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ORIGINAL ARTICLE

Mass, Capacitary Functions, and the Mass-to-Capacity Ratio

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Abstract

We study connections among the ADM mass, positive harmonic functions, and capacity of the boundary on asymptotically flat 3-manifolds of nonnegative scalar curvature. We start with new formulae detecting the mass via positive harmonic functions. Then we derive a family of monotone quantities and geometric inequalities assuming the manifold has simple topology. As a first application, we observe several additional proofs of the 3-dimensional Riemannian positive mass theorem. One proof leads to new, sufficient conditions implying positivity of the mass via C^0-geometry of regions separating the boundary and ∞. A special case of such conditions shows if a region enclosing the boundary has relative small volume, then the mass is positive. As further applications, we obtain integral identities for the mass-to-capacity ratio. We also promote the inequalities to become equality on Schwarzschild manifolds outside rotationally symmetric spheres. Among other things, we show the mass-to-capacity ratio is always bounded below by one minus the square root of the normalized Willmore functional of the boundary. Prompted by these findings, we carry out a study of manifolds satisfying a constraint on the mass-to-capacity ratio in the context of the Bartnik quasi-local mass.

Keywords Harmonic functions · Mass · Capacity · Scalar curvature

Mathematics Subject Classification Primary 53C21; Secondary 83C99

1 Introduction and Statement of Results

On an asymptotically flat 3-manifold (M, g), the ADM mass [3] is a flux integral near ∞ given by

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$$\mathfrak{m} = \lim_{r\to\infty} \frac{1}{16\pi} \int_{|x|=r} \sum_{j,k} (g_{jk,j} - g_{jj,k}) \nu^k .$$

Here $\{x_i\}_{1\le i\le 3}$ is a coordinate chart defining the asymptotic flatness of (M, g) and $\nu = |x|^{-1}x$ denotes the coordinate unit normal to $\{|x| = r\}$. By a result of Bartnik [4], and of Chruściel [11], $\mathfrak{m}$ is independent on the choice of the coordinates $\{x_i\}$.

On an asymptotically flat 3-manifold (M, g) with boundary Σ, the capacity (or the L^2-capacity) of Σ is given by

$$\mathfrak{c}_\Sigma = \inf \left\{ \frac{1}{4\pi} \int_M |\nabla f|^2 \right\} ,$$

where the infimum is taken over all locally Lipschitz functions f that equal 1 at Σ and tend to 0 at ∞. Equivalently, $\mathfrak{c}_\Sigma = \frac{1}{4\pi} \int_M |\nabla \phi|^2 = \frac{1}{4\pi} \int_\Sigma |\nabla \phi|$, where

$$\Delta\phi = 0, \quad \phi|_\Sigma = 1, \quad \text{and } \phi \to 0 \text{ at } \infty.$$

Regarding the mass, a fundamental result is the Riemannian positive mass theorem, first proved by Schoen and Yau [27] and by Witten [31]. The theorem states if (M, g) is a complete, asymptotically flat 3-manifold with nonnegative scalar curvature, without boundary, then

$$\mathfrak{m} \ge 0,$$

and equality holds if and only if (M, g) is isometric to the Euclidean space $\mathbb{R}^3$.

Regarding the mass and the capacity, an important result was due to Bray [6]. Bray showed if (M, g) is a complete, asymptotically flat 3-manifold with nonnegative scalar curvature, with minimal surface boundary $\Sigma = \partial M$, then

$$\mathfrak{m} \ge \mathfrak{c}_\Sigma ,$$

and equality holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside the horizon.

If the mean curvature H of the boundary Σ in (M, g) is not assumed to be zero, using the weak inverse mean curvature flow developed by Huisken and Ilmanen [17], Bray and the author [8] showed

$$\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \ge 1 - \left(\frac{1}{16\pi} \int_\Sigma H^2 \right)^{\frac{1}{2}}$$

under the assumptions $\int_\Sigma H^2 \le 16\pi$ and $H_2(M, \Sigma) = 0$, and equality holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative (constant) mean curvature.

Recently, level sets of harmonic functions have been found to be an efficient tool to study scalar curvature in 3-dimension. A pioneering work of Stern [30] revealed intriguing analogy between the use of such level sets and the use of stable minimal surfaces instituted by Schoen and Yau [27]. On asymptotically flat 3-manifolds, a new proof of the positive mass theorem was given by Bray, Kazaras, Khuri and Stern [7], which made use of harmonic functions asymptotic to a linear coordinate function.

In terms of monotone quantities along the level sets, Munteanu and Wang in [25] established sharp comparison results on complete, nonparabolic 3-manifolds via the discovery of a monotone quantity along level sets of the minimal positive Green's function. In [2], Agostiniani, Mazzieri and Oronzio obtained another proof of the Riemannian positive mass theorem through a different monotone quantity along level sets of the Green's function on asymptotically flat 3-manifolds.

In this paper, we consider harmonic functions u satisfying

$$u(x) = 1 - \mathfrak{c}|x|^{-1} + o(|x|^{-1}), \quad \text{as } x \to \infty,$$

for some constant $\mathfrak{c} > 0$, on an asymptotically flat 3-manifold (M, g). In the case (M, g) has boundary Σ and u is 0 at Σ, $\mathfrak{c} = \mathfrak{c}_\Sigma$ and u is referred as the capacitary function on (M, g). We obtain a sequence of new results relating the mass of (M, g), the capacitary function u, and the capacity $\mathfrak{c}_\Sigma$.

We first find formulae that detect the mass of (M, g) via the level sets of such a u, see Theorem 2.1. In particular, Theorem 2.1 (ii) implies

$$\lim_{t\to 1} \frac{1}{1-t}\left[4\pi - \frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2\right] = 4\pi\, \mathfrak{m}\, \mathfrak{c}^{-1}. \tag{1.1}$$

Here $\Sigma_t = u^{-1}(t)$.

Besides (1.1), in Theorem 2.1 (i), we find

$$\lim_{t\to 1} \frac{1}{1-t}\left[8\pi - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u|\right] = 12\pi\, \mathfrak{m}\, \mathfrak{c}^{-1}. \tag{1.2}$$

Here H denotes the mean curvature of a regular level set Σ_t with respect to $|\nabla u|^{-1}\nabla u$.

An immediate implication of either (1.1) or (1.2) is that the ADM mass $\mathfrak{m}$ is a geometric invariant of (M, g), since the capacitary function and the boundary capacity are independent on the choice of coordinates at ∞.

(1.1) and (1.2) indicate that, as $t \to 1$,

$$8\pi - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| = 3\left[4\pi - \frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2\right] + o\left((1-t)\right).$$

While this asymptotic comparison was made only via information near ∞, we show in Theorem 3.1 that, if M has simple topology and g has nonnegative scalar curvature, then, at each regular level set Σ_t,

$$8\pi - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| \le 3\left[4\pi - \frac{1}{(1-t)^2}\int_{\Sigma_t}|\nabla u|^2\right], \tag{1.3}$$

and "=" holds if and only if (M, g) outside Σ_t is isometric to $\mathbb{R}^3$ minus a round ball.

Inequality (1.3) is derived via a monotone quantity along $\{\Sigma_t\}$, see Lemma 3.1. Among other things, we apply (1.3) to find that the quantities in the mass formulae (1.1) and (1.2) are actually monotone non-decreasing, that is

$$\mathcal{A}(t) := \frac{1}{1-t}\left[8\pi - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u|\right] \nearrow \quad \text{as } t \nearrow, \tag{1.4}$$

and

$$\mathcal{B}(t) := \frac{1}{1-t}\left[4\pi - \frac{1}{(1-t)^2}\int_{\Sigma_t}|\nabla u|^2\right] \nearrow \quad \text{as } t \nearrow, \tag{1.5}$$

see Theorem 3.2. This, combined with Theorem 2.1, then shows

$$8\pi - \int_{\Sigma} H|\nabla u| \le 12\pi\, \mathfrak{m}\, \mathfrak{c}_{\Sigma}^{-1} \tag{1.6}$$

and

$$4\pi - \int_{\Sigma}|\nabla u|^2 \le 4\pi\, \mathfrak{m}\, \mathfrak{c}_{\Sigma}^{-1} \tag{1.7}$$

for the capacitary function u. Furthermore, "=" holds in any of these inequalities if and only if (M, g) is isometric to $\mathbb{R}^3$ minus a round ball.

As an immediate application of (1.1)–(1.7), we observe several new arguments implying the 3-dimensional positive mass theorem, see Sect. 4.

Inequalities (1.6) and (1.7) also give rise to sufficient conditions that imply the positivity of the mass via C^0-geometry of regions separating the boundary and ∞. For instance, as a special case of Theorem 5.1, we show that if M has simply topology and g has nonnegative scalar curvature, then

$$H \le \frac{8\pi L^2}{\mathrm{Vol}(\Omega)} \implies \mathfrak{m} > 0. \tag{1.8}$$

Here Ω is a region whose boundary has two components S_0 and S_1, where S_1 encloses S_0 and S_0 encloses Σ, L is the distance from S_1 to S_0, and $\mathrm{Vol}(\Omega)$ is the volume of (Ω, g). Another such sufficient condition in Theorem 5.2 shows

$$\int_{S_0}|\nabla v|^2 \le 4\pi \implies \mathfrak{m} > 0. \tag{1.9}$$

Here v is the harmonic function on Ω with $v = 0$ at S_0 and $v = 1$ at S_1.

