Mechanics of Soft Materials (2024) 6:4
https://doi.org/10.1007/s42558-024-00059-y

ORIGINAL PAPER

On Itskov (2024) counterexample to the functional basis of isotropic vector and tensor functions by Shariff (2023)

M.H.B.M. Shariff[1]

Received: 15 March 2024 / Accepted: 18 April 2024

Abstract
In the paper Itskov (Mechanics of Soft Materials 6:1, 20242), he gave a counterexample to show that elements of the functional basis by Shariff (Q. J. Mech. Appl. Math. **76**, 143–161, 2023) do not represent isotropic invariants of the vector and tensor arguments and cannot thus be referred to as the functional basis. In this paper, we prove that his counterexample is incorrect.

1 Preliminary

Definition: A functional basis of a vector and tensor system is a set of their isotropic invariants such that every invariant can uniquely be expressed in terms the basis. Since Itskov's [1] paper only deals with two symmetric tensors, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, any scalar-valued isotropic function (invariant) of these tensors should satisfy the condition

$$f(\boldsymbol{A}_1, \boldsymbol{A}_2) = f(\boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T), \quad \forall \boldsymbol{Q} \in Orth^3, \tag{1}$$

where $Orth^3$ denotes a group of all orthogonal transformations within the three-dimensional Euclidean space and $\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$ (the identity tensor). See also the Appendix.

The "classical" functional basis for tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ contains, for example, the isotropic invariants [2]

$$\mathrm{tr}\,\boldsymbol{A}_1, \mathrm{tr}\,\boldsymbol{A}_1^2, \mathrm{tr}\,\boldsymbol{A}_1^3, \mathrm{tr}\,\boldsymbol{A}_2, \mathrm{tr}\,\boldsymbol{A}_2^2, \mathrm{tr}\,\boldsymbol{A}_2^3, \mathrm{tr}\,(\boldsymbol{A}_1\boldsymbol{A}_2), \mathrm{tr}\,(\boldsymbol{A}_1^2\boldsymbol{A}_2), \mathrm{tr}\,(\boldsymbol{A}_1\boldsymbol{A}_2^2), \mathrm{tr}\,(\boldsymbol{A}_1^2, \boldsymbol{A}_2^2), \tag{2}$$

where tr denotes the trace of a tensor.

The spectral functional basis of these two tensors [7] contains the isotropic invariants

$$\lambda_i, \quad A_{ij}^{(2)} = \mathrm{tr}[\boldsymbol{A}_2(\boldsymbol{v}_i \otimes \boldsymbol{v}_j] = \boldsymbol{v}_i \cdot \boldsymbol{A}_2\boldsymbol{v}_j, \tag{3}$$

where λ_i and $\boldsymbol{v}_i$ are, respectively, the eigenvalues and orthonormal eigenvectors of $\boldsymbol{A}_1$, i.e.,

$$\boldsymbol{A}_1 = \sum_{i=1} \lambda_i \boldsymbol{v}_i \otimes \boldsymbol{v}_i \tag{4}$$

✉ M.H.B.M. Shariff
mohd.shariff@ku.ac.ae

1 Department of Applied Mathematics and Science Khalifa University of Science, Technology and Research, Abu Dhabi, United Arab Emirates

Published online: 11 May 2024

and we can express

$$\boldsymbol{A}_2 = \sum_{i,j=1}^{3} A_{ij}^{(2)} \boldsymbol{v}_i \otimes \boldsymbol{v}_j \,. \tag{5}$$

It is shown in Shariff [7] all "classical" invariants such as those given in Eq. 2 can be explicitly expressed in terms of Shariff [7] spectral invariants. For example, the classical invariant

$$\mathrm{tr}\, \boldsymbol{A}_2^2 = \sum_{i,j=1}^{3} A_{ij}^{(2)} A_{ji}^{(2)} \,, \quad \mathrm{tr}\, \boldsymbol{A}_1 \boldsymbol{A}_2 = \sum_{i=1}^{3} \lambda_i A_{ii}^{(2)} \,, \tag{6}$$

Hence, λ_i, $A_{ij}^{(2)}$ are spectral elements of a functional basis for the tensor set $\{\boldsymbol{A}_1, \boldsymbol{A}_2\}$.

In Shariff [6], he showed that spectral invariants (such as λ_i, $A_{ij}^{(2)}$) can be expressed in terms of "classical invariants" (such as those given in Eq. 2).

Itskov [1] claimed, via a counterexample, that the scalars $A_{ij}^{(2)}$ are not isotropic invariants. In this paper, we debunk Itskov's claim, but before we invalidate Itskov's [1] counterexample, a few simple basic examples are given below to facilitate our discussion and for clarity.

