Mechanics of Soft Materials (2024) 6:6
https://doi.org/10.1007/s42558-024-00061-4

ORIGINAL PAPER

Response to Shariff's comments to my paper on his isotropic invariants (Shariff, 2023)

Mikhail Itskov[1,2]

Received: 10 June 2024 / Accepted: 23 June 2024

Abstract

In the paper by Shariff (Q. J. Mech. Appl. Math. **76**, 143–161, 2023) a functional basis of vectors and symmetric tensors based on eigenvectors of the first tensor is proposed. In the paper by Itskov (Mech. Soft Mater. **6**(4), 1–3, 2024) a counterexample is reported demonstrating that the proposed terms do not generally represent isotropic invariants and cannot thus serve as a functional basis of symmetric tensors. In the latest response (Mech. Soft Mater. **6**, 4, 2024) Shariff shows many examples of symmetric tensors for which his functional basis is valid but he does not really oppose the counterexample.

Keywords Functional basis · Symmetric tensors · Isotropic invariants · Eigenvectors · Eigenvalues

In [5] Shariff proposed a functional basis of vectors and symmetric second-order tensors. The basis consists of eigenvalues λ_i of the first symmetric tensor $\mathbf{A}_1$ and components of the given vectors $a_i^{(s)}$ and tensors $A_{ij}^{(r)}$ $(i, j = 1, 2, 3)$ related to the unit eigenvectors $\boldsymbol{v}_i$ $(i = 1, 2, 3)$ of $\mathbf{A}_1$. The proposed (so called spectral) basis should be valid for all symmetric tensors. Indeed, in [5] no restrictions are imposed to the definition domain of the functional basis nor the tensors should be placed in a special order.

The issue of the tensor reordering could play a role since eigenvectors of the first symmetric tensor serve as a basis for all other ones and the corresponding components $A_{ij}^{(r)}$ $(i, j = 1, 2, 3)$ are claimed to represent isotropic invariants. [4] just focuses on the problem where the first symmetric tensor has multiple eigenvalues so that the associated eigenvectors are not uniquely defined. For a counterexample presented in [4] the proposed functional basis is not valid.

In Ch. 1 (Preliminary) [6] Shariff first introduces his functional basis and recalls the classical one for the case of two symmetric second-order tensors $\mathbf{A}_1$ and $\mathbf{A}_2$. In this case, the classical basis consists of ten invariants (see, e.g. [1–3]). Further, he shows that these classical invariants can explicitly be expressed in terms of Shariff's basis. This, however, does not prove that Shariff's terms represent isotropic functions of two symmetric tensors. An inverse expression would do that! Indeed, let us assume that there are unique functions f_{ij} $(i, j = 1, 2, 3)$ of the classical isotropic invariants $I_k(\mathbf{A}_1, \mathbf{A}_2)$ $(k = 1, 2 \ldots, 10)$ such that

$$A_{ij}^{(2)}(\mathbf{A}_1, \mathbf{A}_2) = f_{ij}\big(I_1(\mathbf{A}_1, \mathbf{A}_2), \ldots, I_{10}(\mathbf{A}_1, \mathbf{A}_2)\big), \quad i, j = 1, 2, 3. \tag{1}$$

✉ Mikhail Itskov
itskov@km.rwth-aachen.de

1 Department of Continuum Mechanics, RWTH Aachen University, Eilfschornsteinstr. 18, 52062 Aachen, Germany

2 School of Computer Science & Mathematics, Keele University, ST5 5BG Staffordshire, UK

By the very definition, for any isotropic invariant I_k $(k = 1, 2 \ldots, 10)$

$$I_k\left(\mathbf{QA}_1\mathbf{Q}^{\mathrm{T}}, \mathbf{QA}_2\mathbf{Q}^{\mathrm{T}}\right) = I_k\left(\mathbf{A}_1, \mathbf{A}_2\right), \quad \forall \mathbf{Q} \in Orth^3. \tag{2}$$

Applying this condition to Eq. (1) would yield

$$\begin{aligned} A_{ij}^{(2)}\left(\mathbf{QA}_1\mathbf{Q}^{\mathrm{T}}, \mathbf{QA}_2\mathbf{Q}^{\mathrm{T}}\right) &= f_{ij}\left(I_1(\mathbf{QA}_1\mathbf{Q}^{\mathrm{T}}, \mathbf{QA}_2\mathbf{Q}^{\mathrm{T}}), \ldots, I_{10}(\mathbf{QA}_1\mathbf{Q}^{\mathrm{T}}, \mathbf{QA}_2\mathbf{Q}^{\mathrm{T}})\right) \\ &= f_{ij}\left(I_1(\mathbf{A}_1, \mathbf{A}_2), \ldots, I_{10}(\mathbf{A}_1, \mathbf{A}_2)\right) = A_{ij}^{(2)}(\mathbf{A}_1, \mathbf{A}_2) \end{aligned} \tag{3}$$

