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ORIGINAL ARTICLE

Global Existence and Scattering of the Klein–Gordon–Zakharov System in Two Space Dimensions

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Abstract
We are interested in the Klein–Gordon–Zakharov system in $\mathbb{R}^{1+2}$, which is an important model in plasma physics with extensive mathematical studies. The system can be regarded as semilinear coupled wave and Klein–Gordon equations with nonlinearities violating the null conditions. Without the compactness assumptions on the initial data, we aim to establish the existence of small global solutions, and in addition, we want to illustrate the optimal pointwise decay of the solutions. Furthermore, we show that the Klein–Gordon part of the system enjoys linear scattering, while the wave part has uniformly bounded low-order energy. None of these goals is easy because of the slow pointwise decay nature of the linear wave and Klein–Gordon components in $\mathbb{R}^{1+2}$. We tackle the difficulties by carefully exploiting the properties of the wave and the Klein–Gordon components, and by relying on the ghost weight energy estimates to close higher order energy estimates. This appears to be the first pointwise decay result and the first scattering result for the Klein–Gordon–Zakharov system in $\mathbb{R}^{1+2}$ without compactness assumptions.

Keywords Klein–Gordon–Zakharov system · Pointwise decay · Linear scattering

Mathematics Subject Classification 35L05

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1 Introduction

Model Problem and Main Results We consider the Klein–Gordon–Zakharov model in $\mathbb{R}^{1+2}$, which is an important model in plasma physics with extensive mathematical studies. The model equations are as follows:

$$
\begin{aligned}
-\Box E + E &= -nE, \\
-\Box n &= \Delta |E|^2.
\end{aligned}
\tag{1.1}
$$

The unknowns include the electronic field $E = (E^1, E^2)$ taking values in[1] $\mathbb{R}^2$, and the ion density n taking values in $\mathbb{R}$. The Klein–Gordon–Zakharov equations can be regarded as a semilinear coupled wave and Klein–Gordon system, with Klein–Gordon field E and wave field n. In the spacetime $\mathbb{R}^{1+2}$, we adopt the signature $(-, +, +)$. The wave operator is denoted by $\Box = \partial_\alpha \partial^\alpha$, and $\Delta = \partial_a \partial^a$ represents the Laplace operator. Throughout Greek letters $\alpha, \beta, \cdots \in \{0, 1, 2\}$ denote spacetime indices, while Latin letters $a, b, \cdots \in \{1, 2\}$ are used to represent space indices. The Einstein summation convention is adopted unless otherwise specified. We also apply the Japanese bracket $\langle \rho \rangle := (1 + \rho^2)^{1/2}$, and we use the abbreviation for the L^2 norm $\| \cdot \| := \| \cdot \|_{L^2(\mathbb{R}^2)}$.

We consider the Cauchy problem associated with (1.1) with the initial data on the slice $t = t_0 = 0$

$$
\big(E, \partial_t E\big)(t_0, \cdot) = \big(E_0, E_1\big), \quad \big(n, \partial_t n\big)(t_0, \cdot) = \big(n_0, n_1\big) := \big(\Delta n_0^\Delta, \Delta n_1^\Delta\big),
\tag{1.2}
$$

and the functions $(E_0, E_1, n_0^\Delta, n_1^\Delta)$ are assumed to be sufficiently smooth, but they *do not* need to be compactly supported. The main objective of the present article is the following asymptotic stability result associated with small regular initial data together with the scattering property on the Klein–Gordon components, i.e., the Langmuir wave (stated in the next theorem).

Theorem 1.1 *Consider the Klein–Gordon–Zakharov system in* (1.1)*, and let* $N \geq 14$ *be a large integer. There exists* $\epsilon_0 > 0$*, such that for all initial data satisfying the smallness condition*

$$
\begin{aligned}
&\sum_{0 \leq j \leq N+2} \| \langle |x| \rangle^{N+3} \log\langle 1 + |x| \rangle E_0 \|_{H^j} + \sum_{0 \leq j \leq N+1} \| \langle |x| \rangle^{N+3} \log\langle 1 + |x| \rangle E_1 \|_{H^j} \\
&+ \sum_{0 \leq j \leq N+3} \| \langle |x| \rangle^{j+1} n_0^\Delta \|_{H^j} + \sum_{0 \leq j \leq N+2} \| \langle |x| \rangle^{j+2} n_1^\Delta \|_{H^j} \leq \epsilon < \epsilon_0,
\end{aligned}
\tag{1.3}
$$

the initial value problem (1.1)–(1.2) *admits a global solution* (E, n)*, which enjoys the following optimal pointwise decay results:*

$$
|E(t, x)| \lesssim \langle t \rangle^{-1}, \quad |n(t, x)| \lesssim \langle t \rangle^{-1/2} \langle t - r \rangle^{-1/2}.
\tag{1.4}
$$

[1] Originally, E takes values in $\mathbb{R}^2$, but more general cases of taking values in $\mathbb{C}^{N_0}$ with $N_0 = 1, 2, \cdots$ can also be treated.

Remark 1.2 The global existence for the Klein–Gordon–Zakharov system (with some first order equations) in two space dimensions was proved in [14], but whether the system is stable is unknown. Our result in Theorem 1.1 verifies that the system is not only stable but also asymptotically stable.

Remark 1.3 A similar version of Theorem 1.1 was demonstrated in [5, 32] with compactly supported initial data. Now, we can treat the non-compactly supported initial data with decay at infinity. At this point, we recall the global existence result [39] regarding a quasilinear wave-Klein–Gordon model satisfying the null condition in two space dimensions, where the weights are lower than our result in Theorem 1.1.

Remark 1.4 In Theorem 1.1, we only illustrated the pointwise decay of the global solution (E, n) in (1.4). Besides, the global solution (E, n) has uniformly bounded lower order energy, and more functional properties enjoyed by the global solution can be found in Proposition 4.6 and Definition 4.1.

Remark 1.5 The assumptions on the initial data are needed to guarantee the smallness of the various norms involving the initial data, which are used when we apply the energy estimates or Proposition 2.8 of Georgiev; see Lemma 4.7.

Our next result states that the Klein–Gordon field E scatters linearly.

Theorem 1.6 *Let the same assumptions in Theorem 1.1 hold. Then there exists a pair of functions*

$$(E_0^+, E_1^+) \in H^{N-7} \times H^{N-8},$$

such that

$$\|(E - E^+)(t, \cdot)\|_{H^{N-7}} + \|\partial_t (E - E^+)(t, \cdot)\|_{H^{N-8}} \leq C\langle t\rangle^{-1/4} \to 0, \quad \text{as } t \to +\infty, \tag{1.5}$$

in which E^+ is a linear Klein–Gordon component solving

$$\begin{aligned} &-\Box E^+ + E^+ = 0, \\ &\big(E^+, \partial_t E^+\big)(t_0) = (E_0^+, E_1^+). \end{aligned}$$

Remark 1.7 We want to emphasize that the scattering result in Theorem 1.6 is valid under quite high regularity assumptions on the initial data. As a comparison, we recall that the scattering result of the Zakharov equations in $\mathbb{R}^{1+3}$ in [19] is also obtained with high regularity assumptions on the initial data. Our scattering result is different from the one proved with (radial) initial data in low regularity for model (1.1) in [18] in $\mathbb{R}^{1+3}$, where very different difficulties arise. See in detail below.

Remark 1.8 We note that our method cannot assert whether the wave part n scatters linearly or not in the energy space (i.e., $\sum_{a=1,2} \|\partial_a n\| + \|\partial_t n\|$), and we leave it open. However, we will show that the energy of the wave component n is uniformly bounded in time [see (4.3)], which is necessary to linear scattering.

Remark 1.9 There exist already several global existence results for two-dimensional coupled wave and Klein–Gordon equations with different types of nonlinearities, but most of the results were shown under the assumption that the initial data are compactly supported; see [10, 11, 31] and the references therein for such cases. The ideas and techniques used in proving Theorems 1.1 and 1.6 are expected to have further applications, such as to remove the compactness assumptions on the existing results, or to study coupled wave and Klein–Gordon systems with more general nonlinearities of physical or mathematical interests.

Background and Historical Notes The Klein–Gordon–Zakharov system was originally introduced in [40], which describes the interaction between Langmuir waves and ion sound waves in plasma; see [3] for more of its physical background. The global existence as well as the pointwise decay result on this system in $\mathbb{R}^{1+3}$ was established dating back to [35], and then in many other context (see, for instance, the recent work [7]). However, due to the insufficiency of the decay in lower dimension (see in detail below), the global existence problem in $\mathbb{R}^{1+2}$ is somewhat more challenging. In [5] a global existence result, with pointwise asymptotics of the solution, is established on localized restricted initial data, and then, it is generalized in [12]. These results are established within the vector field method on hyperboloids (which is usually referred to as hyperboloidal method or hyperboloidal foliation method) and thus demand that the initial data being compactly supported. In the present work, we rely on a global iteration framework, which was used for instance in [6], to remove this restriction. Finally, besides the global existence and pointwise asymptotics results, there is also plenty of work concerning other directions on this system. For instance, in $\mathbb{R}^{1+3}$, Ozawa, Tsutaya, and Tsutsumi [36] showed that the Klein–Gordon–Zakharov equations admit global solutions for low regularity initial data under the condition that the propagation speeds are different in two equations. In the work by Shi and Wang [37], a finite time blow-up result was obtained for low regular initial data satisfying certain conditions.

A model closely related to the Klein–Gordon–Zakharov system is the Zakharov system (which includes wave and Schrodinger equations), and in many cases, the progress on one leads to progress on the other. In a series of papers [33, 34], Masmoudi and Nakanishi investigated the limiting system as certain parameters go to $+\infty$ in the Klein–Gordon–Zakharov and the Zakharov equations. In another series of papers, Guo–Nakanishi–Wang [16–18] established scattering for radial solutions with small energy in the low regularity setting for these two systems, while the linear scattering of the Zakharov equations (with high regular initial data without radial assumptions) was illustrated by Hani–Pusateri–Shatah [19].

We next recall some mathematical studies in plasma physics which are relevant to our results. Being a highly important model, the Klein–Gordon–Zakharov system can be derived (with certain assumptions) from Euler–Maxwell system, which can be found in [2, Section 2.1]. The Euler–Maxwell model is one of the most fundamental models in plasma physics, which describes laser–plasma interactions. In the seminal work of Guo–Ionescu–Pausader [15], the two-fluid Euler–Maxwell model was shown to admit smooth solutions in $\mathbb{R}^{1+3}$. We recall that the Euler–Poisson system in $\mathbb{R}^{1+2}$ was proved to have global solutions by Li–Wu [30] and Ionescu–Pausader [21], and later on, the one-fluid Euler–Maxwell system in $\mathbb{R}^{1+2}$ was proved to have global solutions by Deng–Ionescu [4], and both systems can be reduced to Klein–Gordon equations with non-local derivatives on the nonlinear terms.

The coupled wave and Klein–Gordon equations in two space dimensions have received much attention, and continual progress has been made regarding its small data global existence problem in recent years. We are not going be exhaustive here, but instead leading one to the works [9, 10, 31, 39] and the references therein for more discussions.

Major Difficulties and Technical Contributions Concerning the Klein–Gordon–Zakharov system (1.1), there are many difficulties in obtaining the results in Theorems 1.1 and 1.6. Besides the lack of the scaling vector field in the analysis, we demonstrate some key issues encountered in the study of the system (1.1). The first one is the absence of null structure in the nonlinearities of the system (1.1). We remark that the right-hand-side of the system violates the classical null condition of Christodoulou–Klainerman. The second comes from the low decay rate of both wave and Klein–Gordon equations in lower dimension $\mathbb{R}^{1+2}$ and the third one is about dealing with the non-compactly supported initial data. Let us explain in detail these obstacles and our strategies aimed at each of them.

In the research of nonlinear wave systems (including wave-Klein–Gordon systems), the null condition (in the sense of Christodoulou–Klainerman) plays an essential role. Roughly speaking, it provides additional decay near the light cone (i.e., the region t close to $|x|$), where the wave equations fail to have sufficiently fast decay. We note the only existing global existence results on two-dimensional coupled wave and Klein–Gordon equations with non-compactly supported initial data are due to [6, 39], where all of the nonlinearities are assumed to obey the null condition, and thus, the situation we consider here is more difficult. To conquer this difficulty, we observe and take full use of a special structure of the system (1.1): in the right-hand-side of the wave equation, the only quadratic term is a wave-Klein–Gordon mixed one. The Klein–Gordon components enjoy an additional decay rate expressed as $\langle t+r\rangle^{-1}\langle r-t\rangle$ near the light cone (see Proposition 3.5 for more details), which will compensate the absence of the null structure in our analysis.

The insufficiency of decay in the lower dimension brings another difficulty. In $\mathbb{R}^{1+2}$, the free-linear waves decay at the speed of $t^{-1/2}$, while free-linear Klein–Gordon components decay at the speed of t^{-1}. This means that the best we can expect for the nonlinearities is

$$\|nE\|_{L^2(\mathbb{R}^2)} \lesssim t^{-1}, \quad \|\Delta|E|^2\|_{L^2(\mathbb{R}^2)} \lesssim t^{-1}, \tag{1.6}$$

which are non-integrable with respect to time. Thus, under this situation, it is highly non-trivial to prove the sharp pointwise decay results, as well as closing the bootstrap, of E and n. Our strategy of solving this thorny issue of the slow decay in the nE nonlinearity follows. We first reveal a Hessian structure $\Delta n^\Delta E$ with the relation $n = \Delta n^\Delta$ relying on the special structure in the wave equation of n. We only look at the region $r \le 3t$ (as for its complement part one has $\langle t-r\rangle^{-1} \lesssim \langle t+r\rangle^{-1}$), which includes the most subtle region of the light cone. The Hessian structure of Δn^Δ will contribute an extra decay factor of $\langle t - r\rangle^{-1}$ by Proposition 3.1 (roughly, one images that one more derivative give one more decay factor of $\langle t-r\rangle^{-1}$ for wave component), while the Klein–Gordon component E will contribute an extra decay factor of $\langle t+r\rangle^{-1}\langle t-r\rangle$ by Proposition 3.4, and the product of these two factors yields a favorable factor $\langle t+r\rangle^{-1}$ which can give decay in time, and which is favorable.

The third difficulty is the most severe one, and it is the main interest of the present article to tackle. When the initial data are compactly supported, the system (1.1) has been discussed within the hyperboloidal foliation framework; see for example [5, 12, 32]. However, there, due to the essential shortcoming of the hyperboloidal foliation, one cannot analyze the solution outside of the light cone, and thus, the demand on the compactness of the support of initial data became inevitable. Here, we apply another strategy used in [6, 7] which is different from the hyperboloidal foliation. In [6, 7], the decay for the Klein–Gordon components is based on a weighted Sobolev estimate of Georgiev, while the decay for the wave components is based on a special version of Klainerman–Sobolev inequality (as well as some extra decay when second or higher order derivatives hitting on the wave components) where one needs the future information till time $2t$ when deriving the pointwise estimate of a function at time t. To make this Klainerman–Sobolev inequality ready to use, we rely on a global contraction mapping scheme in the proof instead of the usual bootstrap argument. Moreover, one crucial ingredient used in [5, 7] is that we adapt Alinhac's ghost weight method to coupled wave and Klein–Gordon equations, so that one can take advantage of the $\langle t-r\rangle$ decay in the wave component and close the top-order energy estimates. Equipped with these techniques, one manages to treat the whole spacetime in its entirety. This allows us to remove the compactness restriction on the initial data. As a comparison, in the proofs of [6, 7], we close the iteration by relying on a global decay of the Klein–Gordon components. But here, we need to investigate more detailed properties of the Klein–Gordon field E in different spacetime regions [see for instance (4.3)], so that we can manage to close the proof.

