k_{m,liq} \int_0^t \frac{dT_{liq}(0,t)}{dx} dt = \frac{2k_{m,liq}(T_s - T_L)\sqrt{t}}{\sqrt{\alpha_{m,liq}}\pi\text{erf}(\beta)} \tag{2.15}$$

We know that total heat is the sum of sensible heat and latent heat

$$TH = SH + LH \Rightarrow \frac{SH}{LH} = \frac{TH}{LH} - 1$$

Now from Eq. (2.14) and Eq. (2.15) we obtain

$$\frac{TH}{LH} = \frac{c_{p,liq}(T_s - T_L)(\rho c_p)_{m,liq}}{\varphi\sqrt{\pi}\lambda\beta\text{erf}(\beta)(\rho c_p)_{liq}} = \frac{E_1}{\varphi H_{liq}\sqrt{\pi}\beta\text{erf}(\beta)} \tag{2.16}$$

Finally, we have

$$\frac{SH}{LH} = \frac{TH}{LH} - 1 = \frac{E_1}{\varphi\sqrt{\pi}H_{liq}\beta\text{erf}(\beta)} - 1 \tag{2.17}$$

It is clear from Eq. (2.17) that the ratio SH/LH is independent of time. If the temperature at the phase change front and in the vapour phase are the same, i.e., $T_s = T_v$ then the ratio of the temperature contrast Θ_0 ($\Theta_0 = (T_v - T_s)/(T_s - T_L)$) becomes zero. In this case, Eq. (2.13) and Eq. (2.17) yield

$$\frac{SH}{LH} = \exp(\beta^2) - 1 \tag{2.18}$$

In the limit in which H_{liq} is large (small Stefan number), the root β is very small, so Eq. (2.13) yields

$$\beta \simeq \sqrt{\frac{E_1}{2\varphi H_{liq}}} \tag{2.19}$$

The Taylor series expansion of the exponential function gives

$$\exp(\beta^2) \simeq 1 + \beta^2 \simeq 1 + \frac{E_1}{2\varphi H_{liq}} \tag{2.20}$$

Finally, inserting Eq. (2.20) into Eq. (2.18), we obtain

$$\frac{SH}{LH} = \frac{E_1}{2\varphi H_{liq}} \tag{2.21}$$

where $E_1 = \frac{(\rho c_p)_{m,liq}}{(\rho c_p)_{liq}}$, φ is the porosity, and H_{liq} is the reciprocal of the Stefan number (for the liquid phase), $Ste_{liq} = \frac{c_{p,liq}(T_s - T_L)}{\lambda}$. If we assume that $\frac{E_1}{\varphi} = 1$ then Eq. (2.21) yields

$$\frac{SH}{LH} = \frac{Ste_{liq}}{2} \tag{2.22}$$

Equation (2.22) shows that for this particular consideration problem, the Stefan number is approximately twice the ratio of SH/LH, which is also independent of time.

2.2 Effects of density change

This problem is an extension of the problem considered in Section 2.1, by considering advective heat transfer in the vapour region only, and a density difference between the two phases (liquid and vapour). The density difference means that although the liquid is static there is a flow in the vapour region. Usually the density of the liquid is greater than the vapour, i.e., $\rho_{liq} > \rho_{vap}$. To illustrate the effects of the density change, we consider the one-dimensional condensation problem illustrated in Fig. 1. In light of the above assumptions the governing equations along with the boundary conditions are

$$\left\{ \begin{array}{l} (\rho c_p)_{m,liq} \frac{\partial T_{liq}}{\partial t} = k_{m,liq} \frac{\partial^2 T_{liq}}{\partial x^2}, \text{ for } t > 0, 0 < x < S(t), \\ T_{liq}(0) = T_L, T_{liq}(S) = T_s \\ (\rho c_p)_{m,vap} \frac{\partial T_{vap}}{\partial t} + (\rho c_p)_{vap} u_{vap} \frac{\partial T_{vap}}{\partial t} = k_{m,vap} \frac{\partial^2 T_{vap}}{\partial x^2}, \text{ for } t > 0, S(t) < x, \\ T_{vap}(S) = T_s \text{ as } x \rightarrow \infty \text{ then } T_{vap} = T_v \end{array} \right. \tag{2.23}$$

In the absence of any driving force, the velocity of the vapour u_{vap} has to be determined by the mass balance at the interface (Chiareli et al., 1994). Mass conservation across

the liquid–vapour interface $S(t)$ yields

$$u_{\text{vap}} = \varphi \left(1 - \frac{1}{R_1} \right) \frac{dS(t)}{dt}, \text{ where } R_1 = \frac{\rho_{\text{vap}}}{\rho_{\text{liq}}} \quad (2.24)$$