for $\forall \mathbf{Q} \in Orth^3$. As shown in [4] $A_{ij}^{(r)}$ do not represent isotropic invariant. For this reason, we claim that such functions f_{ij} Eq. (1) do not exist! They were also not presented neither in [5] nor in the latest comments by Shariff [6]. He only claims that such expressions can be obtained according to [7]. However, this claim is based in the latter paper on the proposition that the terms $A_{ij}^{(r)}$ $(i, j = 1, 2, 3)$ represent isotropic invariants which we proved to be wrong. Functions f_{ij} $(i, j = 1, 2, 3)$ satisfying (1) are not presented neither in [7].

In Ch. 2 (Examples) Shariff considers some examples of isotropic invariants of symmetric tensors and vectors. Many of these examples (Example 1, 2 and 3) are well-known (see e.g. [2]). Example 4 proves isotropy of $A_{ij}^{(2)}$ $(i, j = 1, 2, 3)$ as functions of $\mathbf{A}_2$ and its orthonormal basis vectors $\boldsymbol{v}_i$ (!). Indeed, $\boldsymbol{v}_i$ $(i = 1, 2, 3)$ are explicitly included into the list of arguments of the isotropic function. In contrast, in the counterexample mentioned above only two symmetric tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are considered as arguments. Thus, this example by Shariff is not relevant for the present case and cannot oppose the counterexample in [4].

In Example 5 Shariff further shows that his terms $A_{ij}^{(2)}$ $(i, j = 1, 2, 3)$ do change when the arguments tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are subject to an orthogonal transformation as $\mathbf{QA}_1\mathbf{Q}^{\mathrm{T}}$ and $\mathbf{QA}_2\mathbf{Q}^{\mathrm{T}}$ but the basis vectors $\boldsymbol{v}_i$ $(i = 1, 2, 3)$ remain unchanged. In the following Example 6 this (evident) statement is illustrated by a particular $\mathbf{Q}$ from the counterexample in [4] and a particular $\mathbf{A}_2$ of a diagonal form with respect to the eigenvectors of $\mathbf{A}_1$. In the further remark 2.7 [6] Shariff discusses the case $\mathbf{A}_1 = \mathbf{I}$ considered in the counterexample. Any vector represents an eigenvector of the identity tensor $\mathbf{I}$ so that $\boldsymbol{v}_i$ $(i = 1, 2, 3)$ can be chosen arbitrarily. However, Shariff claims that these eigenvectors should always be subjected to the orthogonal transformation as $\mathbf{Q}\boldsymbol{v}_i$ in order to keep his terms $A_{ij}^{(2)}$ $(i, j = 1, 2, 3)$ unchanged. This is however wrong since $\boldsymbol{v}_i$ $(i = 1, 2, 3)$ do not belong to the arguments of an isotropic function as e.g. in Example 4. Moreover, in the original definition of the spectral basis by Shariff [5] there are no restrictions or other requirements imposed on the choice of the orthonormal eigenvectors $\boldsymbol{v}_i$ $(i = 1, 2, 3)$. Finally, almost the last paragraph of this remark directly contradicts the result of eq. (30) in [6].

In Ch. 3 (Itskov's counterexample) Shariff further considers the case $\mathbf{A}_1 = \mathbf{I}$ used for the counterexample in [4]. Due to the fact that $\mathbf{A}_1' = \mathbf{QA}_1\mathbf{Q}^{\mathrm{T}} = \mathbf{A}_1,\ \forall \mathbf{Q} \in Orth^3$ and the eigenvectors of $\mathbf{I}$ are not unique as mentioned above, they can be the same for $\mathbf{A}_1$ and $\mathbf{A}_1'$. In this case, the natural choice of the eigenvectors is the orthonormal basis with respect to which components of the tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined in the counterexample. For this reason, the criticism by Shariff of the corresponding statement in [2] appears unreasonable. Further, Shariff shows once more that his components $A_{ij}^{(2)}$ $(i, j = 1, 2, 3)$ of $\mathbf{A}_2$ do remain unchanged only when they are related to the rotated basis $\mathbf{Q}\boldsymbol{v}_i$ $(i = 1, 2, 3)$. This means that these components $A_{ij}^{(2)}$ $(i, j = 1, 2, 3)$ do not represent isotropic invariants especially in the case when eigenvectors of $\mathbf{A}_1$ are not unique as also mentioned above.