Once the global solution is established for the system (1.1), a natural question arises: will the global solution (E, n) scatters to the linear one? This is the second objective of the present work. To show the linear scattering of the Klein–Gordon–Zakharov system (1.1) (or the Zakharov equations) is a tough problem even in $\mathbb{R}^{1+3}$. In [17, 18], the Klein–Gordon–Zakharov system was shown to enjoy the linear scattering for (radial) initial data with low regularity, and later on in [19], the Zakharov system was proved to scatter linearly for initial data with high regularity. All of the works [17–19] are proved in $\mathbb{R}^{1+3}$, and the proofs cannot be directly applied to the two dimensional cases. Based on a scattering result on wave and Dirac equations in [8, 24], we succeed in showing that the Klein–Gordon part E in system (1.1) enjoys linear scattering in its energy space (as illustrated in Theorem 1.6) by adapting the result in [8, 24] to Klein–Gordon equations. We cannot prove whether the wave part n scatters linearly or not, but instead we show that the natural wave energy of n (i.e., $\sum_{a=1,2}\|\partial_a n\| + \|\partial_t n\|$) is uniformly bounded in time, which is a weaker result (also a necessary result to linear scattering). As an interesting comparison, we refer to the tremendous work [22] for the case of Einstein–Klein–Gordon system in $\mathbb{R}^{1+3}$, where neither the wave components nor the Klein–Gordon one scatters linearly.

Further Discussions Many fundamental physical models are governed by the coupled wave and Klein–Gordon equations, including the Dirac–Klein-Gordon equations, the Einstein–Klein–Gordon equations, the Klein–Gordon–Zakharov equations studied in the present paper, the Maxwell–Klein–Gordon equations, etc., and the asymptotic behavior of the equations is relatively well studied in three space dimensions. However, there are few results for these systems of equations in two space dimensions, and the pointwise asymptotics of the solutions (even for small smooth initial data with compact support) are also unknown except the Klein–Gordon–Zakharov equations [5, 12, 32] and some cases of the Dirac–Klein–Gordon equations [10]. We believe that it is of great mathematical and physical significance to show the asymptotic behavior (or some other related results) of such equations in two space dimensions.

We recall that the Klein–Gordon–Zakharov equations were shown to have uniformly bounded (low- and high-order) energy in the recent work [7] in $\mathbb{R}^{1+3}$, where the pointwise decay of the solutions is faster compared to the two-dimensional case. However, in $\mathbb{R}^{1+2}$, due to the critical decay rates of the nonlinearities as illustrated in (1.6), we do not expect the high-order energy to be uniformly bounded in time unless some new observations on system (1.1) are found.

In Theorem 1.6, we are only able to show that the Klein–Gordon component E in the Klein–Gordon–Zakharov (1.1) scatters linearly. For the wave component n, our method cannot say anything about its scattering aspect. Due to the (Klein–Gordon)−(Klein–Gordon) interaction appearing in the n equation and the slow decay nature of the Klein–Gordon component in two space dimensions (recall that even the free Klein–Gordon components decay at speed of t^{-1} in two space dimensions which is a non-integrable quantity), our guess is that the wave component n enjoys nonlinear scattering (also referred to as modified scattering). We expect that methods in [18, 22] can be used to verify our guess.

Last but not least, we briefly discuss about the Zakharov system which is closely related to the present study. The Zakharov system is composed of wave equations and Schrodinger equations, and it might not be easy to apply Klainerman's vector field method due to the lack of Lorentz invariance. Its global existence and scattering were tackled in the three-dimensional case in [19], but they remain open in the two-dimensional case. To make progress in this direction, we believe that it is worth trying the spacetime resonance method used in [19] and the Z-norm method used in [22].

Outline The rest of this article is organized as follows. In Sect. 2, we introduce the preliminaries and some fundamental energy estimates for wave and Klein–Gordon equations. We then explore some extra decay properties for the wave and the Klein–Gordon components in Sect. 3. In Sect. 4, we demonstrate the proof of the global existence and the pointwise decay results for the Klein–Gordon–Zakharov equations in Theorem 1.1 relying on the contraction mapping theorem. Last, we show the scattering result for the Klein–Gordon component in Theorem 1.6 in Sect. 5 with some supporting materials in Appendix A.

2 Preliminaries

2.1 Basic Notation

We work in the (1 + 2) dimensional spacetime with signature $(-, +, +)$, i.e., the Minkowski metric $\eta = \text{diag}\{-1, 1, 1\}$. A point in $\mathbb{R}^{1+2}$ is denoted by $(x_0, x_1, x_2) = (t, x_1, x_2)$, and its spacial radius is written as $r = \sqrt{x_1^2 + x_2^2}$. We use Latin letters to represent space indices $a, b, \cdots \in \{1, 2\}$, while Greek letters are used to denote spacetime indices $\{0, 1, 2\}$, and the indices are raised or lowered by the metric η.

We first recall the vector fields which will be frequently used in the analysis.

- Translations: ∂_α.
- Rotations: $\Omega_{ab} = x_a \partial_b - x_b \partial_a$.
- Lorentz boosts: $L_a = x_a \partial_t + t \partial_a$.
- Scaling vector field: $L_0 = S = t\partial_t + r\partial_r$.

Excluding the scaling vector field L_0, we utilize Γ to denote a general vector field in the set

$$V := \{\partial_\alpha, \Omega_{ab}, L_a\}.$$

For a multi-index $I = (i_1, \ldots, i_6)$ with integers $i_1, \ldots, i_6 \geq 0$, we define $\Gamma^I := \partial_0^{i_1} \partial_1^{i_2} \partial_2^{i_3} \Omega_{12}^{i_4} L_1^{i_5} L_2^{i_6}$, and denote $|I| = i_1 + \cdots + i_6$; other multi-indices are defined in a similar way. Besides, the following good derivatives:

$$G_a := r^{-1}\big(x_a \partial_t + r\partial_a\big),$$

will appear in Alinhac's ghost weight method.

We define (and fix) a smooth cut-off function which is increasing and satisfies

$$\chi(s) := \begin{cases} 0, & s \leq 1, \\ 1, & s \geq 2. \end{cases} \tag{2.1}$$

This will be frequently used to derive energy estimates in different spacetime regions.

2.2 Energy Estimates

We consider the wave-Klein–Gordon equation with $m = 0, 1$

$$-\Box u + m^2 u = F_u. \tag{2.2}$$

We will demonstrate several types of energy estimates for the equation (2.2). We recall the energy functional (with $\delta > 0$)

$$\mathcal{E}_{gst,m}(t, u) := \mathcal{E}_m(t, u) + \sum_a \int_{t_0}^t \int_{\mathbb{R}^2} \frac{\delta |G_a u|^2}{\langle \tau - r\rangle^{1+\delta}} dx d\tau + \int_{t_0}^t \int_{\mathbb{R}^2} \frac{\delta m^2 |u|^2}{\langle \tau - r\rangle^{1+\delta}} dx d\tau, \tag{2.3}$$

in which the natural energy is defined by

$$\mathcal{E}_m(t,u) := \int_{\mathbb{R}^2} \Big(|\partial_t u|^2 + \sum_a |\partial_a u|^2 + m^2 |u|^2 \Big)(t,\cdot)\, dx.$$

The abbreviations $\mathcal{E}(t,u) = \mathcal{E}_0(t,u)$ and $\mathcal{E}_{gst}(t,u) = \mathcal{E}_{gst,0}(t,u)$ will be used. We also remark that when $t = t_0$, $\mathcal{E}_{gst,m}(t_0,u) = \mathcal{E}_m(t_0,u)$.

The natural energy estimates for wave-Klein–Gordon equations read.

Proposition 2.1 *Consider* (2.2) *with* $m = 0, 1$*, and it holds both*

$$\mathcal{E}_m(t,u) \lesssim \mathcal{E}_m(t_0,u) + \int_{t_0}^t \int_{\mathbb{R}^2} \big|F_u\big| \big|\partial_t u\big|\, dx d\tau \tag{2.4}$$

and

$$\mathcal{E}_m(t,u)^{1/2} \lesssim \mathcal{E}_m(t_0,u)^{1/2} + \int_{t_0}^t \big\| F_u \big\|\, d\tau. \tag{2.5}$$

The following energy estimates are due to Alinhac [1], which are referred to as ghost weight energy estimates. Compared with the natural energy estimates, an additional positive spacetime integral can also be controlled.

Proposition 2.2 *Consider* (2.2) *with* $m = 0, 1$*, and we have (with* $\delta > 0$*) both*

$$\mathcal{E}_{gst,m}(t,u) \lesssim \mathcal{E}_{gst,m}(t_0,u) + \int_{t_0}^t \int_{\mathbb{R}^2} \big|F_u\big| \big|\partial_t u\big|\, dx d\tau \tag{2.6}$$

and

$$\mathcal{E}_{gst,m}(t,u)^{1/2} \lesssim \mathcal{E}_{gst,m}(t_0,u)^{1/2} + \int_{t_0}^t \big\| F_u \big\|\, d\tau. \tag{2.7}$$

The following version of ghost weight energy estimates will play a vital role in the proof of the iteration procedure, which was used for instance in [6].

Proposition 2.3 *Consider* (2.2) *with* $m = 1$*, and it holds (with* $\delta, \kappa > 0$*)*

$$\int_{t_0}^t \int_{\mathbb{R}^2} \frac{\kappa |u|^2}{\langle \tau \rangle^{\kappa} \langle \tau - r \rangle^{1+\delta}}\, dx d\tau \lesssim \mathcal{E}_{gst,1}(t_0,u) + \int_{t_0}^t \int_{\mathbb{R}^2} \langle \tau \rangle^{-\kappa} \big|F_u\big| \big|\partial_t u\big|\, dx d\tau. \tag{2.8}$$

In the applications in Sect. 4, we will take $\kappa = \delta/2$.

Proof of Propositions 2.1, 2.2, and 2.3 The above energy estimates in Propositions 2.1, 2.2, and 2.3 are based on the following identity. Let $q(t,r) := \delta \int_{-\infty}^{r-t} \langle s \rangle^{-1-\delta} ds$

which is a uniformly bounded positive function. We apply the multiplier $\langle t\rangle^{-\kappa} e^q \partial_t u$ and obtain

$$\langle t\rangle^{-\kappa} e^q \partial_t u\big(-\Box u + m^2 u\big) = \frac{1}{2}\partial_t\Big(\langle t\rangle^{-\kappa} e^q\Big(\sum_\alpha |\partial_\alpha u|^2 + m^2 u^2\Big)\Big) - \partial_a\big(\langle t\rangle^{\kappa} e^q\, \partial_t u \partial_a u\big)$$
$$+ \frac{\delta}{2}\langle t\rangle^{-\kappa} e^q \langle r-t\rangle^{-1-\delta}\Big(\sum_a |G_a u|^2 + m^2 u^2\Big)$$
$$+ \frac{\kappa}{2} t\langle t\rangle^{-\kappa-2} e^q\Big(\sum_\alpha |\partial_\alpha u|^2 + m^2 u^2\Big).$$

Then, integrating the above identity in $\{0 \leq \tau \leq t\}$ with Stokes formula leads to the desired energy estimates. For (2.4) and (2.5), we take $\delta = 0$, $e^q \equiv 1$ and $\kappa = 0$. For (2.6) and (2.7), we take $\delta > 0$ and $\kappa = 0$. Finally, for (2.8), we fix $\delta, \kappa > 0$. □

2.3 Estimates on Commutators

We first recall the well-known relations

$$\Gamma\Box = \Box\Gamma, \quad \Gamma(\Box - 1) = (\Box - 1)\Gamma;$$

besides, we also need more estimates on commutators. To apply the Klainerman–Sobolev type inequality, we need to bound quantities such as $\|\Gamma^I \partial_\alpha u\|$ by the energies introduced in the last subsection. For this purpose, we first establish the following estimates on commutators.

Lemma 2.4 *For u sufficiently regular, the following bounds hold:*

$$\big|[\partial_\alpha, \Gamma^I]u\big| \leq C(I) \sum_{|J|<|I|} \sum_{\alpha'} |\partial_{\alpha'} \Gamma^J u|, \tag{2.9}$$

$$\big|[\partial_\alpha \partial_\beta, \Gamma^I]u\big| \leq C(I) \sum_{|J|<|I|} \sum_{\alpha', \beta'} |\partial_{\alpha'} \partial_{\beta'} \Gamma^J u|, \tag{2.10}$$

where $C(I)$ is a constant determined by I. When $|I| = 0$, the sums are understood to be zero.

Proof We need to establish the following decomposition:

$$[\partial_\alpha, \Gamma^I] = \sum_{|J|<|I|} \sum \pi_{\alpha J}^{I\beta} \partial_\beta \Gamma^J u, \tag{2.11}$$

where $\pi_{\alpha J}^{I\beta}$ are constants determined by α, I. When $|I| = 0$, the sum is understood to be zero. This can be checked by induction. First, when $|I| = 1$ and $\Gamma^I = \partial_\gamma$, the commutators vanish. When $\Gamma^I = L_b$

$$[\partial_t, L_b] = \partial_b, \quad [\partial_a, L_b] = \delta_{ab}\partial_t,$$

which verify (2.11). Now, suppose that (2.11) holds for $|I| \le k$ and we check the case with $|I| = k+1$. In this case, we write

$$[\partial_\alpha, \Gamma^I]u = [\partial_\alpha, \Gamma^{I_1}\Gamma^{I_2}] = [\partial_\alpha, \Gamma^{I_1}]\Gamma^{I_2}u + \Gamma^{I_1}\big([\partial_\alpha, \Gamma^{I_2}]u\big).$$

Then, suppose that $|I_2| = 1$. By the assumption of induction

$$\begin{aligned}
[\partial_\alpha, \Gamma^I]u &= \sum_{|J_1|<|I_1|}\sum_\beta \pi_{\alpha J_1}^{I_1\beta}\partial_\beta\Gamma^{J_1}\Gamma^{I_2}u + \sum_\beta \Gamma^{I_1}\big(\pi_\alpha^{I_2\beta}\partial_\beta u\big)\\
&= \sum_{|J_1|<|I_1|}\sum_\beta \pi_{\alpha J_1}^{I_1\beta}\partial_\beta\Gamma^{J_1}\Gamma^{I_2}u + \sum_\beta \pi_\alpha^{I_2\beta}\partial_\beta\Gamma^{I_1}u - \sum_\beta \pi_\alpha^{I_2\beta}[\partial_\beta, \Gamma^{I_1}]u\\
&= \sum_{|J_1|<|I_1|}\sum_\beta \pi_{\alpha J_1}^{I_1\beta}\partial_\beta\Gamma^{J_1}\Gamma^{I_2}u + \sum_\beta \pi_\alpha^{I_2\beta}\partial_\beta\Gamma^{I_1}u\\
&\quad + \sum_{|J_1|<|I_1|}\sum_{\beta,\gamma} \pi_\alpha^{I_2\beta}\pi_{\beta J_1}^{I_1\gamma}\partial_\gamma\Gamma^{J_1}.
\end{aligned}$$

Here, remark that in the above three sums, $|J_1| + |I_2| < |I|, |I_1| < |I|$. This closes the induction.