There is also an energy flux condition to be applied at the unknown position of the liquid–vapour interface. The heat jump condition at $S(t)$ is

$$\lambda \rho_{\text{vap}} \left(\varphi \frac{dS(t)}{dt} - u_{\text{vap}} \right) = k_{m,\text{liq}} \frac{\partial T_{\text{liq}}}{\partial x} \Big|_{x=S} - k_{m,\text{vap}} \frac{\partial T_{\text{vap}}}{\partial x} \Big|_{x=S} \quad (2.25)$$

2.2.1 Similarity solution

Adopting the same procedure discussed in Section 2.1.1, the solutions for both the liquid and vapour regions are obtained as

$$F_{\text{liq}}(\eta) = T_L - (T_L - T_S) \frac{\text{erf}(h_2 \eta)}{\text{erf}(h_2 \beta_2)}, \quad \eta < \beta_2 \quad (2.26)$$

$$F_{\text{vap}}(\eta) = T_V + (T_S - T_V) \frac{\text{erfc} \left[\frac{\varphi \beta_2 \left(\frac{1}{R_1} - 1 \right) + \eta}{E_2} \right]}{\text{erfc} \left\{ \beta_2 \left[1 - \frac{\varphi}{E_2} \left(1 - \frac{1}{R_1} \right) \right] \right\}}, \quad \eta > \beta_2$$

where $h_2 = \sqrt{\frac{\alpha_{m,\text{vap}}}{\alpha_{m,\text{liq}}}}$, $E_2 = \frac{(\rho c_p)_{m,\text{vap}}}{(\rho c_p)_{\text{vap}}}$, and the liquid–vapour

front has the position $S(t) = 2\beta_2 \sqrt{\alpha_{m,\text{vap}} t}$. The constant β_2 has to be determined from the following equation.

$$\sqrt{\pi} \varphi \beta_2 H_{\text{liq}} = \frac{E_1}{h_2} \left(\frac{\exp(-h_2^2 \beta_2^2)}{\text{erf}(h_2 \beta_2)} - \frac{\Theta_0}{k} \frac{1}{h_2} \frac{\exp \left\{ -\beta_2^2 \left[1 - \frac{\varphi}{E_2} \left(1 - \frac{1}{R_1} \right) \right]^2 \right\}}{\text{erfc} \left\{ \beta_2 \left[1 - \frac{\varphi}{E_2} \left(1 - \frac{1}{R_1} \right) \right] \right\}} \right) \quad (2.27)$$

Equation (2.27) demonstrates the most general eigenvalue relationship for β_2 in transcendental form. A special case of this result has appeared previously: if $R_1 = 1$, which implies that the densities of the two phases are the same, then Eq. (2.27) reduces to Eq. (2.13). A typical temperature distribution in the liquid–vapour regions with $\alpha_{m,\text{liq}} = 0.12$ and $\alpha_{m,\text{vap}} = 2.4$ is shown in Fig. 2. Pure conduction takes place in the liquid region, thus the temperature profile in the liquid region is nearly linear. In the vapour region the exponential behaviour of the temperature shows convection dominating over conduction. The two regions are separated at $\beta_2 = 0.24$ at interface temperature $T_S = 10$. The interface is moving right into the vapour region with time t , which indicates that condensation is taking place.

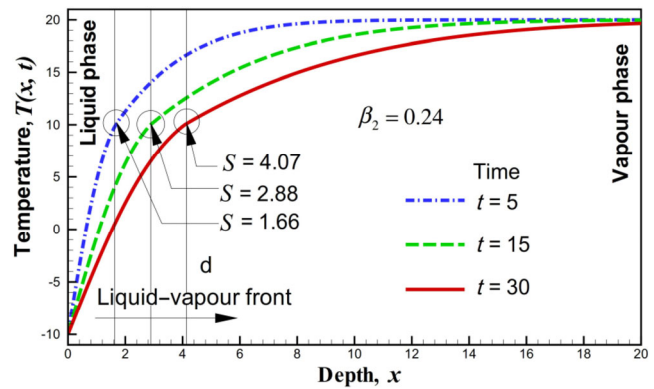


Fig. 2 Similarity solution of temperature profile with $R_1 = 0.1$, $H_{\text{liq}} = 1$, $k = 1$, $h_2 = 4.5$, $E_1 = 1$, $E_2 = 1$, $\Theta_0 = 0.5$, and $\varphi = 0.38$.