In [2], Agostiniani, Mazzieri and Oronzio showed, along $\{\Sigma_t\}$,

$$F(t) := \frac{1}{1-t}\left[4\pi - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| + \frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2\right] \nearrow \quad \text{as } t \nearrow . \tag{1.10}$$

We observe that $\mathcal{A}(t)$, $\mathcal{B}(t)$ in our work are related to $F(t)$ related by

$$F(t) = \mathcal{A}(t) - \mathcal{B}(t). \tag{1.11}$$

In (A.14) and (A.15) of Appendix A, we give integral identities for the differences

$$\mathcal{B}(t_2) - \mathcal{B}(t_1), \quad \mathcal{A}(t_2) - \mathcal{A}(t_1), \quad \text{for } t_1 < t_2.$$

The monotonicity of $F(t)$ can also be seen from (1.11), (A.14) and (A.15). Moreover, as a corollary of (1.1), (1.2) and (1.11), one has $\lim_{t\to 1} F(t) = 8\pi\, \mathfrak{m}\mathfrak{c}_{\Sigma}^{-1}$. Such a limit was shown in [2] in the case that (M, g) is isometric to a spatial Schwarzschild manifold near infinity.

Applying the limits of $\mathcal{A}(t)$, $\mathcal{B}(t)$ as $t \to 1$ and the formulae of their differences at $t_1 < t_2$, we derive integral identities for the mass-to-capacity ratio $\mathfrak{m}\mathfrak{c}_{\Sigma}^{-1}$ in Theorem 6.1. Such integral identities can be compared with the mass identity obtained by Bray, Kazaras, Khuri and Stern [7] via harmonic functions having linear asymptotic.

Inspired by Bray's work [6], in Sect. 7 we promote inequalities (1.3), (1.6) and (1.7) to become equality in spatial Schwarzschild spaces. Among other things, we show in Corollary 7.1 that

$$\left(\frac{1}{\pi}\int_{\Sigma} |\nabla u|^2\right)^{\frac{1}{2}} \le \left(\frac{1}{16\pi}\int_{\Sigma} H^2\right)^{\frac{1}{2}} + 1. \tag{1.12}$$

Moreover, equality holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative mean curvature. In Theorem 7.3, we show, given the same triple (M, g, u),

$$\frac{1}{2}\mathfrak{m}\mathfrak{c}_{\Sigma}^{-1} \ge 1 - \left(\frac{1}{4\pi}\int_{\Sigma} |\nabla u|^2 \, d\sigma\right)^{\frac{1}{2}}, \tag{1.13}$$

and equality holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere. As a result of (1.12) and (1.13), we obtain in Theorem 7.4

$$\mathfrak{m}\mathfrak{c}_{\Sigma}^{-1} \ge 1 - \left(\frac{1}{16\pi}\int_{\Sigma} H^2\right)^{\frac{1}{2}}, \tag{1.14}$$

regardless of the mean curvature H of Σ. (1.14) improves the earlier mentioned result of Bray and the author in [8]. Moreover, if applied to the exterior of small geodesic balls, (1.14) yields another proof of the positive mass theorem, see Remark 7.1.

Prompted by (1.14), in Sect. 8 we carry out a study of manifolds with boundary satisfying a mass-capacity relation

$$\mathfrak{m}\mathfrak{c}_{\Sigma}^{-1} \leq 1. \tag{1.15}$$

Under this assumption, in Theorem 8.1 we promote (1.6) to

$$(2 - \mathfrak{m}\mathfrak{c}_{\Sigma}^{-1})(1 - \mathfrak{m}\mathfrak{c}_{\Sigma}^{-1}) \leq \frac{1}{4\pi} \int_{\Sigma} H|\nabla u|, \tag{1.16}$$

which picks up an intriguing quadratic term $(\mathfrak{m}\mathfrak{c}_{\Sigma}^{-1})^2$. Equality in (1.16) holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative mean curvature.

In Corollary 8.1, we give a capacity-comparison result for manifolds satisfying (1.15) under a condition

$$\mathfrak{m}H_{\max} \leq \frac{2}{3\sqrt{3}}.$$

Here $H_{\max}$ is the maximum of the mean curvature of the boundary and the number $\frac{2}{3\sqrt{3}}$ is the maximum value of $\mathfrak{m}H$ evaluated along rotationally symmetric spheres in a spatial Schwarzschild manifold with positive mass.

In Corollary 8.2, we show manifolds satisfying (1.15) have the mass bounded by

$$\mathfrak{m} \leq \frac{r_{\Sigma}}{2}\left[1 + \left(\frac{1}{16\pi}\int_{\Sigma} H^2\right)^{\frac{1}{2}}\right], \tag{1.17}$$

where r_{Σ} is the area–radius of Σ. Moreover, the capacitary functions u on these manifolds satisfy

$$\int_{\Sigma} |\nabla u|^2 \geq \pi. \tag{1.18}$$

Heuristically, this suggests such manifolds may not have long cylindrical regions shielding the boundary, see Remark 8.6.

Toward the end of Sect. 8, we place condition (1.15) in the context of the Bartnik quasi-local mass [5]. We point out manifolds satisfying (1.15) do not contain closed minimal surfaces enclosing the boundary and static metric extensions with a positive static potential necessarily satisfy (1.15), see Proposition 8.1.

We finish this paper with an appendix, including regularization arguments that can be used to verify various monotonicity in Sect. 3.

2 Detecting the Mass at ∞

Let (M, g) denote an asymptotically flat 3-manifold (with one end) with boundary. By this, we mean there is a compact set $K \subset M$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3$ minus a ball and, with respect to the standard coordinates on $\mathbb{R}^3$, g satisfies

$$g_{ij} = \delta_{ij} + O(|x|^{-\tau}), \quad \partial g_{ij} = O(|x|^{-\tau-1}), \quad \partial\partial g_{ij} = O(|x|^{-\tau-2}) \tag{2.1}$$

for some constant $\tau > \frac{1}{2}$. The scalar curvature R of g is also assumed to be integrable so that the mass $\mathfrak{m}$ of (M, g) exists (see [4, 11] for instance).

Let Σ denote the boundary of M. Let u be the function on (M, g) given by

$$\Delta u = 0 \text{ on } M, \quad u = 0 \text{ at } \Sigma, \quad \text{and } u \to 1 \text{ at } \infty. \tag{2.2}$$

Given any $t \in [0, 1]$, let $\Sigma_t = \{x \in M \mid u(x) = t\}$ denote the level set of u. Below, we collect some basic facts about u and Σ_t.

By the maximum principle, $\max_K u < 1$, hence $|x|$ is defined on Σ_t for t close to 1; moreover, $\min_{\Sigma_t} |x| \to \infty$ as $t \to 1$. Now suppose $\tau \in (\frac{1}{2}, 1)$. As $x \to \infty$, it is known u has an asymptotic expansion (see Lemma A.2 in [22] for instance)

$$u = 1 - \mathfrak{c}_\Sigma |x|^{-1} + O_2(|x|^{-1-\tau}). \tag{2.3}$$

Here $\mathfrak{c}_\Sigma > 0$ is a positive constant equal to the capacity of Σ in (M, g). Let ∇u and $\nabla^2 u$ denote the gradient and the Hessian of u on (M, g), respectively. By (2.3),

$$|\nabla u|^2 = \mathfrak{c}_\Sigma^2 |x|^{-4} + O(|x|^{-4-\tau}), \tag{2.4}$$

$$(\nabla^2 u)_{ij} = \mathfrak{c}_\Sigma |x|^{-3}\big(-3|x|^{-2} x_i x_j + \delta_{ij}\big) + O(|x|^{-3-\tau}). \tag{2.5}$$

Thus, t is a regular value if t is close to 1 and the mean curvature H of Σ_t satisfies

$$H = \operatorname{div}\big(|\nabla u|^{-1} \nabla u\big) = 2|x|^{-1} + O(|x|^{-1-\tau}). \tag{2.6}$$

As a result, for t close to 1, Σ_t has positive mean curvature, and consequently Σ_t is area outer-minimizing as its exterior is foliated by mean-convex surfaces $\{\Sigma_s\}_{s>t}$. Here we say a surface S is area outer-minimizing if every surface $\tilde{S}$ which encloses S has greater area, see [6] for instance.

Lemma 2.1 *Let $|\Sigma_t|$ be the area of Σ_t in (M, g) if t is a regular value of u. Then, as $t \to 1$,*

$$|\Sigma_t| = 4\pi \mathfrak{c}_\Sigma^2 (1 - t)^{-2} + O((1 - t)^{\tau-2}). \tag{2.7}$$

Proof By (2.3), as $t \to 1$,

$$|x| = \mathfrak{c}_\Sigma (1 - t)^{-1} + O((1 - t)^{\tau-1}). \tag{2.8}$$

Let $r_-(t) = \min_{\Sigma_t} |x|$ and $r_+(t) = \max_{\Sigma_t} |x|$. Since Σ_t and the coordinate sphere $S_r := \{|x| = r\}$ are both area outer-minimizing in (M, g), for t close to 1 and large r, respectively, we have

$$|S_{r_-(t)}| \le |\Sigma_t| \le |S_{r_+(t)}|. \tag{2.9}$$

For large r, (2.1) implies

$$|S_r| = 4\pi r^2 + O(r^{2-\tau}). \tag{2.10}$$

Thus, (2.7) follows from (2.8)–(2.10). □

Lemma 2.2 *As* $t \to 1$,

$$\frac{1}{(1-t)} \int_{\Sigma_t} H|\nabla u| = 8\pi + O((1-t)^\tau)$$

and

$$\frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 = 4\pi + O((1-t)^\tau).$$

Proof By (2.6) and (2.8),

$$H = 2\mathfrak{c}_\Sigma^{-1}(1-t) + O((1-t)^{1+\tau}). \tag{2.11}$$

Therefore, using the fact $\int_{\Sigma_t} |\nabla u| = 4\pi \mathfrak{c}_\Sigma$, one has

$$\begin{aligned} \left(\frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| \right) - 8\pi &= \int_{\Sigma_t} \left(\frac{H}{1-t} - \frac{2}{\mathfrak{c}_\Sigma} \right) |\nabla u| \\ &= O((1-t)^\tau). \end{aligned}$$