2 Examples

2.1 Example 1

Let $\boldsymbol{B}$ and $\boldsymbol{C}$ be tensors. The scalar function

$$f(\boldsymbol{B}, \boldsymbol{C}) = \mathrm{tr}(\boldsymbol{B}\boldsymbol{C}) \tag{7}$$

is a scalar-valued isotropic function since

$$f(\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T) = \mathrm{tr}\, \boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^T \boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T = \mathrm{tr}\, \boldsymbol{Q}\boldsymbol{B}\boldsymbol{C}\boldsymbol{Q}^T = \mathrm{tr}\, \boldsymbol{B}\boldsymbol{C}\boldsymbol{Q}^T\boldsymbol{Q} = \mathrm{tr}\, \boldsymbol{B}\boldsymbol{C} = f(\boldsymbol{B}, \boldsymbol{C}) \,, \quad \forall \boldsymbol{Q} \in Orth^3 \,.$$

2.2 Example 2

Let $\boldsymbol{B}$ be a symmetric tensor and $\boldsymbol{C} = \boldsymbol{a} \otimes \boldsymbol{b}$, where $\otimes$ is the dyadic product, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are **arbitrary** vectors. Consider the function

$$f(\boldsymbol{B}, \boldsymbol{a}, \boldsymbol{b}) = \mathrm{tr}[\boldsymbol{B}(\boldsymbol{a} \otimes \boldsymbol{b})] = \boldsymbol{a} \cdot \boldsymbol{B}\boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{B}\boldsymbol{a} \,. \tag{8}$$

It is clear $f(\boldsymbol{B}, \boldsymbol{a}, \boldsymbol{b})$ is a scalar-valued isotropic function since

$$f(\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{a}, \boldsymbol{Q}\boldsymbol{b}) = \boldsymbol{Q}\boldsymbol{a} \cdot \boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{B}\boldsymbol{b} = f(\boldsymbol{B}, \boldsymbol{a}, \boldsymbol{b}) \,, \quad \forall \boldsymbol{Q} \in Orth^3 \,. \tag{9}$$

2.3 Example 3

In view of Eq. 4, the eigenvalues

$$\lambda_i(\boldsymbol{A}_1) = f(\boldsymbol{A}_1, \boldsymbol{v}_i) = \mathrm{tr}[\boldsymbol{A}_1(\boldsymbol{v}_i \otimes \boldsymbol{v}_i)] = \boldsymbol{v}_i \cdot \boldsymbol{A}_1\boldsymbol{v}_i \,, \quad i \text{ not summed} \,. \tag{10}$$

The eigenvalues are isotropic function since

$$f(\boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{v}_i) = \boldsymbol{Q}\boldsymbol{v}_i \cdot \boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{v}_i = \boldsymbol{v}_i \cdot \boldsymbol{A}_1\boldsymbol{v}_i = f(\boldsymbol{A}_1, \boldsymbol{v}_i) \,, \quad \forall \boldsymbol{Q} \in Orth^3 \,. \tag{11}$$

Equation 11 is true even if the eigenvectors of $\boldsymbol{A}_1$ are not unique.

2.4 Example 4

In view of Eq. 3_2, we have

$$A_{ij}^{(2)} = f(\boldsymbol{A}_2, \boldsymbol{v}_i) = \mathrm{tr}[\boldsymbol{A}_2(\boldsymbol{v}_i \otimes \boldsymbol{v}_j)] = \boldsymbol{v}_i \cdot \boldsymbol{A}_2 \boldsymbol{v}_j \,. \tag{12}$$

$A_{ij}^{(2)}$ are isotropic invariants, since

$$f(\boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{v}_i) = \boldsymbol{Q}\boldsymbol{v}_i \cdot \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{v}_j = \boldsymbol{v}_i \cdot \boldsymbol{A}_2\boldsymbol{v}_j = f(\boldsymbol{A}_2, \boldsymbol{v}_i)\,, \quad \forall \boldsymbol{Q} \in Orth^3 \,. \tag{13}$$

Equation 13 is true even if the eigenvectors of $\boldsymbol{A}_1$ are not unique. This example **clearly proves** that $A_{ij}^{(2)}$ are isotropic invariants; hence, no counterexample, including Itskov's counterexample, can prove that they are not isotropic invariants. For example, Pythagoras theorem has been proven to be correct; no counterexample can prove that the theorem is incorrect. This clearly contradicts Itskov's [1] claim that $A_{ij}^{(2)}$ are not isotropic invariants.

2.4.1 Remark

We emphasize that in Examples 3 and 4, λ_i and the scalar $A_{ij}^{(2)}$ are traces of two tensors, i.e.

$$\lambda_i = \mathrm{tr}[\boldsymbol{A}_1(\boldsymbol{v}_i \otimes \boldsymbol{v}_i)]\,, \quad A_{ij}^{(2)} = \mathrm{tr}[\boldsymbol{A}_2(\boldsymbol{v}_i \otimes \boldsymbol{v}_j)]\,. \tag{14}$$

The fact that traces of tensors are isotropic invariants simply proves that λ_i and the scalar $A_{ij}^{(2)}$ are isotropic invariants according to the definition (1).