For (2.10), we need the following decomposition:

$$[\partial_\alpha\partial_\beta, \Gamma^I]u = \sum_{|J|<|I|} \pi_{\alpha\beta J}^{I\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}\Gamma^J u, \tag{2.12}$$

where $\pi_{\alpha\beta J}^{I\alpha'\beta'}$ are constants determined by α, β, I. This is by applying twice (2.11). □

2.4 Global Sobolev Inequalities

Recall that we do not commute the equations with the scaling vector field L_0 when studying the wave–Klein–Gordon systems, so we cannot directly apply the Klainerman–Sobolev inequality with the scaling vector field L_0. Thus, we turn to the special version of the Klainerman–Sobolev inequality (2.13) proved in [25]. The inequality is of vital importance in our study as the scaling vector field L_0 is excluded, even though we have certain price to pay: 1) to get the pointwise decay for a function at time $t > 1$ we need the future information of the function till time $2t$; 2) we do not have any $\langle t-r\rangle$-decay of the function compared with the version of the Klainerman–Sobolev inequality with the scaling vector field L_0. Both of the flaws cause extra difficulties, and we need very delicate analysis to conquer them.

Proposition 2.5 *Let $u = u(t,x)$ be a sufficiently smooth function which decays sufficiently fast at space infinity for each fixed $t \ge 0$. Then, for any $t \ge 0$, $x \in \mathbb{R}^2$, we have*

$$\begin{aligned}
|u(t,x)| &\lesssim \langle t+|x|\rangle^{-1/2}\sup_{0\le s\le 2t, |I|\le 3}\big\|\Gamma^I u(s)\big\|,\\
&\Gamma \in V = \{L_a, \partial_\alpha, \Omega_{ab} = x^a\partial_b - x^b\partial_a\}.
\end{aligned}\tag{2.13}$$

However, when we concentrate on the region outside of the light cone, the situation becomes less complicated. In fact, we have the following version of global inequality:

Proposition 2.6 *Let $u = u(t, x)$ be a sufficiently smooth function which decays sufficiently fast at space infinity for each fixed $t \geq 0$. Then, for all $\eta \in \mathbb{R}$*

$$|\langle r-t\rangle^{\eta} u(t,x)| \lesssim r^{-1/2} \sum_{|I|\leq 1, |J|\leq 1} \|\langle r-t\rangle^{\eta} \Lambda^I \Omega^J u(t,\cdot)\|, \tag{2.14}$$

where Λ represents any of the vector in $\{\partial_r, \Omega = x^1\partial_2 - x^2\partial_1\}$ and Λ^I is a product of these vectors with order $|I|$.

Sketch of Proof We recall a slightly modified version of the classical Sobolev inequality (cf. for instance [25])

$$|u(t,x)| \leq r^{-1/2} \sum_{|I|\leq 1, |J|\leq 1} \|\Lambda^I \Omega^J u(t,\cdot)\|. \tag{2.15}$$

To show (2.15), we first apply the fundamental theorem of calculus on $u^2(t,x)$ for fixed t to get

$$u^2(t,x) = \int_{|x|}^{+\infty} 2u\partial_r u \, dr \lesssim \frac{1}{|x|}\int_{|x|}^{+\infty} \big|u\partial_r u\big| r\, dr \lesssim \frac{1}{|x|}\int_0^{+\infty} \big(u^2 + |\partial_r u|^2\big) r\, dr.$$

Then, we apply the Sobolev inequality on the circle $\mathbb{S}^1$ to get

$$\begin{aligned} u^2(t,x) &\lesssim \frac{1}{|x|}\int_{\mathbb{S}^1}\int_0^{+\infty} \Omega\big(u^2 + |\partial_r u|^2\big) r\, dr d\mathbb{S}^1 \\ &\lesssim \frac{1}{|x|}\int_{\mathbb{S}^1}\int_0^{+\infty} \big(u^2 + |\Omega u|^2 + |\partial_r u|^2 + |\Omega\partial_r u|^2\big) r\, dr d\mathbb{S}^1, \end{aligned}$$

which gives (2.15).

We note $\Omega = \Omega_{12}$ commutes with r, t, while we always get good terms when ∂_r acting on $\langle r-t\rangle^{\eta}$, that is

$$\begin{aligned} \Omega\big(\langle r-t\rangle^{\eta} u(t,x)\big) &= \langle r-t\rangle^{\eta}\Omega u(t,x), \\ \big|\partial_r\big(\langle r-t\rangle^{\eta} u(t,x)\big)\big| &= \big|\langle r-t\rangle^{\eta}\partial_r u(t,x) + \eta(r-t)\langle r-t\rangle^{\eta-2} u(t,x)\big| \\ &\lesssim \langle r-t\rangle^{\eta}\big|\partial_r u(t,x)\big| + \langle r-t\rangle^{\eta-1}|u(t,x)|. \end{aligned}$$

Thus, the proof is done. □

2.5 Pointwise Decay for Klein–Gordon Components

We recall that for linear homogeneous Klein–Gordon equations, the solutions decay at speed $\langle t+r\rangle^{-1}$ in $\mathbb{R}^{1+2}$. Since the Klainerman–Sobolev inequality in Proposition 2.5 gives at best $\langle t+r\rangle^{-1/2}$ decay rate for a given nice function in $\mathbb{R}^{1+2}$, we need the following way to obtain optimal decay for Klein–Gordon components, which was introduced by Georgiev in [13].

We denote $\{p_j\}_0^\infty$ a usual Paley–Littlewood partition of the unity

$$1 = \sum_{j\geq 0} p_j(s), \quad s \geq 0,$$

which is assumed to satisfy

$$0 \leq p_j \leq 1, \quad p_j \in C_0^\infty(\mathbb{R}), \quad \text{for all } j \geq 0,$$

and the supports of the series satisfy

$$\text{supp } p_0 \subset (-\infty, 2], \quad \text{supp } p_j \subset [2^{j-1}, 2^{j+1}], \quad \text{for all } j \geq 1.$$

Now, we are ready to give the statement of the decay result for the Klein–Gordon equations in [13].

Proposition 2.7 *Let v solve the Klein–Gordon equation*

$$-\Box v + v = f,$$

with $f = f(t,x)$ a sufficiently nice function. Then, for all $t \geq 0$, it holds

$$\begin{aligned} \langle t+|x|\rangle |v(t,x)| \lesssim & \sum_{j\geq 0,\, |I|\leq 4} \sup_{0\leq s\leq t} p_j(s) \big\| \langle s+|x|\rangle \Gamma^I f(s,x) \big\| \\ & + \sum_{j\geq 0,\, |I|\leq 5} \big\| \langle |x|\rangle p_j(|x|) \Gamma^I v(0,x) \big\|. \end{aligned} \tag{2.16}$$

As a consequence, we have the following simplified version of Proposition 2.7.

Proposition 2.8 *With the same settings as Proposition 2.7, let $\delta' > 0$ and assume*

$$\sum_{|I|\leq 4} \big\| \langle s+|x|\rangle \Gamma^I f(s,x) \big\| \leq C_f \langle s\rangle^{-\delta'}.$$

Then we have

$$\langle t+|x|\rangle |v(t,x)| \lesssim \frac{C_f}{1-2^{-\delta'}} + \sum_{|I|\leq 5} \big\| \langle |x|\rangle \log\langle 1+|x|\rangle \Gamma^I v(0,x) \big\|. \tag{2.17}$$

3 Extra Decay for Wave and Klein–Gordon Components

In the analysis, we will distinguish the regions $\{(t,x) : |x| \leq 2t\}$ and $\{(t,x) : |x| \geq 2t\}$, because we can take advantage of the extra decay properties of wave and Klein–Gordon components away from the light cone. Therefore, the extra decay properties of wave and Klein–Gordon components in the propositions below will also be demonstrated differently in different spacetime regions.

3.1 Extra Decay for Hessian of Wave Components

Proposition 3.1 *Consider the wave equation*

$$-\Box w = F_w.$$

Then we have

$$|\partial\partial w| \lesssim \frac{1}{\langle t-r\rangle}\big(|\partial\Gamma w| + |\partial w|\big) + \frac{t}{\langle t-r\rangle}|F_w|, \quad \textit{for } |x| \leq 3t, \tag{3.1}$$

as well as

$$|\partial\partial w| \lesssim \frac{r}{\langle t\rangle^2}\big(|\partial\Gamma w| + |\partial w|\big) + |F_w|, \quad \textit{for } |x| \geq \frac{3t}{2}. \tag{3.2}$$

Proof For completeness, we revisit the proof in [28]. Since it is easily seen that the results hold for $t \leq 1$, so we will only consider the case $t \geq 1$.

We first express the wave operator $-\Box$ by ∂_t, L_a to get

$$-\Box = \frac{(t-|x|)(t+|x|)}{t^2}\partial_t\partial_t + \frac{x^a}{t^2}\partial_t L_a - \frac{1}{t}\partial^a L_a + \frac{2}{t}\partial_t - \frac{x^a}{t^2}\partial_a. \tag{3.3}$$

When $t \geq 1$, one has

$$|\partial_t\partial_t w| \lesssim \frac{1}{\langle r-t\rangle}\big(|\partial\Gamma w| + |\partial w|\big) + \frac{t^2}{\langle r-t\rangle\langle r+t\rangle}|F_w|.$$

On the other hand, we note that the following relations hold true:

$$\partial_a\partial_t = -\frac{x_a}{t}\partial_t\partial_t + \frac{1}{t}\partial_t L_a - \frac{1}{t}\partial_a,$$
$$\partial_a\partial_b = \frac{x_a x_b}{t^2}\partial_t\partial_t - \frac{x_a}{t^2}\partial_t L_b + \frac{1}{t}\partial_b L_a - \frac{\delta_{ab}}{t}\partial_t + \frac{x_a}{t^2}\partial_b.$$

This leads to

$$|\partial_\alpha\partial_\beta w| \lesssim \frac{r^2 + \langle t\rangle^2}{\langle t\rangle^2\langle r-t\rangle}\big(|\partial\Gamma w| + |\partial w|\big) + \frac{r+t}{\langle r-t\rangle}|F_w|.$$

Now, when $r \leq 3t$, we remark that $r \leq \langle t \rangle$. Then, the above bound reduces to (3.1). When $r \geq 3t/2$, $\langle r+t \rangle \lesssim \langle r-t \rangle$, the above bound reduces to (3.2). □

3.2 Extra Decay for Wave Components

We recall the smooth and increasing function defined in (2.1)

$$\chi(s) := \begin{cases} 0, & s \leq 1, \\ 1, & s \geq 2. \end{cases}$$

Proposition 3.2 *Consider the wave equation*

$$-\Box w = F_w.$$

Then for any $\eta \geq 0$ *we have*

$$\begin{aligned} &\int_{\mathbb{R}^2} \chi(r-t)\langle r-t \rangle^{2\eta} (\partial w)^2 \, dx \\ &\lesssim \big\| \langle r \rangle^{\eta} \partial w(0,\cdot) \big\|^2 + \int_0^t \int_{\mathbb{R}^2} \chi(r-\tau)\langle r-\tau \rangle^{2\eta} \big| F_w \partial_t w \big| \, dx d\tau. \end{aligned} \tag{3.4}$$

Proof We pick $\chi(r-t)\langle r-t \rangle^{2\eta} \partial_t w$ as the multiplier, and we derive the identity

$$\begin{aligned} &\frac{1}{2}\partial_t \Big(\chi(r-t)\langle r-t \rangle^{2\eta} \Big((\partial_t w)^2 + \sum_a (\partial_a w)^2 \Big) \Big) - \partial_a \big(\chi(r-t)\langle r-t \rangle^{2\eta} \partial^a w \partial_t w \big) \\ &\quad + \frac{1}{2} \big(\chi'(r-t)\langle r-t \rangle^{2\eta} + 2\eta \chi(r-t)\langle r-t \rangle^{2\eta-2}(r-t) \big) \sum_a |G_a u|^2 \\ &= \chi(r-t)\langle r-t \rangle^{2\eta} F_w \partial_t w. \end{aligned}$$

We observe that

$$\big(\chi'(r-t)\langle r-t \rangle^{2\eta} + 2\eta \chi(r-t)\langle r-t \rangle^{2\eta-2}(r-t) \big) \sum_a |G_a u|^2 \geq 0.$$

Then, we are led to the desired energy estimates (3.4) by integrating the above identity over the spacetime region $[0, t] \times \mathbb{R}^2$.

The proof is complete. □

The following energy estimates allow us, in many cases, to gain better t-bound for the energy of the wave component at the expense of losing some $\langle t-r \rangle$-bound inside of the light cone $\{r \leq t\}$. This idea is inspired by the Alinhac's ghost weight energy estimates and was applied for instance in [5] when studying the Klein–Gordon–Zakharov equation in two space dimensions under compactness assumptions, and is now adapted to the non-compact setting.

Proposition 3.3 *Consider the wave equation*

$$-\Box w = F_w.$$

We have for $\eta \geq 0$ *that*

$$\begin{aligned}
&\int_{\mathbb{R}^2} \big((t-r)^{-\eta}\chi(t-r) + 1 - \chi(t-r)\big)\big|\partial w\big|^2 \, dx \\
&\lesssim \mathcal{E}_{gst}(t_0, w) + \int_{t_0}^{t} \big((\tau-r)^{-\eta}\chi(\tau-r) + 1 - \chi(\tau-r)\big)\big|F_w \partial_t w\big| \, dx d\tau.
\end{aligned} \tag{3.5}$$

Remark 3.4 We recall that $\mathcal{E}(t_0, w) = \mathcal{E}_{gst}(t_0, w)$ by (2.3).

Proof Consider the w equation, and take the multiplier

$$\big((t-r)^{-\eta}\chi(t-r) + 1 - \chi(t-r)\big)\partial_t w$$

to have the differential identity

$$\begin{aligned}
&\frac{1}{2}\partial_t\bigg(\big((t-r)^{-\eta}\chi(t-r) + 1 - \chi(t-r)\big)\Big((\partial_t w)^2 + \sum_a (\partial_a w)^2\Big)\bigg) \\
&\quad - \partial_a\Big(\big((t-r)^{-\eta}\chi(t-r) + 1 - \chi(t-r)\big)\partial^a w \partial_t w\Big) \\
&\quad + \frac{\eta}{2}(t-r)^{-\eta-1}\chi(t-r)\sum_a |G_a u|^2 + \frac{1}{2}\big(\chi'(t-r) - (t-r)^{-\eta}\chi'(t-r)\big)\sum_a |G_a u|^2 \\
&= \big((t-r)^{-\eta}\chi(t-r) + 1 - \chi(t-r)\big)\partial_t w F_w.
\end{aligned}$$

We note that

$$\frac{\eta}{2}(t-r)^{-\eta-1}\chi(t-r)\sum_a |G_a u|^2 + \frac{1}{2}\big(\chi'(t-r) - (t-r)^{-\eta}\chi'(t-r)\big)\sum_a |G_a u|^2 \geq 0,$$

so we have

$$\begin{aligned}
&\frac{1}{2}\partial_t\bigg(\big((t-r)^{-\eta}\chi(t-r) + 1 - \chi(t-r)\big)\Big((\partial_t w)^2 + \sum_a (\partial_a w)^2\Big)\bigg) \\
&\quad - \partial_a\Big(\big((t-r)^{-\eta}\chi(t-r) + 1 - \chi(t-r)\big)\partial^a w \partial_t w\Big) \\
&\leq \big((t-r)^{-\eta}\chi(t-r) + 1 - \chi(t-r)\big)\partial_t w F_w.
\end{aligned}$$

Integrating this inequality over the region $[t_0, t] \times \mathbb{R}^2$ yields the desired result.