Similarly, by (2.4) and (2.8),

$$|\nabla u| = \mathfrak{c}_\Sigma^{-1}(1-t)^2 + O((1-t)^{2+\tau}). \tag{2.12}$$

Therefore,

$$\begin{aligned} \left(\frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right) - 4\pi &= \left(\int_{\Sigma_t} \frac{|\nabla u|}{(1-t)^2} - \frac{1}{\mathfrak{c}_\Sigma} \right) |\nabla u| \\ &= O((1-t)^\tau). \end{aligned}$$

□

Lemma 2.3 *As* $t \to 1$, *the gradient of* $|\nabla u|$ *on* Σ_t *satisfies*

$$\left| \nabla_{\Sigma_t} |\nabla u| \right| = O(|x|^{-3-\tau}). \tag{2.13}$$

Proof Write $\nabla u = (\nabla u)^j \partial_j$. By (2.3),

$$(\nabla u)^j = \mathfrak{c}_\Sigma |x|^{-2}|x|^{-1}x_j + O(|x|^{-2-\tau}).$$

Let $V = V^i\partial_i$ denote any unit vector tangent to Σ_t. Then $V^i = O(1)$ and the fact $\langle V, \nabla u\rangle = 0$ shows

$$\sum_i V^i(\nabla u)^i = O(|x|^{-2-\tau}), \quad \text{and hence } \sum_i V^i x_i = O(|x|^{1-\tau}). \tag{2.14}$$

Therefore, by (2.5) and (2.14),

$$\begin{aligned} V(|\nabla u|^2) &= 2(\nabla^2 u)(V, \nabla u) \\ &= 2\mathfrak{c}_\Sigma |x|^{-3}\big[-3|x|^{-2}x_i V^i x_j(\nabla u)^j + \delta_{ij}V^i(\nabla u)^j\big]\big(1 + O(|x|^{-\tau})\big) \\ &= O(|x|^{-5-\tau}). \end{aligned} \tag{2.15}$$

Thus, (2.13) follows from (2.15) and (2.4). □

Lemma 2.4 *As $t \to 1$, the traceless part of the second fundamental form $\mathbb{II}$ of Σ_t, denoted by $\mathring{\mathbb{II}}$, satisfies*

$$|\mathring{\mathbb{II}}| = O(|x|^{-1-\tau}), \tag{2.16}$$

and the Gauss curvature K of Σ_t satisfies $K = |x|^{-2} + O(|x|^{-2-\tau})$.

Proof Let $V = V^i\partial_i$ and $W = W^j\partial_j$ be any two unit vectors tangent to Σ_t at a given point. Then $\delta_{ij}V^iW^j = g(V, W) + O(|x|^{-\tau})$. As V, W are tangential to Σ_t, one has

$$\nabla^2 u(V, W) = \langle \nabla_V \nabla u, W\rangle = |\nabla u|\mathbb{II}(V, W).$$

Hence,

$$\begin{aligned} |\nabla u|\,\mathbb{II}(V, W) &= (\nabla^2 u)_{ij}V^iW^j \\ &= \mathfrak{c}_\Sigma |x|^{-3}\big(-3|x|^{-2}x_iV^ix_jW^j + \delta_{ij}V^iW^j\big)\big(1 + O(|x|^{-\tau})\big) \\ &= \mathfrak{c}_\Sigma |x|^{-3}g(V, W) + O(|x|^{-3-\tau}), \end{aligned} \tag{2.17}$$

where one used (2.5) and (2.14). Therefore, by (2.4),

$$\mathbb{II}(V, W) = |x|^{-1}g(V, W) + O(|x|^{-1-\tau}). \tag{2.18}$$

This combined with (2.6) shows

$$\mathring{\mathbb{II}}(V, W) = \mathbb{II}(V, W) - \frac{1}{2}Hg(V, W) = O(|x|^{-1-\tau}),$$

which proves (2.16). The conclusion on the Gauss curvature follows from (2.18), (2.6) and the Gauss equation. □

Lemma 2.5 *If* (M, g) *satisfies* $\partial\partial\partial g_{ij} = O(|x|^{-3-\tau})$ *in (2.1), then*

$$|D\,\mathring{\mathbb{II}}| = O(|x|^{-2-\tau}). \tag{2.19}$$

Here D *denotes covariant differentiation on* Σ_t.

Proof If g satisfies the higher order derivatives decay assumption, then u satisfies

$$u = 1 - \mathfrak{c}_\Sigma |x|^{-1} + O_3(|x|^{-1-\tau})$$

(see the proof of Lemma A.2 in [22] for instance). The terms $O(|x|^{-3-\tau})$ in (2.5) and $O(|x|^{-1-\tau})$ in (2.6) are then replaced by $O_1(|x|^{-3-\tau})$ and $O_1(|x|^{-1-\tau})$, respectively.

To prove (2.19), let $\{V_\alpha\}_{\alpha=1,2}$ be a local orthonormal frame around a given point p on Σ_t. By definition,

$$(D_{V_\mu}\mathring{\mathbb{II}})(V_\alpha, V_\beta) = \left(D_{V_\mu}\mathbb{II}\right)(V_\alpha, V_\beta) - \frac{1}{2}V_\mu(H)\delta_{\alpha\beta}.$$

By (2.6) and (2.14),

$$V_\mu(H) = 2(-1)|x|^{-2}V_\mu(|x|) + O(|x|^{-2-\tau}) = O(|x|^{-2-\tau}).$$

To estimate $D\mathbb{II}$, one may assume $\{V_\alpha\}$ is normal at p, i.e., $D_{V_\alpha}V_\beta = 0$ at p, then

$$\begin{aligned}(D_{V_\mu}\mathbb{II})(V_\alpha, V_\beta) =&\ V_\mu(\mathbb{II}(V_\alpha, V_\beta)) \\ =&\ V_\mu(|\nabla u|^{-1})\,(\nabla^2 u)_{\alpha\beta} + |\nabla u|^{-1}\, V_\mu\big((\nabla^2 u)_{\alpha\beta}\big).\end{aligned}$$

By (2.13) and (2.17),

$$V_\mu(|\nabla u|^{-1})\,(\nabla^2 u)_{\alpha\beta} = O(|x|^{-2-\tau}).$$

By (2.17) and (2.14),

$$|\nabla u|^{-1}\, V_\mu\big((\nabla^2 u)_{\alpha\beta}\big) = |\nabla u|^{-1}\, O(|x|^{-4-\tau}) = O(|x|^{-2-\tau}).$$

Thus, $(D_{V_\mu}\mathbb{II})(V_\alpha, V_\beta) = O(|x|^{-2-\tau})$. This proves (2.19). □

Let $\mathfrak{m}_H(\Sigma_t)$ denote the Hawking mass [14] of Σ_t if t is a regular value of u. That is

$$\mathfrak{m}_H(\Sigma_t) = \frac{r_t}{2}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H^2\right). \tag{2.20}$$

Here $r_t = \sqrt{\frac{|\Sigma_t|}{4\pi}}$ is the area–radius of Σ_t. By Lemma 2.1,

$$r_t = \mathfrak{c}_\Sigma (1-t)^{-1} + O((1-t)^{\tau-1}). \tag{2.21}$$

Proposition 2.1 *If* $\lim_{t\to 1} \mathfrak{m}_H(\Sigma_t) = \mathfrak{m}$, *where* $\mathfrak{m}$ *is the mass of* (M, g), *then*

$$\lim_{t\to 1} \frac{1}{1-t}\left[8\pi - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u|\right] = 12\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1} \tag{2.22}$$

and

$$\lim_{t\to 1} \frac{1}{1-t}\left[4\pi - \frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2\right] = 4\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}. \tag{2.23}$$

Proof For regular values t, define

$$A(t) = 8\pi - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u|. \tag{2.24}$$

Then

$$-A'(t) = \frac{1}{1-t}\left[-A(t) + 8\pi + \left(\int_{\Sigma_t} H|\nabla u|\right)'\right].$$

Applying formulae for the first and second variation of area, we have

$$\begin{aligned}\left(\int_{\Sigma_t} H|\nabla u|\right)' &= \int_{\Sigma_t} H'|\nabla u| + H|\nabla u|' + H|\nabla u| H |\nabla u|^{-1}\\ &= \int_{\Sigma_t} -|\nabla u|^{-2}\left|\nabla_{\Sigma_t}|\nabla u|\right|^2 + K - \frac{3}{4}H^2 - \frac{1}{2}|\mathring{\mathbb{II}}|^2 - \frac{1}{2}R,\end{aligned} \tag{2.25}$$

where $|\nabla u|' = -H$ and $H' = -\Delta_{\Sigma_t}|\nabla u|^{-1} - (\mathrm{Ric}(\nu,\nu) + |\mathbb{II}|^2)|\nabla u|^{-1}$.