2.5 Example 5

Consider the orthogonally transformed tensors

$$\begin{aligned}
\boldsymbol{A}'_1 &= \boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Q}^T = \boldsymbol{Q}(\sum_{i=1}^{3} \lambda_i \boldsymbol{v}_i \otimes \boldsymbol{v}_i)\boldsymbol{Q}^T = \sum_{i=1}^{3} \lambda_i \boldsymbol{Q}\boldsymbol{v}_i \otimes \boldsymbol{Q}\boldsymbol{v}_i \,, \\
\boldsymbol{A}'_2 &= \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T = \boldsymbol{Q}(\sum_{i,j=1}^{3} A_{ij}^{(2)} \boldsymbol{v}_i \otimes \boldsymbol{v}_j)\boldsymbol{Q}^T = \sum_{i,j=1}^{3} A_{ij}^{(2)} \boldsymbol{Q}\boldsymbol{v}_i \otimes \boldsymbol{Q}\boldsymbol{v}_j \,, \quad \forall \boldsymbol{Q} \in Orth^3 \,.
\end{aligned} \tag{15}$$

Note that $\boldsymbol{A}_1$ in Eq. 4 and $\boldsymbol{A}_2$ in Eq. 5 are both expressed using the basis $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ (reference basis), and $\boldsymbol{A}'_1$ and $\boldsymbol{A}'_2$ in Eq. 15 are expressed using the basis $S' = \{\boldsymbol{Q}\boldsymbol{v}_1, \boldsymbol{Q}\boldsymbol{v}_2, \boldsymbol{Q}\boldsymbol{v}_3\}$ (rotated basis). It is clear from above (and we strongly emphasize) that the S-invariant components λ_i and $A_{ij}^{(2)}$ of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, respectively, are the same as the S'-invariant components of $\boldsymbol{A}'_1$ and $\boldsymbol{A}'_2$, respectively. Furthermore, we can express

$$\boldsymbol{A}'_2 = \sum_{i,j=1}^{3} A'^{(2)}_{ij} \boldsymbol{Q}\boldsymbol{v}_i \otimes \boldsymbol{Q}\boldsymbol{v}_j = \sum_{i,j=1}^{3} \bar{A}'^{(2)}_{ij} \boldsymbol{v}_i \otimes \boldsymbol{v}_j \,, \tag{16}$$

where

$$A'^{(2)}_{ij} = \boldsymbol{Q}\boldsymbol{v}_i \cdot \boldsymbol{A}'_2\boldsymbol{Q}\boldsymbol{v}_j = \boldsymbol{Q}\boldsymbol{v}_i \cdot (\boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T)\boldsymbol{Q}\boldsymbol{v}_j = \boldsymbol{v}_i \cdot \boldsymbol{A}_2\boldsymbol{v}_j = A_{ij}^{(2)} \tag{17}$$

and

$$\bar{A}'^{(2)}_{ij} = \boldsymbol{v}_i \cdot \boldsymbol{A}'_2\boldsymbol{v}_j = \boldsymbol{v}_i \cdot \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T\boldsymbol{v}_j \neq A_{ij}^{(2)} \,. \tag{18}$$

It is obvious in Eq. 18 that the S components $\bar{A}'^{(2)}_{ij}$ of $\boldsymbol{A}'_2$ are not isotropic functions and their values change with respect to different values of $\boldsymbol{Q}$; this fact is important because Itskov's counterexample uses the reference basis S (instead of the rotated basis S') to describe the transformed tensor $\boldsymbol{A}'_2$ and hence obtained the tensor components $\bar{A}'^{(2)}_{ij}$ that are not isotropic functions.

2.6 Example 6

Consider the special case

$$\boldsymbol{A}_2 = \sum_{i=1}^{3} \bar{\lambda}_i \, \boldsymbol{v}_i \otimes \boldsymbol{v}_i = \bar{\lambda}_1 \, \boldsymbol{v}_1 \otimes \boldsymbol{v}_1 + \bar{\lambda}_2 \, \boldsymbol{v}_2 \otimes \boldsymbol{v}_2 + \bar{\lambda}_3 \, \boldsymbol{v}_3 \otimes \boldsymbol{v}_3 \, . \tag{19}$$

We then have

$$\boldsymbol{A}'_2 = \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T = \sum_{k=1}^{3} \bar{\lambda}_k \, \boldsymbol{Q}\boldsymbol{v}_k \otimes \boldsymbol{Q}\boldsymbol{v}_k \, . \tag{20}$$

It is clear from Eqs. 19 and 20 that the S-invariant components $\bar{\lambda}_i$ of $\boldsymbol{A}_2$ and the S'-invariant components of $\boldsymbol{A}'_2$ are the same. This is as expected, since $\bar{\lambda}_i$ are eigenvalues of $\boldsymbol{A}_2$, they are isotropic functions and their values should not change. However if we express $\boldsymbol{A}'_2$ in using the reference basis S, we obtain

$$\boldsymbol{A}'_2 = \sum_{i,j=1}^{3} (\boldsymbol{v}_i \cdot \boldsymbol{A}'_2\boldsymbol{v}_j) \, \boldsymbol{v}_i \otimes \boldsymbol{v}_j \, , \tag{21}$$

where on using Eq. 20

$$\boldsymbol{v}_i \cdot \boldsymbol{A}'_2\boldsymbol{v}_j = \sum_{k=1}^{3} \bar{\lambda}_k (\boldsymbol{v}_i \cdot \boldsymbol{Q}\boldsymbol{v}_k)(\boldsymbol{v}_j \cdot \boldsymbol{Q}\boldsymbol{v}_k) \, . \tag{22}$$