2.2.2 Asymptotic solution

The error functions and exponential terms in the transcendental Eq. (2.27) have the following asymptotic expansions

$$\begin{aligned} \text{erf}(h_2 \beta_2) &= \frac{2}{\sqrt{\pi}} \left[h_2 \beta_2 - \frac{(h_2 \beta_2)^2}{3} + \frac{(h_2 \beta_2)^5}{10} + O(h_2 \beta_2)^6 \right] \\ \frac{e^{-(h_2 \beta_2)^2}}{\text{erf}(h_2 \beta_2)} &= \frac{\sqrt{\pi}}{2h_2 \beta_2} \left[1 + \frac{(h_2 \beta_2)^2}{3} + \frac{(h_2 \beta_2)^4}{90} + O(h_2 \beta_2)^2 \right] \\ &\quad \times \left[1 - (h_2 \beta_2)^2 + \frac{(h_2 \beta_2)^4}{2!} + O(h_2 \beta_2)^6 \right] \\ &= \frac{\sqrt{\pi}}{2h_2 \beta_2} \left[1 - \frac{2}{3}(h_2 \beta_2)^2 + \frac{4}{15}(h_2 \beta_2)^4 + O(h_2 \beta_2)^6 \right] \end{aligned} \quad (2.28)$$

also

$$\text{erfc} \left\{ \beta_2 \left[1 - \frac{\varphi}{E_2} \left(1 - \frac{1}{R_1} \right) \right] \right\} = 1 - \frac{2}{\sqrt{\pi}} \left[\beta_2 h_3 - \frac{1}{3} (\beta_2 h_3)^3 + \frac{1}{10} (\beta_2 h_3)^5 + O(\beta_2 h_3)^7 \right]$$

and

$$\begin{aligned} \frac{e^{-(\beta_2 h_3)^2}}{\text{erfc}(\beta_2 h_3)} &= \left[1 - (\beta_2 h_3)^2 + \frac{1}{2} (\beta_2 h_3)^4 + O(\beta_2 h_3)^6 \right] \\ &\quad \times \left[1 + \frac{2}{\sqrt{\pi}} (\beta_2 h_3) + \frac{4}{\pi} (\beta_2 h_3)^2 - \frac{2(\pi+2)}{3\pi\sqrt{\pi}} (\beta_2 h_3)^3 + O(\beta_2 h_3)^4 \right] \\ &= 1 + \frac{2}{\sqrt{\pi}} (\beta_2 h_3) + \left(\frac{4}{\pi} - 1 \right) (\beta_2 h_3)^2 + O(\beta_2 h_3)^3 \end{aligned} \quad (2.29)$$

Substitution of Eq. (2.28) and Eq. (2.29) into Eq. (2.27)

gives

$$\begin{aligned} \sqrt{\pi}\varphi\beta_2 H_{\text{liq}} = & \frac{E_1}{\hbar_2^2} \left\{ \frac{\sqrt{\pi}}{2\beta_2} \left[1 - \frac{2}{3}(h_2\beta_2)^2 + \frac{4}{15}(\hbar_2\beta_2)^4 \right. \right. \\ & \left. \left. + O(\hbar_2\beta_2)^6 \right] - \frac{\Theta_0}{k} \left[1 + \frac{2}{\sqrt{\pi}}(\beta_2\hbar_3) \right. \right. \\ & \left. \left. + \left(\frac{4}{\pi} - 1 \right) (\beta_2\hbar_3)^2 + O(\beta_2\hbar_3)^3 \right] \right\} \end{aligned} \quad (2.30)$$

If the Stefan number ($Ste_{\text{liq}} = 1/H_{\text{liq}}$) for the liquid phase is close to zero, then the root β_2 is very small, so ignoring the higher order terms in Eq. (2.30), we get

$$\sqrt{\pi}\varphi\beta_2 H_{\text{liq}} = \frac{E_1}{\hbar_2^2} \left(\frac{\sqrt{\pi}}{2\beta_2} - \frac{\Theta_0}{k} \left[1 + \frac{2\beta_2}{\sqrt{\pi}} \left[1 - \frac{\varphi}{E_2} \left(1 - \frac{1}{R_1} \right) \right] \right] \right) \quad (2.31)$$

Figures 3(a) and 3(b) compare the numerical values of β_2 with those given by the asymptotic expansion in Eq. (2.30). In the liquid phase the error is $O(\beta_2 h_2)^6$; if the diffusivity ratio of the two phases $h_2 > 0.5$ then the error due to the solution in the liquid phase is reduced (see Fig. 3(a)). Figure 3(b)

shows that an error of magnitude $O\left\{ \beta_2 \left[1 - \frac{\varphi}{E_2} \left(1 - \frac{1}{R_1} \right) \right] \right\}^2$ in the vapour phase is an insignificant error in the interface energy balance.