By (2.20) and the Gauss–Bonnet theorem,

$$8\pi + \left(\int_{\Sigma_t} H|\nabla u|\right)' = \frac{24\pi\, \mathfrak{m}_H(\Sigma_t)}{r_t} - E(t), \tag{2.26}$$

where

$$E(t) = \int_{\Sigma_t} |\nabla u|^{-2}|\nabla_\Sigma|\nabla u||^2 + \frac{1}{2}|\mathring{\mathbb{II}}|^2 + \frac{1}{2}R.$$

Therefore,

$$-A'(t) = -\frac{A(t)}{1-t} + \frac{24\pi\,\mathfrak{m}_H(\Sigma_t)}{(1-t)r_t} - \frac{1}{1-t}E(t). \tag{2.27}$$

By Lemma 2.2, $\lim_{t\to 1} A(t) = 0$. Hence,

$$A(t) = \frac{1}{1-t}\int_t^1 \left[\frac{24\pi\,\mathfrak{m}_H(\Sigma_s)}{r_s} - E(s)\right]. \tag{2.28}$$

As $t \to 1$, $\mathfrak{m}_H(\Sigma_t) = \mathfrak{m} + o(1)$ by the assumption. Thus, by (2.21),

$$\frac{\mathfrak{m}_H(\Sigma_t)}{r_t} = \mathfrak{m}\,\mathfrak{c}_\Sigma^{-1}(1-t) + (1-t)o(1). \tag{2.29}$$

Consequently,

$$\int_t^1 \frac{\mathfrak{m}_H(\Sigma_s)}{r_s} = \frac{1}{2}\mathfrak{m}\,\mathfrak{c}_\Sigma^{-1}(1-t)^2 + o((1-t)^2). \tag{2.30}$$

To estimate $\int_t^1 E(s)$, we note Lemmas 2.3 and 2.4, combined with (2.8), show

$$|\nabla u|^{-2}\big|\nabla_\Sigma|\nabla u|\big|^2 + \frac{1}{2}|\mathring{\mathbb{II}}|^2 = O(|x|^{-2-2\tau}) = O((1-t)^{2+2\tau}).$$

Thus, by Lemma 2.1,

$$\int_{\Sigma_t} |\nabla u|^{-2}\big|\nabla_\Sigma|\nabla u|\big|^2 + \frac{1}{2}|\mathring{\mathbb{II}}|^2 = O((1-t)^{2\tau}). \tag{2.31}$$

Therefore,

$$\int_t^1\int_{\Sigma_s} |\nabla u|^{-2}\big|\nabla_\Sigma|\nabla u|\big|^2 + \frac{1}{2}|\mathring{\mathbb{II}}|^2 = O((1-t)^{1+2\tau}). \tag{2.32}$$

To handle the scalar curvature term, we use the assumption R is integrable. As $t \to 1$,

$$o(1) = \int_{u\ge t} |R| = \int_t^1\int_{\Sigma_s} |R||\nabla u|^{-1},$$

where we also used the coarea formula. By (2.12), $|\nabla u|^{-1} \ge \frac{1}{2}\mathfrak{c}_\Sigma(1-t)^{-2}$ for t close to 1. Hence,

$$\int_t^1\int_{\Sigma_s} |R||\nabla u|^{-1} \ge \frac{1}{2}\mathfrak{c}_\Sigma(1-t)^{-2}\int_t^1\int_{\Sigma_s} |R|.$$

These imply

$$\int_t^1 \int_{\Sigma_s} |R| = o((1-t)^2). \tag{2.33}$$

It follows from (2.28), (2.30), (2.32) and (2.33) that

$$\frac{1}{1-t} A(t) = 12\pi \, \mathfrak{m} \, \mathfrak{c}_\Sigma^{-1} + o(1) + O((1-t)^{2\tau-1}). \tag{2.34}$$

Since $\tau > \frac{1}{2}$, this proves (2.22).
Similarly, define

$$B(t) = 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2. \tag{2.35}$$

At any regular value t,

$$\begin{aligned} -B'(t) = & \frac{1}{1-t} \left[2(-B(t) + 4\pi) + \frac{1}{1-t} \left(- \int_{\Sigma_t} H |\nabla u| \right) \right] \\ = & \frac{1}{1-t} \left[-2B(t) + A(t) \right]. \end{aligned} \tag{2.36}$$

By Lemma 2.2, $\lim_{t\to 1} B(t) = 0$. Thus,

$$B(t) = \frac{1}{(1-t)^2} \int_t^1 (1-s) A(s).$$

Therefore, as $t \to 1$, by (2.34),

$$\frac{1}{1-t} B(t) = 4\pi \, \mathfrak{m} \, \mathfrak{c}_\Sigma^{-1} + o(1) + O((1-t)^{2\tau-1}).$$

This proves (2.23). □

Theorem 2.1 *Let (M, g) be an asymptotically flat 3-manifold with boundary Σ, with $\partial\partial\partial g_{ij} = O(|x|^{-3-\tau})$ at ∞. Let u be the harmonic function that tends to 1 at ∞ and vanishes at Σ. Then*

(i) $\displaystyle \lim_{t\to 1} \frac{1}{1-t} \left[8\pi - \frac{1}{1-t} \int_{\Sigma_t} H |\nabla u| \right] = 12\pi \, \mathfrak{m} \, \mathfrak{c}_\Sigma^{-1};$

(ii) $\displaystyle \lim_{t\to 1} \frac{1}{1-t} \left[4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] = 4\pi \, \mathfrak{m} \, \mathfrak{c}_\Sigma^{-1}.$

Here $\mathfrak{m}$ is the mass of (M, g) and $\mathfrak{c}_\Sigma$ is the capacity of Σ in (M, g).

Proof It suffices to show $\lim_{t\to 1} \mathfrak{m}_H(\Sigma_t) = \mathfrak{m}$. For t close to 1, let $r_-(t) = \min_{\Sigma_t} |x|$ and $r_+(t) = \max_{\Sigma_t} |x|$. By (2.8), $r_+(t) \le C r_-(t)$. Here and below, $C > 0$ denotes some constant independent on t. By Lemma 2.1, $|\Sigma_t| \le C r_-^2$. By Lemma 2.4, $K \ge C r_-^{-2}$, hence $\mathrm{diam}(\Sigma_t) \le C r_-$. By Lemmas 2.4 and Lemma 2.5, $|\mathring{\mathbb{II}}| \le C r_-^{-1-\tau}$ and $|D\mathring{\mathbb{II}}| \le C r_-^{-2-\tau}$. Hence, $\{\Sigma_t\}$ is a family of nearly round surfaces near ∞ in (M, g) according to Definition 1.3 in [29]. By Theorem 2 in [29], $\lim_{t\to 1} \mathfrak{m}_H(\Sigma_t) = \mathfrak{m}$.

Theorem 2.1 now follows from Proposition 2.1. □

We can indeed interpret the mass-to-capacity ratio as the derivatives at ∞ of the two functions

$$A(t) = 8\pi - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| \quad \text{and} \quad B(t) = 4\pi - \frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2. \tag{2.37}$$

Corollary 2.1 *Let (M, g) be an asymptotically flat 3-manifold with boundary Σ, with $\partial\partial\partial g_{ij} = O(|x|^{-3-\tau})$ at ∞. Let u be the harmonic function that tends to 1 at ∞ and vanishes at Σ. Then the functions $A(t)$ and $B(t)$ have C^1 extensions to $t = 1$ with*

$$A(1) = 0, \quad A'(1) = -12\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}, \quad B(1) = 0, \quad B'(1) = -4\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}.$$

Proof By Lemma 2.2, $A(t)$ and $B(t)$ extend continuously to $t = 1$ with $A(1) = 0$ and $B(1) = 0$. By Theorem 2.1 (i), (2.27), (2.29) and (2.31),

$$\begin{aligned} \lim_{t\to 1} A'(t) &= \lim_{t\to 1}\left[\frac{1}{1-t}A(t) - \frac{24\pi\mathfrak{m}_H(\Sigma_t)}{(1-t)r_t} + \frac{1}{1-t}E(t)\right] \\ &= 12\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1} - 24\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1} \\ &= \lim_{t\to 1}\frac{1}{t-1}A(t). \end{aligned}$$

Similarly, by Theorem 2.1 (i), (ii) and (2.36),

$$\lim_{t\to 1} B'(t) = \lim_{t\to 1}\frac{1}{1-t}\left[2B(t) - A(t)\right] = -4\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1} = \lim_{t\to 1}\frac{1}{t-1}B(t).$$

This shows $A'(t)$ and $B'(t)$ are continuous at $t = 1$ with $A'(1) = -12\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}$ and $B'(1) = -4\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}$. □

Remark 2.1 Proofs in this section essentially only use the structure near ∞ of (M, g). The arguments indeed establish (1.1) and (1.2) for any harmonic function u, satisfying

$$u(x) = 1 - \mathfrak{c}|x|^{-1} + o(|x|^{-1}), \quad \text{as } x \to \infty,$$

for some constant $\mathfrak{c} > 0$. Another way to see this is to derive (1.1) and (1.2) as a corollary of Theorem 2.1. For instance, we may assume $0 < u < 1$ on $\{|x| \ge r\}$ for some large r. Let T be a regular value of u so that $\max_{|x|=r} u < T < 1$. On

$M_T = \{u \geq T\}$, consider $u_T = \frac{1}{1-T}(u - T)$, which is the capacitary function on (M_T, g). By Theorem 2.1 (ii),

$$\lim_{s \to 1} \frac{1}{1-s} \left[4\pi - \frac{1}{(1-s)^2} \int_{\{u_T = s\}} |\nabla u_T|^2 \right] = 4\pi \, \mathfrak{m} \, \mathfrak{c}_T^{-1}, \tag{2.38}$$

where $\mathfrak{c}_T = \frac{1}{1-T}\mathfrak{c}$ is the capacity of $\{u = T\}$. Re-writing (2.38) in terms of u, we then obtain (1.1). Similarly, (1.2) follows from Theorem 2.1 (i).

3 Inequalities Along the Level Sets

In this section, we establish a family of geometric inequalities along $\{\Sigma_t\}$ under assumptions that g has nonnegative scalar curvature and M has simple topology.

We first compare

$$A(t) = 8\pi - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| \quad \text{and} \quad B(t) = 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2.$$

Theorem 3.1 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ. Suppose Σ is connected and $H_2(M, \Sigma) = 0$. Let u be the harmonic function that tends to 1 at ∞ and vanishes at Σ. If g has nonnegative scalar curvature, then*

$$4\pi + \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| \geq \frac{3}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \tag{3.1}$$

for all regular values t, and equality holds at some t if and only if (M, g), outside Σ_t, is isometric to $\mathbb{R}^3$ minus a round ball.