It is clear from Eq. 22 that the tensor components $\boldsymbol{v}_i \cdot \boldsymbol{A}'_2\boldsymbol{v}_j$ are not isotropic functions. For example, consider the special case for $\boldsymbol{Q}$ given in Itskov's [1] Eq. 7, we have

$$\boldsymbol{Q}\boldsymbol{v}_1 = -\boldsymbol{v}_2 \, , \quad \boldsymbol{Q}\boldsymbol{v}_2 = \boldsymbol{v}_1 \, , \quad \boldsymbol{Q}\boldsymbol{v}_3 = \boldsymbol{v}_3 \, , \tag{23}$$

we have from Eqs. 22 and 23

$$\boldsymbol{A}'_2 = \bar{\lambda}_2 \, \boldsymbol{v}_1 \otimes \boldsymbol{v}_1 + \bar{\lambda}_1 \, \boldsymbol{v}_2 \otimes \boldsymbol{v}_2 + \bar{\lambda}_3 \, \boldsymbol{v}_3 \otimes \boldsymbol{v}_3 \, , \tag{24}$$

where the tensor components of $\boldsymbol{A}_2$ in Eq. 19 are not the same as those of $\boldsymbol{A}'_2$ in Eq. 24.

2.7 Important remark

To avoid confusion, especially when two or more eigenvalues have the same value, we write

$$\boldsymbol{A}_1(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = \lambda_1\boldsymbol{v}_1 \otimes \boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 + \lambda_3\boldsymbol{v}_3 \otimes \boldsymbol{v}_3 \, , \tag{25}$$

with the following symmetrical properties,

$$\boldsymbol{A}_1(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = \boldsymbol{A}_1(\lambda_2, \lambda_1, \lambda_3, \boldsymbol{v}_2, \boldsymbol{v}_1, \boldsymbol{v}_3) = \boldsymbol{A}_1(\lambda_3, \lambda_2, \lambda_1, \boldsymbol{v}_3, \boldsymbol{v}_2, \boldsymbol{v}_1) = \text{etc} \, . \tag{26}$$

and

$$QA_1Q^T = \lambda_1 Qv_1 \otimes Qv_1 + \lambda_2 Qv_1 \otimes Qv_2 + \lambda_3 Qv_3 \otimes Qv_3 = A_1(\lambda_1, \lambda_2, \lambda_3, Qv_1, Qv_2, Qv_3)\,. \tag{27}$$

For example, the identity tensor, where the eigenvalues have the same value,

$$I = A_1(\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, v_1, v_2, v_3) \tag{28}$$

and

$$QIQ^T = A_1(\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, Qv_1, Qv_2, Qv_3)\,. \tag{29}$$

Consider the case when Qv_i take the values given in Eq. 23. We then have

$$QIQ^T = A_1(\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, -v_2, v_1, v_3) = A_1(\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, v_1, v_2, v_3)\,. \tag{30}$$

In Itskov's counterexample, discussed below, he used the incorrect relation $A_1(\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, v_1, v_2, v_3)$ for QIQ^T and hence, used the incorrect eigenvectors v_i to evaluate the components of QA_2Q^T instead of the correct eigenvectors Qv_i.

We are now in position to address Itskov's [1] counterexample.

3 Itskov's counterexample

In Itskov's paper, [1] he uses

$$A_1 = I\,, \tag{31}$$

In view of the repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1$, the eigenvectors of A_1 are not unique. A_1 can be represented by **infinitely many different** bases associated with different orthonormal eigenvectors, i.e.,

$$A_1 = I = \sum_{i=1}^{3} v_i \otimes v_i = \sum_{i=1}^{3} \bar{v}_i \otimes \bar{v}_i = \sum_{i=1}^{3} \hat{v}_i \otimes \hat{v}_i = \text{etc.}\,, \tag{32}$$

where the bases $\{v_1, v_2, v_3\}$, $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$, $\{\hat{v}_1, \hat{v}_2, \hat{v}_3\}$ etc. are different.

Consider, for example, the case $\bar{v}_i = Qv_i$. For $Q \neq I$, some of the $\bar{v}_i \neq v_i$ and the bases $S = \{v_1, v_2, v_3\}$ and $S' = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ are **different**. In Itskov's paper, he lets $v_i = e_i$ and Q is given in his Eq. 7. We then obtain

$$\bar{v}_1 = -e_2\,, \quad \bar{v}_2 = e_1\,, \quad \bar{v}_3 = e_3\,, \tag{33}$$

which gives the basis $S' = \{-e_2, e_1, e_3\}$ and it is not the same as the basis $S = \{e_1, e_2, e_3\}$. Itskov expresses

$$A_1 = I = \sum_{i=1}^{3} e_i \otimes e_i\,. \tag{34}$$

We then have

$$A_1' = QA_1Q^T = \sum_{i=1}^{3} Qe_i \otimes Qe_i = \sum_{i=1}^{3} \bar{v}_i \otimes \bar{v}_i = A_1(\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, -e_2, e_1, e_3)$$

$$= A_1(\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, e_1, e_2, e_3) \tag{35}$$

(see Section 2.7).