Ch. 4 is entitled "Proof by contradiction". In this case, one makes an assumption converse to a mathematical statement to be proved and shows that it leads to a contradiction. Instead, Shariff presents one more example of symmetric tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ for which his basis is valid. Specifically, $\mathbf{A}_1 = \mathbf{I}$ while $\mathbf{A}_2$ is of a diagonal form. Since the basis by Shariff [5] should be valid for all symmetric tensors, his statement can be rephrased as "all sheep in the herd are white". The counterexample in [4] demonstrates that in this herd there exists at least one black sheep. A proof by contradiction would show that an existence of a black sheep leads to a self-contradicting result. Instead, in Ch. 4 Shariff presents another although very nice but still dazzling white sheep, which, of course, does not prove that there are no black ones in the herd. In the latest version of his comments [6] Shariff argues by the well-known Pythagoras theorem. Since no counterexamples to it have been found so far, this should also be the case for the functional basis by Shariff. Although in the herd of my famous neighbor there are no black sheep, there can be some in mine.

In the revised version of the comments [6] an additional (second) proof "by contradiction" is presented. In this proof, an arbitrary basis $\boldsymbol{g}_i$ $(i = 1, 2, 3)$ of $\mathbf{A}_2$ is assumed. This basis is again rotated by $\mathbf{Q}$ which is only valid when vectors $\boldsymbol{g}_i$

represent arguments of the isotropic function. However, in the proposed counterexample [4] an isotropic function of only two symmetric tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ is considered.

In Ch. 6 Shariff finally considers the thought experiment of two students in [4]. Accordingly, these students calculate independent of each other the left and right hand side of Eq. (2) for the terms $A_{ij}^{(2)}$ $(i, j = 1, 2, 3)$ by Shariff [5] and generally obtain different results, which once more confirms that these terms do not represent isotropic invariants. Shariff argues that the students would also obtain different results if they calculate different functional bases, which appears as a manipulation. For example, for one symmetric tensor a set of its eigenvalues, principal invariants and principal traces represents each a functional basis. However, the eigenvalues generally differ from the principal invariants or the principal traces, which has nothing to do with their isotropic properties.

Further, Shariff requires that the second student should use the rotated basis $\mathbf{Q}\boldsymbol{v}_i$ $(i = 1, 2, 3)$. In this case, the terms $A_{ij}^{(2)}$ $(i, j = 1, 2, 3)$ calculated by both students will be the same. However, it is possible only accidentally because the student number 2 does not know the tensor $\mathbf{A}_2$ given to the student number 1 and cannot thus calculate $\mathbf{Q}$. Such situation can be crucial when $A_{ij}^{(2)}$ $(i, j = 1, 2, 3)$ are used for material modeling for example as arguments of a strain energy function (as considered by Shariff in Ch. 5 Remark [6]). Indeed, let us imagine two laboratories studying an unknown material and using $A_{ij}^{(2)}$, $(i, j = 1, 2, 3)$ to calculate its strain energy or stresses. If these laboratories use different coordinate systems (material frames) to calculate $A_{ij}^{(2)}$ $(i, j = 1, 2, 3)$, they will most probably obtain different values of the strain energy or stresses in one and the same material and under one the same strain, which is unphysical.

In the last paragraph "Important" added to the revised version [6] Shariff finally accepts that his terms $A_{ij}^{(2)}$ $(i, j = 1, 2, 3)$ "have different values for different coordinate systems (material frames)", which means that they do not represents isotropic invariants.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availability There are no external data supporting the findings of this study.

Declarations

Conflicts of interest The author declares that he has no conflict of interest.

References

1. Boehler, J.P.: On irreducible representations for isotropic scalar functions. J. Appl. Math. Mech. **57**, 323–327 (1977)
2. Itskov, M.: Tensor Algebra and Tensor Analysis for Eng., 5th edn. Springer-Verlag, Berlin (2019)
3. Pennisi, S., Trovato, M.: On the irreducibility of Professor G.F. Smith?s representations for isotropic functions, Int. J. Eng. Sci. 25, 1059–1065 (1987)
4. Itskov, M.: On the functional basis of isotropic vector and tensor functions by Shariff (2023). Mech. Soft Mater. **6**(4), 1–3 (2024)
5. Shariff, M.H.B.M.: On the Smallest Number of Functions Representing Isotropic Functions of Scalars, Vectors and Tensors. Q. J. Mech. Appl. Math. **76**, 143–161 (2023)
6. Shariff, M.H.B.M.: On Itskov (2024) counterexample to the functional basis of isotropic vector and tensor functions by Shariff (2023). Mech. Soft Mater. **6**(4), (2024)
7. Shariff, M.H.B.M.: On the number of independent invariants for m unit vectors and n symmetric second order tensors. Engng. Letters **29**(2), 509–515 (2021)