In particular, at Σ,

$$4\pi + \int_{\Sigma} H|\nabla u| \geq 3 \int_{\Sigma} |\nabla u|^2, \tag{3.2}$$

and equality holds if and only if (M, g) is isometric to $\mathbb{R}^3$ minus a round ball.

Remark 3.1 Inequality (3.1) is equivalent to

$$A(t) \leq 3B(t). \tag{3.3}$$

We will use Theorem 3.1 in this form later to derive other inequalities along $\{\Sigma_t\}$.

Remark 3.2 To the author's knowledge, inequality (3.2) represents a new result even in the 3-dimensional Euclidean space.

To prove Theorem 3.1, we begin with a lemma which may be derived directly from the work of Stern in [30].

Lemma 3.1 *Let (Ω, g) be a compact, orientable, Riemannian 3-manifold with nonnegative scalar curvature, with boundary $\partial\Omega$. Suppose $\partial\Omega$ has two connected components S_1 and S_2. Let u be a harmonic function on (Ω, g) such that $u = c_i$ on S_i, $i = 1, 2$, where c_1, c_2 are constants with $c_1 < c_2 < 1$. If the level set $\Sigma_s := u^{-1}(s)$ is connected for $s \in [c_1, c_2]$, then*

$$\Psi(t) := 4\pi(1-t) + \int_{\Sigma_t} H|\nabla u| - \frac{3}{1-t}\int_{\Sigma_t} |\nabla u|^2 \searrow \quad as\ t \nearrow,$$

i.e., $\Psi(t)$ is monotone nonincreasing. Here $t \in [c_1, c_2]$ denotes a regular value of u and H is the mean curvature of Σ_t with respect to the unit normal $\nu = |\nabla u|^{-1}\nabla u$.

Proof Let $t_1 < t_2$ be two regular values of u. On $\Omega_{[t_1,t_2]} := \{x \in \Omega \mid t_1 \le u(x) \le t_2\}$, one has

$$\int_{\Sigma_{t_1}} H|\nabla u| - \int_{\Sigma_{t_2}} H|\nabla u| \ge \int_{t_1}^{t_2}\int_{\Sigma_t} \frac{1}{2}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + R\right) - 2\pi \int_{t_1}^{t_2} \chi(\Sigma_t). \tag{3.4}$$

Here $\nabla^2 u$, ∇u denote the Hessian, the gradient of u on (M, g), respectively, R is the scalar curvature of g, and $\chi(\Sigma_t)$ is the Euler characteristic of Σ_t. Relation (3.4) is a direct consequence of Stern's computations in Section 2 of [30], and can also be found explicitly from (4.7) in [7] and (2.18) in [16].

Let $\mathbb{II}$ denote the second fundamental form of Σ_t w.r.t. ν. Let X, Y denote vectors tangent to Σ_t. Along Σ_t, one has

$$\nabla^2 u(X, Y) = |\nabla u|\mathbb{II}(X, Y), \quad \nabla^2 u(X, \nu) = X(|\nabla u|), \quad \nabla^2 u(\nu, \nu) = -H|\nabla u|. \tag{3.5}$$

Here the first two equations follow from definitions of $\nabla^2 u$ and $\mathbb{II}$, and the last equation follows from $0 = \Delta u = \Delta_{\Sigma_t} u + H\partial_\nu u + \nabla^2 u(\nu, \nu)$. As a result,

$$|\nabla u|^{-2}\,|\nabla^2 u|^2 = |\mathbb{II}|^2 + 2|\nabla u|^{-2}\left|\nabla_{\Sigma_t}|\nabla u|\right|^2 + H^2. \tag{3.6}$$

Under the assumption Σ_t is connected, it follows from (3.4) and (3.6) that

$$\begin{aligned} &4\pi(t_2 - t_1) + \int_{\Sigma_{t_1}} H|\nabla u| - \int_{\Sigma_{t_2}} H|\nabla u| \\ &\ge \int_{t_1}^{t_2}\int_{\Sigma_t} \frac{1}{2}|\mathring{\mathbb{II}}|^2 + |\nabla u|^{-2}\left|\nabla_{\Sigma_t}|\nabla u|\right|^2 + \frac{3}{4}H^2 + \frac{1}{2}R, \end{aligned} \tag{3.7}$$

where $\mathring{\mathbb{II}}$ denotes the traceless part of $\mathbb{II}$.

To handle the term of H^2 in (3.7), we follow the idea in [2, 25] to replace it with $\left(H - 2|\nabla u|(1-u)^{-1}\right)^2$. A motivation to this may be seen in the model case in which

$\Omega = \{R_1 \leq |x| \leq R_2\} \subset \mathbb{R}^3$ and $u = 1 - |x|^{-1}$. In this special setting, H and $|\nabla u|$ satisfy $H = 2|\nabla u|(1-u)^{-1}$ along any level set sphere.

Thus, one can rewrite (3.7) as

$$\begin{aligned} &4\pi(t_2 - t_1) + \int_{\Sigma_{t_1}} H|\nabla u| - \int_{\Sigma_{t_2}} H|\nabla u| \\ &\quad + 3\int_{t_1}^{t_2} \left[-\frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| + \frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2 \right] \\ &\geq \int_{t_1}^{t_2}\int_{\Sigma_t} \frac{1}{2}|\mathring{\mathbb{II}}|^2 + |\nabla u|^{-2}\big|\nabla_{\Sigma_t}|\nabla u|\big|^2 + \frac{3}{4}\left(H - \frac{2|\nabla u|}{1-u}\right)^2 + \frac{1}{2}R. \end{aligned} \tag{3.8}$$

At each regular value t, one has $\left(\int_{\Sigma_t} |\nabla u|^2\right)' = -\int_{\Sigma_t} H|\nabla u|$, and therefore,

$$-\frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| + \frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{1-t}\int_{\Sigma_t} |\nabla u|^2\right).$$

Thus, if $[t_1, t_2]$ has no critical values, the above directly shows

$$\begin{aligned} &\int_{t_1}^{t_2}\left(-\frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| + \frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2\right) \\ &= \frac{1}{1-t_2}\int_{\Sigma_{t_2}} |\nabla u|^2 - \frac{1}{1-t_1}\int_{\Sigma_{t_1}} |\nabla u|^2. \end{aligned} \tag{3.9}$$

In general, if $[t_1, t_2]$ has critical values, one may use a regularization argument to still obtain (3.9). For instance, applying Lemma A.1 of Appendix A to u on $\Omega_{[t_1,t_2]}$, one has

$$\begin{aligned} &\frac{1}{1-t_2}\int_{\Sigma_{t_2}} |\nabla u|^2 - \frac{1}{1-t_1}\int_{\Sigma_{t_1}} |\nabla u|^2 \\ &= \int_{\Omega_{[t_1,t_2]}} \frac{|\nabla u|^3}{(1-u)^2} + \int_{\{\nabla u \neq 0\} \subset \Omega_{[t_1,t_2]}} \frac{1}{1-u}|\nabla u|^{-1}\nabla^2 u(\nabla u, \nabla u). \end{aligned} \tag{3.10}$$

This, together with the coarea formula and (3.5), gives (3.9).

By (3.8) and (3.9),

$$\Psi(t_1) - \Psi(t_2) \geq \int_{t_1}^{t_2}\int_{\Sigma_t} \frac{1}{2}|\mathring{\mathbb{II}}|^2 + |\nabla u|^{-2}\big|\nabla_{\Sigma_t}|\nabla u|\big|^2 + \frac{3}{4}\left(H - \frac{2|\nabla u|}{1-u}\right)^2 + \frac{1}{2}R. \tag{3.11}$$

For the later purpose in Appendix A, we note that (3.11) holds without assumptions on the scalar curvature R.

If the scalar curvature R is nonnegative, then (3.11) implies $\Psi(t_1) \geq \Psi(t_2)$, which proves the lemma. □

In the context of Theorem 3.1, the assumption Σ is connected and $H_2(M, \Sigma) = 0$ is a sufficient condition to ensure $\chi(\Sigma_t) \le 2$ for a regular Σ_t. Under this condition, u being harmonic and the maximum principle guarantee Σ_t is connected. (The same assumption was used by Bray and the author [8] in estimating the capacity of Σ in (M, g) via the solution to the weak inverse mean curvature $(1/H)$ flow [17]. In that setting, a different reasoning shows the level set of the $1/H$ flow is connected.)

Proof of Theorem 3.1 Let $\Psi(t)$ be given from Lemma 3.1. On an asymptotically flat (M, g), a corollary of Lemma 2.2 shows

$$\lim_{t \to 1} \Psi(t) = 0.$$

Thus, letting $t_2 \to 1$ in (3.11) gives

$$\Psi(t) \ge \int_t^1 \int_{\Sigma_s} \frac{1}{2}|\mathring{\mathbb{II}}|^2 + |\nabla u|^{-2}\big|\nabla_{\Sigma_t}|\nabla u|\big|^2 + \frac{3}{4}\left(H - \frac{2|\nabla u|}{1-u}\right)^2 + \frac{1}{2}R \tag{3.12}$$

for every regular value t. In particular, if $R \ge 0$, then $\Psi(t) \ge 0$.

Inequality (3.1) follows from (3.12) by noting that

$$\frac{1}{1-t}\Psi(t) = 3B(t) - A(t). \tag{3.13}$$

To show the rigidity case of (3.1), it suffices to establish it for the case $t = 0$. Suppose the equality in (3.2) holds, then, by (3.12) and its proof, for every regular value $t \in [0, 1]$, Σ_t is connected (orientable) with $\chi(\Sigma_t) = 2$, hence Σ_t is a 2-sphere; moreover, $R = 0$, $|\nabla u|$ only depends on t, Σ_t is totally umbilic, and $H = \frac{2}{1-t}|\nabla u|$.