In view of Eq. 32, we have $\boldsymbol{A}'_1 = \boldsymbol{A}_1$. However, it is important to note that although $\boldsymbol{A}'_1 = \boldsymbol{A}_1$, **their bases are not unique**. This contradicts Itskov's statement ST:
"*Due to the fact that $\boldsymbol{A}'_1 = \boldsymbol{A}_1$ the eigenvectors (5) remain unchanged*."

Hence, Itskov's statement ST is false and he contradicts his own statement in the last paragraph of his paper stating that "*Alternatively, the students can use arbitrary orthonormal bases since every of them represents a set of eigenvectors of the identity tensor $\boldsymbol{A}_1$*."

Alternatively, without using basis, we can show that $\boldsymbol{A}'_1 = \boldsymbol{A}_1$, via the relations $\boldsymbol{A}'_1 = \boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Q}^T = \boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T = \boldsymbol{I} = \boldsymbol{A}_1$. In Itskov's paper,

$$\boldsymbol{A}_2 = \sum_{i=1}^{3} A^{(2)}_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j\,, \quad A^{(2)}_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{A}_2\boldsymbol{e}_j\,. \tag{36}$$

For $\boldsymbol{A}'_2$, we have

$$\boldsymbol{A}'_2 = \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T = \sum_{i=1}^{3} A^{(2)}_{ij}\bar{\boldsymbol{v}}_i \otimes \bar{\boldsymbol{v}}_j\,. \tag{37}$$

It is clear from Eqs. 36_2 and 37 that $A^{(2)}_{ij}$ remain unchanged under the orthogonal transformation given in Eq. 7 of Itskov's paper. In fact Section 2.4 proves that the values of $A^{(2)}_{ij}$ remain unchanged under **any** orthogonal transformation, provided that $\boldsymbol{A}'_2$ is described by the basis $S' = \{\boldsymbol{Q}\boldsymbol{v}_1, \boldsymbol{Q}\boldsymbol{v}_2, \boldsymbol{Q}\boldsymbol{v}_3\}$. However, we strongly make a point that Itskov incorrectly uses the basis S (instead of correctly using the basis S' for $\boldsymbol{A}'_1$—see Section 2.7) to represent $\boldsymbol{A}'_2$, i.e.,

$$\boldsymbol{A}'_2 = \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T = \sum_{i=1}^{3} B_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j\,, \quad B_{ij} = \boldsymbol{e}_i \cdot (\boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T)\boldsymbol{e}_j\,. \tag{38}$$

The **matrix given in the author's Eq.** 8 is B_{ij} (there is a typo in Itskov's Eq. 8, $\boldsymbol{A}'_2 = \boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Q}^T$ should be $\boldsymbol{A}'_2 = \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T$). Note that the "tensor" components $A^{(2)}_{ij}$ are isotropic invariants, and it is clear from Eq. 38_2 that B_{ij} are not isotropic invariants, $B_{ij} \neq A^{(2)}_{ij}$ and, clearly, the B_{ij} values depend on the values of $\boldsymbol{Q}$, as exemplified in Itskov's Eq. 8.

4 Proof by contradiction

4.1 Proof 1

In this section, we use

$$\boldsymbol{A}_1 = \sum_{i=1}^{3} \boldsymbol{e}_i \otimes \boldsymbol{e}_i\,, \quad \boldsymbol{A}_2 = \sum_{i=1}^{3} \bar{\lambda}_i\boldsymbol{e}_i \otimes \boldsymbol{e}_i\,, \tag{39}$$

instead of $\boldsymbol{A}_2$ given in Itskov's Eq. 4 to invalidate his counterexample by proving (via contradiction) that if Itskov's counterexample is correct then the eigenvalues $\bar{\lambda}_i$ are **not** isotropic functions. For simplicity, we let the eigenvalues $\bar{\lambda}_i$ to take the values

$$\bar{\lambda}_1 = 1\,, \quad \bar{\lambda}_2 = 2\,, \quad \bar{\lambda}_3 = 3 \tag{40}$$

We note that

$$\bar{\lambda}_i = A^{(2)}_{ii}\,, \quad A^{(2)}_{ij} = 0\,, \quad i \neq j. \tag{41}$$