The proof is done. □

3.3 Extra Decay for Klein–Gordon Components

Proposition 3.5 *Consider the Klein–Gordon equation*

$$-\Box v + v = F_v,$$

then for $t \geq 1$*, we have*

$$|v| \lesssim \frac{|t-r|}{\langle t \rangle}|\partial\partial v| + \frac{1}{\langle t \rangle}|\partial\Gamma v| + \frac{1}{\langle t \rangle}|\partial v| + |F_v|, \quad \text{for } |x| \leq 3t. \tag{3.6}$$

Proof This phenomena was detected in [20, 27]. The following proof can be found in [31] in the light cone. Here, we give a generalization in a larger region of spacetime. Recalling the expression of the wave operator in (3.3), we find

$$-\Box v + v = \frac{(t-|x|)(t+|x|)}{t^2}\partial_t\partial_t v + \frac{x^a}{t^2}\partial_t L_a v - \frac{1}{t}\partial^a L_a v + \frac{2}{t}\partial_t v - \frac{x^a}{t^2}\partial_a v + v,$$

which leads us to

$$|v| \lesssim \frac{|(t-|x|)(t+|x|)|}{t^2}|\partial_t\partial_t v| + \frac{|x^a|}{t^2}|\partial_t L_a v| + \frac{1}{t}|\partial^a L_a v| + \frac{2}{t}|\partial_t v| + \frac{|x^a|}{t^2}|\partial_a v| + |F_v|.$$

If $|x| \leq 3t$, we further have

$$|v| \lesssim \frac{\big|t-|x|\big|}{t}|\partial_t\partial_t v| + \sum_a \frac{1}{t}|\partial L_a v| + \frac{1}{t}|\partial v| + |F_v|,$$

which finishes the proof. □

We recall the smooth and increasing function χ defined in (2.1).

Proposition 3.6 *Consider the Klein–Gordon equation*

$$-\Box v + v = F_v.$$

Then for all $\eta \geq 0$ *we have*

$$\begin{aligned}
&\int_{\mathbb{R}^2} \chi(r-t)\langle r-t\rangle^{2\eta}\big((\partial v)^2 + v^2\big)\,dx \\
&\quad + \int_0^t \int_{\mathbb{R}^2} \big(\chi'(r-t)\langle r-t\rangle^{2\eta} + \eta\chi(r-t)\langle r-t\rangle^{2\eta-2}(r-t)\big)v^2\,dx\,d\tau \\
&\lesssim \big\|\langle r\rangle^\eta \partial v(0,\cdot)\big\|^2 + \big\|\langle r\rangle^\eta v(0,\cdot)\big\|^2 + \int_0^t \int_{\mathbb{R}^2} \chi(r-\tau)\langle r-\tau\rangle^{2\eta}\big|F_v\partial_t v\big|\,dxd\tau.
\end{aligned} \tag{3.7}$$

Proof The proof is almost the same as the proof of Proposition 3.2. We choose

$$\chi(r-t)\langle r-t\rangle^{2\eta}\partial_t v$$

to be the multiplier, and we derive the identity

$$\begin{aligned}
&\frac{1}{2}\partial_t\Big(\chi(r-t)\langle r-t\rangle^{2\eta}\Big((\partial_t v)^2+\sum_a(\partial_a v)^2+v^2\Big)\Big)-\partial_a\big(\chi(r-t)\langle r-t\rangle^{2\eta}\partial^a v\partial_t v\big)\\
&\quad+\frac{1}{2}\big(\chi'(r-t)\langle r-t\rangle^{2\eta}+2\eta\chi(r-t)\langle r-t\rangle^{2\eta-2}(r-t)\big)\Big(\sum_a|G_a v|+v^2\Big)\\
&=\chi(r-t)\langle r-t\rangle^{2\eta}F_v\partial_t v.
\end{aligned}$$

We observe that

$$\big(\chi'(r-t)\langle r-t\rangle^{2\eta}+2\eta\chi(r-t)\langle r-t\rangle^{2\eta-2}(r-t)\big)\sum_a|G_a v|\geq 0.$$

Then, we are led to the desired energy estimates (3.7) by integrating the above identity over the spacetime region $[0,t]\times\mathbb{R}^2$.

The proof is complete. □

4 Global Existence

4.1 Solution Space and Solution Mapping

In many cases, to prove the global existence of a nonlinear system one relies on a bootstrap argument. In our case, due to the utilization of the Klainerman–Sobolev inequality without scaling vector field stated in Proposition 2.5, where one requires the future information till time $2t$ when deriving the pointwise estimate for the function at time t, we turn to the aid of the contraction mapping theorem, and thus an iteration procedure. For easy readability, we will use capital letters (like Ψ, V) to denote Klein–Gordon components, while small letters (like ϕ, u) are used to represent wave components.

We now define the solution space X with some small $0<\delta\ll 1$ (recall the regularity index $N\geq 14$ below). We recall the cut-off function χ, which is smooth and increasing, defined in (2.1)

$$\chi(s):=\begin{cases}0, & s\leq 1,\\ 1, & s\geq 2.\end{cases}$$

We first define various groups of norms for a pair of functions (V,u)

- Group I:

$$
\begin{aligned}
\|(V,u)\|_{\mathrm{I}} := & \sup_{t\geq 0,\, |I|\leq N+1} \langle t\rangle^{-\delta} \mathcal{E}_{gst,1}(t, \Gamma^I V)^{1/2} \\
& + \sup_{t\geq 0,\, |I|\leq N+1} \langle t\rangle^{-\delta/2} \left(\int_0^t \left\| \frac{\Gamma^I V}{\langle \tau\rangle^{\delta/2} \langle \tau - r\rangle^{1/2+\delta/2}} \right\|^2 d\tau \right)^{1/2} \\
& + \sup_{t\geq 0,\, |I|\leq N} \langle t\rangle^{-\delta} \left\| \frac{\langle t+r\rangle}{\langle t-r\rangle} \Gamma^I V \right\|.
\end{aligned}
$$

- Group II:

$$
\|(V,u)\|_{\mathrm{II}} := \sup_{t,r\geq 0,\, |I|\leq N-5} \langle t+r\rangle \left|\Gamma^I V\right| + \sup_{t,r\geq 0,\, |I|\leq N-7} \langle t-r\rangle^{-1} \langle t+r\rangle^2 \left|\Gamma^I V\right|.
$$

- Group III:

$$
\begin{aligned}
\|(V,u)\|_{\mathrm{III}} := & \sup_{t\geq 0,\, |I|\leq N-1} \langle t\rangle^{-1/2-\delta} \Big(\left\| \chi^{1/2}(r-t) \langle t-r\rangle \partial \Gamma^I V \right\| \\
& + \left\| \chi^{1/2}(r-t) \langle t-r\rangle \Gamma^I V \right\| \Big) \\
& + \sup_{t\geq 0,\, |I|\leq N-5} \langle t\rangle^{-\delta} \Big(\left\| \chi^{1/2}(r-t) \langle t-r\rangle \partial \Gamma^I V \right\| \\
& + \left\| \chi^{1/2}(r-t) \langle t-r\rangle \Gamma^I V \right\| \Big) \\
& + \sup_{t,r\geq 0,\, |I|\leq N-8} \chi(r-t) \langle t+r\rangle^{5/4-\delta} \left|\Gamma^I V\right|.
\end{aligned}
$$

- Group IV:

$$
\begin{aligned}
\|(V,u)\|_{\mathrm{IV}} := & \sup_{t\geq 0,\, |I|\leq N+1} \langle t\rangle^{-\delta} \left\| \Gamma^I u \right\| \\
& + \sup_{t\geq 0,\, |I|\leq N-1} \langle t\rangle^{-1/2-\delta} \left\| \chi^{1/2}(r-t) \langle t-r\rangle \Gamma^I u \right\| \\
& + \sup_{t\geq 0,\, |I|\leq N-5} \langle t\rangle^{-\delta} \left\| \chi^{1/2}(r-t) \langle t-r\rangle \Gamma^I u \right\| \\
& + \sup_{t\geq 0,\, |I|\leq N} \langle t\rangle^{-\delta} \left\| \big(1-\chi(r-2t)\big)^{1/2} \langle t-r\rangle \Gamma^I u \right\| \\
& + \sup_{t\geq 0,\, |I|\leq N-2} \left\| \big(1-\chi(r-2t)\big)^{1/2} \langle t-r\rangle^{1-\delta} \Gamma^I u \right\|.
\end{aligned}
$$

- Group V:

$$
\|(V,u)\|_{\mathrm{V}} := \sup_{t,r\geq 0,\, |I|\leq N-8} \langle t-r\rangle^{1-\delta} \langle t+r\rangle^{1/2} \left|\Gamma^I u\right|.
$$

- Group VI:

$$\|(V,u)\|_{\mathrm{VI}} := \sup_{t\geq 0,\,|I|\leq N-2} \mathcal{E}_{gst}(t,\Gamma^I u)^{1/2} + \sup_{t\geq 0,\,|I|\leq N-1} \mathcal{E}_{gst,1}(t,\Gamma^I V)^{1/2}.$$

Definition 4.1 Let $\Psi = \Psi(t,x)$, $\phi = \phi(t,x)$ be sufficiently regular functions, in which Ψ is an $\mathbb{R}^2$-valued function, while ϕ is a scalar-valued function, and we say (Ψ,ϕ) belongs to the metric space X if

- It satisfies

$$\big(\Psi, \partial_t\Psi, \phi, \partial_t\phi\big)(0,\cdot) = \big(E_0, E_1, n_0, n_1\big). \tag{4.1}$$

- It satisfies

$$\big\|(\Psi,\phi)\big\|_X \leq C_1\epsilon, \tag{4.2}$$

  in which $C_1 \gg 1$ is some big constant to be determined, the size of the initial data $\epsilon \ll 1$ is small enough, such that $C_1\epsilon \ll 1$, and the norm $\|\cdot\|_X$ for a pair of $\mathbb{R}^2\times\mathbb{R}$-valued functions (V,u) is defined by (with $0<\delta\ll 1$)

$$\begin{aligned}\big\|(V,u)\big\|_X := \big\|(V,u)\big\|_{\mathrm{I}} + \big\|(V,u)\big\|_{\mathrm{II}} + \big\|(V,u)\big\|_{\mathrm{III}}\\ + \big\|(V,u)\big\|_{\mathrm{IV}} + \big\|(V,u)\big\|_{\mathrm{V}} + \big\|(V,u)\big\|_{\mathrm{VI}}.\end{aligned} \tag{4.3}$$

Remark 4.2 In general, one may follow two different strategies when applying the contraction mapping (or bootstrap) argument. The first is to include as few norms in the construction of working space as possible, and the second is the inverse. Essentially speaking neither is simpler than the other and both have their advantages. However, in the present situation, we prefer the second. Our main considerations are the following two points.

- Visualizing the functional properties of the global solution. It is clear that once the global solution is constructed, all the bounds described by the 18 pieces of norms are valid.
- Making the proof more "delayering". The estimate on each quantity is directly based on the assumptions on $\|(\Psi,\phi)\|_X < C_1\epsilon$, and one need not estimate "intermediate" quantities.

Remark 4.3 The choice for the norms included in $\|\cdot\|_X$ is determined by the feature of the model equation (1.1) and those various estimates introduced in Sects. 2 and 3. It is not a trivial task to decide which norms are included in $\|\cdot\|_X$, and we give here a simple heuristic explanation. The slow increasing rate $\langle t\rangle^\delta$ in the top-order energy of E (the first line in $\|\cdot\|_{\mathrm{I}}$) and n (the first line in $\|\cdot\|_{\mathrm{III}}$) is due to our previous work [5, 12] when treating compactly supported initial data and seems inevitable in the analysis. To close this, we find it suffices to show sharp time decay $\langle t+r\rangle^{-1}$ for lower order derivatives of E and to show $\langle t+r\rangle^{-1/2}\langle t-r\rangle^{-1/2-}$ for lower order derivatives of n (here $-1/2-$ means some number strictly smaller than $-1/2$). Then, several energy estimates and "extra decay" properties for the system are implemented to achieve this, and thus, the various types of norms are included in $\|\cdot\|_X$.

Remark 4.4 The way we divide norms into six groups allows us to prove the estimates of $\|\cdot\|_X$ in a chronological order. Besides, the proofs for the pieces of norms in the same group are connected, and for instance, the proofs of the second and third lines of $\|\cdot\|_{\mathrm{I}}$ are based on the proof of the first line of $\|\cdot\|_{\mathrm{I}}$. Moreover, the numbers of pieces of norms included in a group are reasonably large, so that the quantities can be easily referred in the proofs below.

It is easy to see that the solution space X is complete with respect to the metric induced from the norm $\|\cdot\|_X$. Next, we want to construct a contraction mapping. To achieve this, we first define a solution mapping, and then prove it is also a contraction mapping by carefully choosing the size of the parameters C_1, ϵ. We recall that $A \lesssim B$ means $A \le CB$ with C independent of C_1, ϵ.

Definition 4.5 Given a pair of functions $(\Psi, \phi) \in X$, the solution mapping T maps it to the unique pair of functions $\big(\widetilde{\Phi}, \widetilde{\phi}\big)$, which is the solution to the following linear equations:

$$
\begin{aligned}
& -\Box\widetilde{\Psi} + \widetilde{\Psi} = -\phi\Psi, \\
& -\Box\widetilde{\phi} = \Delta|\Psi|^2, \\
& \big(\widetilde{\Phi}, \partial_t\widetilde{\Phi}, \widetilde{\phi}, \partial_t\widetilde{\phi}\big)(t_0) = (E_0, E_1, n_0, n_1),
\end{aligned}
\tag{4.4}
$$

and we will write $\big(\widetilde{\Psi}, \widetilde{\phi}\big) = T(\Psi, \phi)$.

4.2 Contraction Mapping and Global Existence

The goal of this part is to show the solution mapping T is a contraction mapping from the solution space X to X.

Proposition 4.6 *With suitably chosen large C_1 and small ϵ, we have the following.*

- *Given a pair of functions $(\Psi, \phi) \in X$, we have*

$$
T(\Psi, \phi) \in X. \tag{4.5}
$$

- *For any $(\Psi, \phi), (\Psi', \phi') \in X$, it holds*

$$
\big\|T(\Psi, \phi) - T(\Psi', \phi')\big\|_X \le \frac{1}{2}\big\|(\Psi, \phi) - (\Psi', \phi')\big\|_X. \tag{4.6}
$$

Since system (1.1) is a semilinear system, there is no derivative loss in the analysis, and (4.6) can be shown in the same way as we prove (4.5).

We rewrite (4.4) to take advantage of the special structure (i.e., a hidden divergence form structure) of the nonlinearities appearing in the wave equation of $\widetilde{\phi}$, which read

$$\begin{aligned}
&-\Box\widetilde{\Psi}+\widetilde{\Psi}=-\phi\Psi,\\
&-\Box\widetilde{\phi}^{\Delta}=|\Psi|^2,\quad \widetilde{\phi}=\Delta\widetilde{\phi}^{\Delta},\\
&\big(\widetilde{\Phi},\partial_t\widetilde{\Phi},\widetilde{\phi}^{\Delta},\partial_t\widetilde{\phi}^{\Delta}\big)(t_0)=(E_0,E_1,n_0^{\Delta},n_1^{\Delta}).
\end{aligned}\tag{4.7}$$

We note that this kind of reformulation has been used before; see for instance [23]. To estimate higher order energy, we act Γ^I to (4.7) to get (recall the commutator relations in Sect. 2.3)

$$\begin{aligned}
&-\Box\Gamma^I\widetilde{\Psi}+\Gamma^I\widetilde{\Psi}=-\Gamma^I\big(\phi\Psi\big),\\
&-\Box\Gamma^I\widetilde{\phi}^{\Delta}=\Gamma^I\big(|\Psi|^2\big).
\end{aligned}\tag{4.8}$$

In the sequel, we will write $\big(\widetilde{\Psi},\widetilde{\phi}\big)=T(\Psi,\phi)$.