To show (M, g) is isometric to $\mathbb{R}^3$ minus a round ball, we start from a neighborhood of the boundary Σ. For convenience, we normalize (M, g) so that $|\Sigma| = 4\pi$. It follows from the equality

$$4\pi + \int_\Sigma H|\nabla u| = 3\int_\Sigma |\nabla u|^2$$

that $|\nabla u| = 1$ and $H = 2$ at $\Sigma = \Sigma_0$. Locally, g takes the form of $g = \eta(t)^{-2}\mathrm{d}t^2 + \gamma_t$ near Σ_0, where $t = u$, $\eta(t) = |\nabla u|$ and γ_t denotes the induced metric on Σ_t, which satisfies $\partial_t \gamma_t = 2\eta(t)^{-1}\mathbb{II}_t = \eta(t)^{-1}H\gamma_t = 2(1-t)^{-1}\gamma_t$. Thus, $(1-t)^2\gamma_t =$ a fixed metric. Similarly, since $|\nabla u|' = -H$, $\eta(t)$ satisfies $\eta'(t) = \frac{-2}{1-t}\eta(t)$. Hence, $(1-t)^{-2}\eta =$ a constant. As $|\nabla u| = 1$ at Σ, we thus have $\eta = (1-t)^2$ and $g = (1-t)^{-4}\mathrm{d}t^2 + (1-t)^{-2}\sigma_o$ for some fixed metric σ_o on the 2-sphere Σ. Invoking the fact $R = 0$ near Σ, we see σ_o is a round metric with Gauss curvature 1 on Σ.

Now, if u has a critical value, let $t_0 \in (0, 1)$ be the smallest critical value of u. The above argument then shows $(u^{-1}([0, t_0)), g)$ is isometric to

$$\left(\Sigma \times [0, t_0), (1-t)^{-4}\mathrm{d}t^2 + (1-t)^{-2}\sigma_o\right).$$

In particular, this implies $|\nabla u| = (1-t_0)^2 \neq 0$ on the set $\partial\{u < t_0\} = \partial\{u \geq t_0\}$. As a result, $\partial\{u \geq t_0\}$ is an embedded surface in M. Therefore, $\partial\{u \geq t_0\} = \{u = t_0\}$ by the strong maximum principle. In summary, this shows $\nabla u \neq 0$ on the set $\{u = t_0\}$, which contradicts to the assumption t_0 is a critical value. Hence, u has no critical values. We conclude (M, g) is isometric to

$$\left(\Sigma \times [0,1), (1-t)^{-4}\mathrm{d}t^2 + (1-t)^{-2}\sigma_o\right),$$

which, upon a change of variable $1 - t = r^{-1}$, is isometric to $\mathbb{R}^3$ minus a unit ball. □

Remark 3.3 In Theorem 3.1, if $u = c$ at Σ for some constant $c < 1$, (3.1) and its equality case still hold. This can be seen either from the above proof, or from considering the function $\frac{1}{1-c}(u - c)$.

Theorem 3.1 implies an upper bound of $\frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2$ via $\int_{\Sigma_t} H^2$.

Corollary 3.1 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ. Suppose Σ is connected and $H_2(M, \Sigma) = 0$. Let u be the harmonic function such that $u = 0$ at Σ and $u \to 1$ at ∞. If g has nonnegative scalar curvature, then*

$$\frac{1}{4\pi}\int_\Sigma |\nabla u|^2 \leq \frac{1}{9}\left[2W + 2\sqrt{W^2 + 3W} + 3\right], \tag{3.14}$$

where $W = \frac{1}{16\pi}\int_\Sigma H^2$, and equality holds if and only if (M, g) is isometric to $\mathbb{R}^3$ minus a round ball.

Proof Let $z = \left(\int_\Sigma |\nabla u|^2\right)^{\frac{1}{2}}$. By Theorem 3.1 and Hölder's inequality,

$$4\pi + \sqrt{16\pi W} z \geq 3z^2.$$

This implies the bound of z in (3.14) by elementary reason. The equality case follows from the equality case in Theorem 3.1. □

We next apply Theorem 3.1 to show that the quantities in Theorem 2.1, which approach to constant multiples of $\mathfrak{m}\mathfrak{c}_\Sigma^{-1}$ at ∞, are actually monotone.

Theorem 3.2 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ. Suppose Σ is connected and $H_2(M, \Sigma) = 0$. Let u be the harmonic function such that $u = 0$ at Σ and $u \to 1$ at ∞. If g has nonnegative scalar curvature, then*

(i) $\mathcal{A}(t) := \frac{1}{1-t}\left[8\pi - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u|\right] \nearrow$ *as $t \nearrow$, i.e., $\mathcal{A}(t)$ is monotone non-decreasing in t. If in addition $\partial\partial\partial g_{ij} = O(|x|^{-3-\tau})$ at ∞, then $\mathcal{A}(t) \leq 12\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}$. In particular, at Σ,*

$$8\pi - \int_\Sigma H|\nabla u| \leq 12\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}, \tag{3.15}$$

and equality holds if and only if (M, g) *is isometric to* $\mathbb{R}^3$ *minus a round ball.*

(ii) $\mathcal{B}(t) := \frac{1}{1-t}\left[4\pi - \frac{1}{(1-t)^2}\int_{\Sigma_t}|\nabla u|^2\right] \nearrow$ *as* $t \nearrow$, *i.e.,* $\mathcal{B}(t)$ *is monotone non-decreasing in* t. *If in addition* $\partial\partial\partial g_{ij} = O(|x|^{-3-\tau})$ *at* ∞, *then* $\mathcal{B}(t) \leq 4\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}$. *In particular, at* Σ,

$$4\pi - \int_\Sigma |\nabla u|^2 \leq 4\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}, \tag{3.16}$$

and equality holds if and only if (M, g) *is isometric to* $\mathbb{R}^3$ *minus a round ball.*

Proof We first show (ii) as it is more straightforward. By (2.36) and (3.3), at every regular value t, we have

$$-B'(t) = \frac{1}{1-t}\left[-2B(t) + A(t)\right] \leq \frac{1}{1-t}B(t).$$

Therefore, $\left[\frac{1}{1-t}B(t)\right]' \geq 0$, which implies the monotonicity of $\mathcal{B}(t) = \frac{1}{1-t}B(t)$ in the case that u has no critical values. If u has critical values, we may again apply a regularization argument to show that $\mathcal{B}(t_2) - \mathcal{B}(t_1) \geq 0$ for $t_2 > t_1$, see Proposition A.1 in Appendix A for details.

By Theorem 2.1 (ii),

$$\lim_{t\to 1}\mathcal{B}(t) = 4\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}.$$

Therefore, the monotonicity of $\mathcal{B}(t)$ shows

$$\mathcal{B}(t) \leq 4\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}.$$

At $t = 0$, this gives

$$\mathcal{B}(0) = 4\pi - \int_\Sigma |\nabla u|^2 \leq 4\pi\, \mathfrak{m}\, \mathfrak{c}_\Sigma^{-1}.$$

The rigidity part follows from the rigidity part of Theorem 3.1.

To show (i), we calculate, at a regular value t,

$$\mathcal{A}'(t) = \frac{1}{(1-t)^2}\left[A(t) - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| - \left(\int_{\Sigma_t} H|\nabla u|\right)'\right].$$

By (2.25) and the Gauss–Bonnet theorem,

$$\begin{aligned}\mathcal{A}'(t) &\geq \frac{1}{(1-t)^2}\left[A(t) - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| + \int_{\Sigma_t}\left(-K + \frac{3}{4}H^2\right)\right] \\ &\geq \frac{1}{(1-t)^2}\left[A(t) - \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| - 4\pi + \frac{3}{4}\int_{\Sigma_t} H^2\right] \\ &= \frac{1}{(1-t)^2}\left[\underbrace{4\pi + \frac{1}{1-t}\int_{\Sigma_t} H|\nabla u| - \frac{3}{(1-t)^2}\int_{\Sigma_t}|\nabla u|^2}_{I(t)}\right. \\ &\quad \left. + \frac{3}{4}\int_{\Sigma_t}\left(H - \frac{2|\nabla u|}{1-u}\right)^2\right].\end{aligned}$$

By (3.3),

$$I(t) = 3B(t) - A(t) \geq 0.$$

Therefore, $\mathcal{A}'(t) \geq 0$, which implies the monotonicity of $\mathcal{A}(t)$ in the absence of critical values. The general case can be again handled by a regularization argument that shows $\mathcal{A}(t_2) - \mathcal{A}(t_1) \geq 0$ for $t_2 > t_1$, see Proposition A.1 in Appendix A.

The remaining conclusions in (i) follow from Theorem 2.1 (i) and Theorem 3.1. □

Remark 3.4 If $u = c$ at Σ for some constant $c < 1$, Theorem 3.2 still holds with $\mathfrak{c}_\Sigma$ replaced by $\mathfrak{c}$, the constant in the asymptotic expansion $u = 1 - \mathfrak{c}|x|^{-1} + o(|x|^{-1})$ as $x \to \infty$.

Remark 3.5 In the above proof of (3.15) and (3.16), we assumed $\partial^3 g_{ij} = O(|x|^{-\tau-3})$ to obtain $\lim_{t\to 1}\mathcal{A}(t) = 12\pi\mathfrak{m}\mathfrak{c}_\Sigma^{-1}$ and $\lim_{t\to 1}\mathcal{B}(t) = 4\pi\mathfrak{m}\mathfrak{c}_\Sigma^{-1}$ by applying Theorem 2.1. We mention that this assumption on $\partial^3 g_{ij}$ can indeed be dropped, due to some recent developments since the appearance of this work. In [15], it was shown that, under the general asymptotic condition (2.1), one has

$$\limsup_{t\to 1}\mathcal{A}(t) \leq 12\pi\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \quad \text{and} \quad \limsup_{t\to 1}\mathcal{B}(t) \leq 4\pi\mathfrak{m}\mathfrak{c}_\Sigma^{-1}.$$

(See [15, Proposition 4.1], which made use of [1, Lemma 2.5].) Combined with the monotonicity, such inequalities are sufficient to obtain (3.15) and (3.16) in Theorem 3.2. For this reason, theorems in later sections of this paper, which made use of (3.15) and (3.16), do not need the assumption on $\partial^3 g_{ij}$ neither.