Since $\bar{\lambda}_i$ are isotropic invariants, $A^{(2)}_{ii}$ are also isotropic invariants (another example showing that $A^{(2)}_{ij}$ are isotropic scalar functions). With respect to the basis $S' = \{\boldsymbol{Q}\boldsymbol{e}_1, \boldsymbol{Q}\boldsymbol{e}_2, \boldsymbol{Q}\boldsymbol{e}_3\}$, we have, for the transformed tensor,

$$\boldsymbol{A}_2' = \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T = \sum_{i=1}^{3} \bar{\lambda}_i\, \boldsymbol{Q}\boldsymbol{e}_i \otimes \boldsymbol{Q}\boldsymbol{e}_i = 1\,\boldsymbol{Q}\boldsymbol{e}_1 \otimes \boldsymbol{Q}\boldsymbol{e}_1 + 2\,\boldsymbol{Q}\boldsymbol{e}_2 \otimes \boldsymbol{Q}\boldsymbol{e}_2 + 3\,\boldsymbol{Q}\boldsymbol{e}_3 \otimes \boldsymbol{Q}\boldsymbol{e}_3\,, \tag{42}$$

for $\boldsymbol{Q}$ given in Itskov's (7). Since $\bar{\lambda}_i$ are isotropic invariants, their values are not expected to change in the transformed tensor, as shown in Eq. 42. However, Itskov uses the basis $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ for $\boldsymbol{A}_2'$. Following the work given in Section 2.6, we obtain

$$\boldsymbol{A}_2' = \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T = 2\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + 1\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + 3\boldsymbol{e}_3 \otimes \boldsymbol{e}_3\,, \tag{43}$$

which implies

$$\bar{\lambda}_1 = 2\,, \quad \bar{\lambda}_2 = 1\,, \quad \bar{\lambda}_3 = 3\,. \tag{44}$$

Hence, if the basis S is assumed to be the corret basis for the transformed tensor, then the different values of the eigenvalues given in Eqs. 40 and 44 show that the eigenvalues $\bar{\lambda}_i$ are **not isotropic invariants**: This false conclusion further invalidate Itskov's counterexample.

4.2 Proof 2

In this section, we prove that an existence of a non-isotropic function $A^{(2)}_{ij}$ leads to a self-refuting result.

Proof. Assuming, as claimed by Itskov, there exist a set of eigenvectors $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ that indicates that $A^{(2)}_{ij}(\boldsymbol{g}_i \otimes \boldsymbol{g}_j, \boldsymbol{A}_2)$ are not isotropic functions. Since

$$A^{(2)}_{ij}(\boldsymbol{g}_i \otimes \boldsymbol{g}_j, \boldsymbol{A}_2) = \boldsymbol{g}_i \cdot \boldsymbol{A}_2\boldsymbol{g}_j = \boldsymbol{Q}\boldsymbol{g}_i \cdot \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{g}_j = A^{(2)}_{ij}(\boldsymbol{Q}\boldsymbol{g}_i \otimes \boldsymbol{g}_j\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T)$$

which proves that $A^{(2)}_{ij}$ are isotropic invariants.

5 Remark

Let, for example, a free energy function

$$W_e = \bar{W}(\boldsymbol{A}_1, \boldsymbol{A}_2) = \bar{W}(\boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T) = W(\lambda_i, A^{(2)}_{ij})\,. \tag{45}$$

As explained in the literature [3, 4, 7, 8], it is important to note W must contain spectral invariants that satisfy the P-property, to deal with symmetry and coalescence of eigenvalues. For example, a general spectral invariant to describe W is of the form [5]

$$I = \sum_{i=1}^{3} g(\lambda_i)\boldsymbol{v}_i \cdot \boldsymbol{T}\boldsymbol{v}_i\,, \tag{46}$$

where $\boldsymbol{T}$ is a second-order tensor. For example, the invariants

$$\mathrm{tr}\,\boldsymbol{A}_2^2 = \sum_{i,j=1}^{3} A^{(2)}_{ij}A^{(2)}_{ji}\,, \quad g(\lambda_i) = 1\,, \quad \boldsymbol{T} = \boldsymbol{A}_2^2\,, \tag{47}$$

$$\mathrm{tr}\,(\boldsymbol{A}_1\boldsymbol{A}_2) = \sum_{i=1}^{3} \lambda_i A^{(2)}_{ii}\,, \quad g(\lambda_i) = \lambda_i\,, \quad \boldsymbol{T} = \boldsymbol{A}_2\,, \tag{48}$$

$$\mathrm{tr}\,\boldsymbol{A}_1^2 = \sum_{i=1}^{3} \lambda_i^2\,, \quad g(\lambda_i) = \lambda_i^2\,, \quad \boldsymbol{T} = \boldsymbol{I}\,. \tag{49}$$

In Shariff [8], it is shown that all “classical" invariants can be explicitly expressed in terms of the spectral invariants developed in [8].