In Fig. 4(a) and Fig. 4(b), the similarity front parameter β_2 is plotted against the diffusivity ratio h_2 and temperature contrast Θ_0 , for various values of the density ratio R_1 . The results for the special case (same density ratio, $R_1 = 1$ in the transcendental Eq. (2.13)) also presented in Fig. 4. Figures 4(a) and 4(b) show that as either the diffusivity ratio h_2 or the temperature contrast ratio Θ_0 increases, the similarity front parameter β_2 decreases and thus the liquid vapour front moves more slowly.

3 Conclusions

The focus of the current analysis is to investigate the unsteady two-phase flow problem in a porous medium. Two cases are examined firstly, when the densities of liquid and vapour phases remain the same, i.e., $\rho_{\text{liq}} = \rho_{\text{vap}}$ and secondly, when the densities of liquid and vapour regions are not the same, i.e., $\rho_{\text{liq}} \neq \rho_{\text{vap}}$. Typically, we consider that the density of

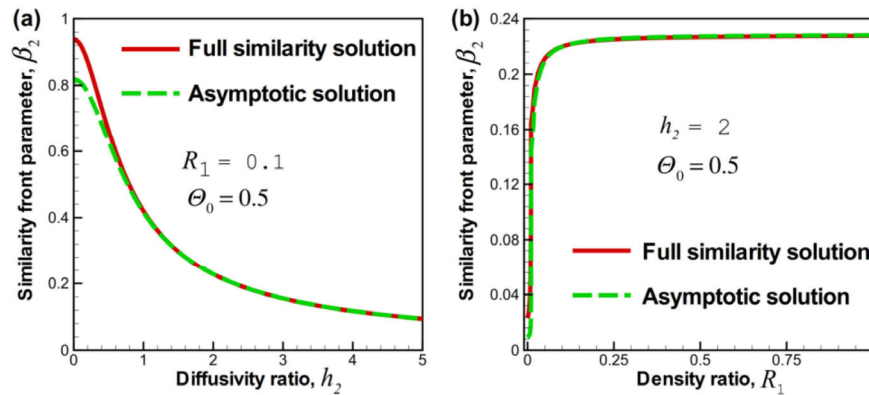


Fig. 3 Similarity front parameter β_2 as a function of the diffusivity ratio h_2 (a) and the density ratio R_1 (b), where $H_{\text{liq}} = 5$, $k = 4$, $E_1 = 1$, $E_2 = 1$, $\Theta_0 = 0.5$, and $\varphi = 0.38$.

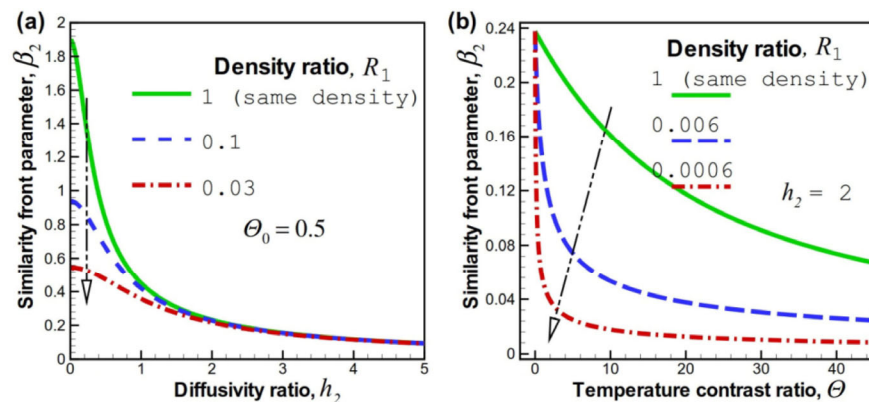


Fig. 4 Similarity front parameter β_2 as a function of the diffusivity ratio h_2 (a) and the temperature contrast ratio Θ_0 (b), where $H_{\text{liq}} = 5$, $k = 4$, $E_1 = 1$, $E_2 = 1$, $\Theta_0 = 0.5$, and $\varphi = 0.38$.

liquid is greater than the vapour, i.e., $\rho_{\text{liq}} > \rho_{\text{vap}}$. Such analysis has not been investigated for the unsteady form of two-phase model embedded in a porous layer with variations in densities of vapour and liquid regions. The system of PDEs of the two-dimensional heat conduction model has been converted into a system of ODEs with the help of similarity transformation. The similarity solution assumes that when time $t = 0$, the porous layer is filled with one phase (vapour). As the process starts, heat transport (conduction and advection in the vapour phase) takes place and the vapour temperature decreases with time, the liquid–vapour interface starts to exist. The interface moves forward into the vapour phase with time, which indicates that condensation takes place. Both the asymptotic and similarity solutions agree well.

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