Remark 3.6 Comparing (3.2), (3.15) and (3.16), we have (3.2) + (3.16) ⇒ (3.15).

4 Proofs of the Positive Mass Theorem

The 3-dimensional Riemannian positive mass theorem (PMT), first proved by Schoen–Yau [27] and later by Witten [31], asserts that if (M, g) is a complete, asymptotically flat 3-manifold without boundary, with nonnegative scalar curvature, then $\mathfrak{m} \geq 0$, and $\mathfrak{m} = 0$ if and only if (M^3, g) is isometric to $\mathbb{R}^3$.

Since the work of Schoen–Yau and Witten, other proofs of this theorem have been given by Huisken–Ilmanen [17], by Li [21], by Bray–Kazaras–Khuri–Stern [7], and by Agostiniani–Mazzieri–Oronzio [2]. (Agostiniani–Mantegazza–Mazzieri–Oronzio [1] also gave a new proof of the Riemannian Penrose inequality, first proved by Bray [6] and Huisken–Ilmanen [17].)

To show the 3-dimensional PMT, it is known it suffices to assume M is topologically $\mathbb{R}^3$, see [7, Section 2] for instance; it also suffices to assume g has higher order derivatives decay near ∞, see [26, 28]. For this reason, one can make these assumptions in proving $\mathfrak{m} \geq 0$. Once the inequality is shown, the rigidity case of $m = 0$ follows from a variational argument, see [27].

As applications of Theorems 2.1 and 3.2, we observe a few additional arguments that demonstrate $\mathfrak{m} \geq 0$. We first outline the tools and features of the proofs to be given:

- Proof I uses Theorem 2.1 (ii) and a result of Munteanu–Wang [25].
- Proof II is self-contained. It makes use of Theorems 2.1 and 3.2.
- Proof III is self-contained. It uses the inequalities in Theorem 3.2. Proof III leads to new sufficient conditions that guarantee the positivity of the mass, see Sect. 5.

In what follows, let (M, g) be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature. Suppose M is topologically $\mathbb{R}^3$.

Proof I Take $p \in M$. Let $G(x)$ be the minimal positive Green's function with a pole at p, with $G(x) \to 0$ as $x \to \infty$. Let $u = 1 - G$. By Theorem 1.1 of Muntenau–Wang [25],

$$4\pi(1-t) - \frac{1}{1-t}\int_{\Sigma_t} |\nabla u|^2 \searrow \quad \text{as } t \nearrow,$$

i.e., it is monotone non-increasing in t.

As $x \to \infty$, $u = 1 - \frac{1}{4\pi}|x|^{-1} + O(|x|^{-1-\tau})$. By Lemma 2.2, $\frac{1}{1-t}\int_{\Sigma_t} |\nabla u|^2 \to 0$ as $t \to 1$. Hence,

$$4\pi(1-t) - \frac{1}{1-t}\int_{\Sigma_t} |\nabla u|^2 \geq 0.$$

Consequently,

$$\frac{1}{(1-t)}\left[4\pi - \frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2\right] \geq 0.$$

By Theorem 2.1 (ii),

$$\lim_{t\to 1} \frac{1}{(1-t)} \left[4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] = (4\pi)^2 \mathfrak{m}.$$

Therefore, $\mathfrak{m} \geq 0$. □

Proof II Take $p \in M$. Let $G(x)$ be the minimal positive Green's function with a pole at p. Let $d(x)$ denote the distance from x to p in (M, g). As $x \to p$, it is known

$$G(x) = \frac{1}{4\pi} d(x)^{-1} + o(d(x)^{-1}), \quad |\nabla G(x)| = \frac{1}{4\pi} d(x)^{-2} + o(d(x)^{-2}). \quad (4.1)$$

(See [25, Equation (3.3)] or [23, Theorem 2.4] for instance.)

Consider $u = 1 - G$. By Theorem 3.2 (ii),

$$\mathcal{B}(t) = \frac{1}{1-t} \left[4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right] \nearrow \quad \text{as } t \nearrow, \qquad (4.2)$$

i.e., it is monotone non-decreasing in t. Note this is different from the monotonicity of Munteanu–Wang [25]. The latter asserts $(1-t)^2 \mathcal{B}(t)$ is monotone non-increasing.

As $t \to -\infty$, by (4.1), $\frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2$ is bounded, hence $\frac{1}{(1-t)^3} \int_{\Sigma_t} |\nabla u|^2 \to 0$. Thus,

$$\lim_{t \to -\infty} \mathcal{B}(t) = 0. \qquad (4.3)$$

Therefore, by (4.2) and (4.3), $\mathcal{B}(t) \geq 0$. By Theorem 2.1 (ii),

$$(4\pi)^2 \, \mathfrak{m} = \lim_{t \to 1} \mathcal{B}(t) \geq 0.$$

□

Remark 4.1 Proof II is similar to that of Agostiniani–Mazzieri–Oronzio [2]. The difference is the use of different monotone quantities, i.e., $\mathcal{B}(t)$ compared to $F(t)$. A feature of $\mathcal{B}(t)$ used here is that it does not involve derivatives of the metric.

Remark 4.2 One can also work with $\mathcal{A}(t)$, and apply Theorems 3.2 (i) and 2.1 (i). In this case, one checks $\lim_{t\to -\infty} \frac{1}{(1-t)^2} \int_{\Sigma_t} H|\nabla u| = 0$, which follows from known estimates on $\nabla^2 G$ near the pole (see [2, 23, 25] for instance).

Proof III Take $p \in M$. Given a small $r > 0$, let B_r denote the geodesic ball of radius r centered at p. Let $\Sigma_r = \partial B_r$ and $u = u_r$ be the harmonic function with $u = 0$ at Σ_r and $u \to 1$ at ∞. Let $\mathfrak{c}_r$ be the capacity of Σ_r in (M, g).

Applying (3.15) of Theorem 3.2 (i) to $(M \setminus B_r, g)$, we have

$$\mathfrak{c}_r \left(8\pi - \int_{\Sigma_r} H |\nabla u| \right) \leq 12\pi \, \mathfrak{m}. \qquad (4.4)$$

It remains to check, as $r \to 0$,

$$c_r = O(r) \quad \text{and} \quad \int_{\Sigma_r} H|\nabla u| = O(1). \tag{4.5}$$

A conclusion $\mathfrak{m} \geq 0$ will follow from (4.4) and (4.5).

To estimate c_r, we may use the variational characterization of the capacity, i.e.,

$$c_r = \inf_f \left\{ \frac{1}{4\pi} \int_{M \setminus B_r} |\nabla f|^2 \right\}, \tag{4.6}$$

where f is a Lipschitz function with $f = 0$ at Σ_r and $f \to 1$ at ∞. Consider a test function $f(x) = r^{-1}(d(x) - r)$ in $B_{2r} \setminus B_r$ and extend f to be 1 outside B_{2r}. Here $d(x)$ is the distance from x to p. Then

$$c_r \leq \frac{1}{4\pi} \int_{B_{2r} \setminus B_r} |\nabla f|^2 = \frac{1}{4\pi r^2} \mathrm{Vol}(B_{2r} \setminus B_r) = O(r). \tag{4.7}$$

For $\int_{\Sigma_r} H|\nabla u|$, we have

$$\left| \int_{\Sigma_r} H|\nabla u| \right| \leq \max_{\Sigma_r} |H| \int_{\Sigma_r} |\nabla u| = \max_{\Sigma_r} |H| \, \mathfrak{c}_r = O(1)$$

by (4.7) and the fact $H = 2r^{-1} + O(r)$ (see (3.34) in [13] for instance).

This verifies (4.5) and completes the proof. □

Remark 4.3 In the above proof, we estimated $\mathfrak{c}_r$ by the so-called relative capacity of Σ_r in B_{2r}. By a result of Jauregui [18], one can indeed check

$$\limsup_{r \to 0} \left(8\pi - \int_{\Sigma_r} H|\nabla u| \right) \geq 0.$$

Remark 4.4 Alternatively one may use (3.16) of Theorem 3.2 (ii) to have

$$c_r \left(4\pi - \int_{\Sigma_r} |\nabla u|^2 \right) \leq 4\pi \, \mathfrak{m}$$

and check $\int_{\Sigma_r} |\nabla u|^2 = O(1)$. For instance, by the maximum principle, $|\nabla u| \leq |\nabla v|$ at ∂B_r, where v is the harmonic function with $v = 0$ at ∂B_r and $v = 1$ at ∂B_{2r}. By scaling and elliptic boundary estimates, $\int_{\partial B_r} |\nabla v|^2 = O(1)$ which shows $\int_{\Sigma_r} |\nabla u|^2 = O(1)$.

In addition to the above proofs, we want to mention there is a more geometric proof of PMT, which makes use of the full strength of our mass-capacity inequality Theorem 7.4 in Sect. 7. We give that proof in Remark 7.1.

5 Positive Mass Theorems with Boundary

Inspired by Proof III in the preceding section, we give some sufficient conditions that imply positive mass on manifolds with boundary.