6 Subjective student scenario

We personally believe that if one has given a robust and correct counterexample, then one need not give a subjective scenario, such as the student scenario given in [1], to further justify one's counterexample. Since, the student scenario is subjective, we are reluctant to address it. However, for the readers' benefit, we will discuss it in this section. We strongly emphasize that such a subjective scenario does not prove whether a scalar function is isotropic or not. For example, consider two students who do not know each other and are asked independently to construct the “classical" functional basis for $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. Since they do not know each other, there is a possibility that student 1 will construct the functional basis given in Eq. 2 and student 2 will construct the functional basis

$$I_1\,,\, I_2\,,\, I_3\,,\, \operatorname{tr}\boldsymbol{A}_2^3\,,\, \operatorname{tr}(\boldsymbol{A}_1\boldsymbol{A}_2)\,,\, \operatorname{tr}(\boldsymbol{A}_1^2\boldsymbol{A}_2)\,,\, \operatorname{tr}(\boldsymbol{A}_1\boldsymbol{A}_2^2)\,,\, \operatorname{tr}(\boldsymbol{A}_1^2,\, \boldsymbol{A}_2^2)\,, \tag{50}$$

where

$$I_1 = \operatorname{tr}\boldsymbol{A}_1\,, \quad I_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\,, \quad I_3 = \lambda_1\lambda_2\lambda_3\,. \tag{51}$$

They were then asked to evaluate the values of their invariants for $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ given in Itskov (4). Student 1 will obtain the values

$$\operatorname{tr}\boldsymbol{A}_1 = \operatorname{tr}\boldsymbol{A}_1^2 = \operatorname{tr}\boldsymbol{A}_1^3 = 3\,, \quad \text{etc.} \tag{52}$$

and student 2 will obtain the values

$$I_1 = I_2 = 3\,, \quad I_3 = 1\,, \quad \text{etc.}. \tag{53}$$

The different numerical values in Eqs. 52 and 53 may make the students think that elements of their functional basis are not isotropic functions. The above different values do not prove, at all, that the classical invariants are not isotropic functions. A similar but somewhat different situation occurs in Itskov's student scenario.

Itskov constructs a scenario, where student 1 is asked to obtain the components of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, and student 2 to obtain the components of $\boldsymbol{A}_1 (= \boldsymbol{A}'_1)$ and $\boldsymbol{A}'_2$. He has given the condition the students do not know each other and are asked independently to calculate the spectral invariants (3). We discuss this via the following two scenarios, taking note that $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are those given in Itskov (4) and $\boldsymbol{Q}$ is given in Itskov (7).

Scenario 1: Students 1 and 2, by chance (or cheated), both use the same eigenvectors $\boldsymbol{e}_i$ for $\boldsymbol{A}_1 = \boldsymbol{I}$.

Student 1, using the reference basis $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, will obtain the spectral invariants $\lambda_i = 1\,,\quad A^{(2)}_{ij}$, for, respectively, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, and student 2, using the correct the rotated basis $S' = \{\boldsymbol{Q}\boldsymbol{e}_1,\, \boldsymbol{Q}\boldsymbol{e}_2,\, \boldsymbol{Q}\boldsymbol{e}_3\}$, will also obtain the same spectral invariants $\lambda_i = 1$ and $A^{(2)}_{ij}$ for, respectively, $\boldsymbol{A}'_1$ and $\boldsymbol{A}'_2$. However, if by mistake student 2 (thinking that the identity tensor $\boldsymbol{I}$ has a unique basis and assuming that the bases S and S' are the same) uses the unrotated basis S for $\boldsymbol{A}'_1$ and $\boldsymbol{A}'_2$, he/she will then obtain different tensor components $B_{ij} = \boldsymbol{e}_i \cdot (\boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T)\boldsymbol{e}_j \neq A^{(2)}_{ij}$ for $\boldsymbol{A}_2$. This scenario does not prove that $A^{(2)}_{ij}$ are not isotropic invariants

Scenario 2: Since $\boldsymbol{A}_1 = \boldsymbol{I}$ has infinitely many bases, there is a high probability that students 1 and 2 will use different eigenvectors to describe $\boldsymbol{A}_1$. For example, student 1 will use

$$\boldsymbol{A}_1 = \sum_{i=1}^{3} \boldsymbol{e}_i \otimes \boldsymbol{e}_i \tag{54}$$

and obtain the components $A^{(2)}_{ij}$ for $\boldsymbol{A}_2$. Student 2 will use

$$\boldsymbol{A}_1 = \sum_{i=1}^{3} \bar{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}_i \,, \quad \bar{\boldsymbol{e}}_i \neq \boldsymbol{e}_i \tag{55}$$

and obtain $\bar{A}^{(2)}_{ij} = \bar{\boldsymbol{e}}_i \cdot \boldsymbol{A}_2 \bar{\boldsymbol{e}}_j$ for $\boldsymbol{A}_2$. Student 2, using the correct rotated basis $S' = \{\boldsymbol{Q}\bar{\boldsymbol{e}}_1, \boldsymbol{Q}\bar{\boldsymbol{e}}_2, \boldsymbol{Q}\bar{\boldsymbol{e}}_3\}$, will also obtain $\bar{A}^{(2)}_{ij}$ for $\boldsymbol{A}'_2$. Although $\bar{A}^{(2)}_{ij} \neq A^{(2)}_{ij}$, this scenario does not prove that invariants $\bar{A}^{(2)}_{ij}$ and $A^{(2)}_{ij}$ are not isotropic invariants: **Both of them are isotropic invariants** as exemplified in Section 2. It is important to note that, as explained in Section 5 (see also reference [5]), to satisfy the P-property the spectral invariants must be of the form Eq. 46. Consider the following examples (for $\lambda_i = 1$):