Preliminary Estimates

Lemma 4.7 *If (Ψ,ϕ) lies in the solution space X and the initial data $(E_0,E_1,n_0=\Delta n_0^{\Delta},n_1=\Delta n_1^{\Delta})$ satisfy the bound (1.3), that is*

$$\begin{aligned}
&\sum_{0\le j\le N+2}\|\langle|x|\rangle^{N+3}\log\langle 1+|x|\rangle E_0\|_{H^j}+\sum_{0\le j\le N+1}\|\langle|x|\rangle^{N+3}\log\langle 1+|x|\rangle E_1\|_{H^j}\\
&\quad+\sum_{0\le j\le N+3}\|\langle|x|\rangle^{j+1}n_0^{\Delta}\|_{H^j}+\sum_{0\le j\le N+2}\|\langle|x|\rangle^{j+2}n_1^{\Delta}\|_{H^j}\le\epsilon,
\end{aligned}$$

then we have

$$\begin{aligned}
&\mathcal{E}_{gst,1}(t_0,\Gamma^I\widetilde{\Psi})^{1/2}\lesssim\epsilon, && |I|\le N+1,\\
&\mathcal{E}(t_0,\partial\Gamma^I\widetilde{\phi}^{\Delta})^{1/2}\lesssim\epsilon, && |I|\le N+1,\\
&\|\langle|x|\rangle\log\langle 1+|x|\rangle\Gamma^I\widetilde{\Psi}(t_0,x)\|\lesssim\epsilon, && |I|\le N,\\
&\|\langle|x|\rangle\Gamma^I\widetilde{\Psi}(t_0,\cdot)\|\lesssim\epsilon, && |I|\le N,\\
&\|\langle|x|\rangle\partial\partial\Gamma^I\widetilde{\phi}^{\Delta}(t_0,\cdot)\|\lesssim\epsilon, && |I|\le N-1.
\end{aligned}\tag{4.9}$$

With Lemma 4.7, the contribution from the initial data is always a favorable quantity of size ϵ, and we will not specify it in the following analysis.

Lemma 4.8 *If (Ψ,ϕ) lies in the solution space X, then the following estimates hold:*

$$\begin{aligned}
&\frac{\langle t+r\rangle}{\langle t-r\rangle}\big|\Gamma^I\Psi\big|\le C_1\epsilon\langle t+r\rangle^{-1}, && |I|\le N-7,\\
&\langle t-r\rangle\big|\Gamma^I\phi\big|\le C_1\epsilon\langle t+r\rangle^{-1/2+\delta}, && |I|\le N-8.
\end{aligned}\tag{4.10}$$

Proof The first estimate is from the definition of the norm $\|\cdot\|_{\mathrm{II}}$. The second estimate is from the last line of the norm $\|\cdot\|_{\mathrm{V}}$ and the fact $\langle t-r\rangle\lesssim\langle t+r\rangle$. □

Verification of Norms in Group I

Lemma 4.9 *Let (Ψ, ϕ) lie in the solution space X. Then we have*

$$\mathcal{E}_{gst,1}(t, \Gamma^I \widetilde{\Psi})^{1/2} \lesssim \epsilon + (C_1\epsilon)^{3/2} \langle t \rangle^{\delta}, \quad |I| \le N+1, \tag{4.11}$$

$$\left(\int_0^t \left\| \frac{\Gamma^I \widetilde{\Psi}}{\langle \tau \rangle^{\delta/2} \langle \tau - r \rangle^{1/2+\delta/2}} \right\|^2 d\tau \right)^{1/2} \lesssim \epsilon + (C_1\epsilon)^{3/2} \langle t \rangle^{\delta/2}, \quad |I| \le N+1. \tag{4.12}$$

Proof We first show (4.11). Consider (4.8) and apply the ghost weight energy estimates (2.7), and for $|I| \le N+1$, we find

$$\mathcal{E}_{gst,1}(t, \Gamma^I \widetilde{\Psi})^{1/2} \lesssim \mathcal{E}_{gst,1}(t_0, \Gamma^I \widetilde{\Psi})^{1/2} + \int_0^t \left\| \Gamma^I \big(\phi \Psi\big)(\tau) \right\| d\tau.$$

Recall that Leibniz rule yields

$$\Gamma^I \big(\phi \Psi\big) = \sum_{I_1 + I_2 = I} \Gamma^{I_1} \phi \, \Gamma^{I_2} \Psi,$$

and we thus have

$$\begin{aligned} \left\| \Gamma^I \big(\phi \Psi\big) \right\| &\lesssim \sum_{|I_1| + |I_2| = |I|} \left\| \Gamma^{I_1} \phi \, \Gamma^{I_2} \Psi \right\| \\ &\lesssim \sum_{\substack{|I_2| \le N-5 \\ |I_1| + |I_2| \le |I|}} \left\| \Gamma^{I_1} \phi \, \Gamma^{I_2} \Psi \right\| + \sum_{\substack{|I_1| \le N-8 \\ |I_1| + |I_2| \le |I|}} \left\| \Gamma^{I_1} \phi \, \Gamma^{I_2} \Psi \right\|, \end{aligned} \tag{4.13}$$

in which we used $N \ge 14$ in the last inequality.

Next, we estimate those two quantities in the above inequality. We start with

$$\sum_{\substack{|I_2| \le N-5 \\ |I_1| + |I_2| \le |I|}} \left\| \Gamma^{I_1} \phi \, \Gamma^{I_2} \Psi \right\| \lesssim \sum_{\substack{|I_2| \le N-5 \\ |I_1| \le |I|}} \left\| \Gamma^{I_1} \phi \right\| \left\| \Gamma^{I_2} \Psi \right\|_{L^\infty} \lesssim (C_1 \epsilon)^2 t^{-1+\delta}, \tag{4.14}$$

and in the last inequality, we used the estimate from $\| \cdot \|_{\mathrm{II}}$ and the one in the first line of $\| \cdot \|_{\mathrm{IV}}$. As for the other one, we have

$$\begin{aligned} &\sum_{\substack{|I_1| \le N-8 \\ |I_1| + |I_2| \le |I|}} \left\| \Gamma^{I_1} \phi \, \Gamma^{I_2} \Psi \right\| \lesssim \sum_{\substack{|I_1| \le N-8 \\ |I_1| + |I_2| \le |I|}} \left\| \langle t \rangle^{\delta/2} \langle t - r \rangle^{1/2+\delta/2} \Gamma^{I_1} \phi \right\|_{L^\infty} \left\| \frac{\Gamma^{I_2} \Psi}{\langle t \rangle^{\delta/2} \langle t - r \rangle^{1/2+\delta/2}} \right\| \\ &\lesssim C_1 \epsilon \langle t \rangle^{-1/2+\delta/2} \sum_{|I_2| \le N+1} \left\| \frac{\Gamma^{I_2} \Psi}{\langle t \rangle^{\delta/2} \langle t - r \rangle^{1/2+\delta/2}} \right\|, \end{aligned} \tag{4.15}$$

in which we used the estimate from $\|\cdot\|_V$ and the smallness of δ.

Gathering the estimates leads us to (4.11)

$$
\begin{aligned}
&\mathcal{E}_{gst,1}(t,\Gamma^I\widetilde{\Psi})^{1/2}\\
&\lesssim \epsilon+\int_0^t \Big((C_1\epsilon)^2\tau^{-1+\delta}+C_1\epsilon\langle\tau\rangle^{-1/2+\delta/2}\sum_{|I_2|\le N+1}\Big\|\frac{\Gamma^{I_2}\Psi}{\langle\tau\rangle^{\delta/2}\langle\tau-r\rangle^{1/2+\delta/2}}\Big\|\Big)\,d\tau\\
&\lesssim \epsilon+(C_1\epsilon)^2\langle t\rangle^{\delta}+C_1\epsilon\Big(\int_0^t\langle\tau\rangle^{-1+\delta}\,d\tau\Big)^{1/2}\sum_{|I_2|\le N+1}\Big(\int_0^t\Big\|\frac{\Gamma^{I_2}\Psi}{\langle\tau\rangle^{\delta/2}\langle\tau-r\rangle^{1/2+\delta/2}}\Big\|^2 d\tau\Big)^{1/2}\\
&\lesssim \epsilon+(C_1\epsilon)^2\langle t\rangle^{\delta},
\end{aligned}
$$

and in the last inequality, we used the estimate from the second line of $\|\cdot\|_I$.

Finally, we want to deduce (4.12). The ghost weight energy estimates (2.8) indicate

$$
\begin{aligned}
&\int_0^t\Big\|\frac{\Gamma^I\widetilde{\Psi}}{\tau^{\delta/2}\langle\tau-r\rangle^{1/2+\delta/2}}\Big\|^2 d\tau\\
&\quad\lesssim \mathcal{E}_{gst,1}(t_0,\Gamma^I\widetilde{\Psi})+\int_0^t\int_{\mathbb{R}^2}\langle\tau\rangle^{-\delta}\big|\Gamma^I\big(\phi\Psi\big)\partial_t\Gamma^I\widetilde{\Psi}\big|(\tau,x)\,dxd\tau.
\end{aligned}
$$

With the estimates (4.11) we just proved, we have

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^2}\langle\tau\rangle^{-\delta}\big|\Gamma^I\big(\phi\Psi\big)\partial_t\Gamma^I\widetilde{\Psi}\big|(\tau,x)\,dxd\tau\\
&\lesssim\int_0^t\langle\tau\rangle^{-\delta}\big\|\Gamma^I\big(\phi\Psi\big)\big\|\,\big\|\partial_t\Gamma^I\widetilde{\Psi}\big\|\,d\tau\\
&\lesssim\Big(\int_0^t\langle\tau\rangle^{-\delta+1}\big\|\Gamma^I\big(\phi\Psi\big)\big\|^2 d\tau\Big)^{1/2}\Big(\int_0^t\langle\tau\rangle^{-\delta-1}\big\|\partial_t\Gamma^I\widetilde{\Psi}\big\|^2 d\tau\Big)^{1/2}\\
&\lesssim C_1\epsilon\langle t\rangle^{\delta/2}\Big(\int_0^t\langle\tau\rangle^{-\delta+1}\big\|\Gamma^I\big(\phi\Psi\big)\big\|^2 d\tau\Big)^{1/2}.
\end{aligned}
$$

To proceed, we find

$$
\begin{aligned}
&\Big(\int_0^t\langle\tau\rangle^{-\delta+1}\big\|\Gamma^I\big(\phi\Psi\big)\big\|^2 d\tau\Big)^{1/2}\\
&\lesssim\Big(\int_0^t\Big((C_1\epsilon)^2\langle\tau\rangle^{-1+\delta}+C_1\epsilon\sum_{|I_2|\le N+1}\Big\|\frac{\Gamma^{I_2}\Psi}{\langle\tau\rangle^{\delta/2}\langle\tau-r\rangle^{1/2+\delta/2}}\Big\|^2\Big)\,d\tau\Big)^{1/2}\\
&\lesssim (C_1\epsilon)^2\langle t\rangle^{\delta/2},
\end{aligned}
$$

which further gives us

$$
\int_0^t\Big\|\frac{\Gamma^I\widetilde{\Psi}}{\tau^{\delta/2}\langle\tau-r\rangle^{1/2+\delta/2}}\Big\|^2 d\tau\lesssim\epsilon^2+(C_1\epsilon)^3\langle t\rangle^{\delta},
$$

and hence (4.12).

We complete the proof. □

Lemma 4.10 *The following estimates are valid:*

$$\left\| \frac{\langle t+r\rangle}{\langle t-r\rangle} \Gamma^I \widetilde{\Psi} \right\| \lesssim \epsilon + (C_1\epsilon)^{3/2} \langle t\rangle^{\delta}, \quad |I| \le N. \tag{4.16}$$

Proof In the region $\{r \geq 2t\}$, it holds $\langle t+r\rangle \lesssim \langle t-r\rangle$, and thus, we only need to consider the region $\{r \leq 2t\}$. Recall that Lemma 4.9 provides us with

$$\left\|\partial \Gamma^J \widetilde{\Psi}\right\| + \left\|\Gamma^J \widetilde{\Psi}\right\| \lesssim \epsilon + (C_1\epsilon)^2 \langle t\rangle^{\delta}, \quad |I| \le N+1.$$

Thus, the proof for the region $\{r \leq 2t\}$ follows from Lemma 4.9 and Proposition 3.5. □

Verification of Norms in Group II

Lemma 4.11 *The following estimates hold:*

$$\left|\Gamma^I \widetilde{\Psi}\right| \lesssim \left(\epsilon + (C_1\epsilon)^2\right) \langle t+r\rangle^{-1}, \quad |I| \le N-5. \tag{4.17}$$

Proof By Proposition 2.8, we need to bound the quantity

$$\sum_{|I|\le N-1} \left\|\langle t+r\rangle \Gamma^I \left(\phi\Psi\right)\right\|.$$

We have

$$\begin{aligned}
&\sum_{|I|\le N-1} \left\|\langle t+r\rangle \Gamma^I \left(\phi\Psi\right)\right\| \\
&\lesssim \sum_{\substack{|I_1|\le N-1\\ |I_2|\le N-8}} \left\| \frac{\langle t+r\rangle}{\langle t-r\rangle}\Gamma^{I_1}\Psi \right\| \left\|\langle t-r\rangle \Gamma^{I_2}\phi\right\|_{L^\infty} + \sum_{\substack{|I_1|\le N-7\\ |I_2|\le N-1}} \left\| \frac{\langle t+r\rangle}{\langle t-r\rangle}\Gamma^{I_1}\Psi \right\|_{L^\infty} \left\|\langle t-r\rangle \Gamma^{I_2}\phi\right\| \\
&\lesssim (C_1\epsilon)^2 \langle t\rangle^{-1/2+3\delta},
\end{aligned}$$

in which we used the estimates from $\|\cdot\|_{\mathrm{I}}$, $\|\cdot\|_{\mathrm{IV}}$, and Lemma 4.8 in the last inequality. Then, Proposition 2.8 yields the desired result (4.17). □

Lemma 4.12 *It holds*

$$\left|\Gamma^I \widetilde{\Psi}\right| \lesssim \left(\epsilon + (C_1\epsilon)^2\right) \frac{\langle t-r\rangle}{\langle t+r\rangle^2}, \quad |I| \le N-7. \tag{4.18}$$

Proof The proof follows from Lemma 4.11 and Proposition 3.5. □

Verification of Norms in Group III

Lemma 4.13 *We have*

$$\big\|\chi^{1/2}(r-t)\langle r-t\rangle\partial\Gamma^I\widetilde{\Psi}\big\| + \big\|\chi^{1/2}(r-t)\langle r-t\rangle\Gamma^I\widetilde{\Psi}\big\| \lesssim \begin{cases}\epsilon + (C_1\epsilon)^2\langle t\rangle^{1/2+\delta}, & |I|\le N-1,\\ \epsilon + (C_1\epsilon)^2\langle t\rangle^{\delta}, & |I|\le N-5.\end{cases} \tag{4.19}$$

Proof We apply the energy estimates in Proposition 3.6 with $\eta = 1$ to the $\Gamma^I\widetilde{\Psi}$ equation in (4.8) with $|I|\le N-1$, and obtain

$$\begin{aligned}&\big\|\chi^{1/2}(r-t)\langle r-t\rangle\partial\Gamma^I\widetilde{\Psi}\big\| + \big\|\chi^{1/2}(r-t)\langle r-t\rangle\Gamma^I\widetilde{\Psi}\big\|\\ &\lesssim \big\|\langle r\rangle\partial\Gamma^I\widetilde{\Psi}(t_0,\cdot)\big\| + \big\|\langle r\rangle\Gamma^I\widetilde{\Psi}(t_0,\cdot)\big\| + \int_{t_0}^t \big\|\chi^{1/2}(r-\tau)\langle r-\tau\rangle\Gamma^I(\phi\Psi)\big\|\,d\tau.\end{aligned}$$

We note that

$$\begin{aligned}&\big\|\chi^{1/2}(r-\tau)\langle r-\tau\rangle\Gamma^I(\phi\Psi)\big\|\\ &\lesssim \sum_{|I_1|\le N-8,\,|I_2|\le N-1}\big\|\chi^{1/2}(r-\tau)\langle r-\tau\rangle\Gamma^{I_1}\phi\big\|_{L^\infty}\big\|\Gamma^{I_2}\Psi\big\|\\ &\quad+\sum_{|I_1|\le N-1,\,|I_2|\le N-5}\big\|\chi^{1/2}(r-\tau)\langle r-\tau\rangle\Gamma^{I_1}\phi\big\|\big\|\Gamma^{I_2}\Psi\big\|_{L^\infty}\\ &\lesssim (C_1\epsilon)^2\langle\tau\rangle^{-1/2+\delta},\end{aligned}$$

where, in the last step, we used the estimates from $\|\cdot\|_{\mathrm{II}}$, $\|\cdot\|_{\mathrm{IV}}$, $\|\cdot\|_{\mathrm{VI}}$, and Lemma 4.8, and which gives the first inequality in (4.19).