$$I = \sum_{i=1}^{3} g(\lambda_i) A^{(2)}_{ii} = \sum_{i=1}^{3} g(\lambda_i) \bar{A}^{(2)}_{ii} = g(1) \mathrm{tr}\, \boldsymbol{A}_2 \,,$$

$$\mathrm{tr}\,(\boldsymbol{A}_1 \boldsymbol{A}_2) = \sum_{i=1}^{3} \lambda_i A^{(2)}_{ii} = \sum_{i=1}^{3} \lambda_i \bar{A}^{(2)}_{ii} = \sum_{i=1}^{3} A^{(2)}_{ii} = \sum_{i=1}^{3} \bar{A}^{(2)}_{ii} = \mathrm{tr}\, \boldsymbol{A}_2 \,, \tag{56}$$

where they have the same values even though $\bar{A}^{(2)}_{ij} \neq A^{(2)}_{ij}$.

Important: Even though $A^{(2)}_{ij}$ have different values for different coordinate systems (material frames), in view of the P-property [5], they will give the same values for the strain energy or stresses at the same strain. Spectral invariants, such as $A^{(2)}_{ij}$, together with the P-property, have been used in the literature (see reference [8] and references within) to model mechanical behaviors in solid mechanics.

7 Conclusion

Itskov's [1] objective is to prove, via a counterexample, that Shariff spectral invariants [7] are not isotropic scalar functions. In Section 3 (also in Section 2.4), we prove that his counterexample is wrong because, from his statement "*Due to the fact that* $\boldsymbol{A}'_1 = \boldsymbol{A}_1$ *the eigenvectors (5) remain unchanged*," he incorrectly assumes that the basis for the identity tensor $\boldsymbol{I}$ is unique; hence, he uses the reference basis (instead of the rotated basis) to evaluate the transformed tensors and obtains an incorrect result. In Section 2 and in the literature [6, 7], we have shown that Shariff spectral invariants are isotropic scalar functions. In Section 6, we address the subjective students' scenario.

Appendix

In view of Eq. 4, we can write

$$f(\boldsymbol{A}_1, \boldsymbol{A}_2) = g(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{A}_2) \,, \tag{A1}$$

with the symmetry property

$$g(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{A}_2) = g(\lambda_2, \lambda_1, \lambda_3, \boldsymbol{v}_2, \boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{A}_2) = g(\lambda_3, \lambda_2, \lambda_1, \boldsymbol{v}_3, \boldsymbol{v}_2, \boldsymbol{v}_1, \boldsymbol{A}_2) = \text{etc.} \,. \tag{A2}$$

Since g is an isotropic scalar function, we have,

$$g(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{A}_2) = g(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{Q}\boldsymbol{v}_1, \boldsymbol{Q}\boldsymbol{v}_2, \boldsymbol{Q}\boldsymbol{v}_3, \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T) \,, \quad \forall \boldsymbol{Q} \in Orth^3 \,. \tag{A3}$$

Since $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \delta_{ij}$ (the Kronecker delta), we then have, the basis containing the spectral elements

$$\lambda_i \,, \quad A^{(2)}_{ij} \tag{A4}$$

that are isotropic functions.

Data Availability There are no external data supporting the findings of this study.

Declarations

Conflict of interest The author declares no competing interests.

References

1. Itskov, M.: On the functional basis of isotropic vector and tensor functions by Sharif (2023). Mech. Soft Mater **6**, 1 (2024)
2. Pennisi, S., Trovato, M.: On the irreducibility of Professor G.F. Smith's representations for isotropic functions. Int. J. Engng Sci. **25**(8), 1059–1065 (1987)
3. Shariff, M.H.B.M.: Spectral derivatives in continuum mechanics. Q. J. Mech. Appl. Math. **70**(4), 479–496 (2017)
4. Shariff, M.H.B.M.: On the spectral constitutive modelling of transversely isotropic soft tissue: physical invariants. Int. J. Engng. Sci. **120**, 199–219 (2017)
5. Shariff, M.H.B.M.: A general spectral nonlinear elastic consistent tangent modulus tensor formula for finite element software. Results Appl. Math. **7**, 100113 (2020). https://doi.org/10.1016/j.rinam.2020.100113
6. Shariff, M.H.B.M.: On the number of independent invariants for m unit vectors and n symmetric second order tensors. Engng. Letters **29**(2), 509–515 (2021)
7. Shariff, M.H.B.M.: On the smallest number of functions representing isotropic functions of scalars, vectors and tensors. Q. J. Mech. Appl. Math. **76**, 143–161 (2023)
8. Shariff, M.H.B.M., Bustamante, R., Merodio, J.: Spectral formulations in nonlinear solids: a brief summary. Math. Mech. Solids (2023). https://doi.org/10.1177/10812865231204