Analogously, for the case of $|I|\le N-5$, we only need to bound

$$\big\|\chi^{1/2}(r-\tau)\langle r-\tau\rangle\Gamma^I(\phi\Psi)\big\|.$$

Our strategy is to always take L^∞-norm on the Ψ part, and by the estimates from $\|\cdot\|_{\mathrm{II}}$, $\|\cdot\|_{\mathrm{IV}}$, we find

$$\begin{aligned}\big\|\chi^{1/2}(r-\tau)\langle r-\tau\rangle\Gamma^I(\phi\Psi)\big\| &\lesssim \sum_{|I_1|\le N-5,\,|I_2|\le N-5}\big\|\chi^{1/2}(r-\tau)\langle r-\tau\rangle\Gamma^{I_1}\phi\big\|\big\|\Gamma^{I_2}\Psi\big\|_{L^\infty}\\ &\lesssim (C_1\epsilon)^2\langle\tau\rangle^{-1+\delta},\end{aligned}$$

which finishes the proof. □

Lemma 4.14 *We get*

$$\chi(r-t)\big|\Gamma^I\widetilde{\Psi}\big| \lesssim \big(\epsilon + (C_1\epsilon)^{3/2}\big)\langle t+r\rangle^{-5/4+\delta},\quad |I|\le N-8. \tag{4.20}$$

Proof We will rely on the weighted energy estimates in Lemma 4.13 and the weighted Sobolev inequality in Proposition 2.6 to derive the pointwise estimates in (4.20).

Applying the weighted Sobolev inequality in Proposition 2.6 implies (note we have $\langle r\rangle \lesssim r$ within the support of $\chi(r-t)$)

$$\chi(r-t)\langle t-r\rangle\langle r\rangle^{1/2}\big|\Gamma^I\widetilde{\Psi}\big| \lesssim \sum_{|J_2|\le 1,|J_1|\le 1}\big\|\Lambda^{J_2}\Omega^{J_1}\big(\chi(r-t)\langle t-r\rangle\Gamma^I\widetilde{\Psi}\big)\big\|,$$

for $\Lambda \in \{\partial_r, \Omega = \Omega_{12}\}$. We note within the support of $\chi'(r-t)$, it holds that $1 \lesssim \langle t-r\rangle \lesssim 1$, and by the commutator estimates $\Omega_{12}r = \Omega_{12}t = 0$, we find

$$\sum_{|J_2|\le 1,|J_1|\le 1,|I|\le N-8}\big\|\Lambda^{J_2}\Omega^{J_1}\big(\chi(r-t)\langle t-r\rangle\Gamma^I\widetilde{\Psi}\big)\big\| \lesssim \sum_{|I_1|\le N-5}\big\|\chi(r-t)\langle t-r\rangle\Gamma^{I_1}\widetilde{\Psi}\big\|.$$

Then, by Lemma 4.13, we get

$$\sum_{|J_2|\le 1,|J_1|\le 1,|I|\le N-8}\big\|\Lambda^{J_2}\Omega^{J_1}\big(\chi(r-t)\langle t-r\rangle\Gamma^I\widetilde{\Psi}\big)\big\| \lesssim \epsilon + (C_1\epsilon)^2\langle t\rangle^{\delta},$$

and hence

$$\begin{aligned}\chi(r-t)\big|\Gamma^I\widetilde{\Psi}\big| &\lesssim \big(\epsilon + (C_1\epsilon)^2\big)\langle t-r\rangle^{-1}\langle r\rangle^{-1/2+\delta}\\ &\lesssim \big(\epsilon + (C_1\epsilon)^2\big)\langle t-r\rangle^{-1}\langle t+r\rangle^{-1/2+\delta}, \quad |I|\le N-8.\end{aligned}$$

Finally, by the aid of Lemma 4.12, we are led to

$$\begin{aligned}\chi(r-t)\big|\Gamma^I\widetilde{\Psi}\big| &\lesssim \chi^{1/2}(r-t)\big|\Gamma^I\widetilde{\Psi}\big|^{1/2}\big|\Gamma^I\widetilde{\Psi}\big|^{1/2}\\ &\lesssim \big(\epsilon + (C_1\epsilon)^2\big)\langle t-r\rangle^{-1/2}\langle t+r\rangle^{-1/4+\delta/2}\langle t-r\rangle^{1/2}\langle t+r\rangle^{-1}\\ &\lesssim \big(\epsilon + (C_1\epsilon)^2\big)\langle t+r\rangle^{-5/4+\delta}, \quad |I|\le N-8.\end{aligned}$$

The proof is done. □

Verification of Norms in Group IV

Lemma 4.15 *We get*

$$\big\|\Gamma^I\widetilde{\phi}\big\| \lesssim \epsilon + (C_1\epsilon)^2\langle t\rangle^{\delta}, \quad |I|\le N+1. \tag{4.21}$$

Proof Our strategy is to first prove the bounds for $\widetilde{\phi}^{\Delta}$, and then pass them to $\widetilde{\phi}$ according to the relation

$$\widetilde{\phi} = \Delta\widetilde{\phi}^{\Delta}.$$

We act $\partial\Gamma^I$ with $|I|\le N+1$ to the $\widetilde{\phi}^{\Delta}$ equation to get

$$-\Box\partial\Gamma^I\widetilde{\phi}^{\Delta} = \partial\Gamma^I\big(|\Psi|^2\big).$$

Then, the energy estimates (2.5) imply

$$\mathcal{E}(t, \partial\Gamma^I\widetilde{\phi}^\Delta)^{1/2} \lesssim \mathcal{E}(t_0, \partial\Gamma^I\widetilde{\phi}^\Delta)^{1/2} + \int_0^t \big\|\partial\Gamma^I\big(|\Psi|^2\big)\big\|(\tau)\,d\tau.$$

Simple analysis shows

$$\begin{aligned}\big\|\partial\Gamma^I\big(|\Psi|^2\big)\big\| &\lesssim \sum_{\substack{|I_1|\le|I|\\|I_2|\le N-5}} \big\|\partial\Gamma^{I_1}\Psi\big\|\big\|\Gamma^{I_2}\Psi\big\|_{L^\infty} + \sum_{\substack{|I_1|\le|I|\\|I_2|\le N-6}} \big\|\Gamma^{I_1}\Psi\big\|\big\|\partial\Gamma^{I_2}\Psi\big\|_{L^\infty}\\ &\lesssim (C_1\epsilon)^2\langle t\rangle^{-1+\delta},\end{aligned}$$

in which we used the estimates from $\|\cdot\|_{\mathrm{I}}, \|\cdot\|_{\mathrm{II}}$.

Thus, we have

$$\mathcal{E}(t, \partial\Gamma^I\widetilde{\phi}^\Delta)^{1/2} \lesssim \epsilon + (C_1\epsilon)^2\langle t\rangle^\delta, \quad |I| \le N+1. \tag{4.22}$$

Finally, we observe for $|I| \le N+1$

$$\big\|\Gamma^I\widetilde{\phi}\big\| = \big\|\Gamma^I\Delta\widetilde{\phi}^\Delta\big\| \lesssim \sum_{|I_1|\le N+1} \big\|\partial\partial\Gamma^{I_1}\widetilde{\phi}^\Delta\big\| \lesssim \epsilon + (C_1\epsilon)^2\langle t\rangle^\delta,$$

which finishes the proof. □

Lemma 4.16 *The following holds:*

$$\big\|\chi^{1/2}(r-t)\langle r-t\rangle\Gamma^I\widetilde{\phi}\big\| \lesssim \begin{cases}\epsilon + (C_1\epsilon)^2\langle t\rangle^{1/2+\delta}, & |I| \le N-1,\\ \epsilon + (C_1\epsilon)^2\langle t\rangle^\delta, & |I| \le N-5.\end{cases} \tag{4.23}$$

Proof As before, our strategy is to first derive the estimates for $\widetilde{\phi}^\Delta$, and then transform the estimates to $\widetilde{\phi}$ via the relation

$$\widetilde{\phi} = \Delta\widetilde{\phi}^\Delta.$$

Consider the $\partial\Gamma^I\widetilde{\phi}^\Delta$ equation in (4.8) with $|I| \le N-1$, and we apply the energy estimates in Proposition 3.2 with $\eta = 1$ to get

$$\begin{aligned}\big\|\chi^{1/2}(r-t)\langle r-t\rangle\partial\partial\Gamma^I\widetilde{\phi}^\Delta\big\| &\lesssim \big\|\langle r\rangle\partial\partial\Gamma^I\widetilde{\phi}^\Delta(t_0,\cdot)\big\|\\ &\quad + \int_0^t \big\|\chi^{1/2}(r-\tau)\langle r-\tau\rangle\partial\Gamma^I\big|\Psi\big|^2\big\|\,d\tau.\end{aligned}$$

In succession, we have

$$\begin{aligned}\big\|\chi^{1/2}(r-t)\langle r-t\rangle\partial\Gamma^I\big|\Psi\big|^2\big\| &\lesssim \sum_{|I_1|\le N-1, |I_2|\le N-5} \big\|\chi^{1/2}(r-t)\langle r-t\rangle\partial\Gamma^{I_1}\Psi\big\|\big\|\Gamma^{I_2}\Psi\big\|_{L^\infty}\\ &+ \sum_{|I_1|\le N-1, |I_2|\le N-5} \big\|\chi^{1/2}(r-t)\langle r-t\rangle\Gamma^{I_1}\Psi\big\|\big\|\Gamma^{I_2}\Psi\big\|_{L^\infty} \lesssim (C_1\epsilon)^2\langle t\rangle^{-1/2+\delta},\end{aligned}$$

in which we used the estimates from $\|\cdot\|_{\mathrm{II}}$, $\|\cdot\|_{\mathrm{III}}$, and which further gives us

$$\begin{aligned}\big\|\chi^{1/2}(r-t)\langle r-t\rangle\partial\partial\Gamma^I\widetilde{\phi}^\Delta\big\| &\lesssim \epsilon+(C_1\epsilon)^2\int_0^t\langle\tau\rangle^{-1/2+\delta}\,d\tau\\ &\lesssim \epsilon+(C_1\epsilon)^2\langle t\rangle^{1/2+\delta},\quad |I|\le N-1.\end{aligned}$$

Thus, the first inequality in (4.23) is verified due to

$$\begin{aligned}\big\|\chi^{1/2}(r-t)\langle r-t\rangle\Gamma^I\widetilde{\phi}\big\| &= \big\|\chi^{1/2}(r-t)\langle r-t\rangle\Gamma^I\Delta\widetilde{\phi}^\Delta\big\|\\ &\lesssim \sum_{|I_1|\le|I|}\big\|\chi^{1/2}(r-t)\langle r-t\rangle\partial\partial\Gamma^{I_1}\widetilde{\phi}^\Delta\big\|\\ &\lesssim \epsilon+(C_1\epsilon)^2\langle t\rangle^{1/2+\delta},\quad |I|\le N-1.\end{aligned}$$

The case of $|I|\le N-5$ can be derived in the same manner. The proof is complete. □

Lemma 4.17 *We have the following estimates:*

$$\big\|\big(1-\chi(r-2t)\big)^{1/2}\langle t-r\rangle\Gamma^I\widetilde{\phi}\big\|\lesssim\big(\epsilon+(C_1\epsilon)^2\big)\langle t\rangle^\delta,\quad |I|\le N. \tag{4.24}$$

Proof Again, we will first show the bounds for $\widetilde{\phi}^\Delta$, and then pass them to $\widetilde{\phi}$. We only consider the region $\{r\le 3t\}$ for large t in the following.

Similar to (4.22), we have

$$\mathcal{E}(t,\Gamma^I\widetilde{\phi}^\Delta)^{1/2}\lesssim\epsilon+(C_1\epsilon)^2\langle t\rangle^\delta,\quad |I|\le N+1. \tag{4.25}$$

Recall that we can obtain some extra decay for the Hessian form of the wave components as illustrated in Proposition 3.1, which, for the $\widetilde{\phi}^\Delta$ component in Eq. (4.7), reads

$$|\partial\partial\widetilde{\phi}^\Delta|\lesssim\frac{1}{\langle t-r\rangle}\big(|\partial\Gamma\widetilde{\phi}^\Delta|+|\partial\widetilde{\phi}^\Delta|\big)+\frac{t}{\langle t-r\rangle}|\Psi|^2,\quad r\le 3t,$$

in which we used the relation $r\lesssim t$. To proceed, we have

$$\begin{aligned}&\big(1-\chi(r-2t)\big)^{1/2}\langle t-r\rangle|\partial\partial\widetilde{\phi}^\Delta|\\ &\lesssim\big(1-\chi(r-2t)\big)^{1/2}\big(|\partial\Gamma\widetilde{\phi}^\Delta|+|\partial\widetilde{\phi}^\Delta|\big)+\big(1-\chi(r-2t)\big)^{1/2}t|\Psi|^2.\end{aligned}$$

Taking L^2-norm and using the simple triangle inequality yield

$$\begin{aligned}\big\|\big(1-\chi(r-2t)\big)^{1/2}\langle t-r\rangle|\partial\partial\widetilde{\phi}^\Delta|\big\| &\lesssim\big\|\partial\Gamma\widetilde{\phi}^\Delta\big\|+\big\|\partial\widetilde{\phi}^\Delta\big\|+\big\|t|\Psi|^2\big\|\\ &\lesssim\epsilon+(C_1\epsilon)^2\langle t\rangle^\delta,\end{aligned}$$

in which we used (4.25) in the last step. Thus, we obtain

$$\begin{aligned}\big\|\big(1-\chi(r-2t)\big)^{1/2}\langle t-r\rangle\widetilde{\phi}\big\| &\lesssim \big\|\big(1-\chi(r-2t)\big)^{1/2}\langle t-r\rangle|\partial\partial\widetilde{\phi}^{\Delta}|\big\| \\ &\lesssim \epsilon+(C_1\epsilon)^2\langle t\rangle^{\delta}.\end{aligned}$$

In the same way [with (4.25)], we get (4.24). The proof is done. □

Lemma 4.18 *The following bounds hold true:*

$$\big\|\big(1-\chi(r/2t)\big)\langle t-r\rangle^{1-\delta}\Gamma^J\widetilde{\phi}\big\| \lesssim \epsilon+(C_1\epsilon)^2, \quad |J|\le N-2. \tag{4.26}$$

Proof We work with the $\Gamma^I\widetilde{\phi}^{\Delta}$ equation with $|I|\le N-1$. The energy estimates (3.5) give us

$$\begin{aligned}&\big\|\big((t-r)^{-2\delta}\chi(t-r)+1-\chi(t-r)\big)^{1/2}\partial\Gamma^I\widetilde{\phi}^{\Delta}\big\|^2 \\ &\lesssim \mathcal{E}_{gst}(t_0,\Gamma^I\widetilde{\phi}^{\Delta})+\int_{t_0}^t\int_{\mathbb{R}^2}\Big|\big((\tau-r)^{-2\delta}\chi(\tau-r)+1-\chi(\tau-r)\big)\Gamma^I|\Psi|^2\partial_t\Gamma^I\widetilde{\phi}^{\Delta}\Big|\,dxd\tau.\end{aligned}$$

We need to bound the above spacetime integral, and we find

$$\begin{aligned}&\int_{t_0}^t\int_{\mathbb{R}^2}\Big|\big((\tau-r)^{-2\delta}\chi(\tau-r)+1-\chi(\tau-r)\big)\Gamma^I|\Psi|^2\partial_t\Gamma^I\widetilde{\phi}^{\Delta}\Big|\,dxd\tau \\ &\lesssim \int_{t_0}^t\Big\|\big((\tau-r)^{-2\delta}\chi(\tau-r)+1-\chi(\tau-r)\big)\Gamma^I|\Psi|^2\Big\|\,\big\|\partial_t\Gamma^I\widetilde{\phi}^{\Delta}\big\|\,d\tau \\ &\lesssim C_1\epsilon\int_{t_0}^t\Big\|\big((\tau-r)^{-2\delta}\chi(\tau-r)+1-\chi(\tau-r)\big)\Gamma^I|\Psi|^2\Big\|\langle\tau\rangle^{\delta}\,d\tau.\end{aligned}$$

We then do the estimates in different regions (note that the relation $1\lesssim\langle t-r\rangle\lesssim 1$ holds when $|t-r|\lesssim 1$), and we proceed to have

$$\begin{aligned}&\int_{t_0}^t\Big\|\big((\tau-r)^{-2\delta}\chi(\tau-r)+1-\chi(\tau-r)\big)\Gamma^I|\Psi|^2\Big\|\langle\tau\rangle^{\delta}\,d\tau \\ &\lesssim \int_{t_0}^t\big\|\langle\tau-r\rangle^{-2\delta}\Gamma^I|\Psi|^2\big\|\langle\tau\rangle^{\delta}\,d\tau+\int_{t_0}^t\big\|\chi(r-\tau)\Gamma^I|\Psi|^2\big\|\langle\tau\rangle^{\delta}\,d\tau \\ &=: A_1+A_2.\end{aligned}$$

To estimate A_1, we utilize the spacetime integral bounds in the ghost weight energy estimates to get

$$\begin{aligned}A_1 &\lesssim \sum_{|I_1|\le N-1,|I_2|\le N-7}\int_{t_0}^t\big\|\Gamma^{I_1}\Psi\big\|\,\big\|\langle\tau-r\rangle^{-2\delta}\Gamma^{I_2}\Psi\big\|_{L^\infty}\langle\tau\rangle^{\delta}\,d\tau \\ &\lesssim (C_1\epsilon)^2\int_{t_0}^t\langle\tau\rangle^{-1-\delta}\,d\tau \\ &\lesssim (C_1\epsilon)^2,\end{aligned}$$

and we used the estimates from $\|\cdot\|_{\mathrm{VI}}$ and Lemma 4.8 in the second step. For the term A_2, we apply the estimates from $\|\cdot\|_{\mathrm{III}}$, $\|\cdot\|_{\mathrm{VI}}$ to have

$$
\begin{aligned}
A_2 &\lesssim \sum_{|I_1|\le N-1, |I_2|\le N-8} \int_{t_0}^{t} \|\Gamma^{I_1}\Psi\| \big\|\chi(r-\tau)\Gamma^{I_2}\Psi|\big\|_{L^\infty} \langle\tau\rangle^{\delta}\, d\tau \\
&\lesssim (C_1\epsilon)^2 \int_{t_0}^{t} \langle\tau\rangle^{-5/4+3\delta}\, d\tau \lesssim (C_1\epsilon)^2.
\end{aligned}
$$

Gathering the above estimates, we arrive at

$$
\big\|\big((t-r)^{-2\delta}\chi(t-r)+1-\chi(t-r)\big)^{1/2}\partial\Gamma^I\widetilde{\phi}^{\Delta}\big\| \lesssim \epsilon + (C_1\epsilon)^{3/2}, \quad |I|\le N-1.
$$

Finally, recall again the estimates for the Hessian of wave component in Proposition 3.1

$$
\big|\partial\partial\Gamma^J\widetilde{\phi}^{\Delta}\big| \lesssim \frac{1}{\langle t-r\rangle}\sum_{|I|\le|J|+1}\big|\partial\Gamma^J\widetilde{\phi}^{\Delta}\big| + \frac{\langle t\rangle}{\langle t-r\rangle}\big|\Gamma^J|\Psi|^2\big|, \quad r\le 3t,
$$

and we further obtain (for $|J|\le N-2$)

$$
\begin{aligned}
&\big\|\big(1-\chi(r/2t)\big)\langle t-r\rangle^{1-\delta}\Gamma^J\widetilde{\phi}\big\| \\
&\quad\lesssim \big\|\big((t-r)^{-2\delta}\chi(t-r)+1-\chi(t-r)\big)^{1/2}\langle t-r\rangle\Gamma^J\widetilde{\phi}\big\| \\
&\quad\lesssim \sum_{|J_1|\le N-2}\big\|\big((t-r)^{-2\delta}\chi(t-r)+1-\chi(t-r)\big)^{1/2}\langle t-r\rangle\partial\partial\Gamma^{J_1}\widetilde{\phi}^{\Delta}\big\| \\
&\quad\lesssim \epsilon + (C_1\epsilon)^2.
\end{aligned}
$$

The proof is done. □

Verification of Norms in Group V

Lemma 4.19 *We have for* $|I|\le N-5$

$$
\big|\Gamma^I\widetilde{\phi}\big| \lesssim \big(\epsilon+(C_1\epsilon)^2\big)\langle t-r\rangle^{-1+\delta}\langle t+r\rangle^{-1/2}, \quad \frac{3t}{4}\le r\le 2t. \tag{4.27}
$$

Proof The proof follows from Lemma 4.18 and the weighted Sobolev inequality in Proposition 2.6. We note that (4.27) is true for $r\le 1$, so in the following, we only consider $r\ge 1$.

For $|I|\le N-5$, we have (for $\Lambda\in\{\partial_r, \Omega=\Omega_{12}\}$)

$$
\begin{aligned}
&\langle r\rangle^{1/2}\big|\big(1-\chi(r/2t)\big)\chi(1/2+2r/t)\langle t-r\rangle^{1-\delta}\Gamma^I\widetilde{\phi}\big| \\
&\quad\lesssim \sum_{|J|\le 2}\sup_{t\ge t_0}\Big\|\Lambda^J\Big(\big(1-\chi(r/2t)\big)\chi(1/2+2r/t)\langle t-r\rangle^{1-\delta}\Gamma^I\widetilde{\phi}\Big)\Big\|.
\end{aligned}
$$

Note that the rotation vector field Ω commutes with r, t (see also Proposition 2.6), which gives us

$$\begin{aligned}
&\sum_{|J|\leq 2} \Big\| \Lambda^J \Big(\big(1-\chi(r/2t)\big)\chi(1/2+2r/t)\langle t-r\rangle^{1-\delta}\Gamma^I\widetilde{\phi}\Big)\Big\| \\
&\lesssim \sum_{|I_1|\leq N-2} \Big\| \big(1-\chi(r/2t)\big)\chi(1/2+2r/t)\langle t-r\rangle^{1-\delta}\Gamma^{I_1}\widetilde{\phi}\Big\| \\
&\quad + \sum_{|I_1|\leq N-3} \Big\| \langle t\rangle^{-1}\langle t-r\rangle^{1-\delta}\Gamma^{I_1}\widetilde{\phi}\Big\| \\
&\lesssim \epsilon + (C_1\epsilon)^2,
\end{aligned}$$

which leads us to

$$\begin{aligned}
&\big|\big(1-\chi(r/2t)\big)\chi(1/2+2r/t)\langle t-r\rangle^{1-\delta}\Gamma^I\widetilde{\phi}\big| \\
&\lesssim \big(\epsilon+(C_1\epsilon)^2\big)\langle r\rangle^{-1/2}, \quad |I|\leq N-5.
\end{aligned}$$

The proof is complete by noting $\langle t+r\rangle \lesssim \langle r\rangle \lesssim \langle t+r\rangle$ when $r\geq 3t/4$. □

Lemma 4.20 *The following pointwise bounds are valid:*

$$\big|\Gamma^I\widetilde{\phi}\big| \lesssim \big(\epsilon+(C_1\epsilon)^2\big)\langle t-r\rangle^{-1+\delta}\langle t+r\rangle^{-1/2}, \quad |I|\leq N-8. \tag{4.28}$$

Proof Thanks to the estimates in Lemma 4.19, we only need to show (4.28) holds in the regions $\{r\leq 3t/4\}$ and $\{r\geq 2t\}$ for large t.

Case I: $\{r\leq 3t/4\}$. Consider first the $\Gamma^{J_1}\widetilde{\phi}^\Delta$ equations with $|J_1|\leq N-1$, and the energy estimates (2.7) yields

$$\mathcal{E}_{gst}(t,\Gamma^{J_1}\widetilde{\phi}^\Delta)^{1/2} \lesssim \mathcal{E}_{gst}(t_0,\Gamma^{J_1}\widetilde{\phi}^\Delta)^{1/2} + \int_{t_0}^t \big\|\Gamma^{J_1}|\Psi|^2\big\|\,d\tau.$$

We proceed to bound

$$\big\|\Gamma^{J_1}|\Psi|^2\big\| \lesssim \sum_{|I_1|\leq N-1,\,|I_2|\leq N-5} \big\|\Gamma^{I_1}\Psi\big\|\,\big\|\Gamma^{I_2}\Psi\big\|_{L^\infty} \lesssim (C_1\epsilon)^2\langle\tau\rangle^{-1},$$

in which we used the estimates from $\|\cdot\|_{\mathrm{II}}$, $\|\cdot\|_{\mathrm{VI}}$. Thus, we have

$$\mathcal{E}_{gst}(t,\Gamma^{J_1}\widetilde{\phi}^\Delta)^{1/2} \lesssim \epsilon + (C_1\epsilon)^2\int_{t_0}^t \langle\tau\rangle^{-1}\,d\tau \lesssim \epsilon+(C_1\epsilon)^2\langle t\rangle^{\delta/2}, \quad |J_1|\leq N-1.$$

Next, we apply the Klainerman–Sobolev inequality in Proposition 2.5 and the commutator estimates in Lemma 2.4 to get

$$\big|\partial\Gamma^J\widetilde{\phi}^\Delta\big| \lesssim \big(\epsilon+(C_1\epsilon)^2\big)\langle t+r\rangle^{-1/2+\delta/2}, \quad |J|\leq N-4.$$

Then, Proposition 3.1 allows us to obtain extra $\langle t-r\rangle$-decay with one more derivative, that is

$$\big|\partial\partial\Gamma^J\widetilde{\phi}^\Delta\big| \lesssim \big(\epsilon+(C_1\epsilon)^2\big)\langle t-r\rangle^{-1}\langle t+r\rangle^{-1/2+\delta/2}, \quad |J|\leq N-5.$$

Finally, recalling the relation $\widetilde{\phi}=\Delta\widetilde{\phi}^\Delta$ gives us

$$\begin{aligned}\big|\Gamma^J\widetilde{\phi}\big| = \big|\Gamma^J\Delta\widetilde{\phi}^\Delta\big| &\lesssim \sum_{|I_1|\leq|J|}\big|\partial\partial\Gamma^I\widetilde{\phi}^\Delta\big| \\ &\lesssim \big(\epsilon+(C_1\epsilon)^2\big)\langle t-r\rangle^{-1}\langle t+r\rangle^{-1/2+\delta}, \quad |J|\leq N-5,\end{aligned}$$

and hence, for $|J|\leq N-5$, we have

$$\big|\Gamma^J\widetilde{\phi}\big| \lesssim \big(\epsilon+(C_1\epsilon)^2\big)\langle t-r\rangle^{-1+\delta}\langle t+r\rangle^{-1/2}, \quad \text{for } r\leq\frac{3t}{4}.$$

Case II: $\{r\geq 2t\}$. We only pay attention to the region $r\geq 1$ as the result is true for $r\leq 1$. Recall the estimates in Lemma 4.16

$$\big\|\chi^{1/2}(r-t)\langle r-t\rangle\Gamma^J\widetilde{\phi}\big\| \lesssim \epsilon+(C_1\epsilon)^2\langle t\rangle^\delta, \quad |J|\leq N-5,$$

and this deduces that for large t, it holds

$$\big\|\chi(-6t/r+5)\langle r-t\rangle\Gamma^J\widetilde{\phi}\big\| \lesssim \epsilon+(C_1\epsilon)^2\langle t\rangle^\delta, \quad |J|\leq N-5,$$

in which $\chi(-6t/r+5)$ is 1 for $r\geq 2t$, and 0 for $r\leq 3t/2$. Then, we apply again the weighted Sobolev inequality in Proposition 2.6 to derive

$$\chi(-6t/r+5)\langle r-t\rangle\big|\Gamma^I\widetilde{\phi}\big| \lesssim \big(\epsilon+(C_1\epsilon)^2\big)\langle r\rangle^{-1/2+\delta}, \quad |I|\leq N-8.$$

Finally, combining the afore-obtained results in Lemma 4.19, we finish the proof. □

Verification of Norms in Group VI

Lemma 4.21 *We have the following uniform bounds:*

$$\begin{aligned}&\mathcal{E}_{gst,1}(t,\Gamma^I\widetilde{\Psi})^{1/2} \lesssim \epsilon+(C_1\epsilon)^{3/2}, \quad |I|\leq N-1,\\ &\mathcal{E}_{gst}(t,\Gamma^I\widetilde{\phi})^{1/2} \lesssim \epsilon+(C_1\epsilon)^{3/2}, \quad |I|\leq N-2.\end{aligned} \tag{4.29}$$

Proof Our strategy is to divide the spacetime region roughly into two parts $\{r\geq 2t\}$, $\{r\leq 2t\}$, and then conduct the estimates in different parts.

For the Klein–Gordon part $\widetilde{\Psi}$, the energy estimates (2.7) give us for $|I|\leq N-1$

$$\mathcal{E}_{gst,1}(t,\Gamma^I\widetilde{\Psi})^{1/2} \lesssim \mathcal{E}_{gst,1}(t_0,\Gamma^I\widetilde{\Psi})^{1/2}+\int_{t_0}^t\big\|\Gamma^I(\phi\,\Psi)\big\|\,d\tau.$$

We have

$$\int_{t_0}^{t} \|\Gamma^I(\phi\,\Psi)\|\,d\tau \lesssim \int_{t_0}^{t} \|\chi(r-2\tau)\Gamma^I(\phi\,\Psi)\|\,d\tau + \int_{t_0}^{t} \|\big(1-\chi(r-2\tau)\big)\Gamma^I(\phi\,\Psi)\|\,d\tau$$
$$=: A_1 + A_2.$$

We next bound these two terms separately. On the one hand, we find

$$\begin{aligned}
A_1 &\lesssim \sum_{|I_1|+|I_2|\le|I|} \int_{t_0}^{t} \|\chi(r-2\tau)\Gamma^{I_1}\phi\,\Gamma^{I_2}\Psi)\|\,d\tau\\
&\lesssim \sum_{|I_1|\le|I|,|I_2|\le N-5} \int_{t_0}^{t} \|\chi(r-2\tau)\langle r-\tau\rangle\Gamma^{I_1}\phi\|\,\|\langle\tau\rangle^{-1}\Gamma^{I_2}\Psi\|_{L^\infty}\,d\tau\\
&\quad+ \sum_{|I_1|\le N-8,|I_2|\le|I|} \int_{t_0}^{t} \|\chi(r-2\tau)\Gamma^{I_1}\phi\|_{L^\infty}\,\|\Gamma^{I_2}\Psi\|\,d\tau\\
&\lesssim (C_1\epsilon)^2 \int_{t_0}^{t} \langle\tau\rangle^{-3/2+2\delta}\,d\tau \lesssim (C_1\epsilon)^2,
\end{aligned}$$

in which we used the estimates from $\|\cdot\|_{\mathrm{II}}$, $\|\cdot\|_{\mathrm{IV}}$, $\|\cdot\|_{\mathrm{V}}$, and $\|\cdot\|_{\mathrm{VI}}$ in the last but one step. On the other hand, we have

$$\begin{aligned}
A_2 &\lesssim \sum_{|I_1|+|I_2|\le|I|} \int_{t_0}^{t} \|\big(1-\chi(r-2\tau)\big)\Gamma^{I_1}\phi\,\Gamma^{I_2}\Psi\|\,d\tau\\
&\lesssim \sum_{\substack{|I_1|\le|I|\\ |I_2|\le N-7}} \int_{t_0}^{t} \|\big(1-\chi(r-2\tau)\big)^{1/2}\langle r-\tau\rangle\Gamma^{I_1}\phi\|\,\|\big(1-\chi(r-2\tau)\big)^{1/2}\langle r-\tau\rangle^{-1}\Gamma^{I_2}\Psi\|_{L^\infty}\,d\tau\\
&\quad+ \sum_{\substack{|I_1|\le N-8\\ |I_2|\le|I|}} \int_{t_0}^{t} \Big\|\big(1-\chi(r-2\tau)\big)^{1/2}\frac{\langle r-\tau\rangle}{\langle r+\tau\rangle}\Gamma^{I_1}\phi\Big\|_{L^\infty}\Big\|\big(1-\chi(r-2\tau)\big)^{1/2}\frac{\langle r+\tau\rangle}{\langle r-\tau\rangle}\Gamma^{I_2}\Psi\Big\|\,d\tau\\
&\lesssim (C_1\epsilon)^2 \int_{t_0}^{t} \langle\tau\rangle^{-3/2+2\delta}\,d\tau \lesssim (C_1\epsilon)^2,
\end{aligned}$$

in which we used the estimates from $\|\cdot\|_{\mathrm{I}}$, $\|\cdot\|_{\mathrm{IV}}$, $\|\cdot\|_{\mathrm{V}}$, and Lemma 4.8 in the last but one step.

Thus, we are led to

$$\mathcal{E}_{gst,1}(t,\Gamma^I\widetilde{\Psi})^{1/2} \lesssim \epsilon + (C_1\epsilon)^2, \quad |I|\le N-1.$$

For the wave part $\widetilde{\phi}$, we have, according to the energy estimates (2.6), that

$$\mathcal{E}_{gst}(t,\Gamma^I\widetilde{\phi}) \lesssim \mathcal{E}_{gst}(t_0,\Gamma^I\widetilde{\phi}) + \int_{t_0}^{t}\int_{\mathbb{R}^2} \big|\Gamma^I\Delta|\Psi|^2\,\partial_t\Gamma^I\widetilde{\phi}\big|\,dxd\tau.$$

Recalling the estimates in Lemmas 4.16 and 4.17, for $|I| \le N-2$, we have

$$\begin{aligned}&\big\|\langle\tau-r\rangle\partial_t\Gamma^I\widetilde{\phi}\big\|\\&\quad\lesssim\big\|\chi(r-2\tau)\langle\tau-r\rangle\partial_t\Gamma^I\widetilde{\phi}\big\|+\big\|\big(1-\chi(r-2\tau)\big)\langle\tau-r\rangle\partial_t\Gamma^I\widetilde{\phi}\big\|\lesssim C_1\epsilon\langle\tau\rangle^{1/2+\delta}.\end{aligned}$$

Successively, we obtain

$$\begin{aligned}&\int_{t_0}^t\int_{\mathbb{R}^2}\big|\Gamma^I\Delta|\Psi|^2\,\partial_t\Gamma^I\widetilde{\phi}\big|\,dxd\tau\\&\quad\lesssim\int_{t_0}^t\big\|\langle\tau-r\rangle^{-1}\Gamma^I\Delta|\Psi|^2\big\|\big\|\langle\tau-r\rangle\partial_t\Gamma^I\widetilde{\phi}\big\|\,d\tau\\&\quad\lesssim C_1\epsilon\int_{t_0}^t\langle\tau\rangle^{1/2+\delta}\big\|\langle\tau-r\rangle^{-1}\Gamma^I\Delta|\Psi|^2\big\|\,d\tau.\end{aligned}$$

By the pointwise decay for the Klein–Gordon field Ψ in Lemma 4.8, we get

$$\sum_{|I_1|\le N-7}\big\|\langle\tau-r\rangle^{-1}\Gamma^{I_1}\Psi\big\|_{L^\infty}\lesssim C_1\epsilon\langle\tau\rangle^{-2},$$

which further gives us

$$\big\|\langle\tau-r\rangle^{-1}\Gamma^I\Delta|\Psi|^2\big\|\lesssim\sum_{\substack{|I_1|\le N-7\\|I_2|\le N}}\big\|\langle\tau-r\rangle^{-1}\Gamma^{I_1}\Psi\big\|_{L^\infty}\big\|\Gamma^{I_2}\Psi\big\|\lesssim(C_1\epsilon)^2\langle\tau\rangle^{-2+\delta}.$$

Thus, we arrive at

$$\mathcal{E}_{gst}(t,\Gamma^I\widetilde{\phi})\lesssim\epsilon^2+(C_1\epsilon)^3\int_{t_0}^t\langle\tau\rangle^{-3/2+2\delta}\,d\tau\lesssim\epsilon^2+(C_1\epsilon)^3,\quad|I|\le N-2.$$

The proof is done. □

Proof of Theorem 1.1

Proof of Proposition 4.6 Gathering the results obtained in Lemmas 4.9–4.21, we get

$$\big\|\big(\widetilde{\Psi},\widetilde{\phi}\big)\big\|_X\le C\epsilon+C(C_1\epsilon)^2.$$

By choosing large $C_1\gg1$, and sufficiently small $\epsilon\ll1$, such that $C_1\epsilon\ll1$, we arrive at

$$\big\|\big(\widetilde{\Psi},\widetilde{\phi}\big)\big\|_X\le\frac{1}{2}C_1\epsilon,$$

which means that

$$\big(\widetilde{\Psi},\widetilde{\phi}\big)\in X.$$

Thanks to the semilinear nature of the system (1.1), we can show (4.6) (we might further shrink the size of ϵ) in almost the same manner. To be more concrete, for two elements $(\Psi, \phi), (\Psi', \phi') \in X$ with $(\widetilde{\Psi}, \widetilde{\phi}) = T(\Psi, \phi)$, $(\widetilde{\Psi'}, \widetilde{\phi'}) = T(\Psi', \phi')$, we need to consider

$$\begin{aligned}
&-\Box \widetilde{P} + \widetilde{P} = -\phi P - q\Psi', \\
&-\Box \widetilde{q}^{\Delta} = \Psi \cdot P + P \cdot \Psi', \quad \widetilde{q} = \Delta \widetilde{q}^{\Delta}, \\
&\big(\widetilde{P}, \partial_t \widetilde{P}, \widetilde{q}^{\Delta}, \partial_t \widetilde{q}^{\Delta}\big)(t_0) = (0, 0, 0, 0),
\end{aligned} \tag{4.30}$$

in which

$$\begin{aligned}
&P = \Psi - \Psi', \quad \widetilde{P} = \widetilde{\Psi} - \widetilde{\Psi'}, \\
&q = \phi - \phi', \quad \widetilde{q} = \widetilde{\phi} - \widetilde{\phi'}.
\end{aligned}$$

In view of the analysis in proving Lemmas 4.9–4.21 (and the fact that the initial data in (4.30) are all zero), we can show (for some constant C)

$$\|(\widetilde{P}, \widetilde{q})\|_X \lesssim \big(\|(\Psi, \phi)\|_X + \|(\Psi', \phi')\|_X\big)\|(P, q)\|_X \leq C\epsilon\|(P, q)\|_X, \tag{4.31}$$

then the smallness of ϵ ensures (4.6) [recall this ϵ is from the smallness of initial data, i.e., (1.3)].

The proof is complete. □

Proof of Theorem 1.1 The Banach fixed point theorem together with Proposition 4.6 leads us to Theorem 1.1. □

5 Scattering

In this section, we briefly discuss about the scattering of the Klein–Gordon–Zakharov system (1.1) in $\mathbb{R}^{1+2}$. We show that the Klein–Gordon field E scatters to a linear Klein–Gordon equation in its high-order energy space (i.e., $\|E\|_{H^{N-7}} + \|\partial_t E\|_{H^{N-8}}$), but whether the wave field n scatters (linearly or nonlinearly) is unknown. We also note that this is different from the scattering result obtained in [18] for Klein–Gordon–Zakharov equations in $\mathbb{R}^{1+3}$, where the initial data are assumed to lie in the low regularity space and different difficulties arise.

We need one key fundamental result from [24] (Lemma 6.12 there), which originally provides a sufficient condition for the linear scattering of wave equations, but it extends to Klein–Gordon cases with similar proof. We now give the statement of the fundamental result and its proof can be found in either [24] or Appendix A.

Lemma 5.1 *Consider the Klein–Gordon equation*

$$\begin{aligned}
&-\Box u + u = Q(t, x), \\
&\big(u, \partial_t u\big)(t_0 = 0) = (u_0, u_1).
\end{aligned}$$

If it holds (with $N_1 \geq 1$ an integer)

$$\int_0^{+\infty} \|Q(\tau,\cdot)\|_{H^{N_1}}\, d\tau < +\infty, \tag{5.1}$$

then there exist $(u_0^+, u_1^+) \in H^{N_1+1} \times H^{N_1}$ and a free Klein–Gordon component u^+ satisfying

$$\begin{aligned} &-\Box u^+ + u^+ = 0,\\ &\big(u^+, \partial_t u^+\big)(t_0 = 0) = (u_0^+, u_1^+), \end{aligned}$$

such that u scatters to u^+, that is

$$\begin{aligned} &\|(u-u^+)(t)\|_{H^{N_1+1}} + \|\partial_t(u-u^+)(t)\|_{H^{N_1}}\\ &\leq C\int_t^{+\infty} \|Q(\tau,\cdot)\|\, d\tau \to 0, \quad \text{as } t \to +\infty. \end{aligned} \tag{5.2}$$

Remark 5.2 By Lemma 5.1 and the results obtained in [7], we know that the Klein–Gordon–Zakharov equations enjoy linear scattering in $\mathbb{R}^{1+3}$, which was also shown in [35] with high regular initial data. But again, we want to emphasize that there are different difficulties arising in obtaining scattering results for data lying in low regularity space as studied in [18] on Klein–Gordon–Zakharov equations.

Proof of Theorem 1.6 According to Lemma 5.1, we only need to verify

$$\big\|nE\big\|_{H^{N-8}} \lesssim (C_1\epsilon)^2\langle t\rangle^{-5/4}. \tag{5.3}$$

By the definition of the $\|\cdot\|_X$ norm in (4.3) (more precisely $\|\cdot\|_{\mathrm{I}}$, $\|\cdot\|_{\mathrm{V}}$), we find

$$\begin{aligned} \big\|nE\big\|_{H^{N-8}} &\lesssim \sum_{|J|\leq N-8} \big\|\Gamma^J(nE)\big\|\\ &\lesssim \sum_{|J_1|,|J_2|\leq N-8} \Big\|\frac{\langle t-r\rangle}{\langle t+r\rangle}\Gamma^{J_1} n\Big\|_{L^\infty} \Big\|\frac{\langle t+r\rangle}{\langle t-r\rangle}\Gamma^{J_2} E\Big\|\\ &\lesssim (C_1\epsilon)^2\langle t\rangle^{-3/2+2\delta}. \end{aligned}$$

By the smallness of δ, we thus arrive at (5.3), and hence Theorem 1.6. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

Appendix A: Proof of Lemma 5.1

By the linear theory on wave equations, the free-linear Klein–Gordon equation generates a strongly continuous semi-group acting on $H^{N_1+1} \times H^{N_1}$ as follows (with $N_1 \geq 1$ an integer). Let $(u_0, u_1) \in H^{N_1+1} \times H^{N_1}$. Then, the Cauchy problem

$$-\Box u + u = 0, \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x)$$

generates a unique global solution $(u(x), \partial_t u(x)) \in C([0, \infty), H^{N_1+1} \times H^{N_1}) \cap C^1([0, \infty), H^{N_1} \times H^{N_1-1})$ (a detailed proof can be found in [38]). This leads to

$$\mathcal{S}_1(t) : (u_0, u_1) \mapsto \big(u(t, \cdot), \partial_t u(t, \cdot)\big) \in H^{N_1+1} \times H^{N_1},$$

with (T below means the transpose of a matrix)

$$\mathcal{S}_1(t)(u_0, u_1) = \left(e^{t\mathcal{A}_1} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right)^T, \quad \mathcal{A}_1 = \begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix}.$$

By energy identity, $\mathcal{S}_1(t)$ is unitary for all $t \geq 0$. By the invariance under time translation and global uniqueness

$$\mathcal{S}_1(t + s) = \mathcal{S}_1(t) \circ \mathcal{S}_1(s).$$

By the fact that $(u(x), \partial_t u(x)) \in C([0, \infty), H^{N_1+1} \times H^{N_1})$

$$\mathcal{S}_1(t) : (u_0, u_1) \mapsto (u, \partial_t u) \quad \text{in } H^{N_1+1} \times H^{N_1}.$$

That is, $\mathcal{S}_1(t)$ is a strongly continuous semi-group. Next, we consider a non-homogeneous case

$$\begin{aligned} -\Box u + u &= Q(t, x), \\ \big(u, \partial_t u\big)(t_0 = 0) &= (u_0, u_1), \end{aligned} \tag{A.1}$$

where $Q(t, \cdot)$ is supposed to be in $L^1([0, \infty), H^{N_1})$. This equation has a unique global solution in $C([0, \infty), H^{N_1+1} \times H^{N_1}) \cap C^1([0, \infty), H^{N_1} \times H^{N_1-1})$ (see also a detailed proof in [38]). By Duhamel's principle (which is guaranteed by the strong continuity of $\mathcal{S}_1(t)$), the associated global solution can be written as

$$(u, \partial_t u)(t) = \mathcal{S}_1(t)(u_0, u_1) + \int_0^t \mathcal{S}_1(t - \tau)(0, Q(\tau))d\tau.$$

Inspired by this formula, we set the initial data (u_0^+, u_1^+) to be

$$(u_0^+, u_1^+) = (u_0, u_1) + \int_0^{+\infty} \mathcal{S}_1(-\tau)(0, Q(\tau, x))\, d\tau, \tag{A.2}$$

which is well defined in $H^{N_1+1} \times H^{N_1}$ as long as

$$\int_0^{+\infty} \|Q(\tau, x)\|_{H^{N_1}}\, d\tau < +\infty.$$

Then, we observe that

$$\begin{aligned}\|(u, \partial_t u) - \mathcal{S}_1(t)(u_0^+, u_1^+)\|_{H^{N_1+1}\times H^{N_1}} &= \left\| \int_t^{+\infty} \mathcal{S}_1(t-\tau)(0, Q(\tau, x))\, d\tau \right\|_{H^{N_1+1}\times H^{N_1}} \\ &\lesssim \int_t^{+\infty} \|Q(\tau, x)\|_{H^{N_1}}\, d\tau,\end{aligned} \tag{A.3}$$

which finishes the proof of Lemma 5.1.

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