Theorem 5.1 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with nonnegative scalar curvature, with boundary Σ. Suppose Σ is connected and $H_2(M, \Sigma) = 0$. Let $\Omega \subset M$ be a bounded region separating Σ and ∞. More precisely, this means $\partial\Omega$ has two connected components S_0 and S_1, where S_0 encloses Σ (and is allowed to coincide with Σ) and S_1 encloses S_0. Let u_Ω be the function on Ω with*

$$\Delta u_\Omega = 0, \quad u_\Omega|_{S_0} = 0, \quad \text{and } u_\Omega|_{S_1} = 1.$$

Let $\mathfrak{c}(\Omega) = \frac{1}{4\pi}\int_\Omega |\nabla u_\Omega|^2 = \frac{1}{4\pi}\int_{S_0} |\nabla u_\Omega|$. Then

$$H \leq \frac{2}{\mathfrak{c}(\Omega)} \implies \mathfrak{m} > 0. \tag{5.1}$$

In particular, this implies

$$H \leq \frac{8\pi L^2}{\mathrm{Vol}(\Omega)} \implies \mathfrak{m} > 0. \tag{5.2}$$

Here H is the mean curvature of Σ in (M, g), $\mathrm{Vol}(\Omega)$ is the volume of (Ω, g), and L is the distance between S_0 and S_1.

Proof Let u be the harmonic function on M with $u = 0$ at Σ and $u \to 1$ at ∞. By (3.15) of Theorem 3.2 (i),

$$\begin{aligned} 12\pi\, \mathfrak{m}\mathfrak{c}_\Sigma^{-1} &\geq 8\pi - \int_\Sigma H|\nabla u| \\ &\geq 4\pi\Big(2 - \mathfrak{c}_\Sigma \max_\Sigma H\Big), \end{aligned} \tag{5.3}$$

where $\mathfrak{c}_\Sigma$ is the capacity of Σ in (M, g). This shows

$$\max_\Sigma H \leq 2\mathfrak{c}_\Sigma^{-1} \Longrightarrow \mathfrak{m} \geq 0, \quad \max_\Sigma H < 2\mathfrak{c}_\Sigma^{-1} \Longrightarrow \mathfrak{m} > 0, \tag{5.4}$$

respectively.

Let D denote the region enclosed by S_1 with Σ. Extending u_Ω to be 1 on $M \setminus D$ and to be 0 on $D \setminus \Omega$. By the variational characterization of the capacity,

$$\mathfrak{c}_\Sigma < \frac{1}{4\pi}\int_M |\nabla u_\Omega|^2 = \mathfrak{c}(\Omega). \tag{5.5}$$

Therefore, (5.1) follows from (5.4) and (5.5).

To see (5.2), it suffices to estimate $\mathfrak{c}(\Omega)$. On Ω, consider a test function $f(x)$ which equals $L^{-1}d(x)$ if $d(x) \le L$ and is identically 1 if $d(x) \ge L$. Here $d(x)$ denotes the distance from x to S_0. Then

$$\mathfrak{c}(\Omega) \le \frac{1}{4\pi}\int_{\Omega} |\nabla f|^2 \le \frac{1}{4\pi L^2}\mathrm{Vol}(\Omega). \tag{5.6}$$

Hence, (5.2) follows from (5.1) and (5.6). □

The next result does not involve the mean curvature of the boundary. It makes use of (3.16) in Theorem 3.2 (ii).

Theorem 5.2 *Let (M, g), Ω, S_0, S_1 and u_Ω be given as in Theorem 5.1. Then*

$$\int_{S_0} |\nabla u_\Omega|^2 \le 4\pi \Longrightarrow \mathfrak{m} > 0. \tag{5.7}$$

Proof Let $\tilde{M}$ denote the region outside S_0. Let $\tilde{u}$ be the harmonic function on $\tilde{M}$ with $\tilde{u} = 0$ at S_0 and $\tilde{u} \to 1$ at ∞. Applying (3.16) of Theorem 3.2 (ii) to $(\tilde{M}, g, \tilde{u})$, we have

$$\int_{S_0} |\nabla \tilde{u}|^2 \le 4\pi \Longrightarrow \mathfrak{m} \ge 0, \quad \int_{S_0} |\nabla \tilde{u}|^2 < 4\pi \Longrightarrow \mathfrak{m} > 0, \tag{5.8}$$

respectively. On (Ω, g), the maximum principle shows

$$|\nabla \tilde{u}| < |\nabla u_\Omega| \quad \text{at } S_0. \tag{5.9}$$

Therefore, (5.7) follows from (5.8) and (5.9). □

Remark 5.1 It may be worthy of noting that the condition in (5.7) and the upper bound of H in (5.1) only involve the C^0-geometry of (Ω, g).

It is conceivable that Theorems 5.1 and 5.2 may be used to study the mass of incomplete asymptotically flat 3-manifolds. Recently Cecchini–Zeidler [10] and Lee–Lesourd–Unger [19] have given sufficient conditions, involving a positive lower bound of the scalar curvature on suitable regions in a manifold (M^n, g) that is spin or of dimension $3 \le n \le 7$, which guarantee the positivity of the mass. If such conditions are interpreted as shielding the incomplete part by regions with sufficiently positive scalar curvature, conditions in (5.1), (5.2) and (5.7) may be thought as shielding conditions in terms of the C^0-geometry of a separating region.

We end this section with the following proposition which was known and proved previously via the weak inverse mean curvature $(1/H)$ flow developed by Huisken–Ilmanen [17]. We include it here to show that the result can also be proved using harmonic functions.

Proposition 5.1 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ. Suppose Σ is connected and $H_2(M, \Sigma) = 0$. If g has nonnegative scalar curvature, then*

$$\int_\Sigma H^2 \leq 16\pi \implies \mathfrak{m} \geq 0,$$

and $\mathfrak{m} = 0$ if and only if (M, g) is isometric to $\mathbb{R}^3$ minus a round ball.

Proof By Corollary 3.1,

$$\int_\Sigma H^2 \leq 16\pi \implies \int_\Sigma |\nabla u|^2 \leq 4\pi.$$

Hence, $\mathfrak{m} \geq 0$ by (3.16). The rigidity case follows from that of Corollary 3.1. □

6 Integral Identities for the Mass-to-Capacity Ratio

In [7], Bray–Kazaras–Khuri–Stern found an integral identity for the mass of an asymptotically flat manifold. More precisely, if (E, g) denotes the exterior region of a complete, asymptotically flat Riemannian 3-manifold (M, g) with mass $\mathfrak{m}$, then

$$16\pi\mathfrak{m} \geq \int_E \left(\frac{|\nabla^2 u|}{|\nabla u|} + R|\nabla u| \right), \tag{6.1}$$

where u is a harmonic function on (E, g) satisfying Neumann boundary conditions at ∂E, and which is asymptotic to one of the asymptotically flat coordinate functions at ∞. In particular, if the scalar curvature is nonnegative, then $\mathfrak{m} \geq 0$.

In this section, we derive mass identities analogous to (6.1) with u being a harmonic function that equals 0 at the boundary and is asymptotic to 1 at ∞.

Theorem 6.1 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ. Suppose Σ is connected and $H_2(M, \Sigma) = 0$. Let u be the harmonic function such that $u = 0$ at Σ and $u \to 1$ at ∞. Let Φ_u be a symmetric $(0, 2)$ tensor given by*

$$\Phi_u = \frac{|\nabla u|^2}{1-u} g - \frac{3 \mathrm{d}u \otimes \mathrm{d}u}{1-u}.$$

Let $\mathfrak{m}$ be the mass of (M, g) and $\mathfrak{c}_\Sigma$ be the capacity of Σ in (M, g). Then

$$\begin{aligned} &\mathfrak{m}\mathfrak{c}_\Sigma^{-1} - \left(1 - \frac{1}{4\pi}\int_\Sigma |\nabla u|^2\right) \\ &\geq \frac{1}{16\pi}\int_M \left[\frac{1}{(1-u)^2} - 1\right]\left(\frac{|\nabla^2 u - \Phi_u|^2}{|\nabla u|} + R|\nabla u|\right) \end{aligned} \tag{6.2}$$

and

$$\begin{aligned}
&\mathfrak{m}\mathfrak{c}_{\Sigma}^{-1} - \frac{2}{3}\left(1 - \frac{1}{8\pi}\int_{\Sigma} H|\nabla u|\right) \\
&\geq \frac{1}{16\pi}\int_M \left[\frac{1}{(1-u)^2} - \frac{1}{3}\right]\left(\frac{|\nabla^2 u - \Phi_u|^2}{|\nabla u|} + R|\nabla u|\right).
\end{aligned} \tag{6.3}$$

Proof By (3.5), along a regular level set Σ_t, $(\nabla^2 u - \Phi_u)$ satisfies

$$\begin{aligned}
\left(\nabla^2 u - \Phi_u\right)(\nu, \nu) &= -H|\nabla u| + \frac{2|\nabla u|^2}{1-u}, \\
\left(\nabla^2 u - \Phi_u\right)(\nu, \cdot)|_{\Sigma_t} &= \langle \nabla_{\Sigma_t}|\nabla u|, \cdot\rangle, \\
\left(\nabla^2 u - \Phi_u\right)(\cdot, \cdot)|_{\Sigma_t} &= |\nabla u|\left(\mathbb{II} - \frac{|\nabla u|}{1-u}\gamma\right),
\end{aligned}$$

where γ denotes the induced metric on Σ_t. Therefore,

$$\begin{aligned}
&|\nabla u|^{-2}\left|\nabla^2 u - \Phi_u\right|^2 \\
&= \frac{3}{2}\left(H - \frac{2|\nabla u|}{1-u}\right)^2 + 2|\nabla u|^{-2}\left|\nabla_{\Sigma_t}|\nabla u|\right|^2 + |\mathring{\mathbb{II}}|^2.
\end{aligned} \tag{6.4}$$

Given two regular values $t_1 < t_2$, by (A.14) in Proposition A.1 of Appendix A, we have

$$\mathcal{B}(t_2) - \mathcal{B}(t_1) = \int_{t_1}^{t_2} \frac{1}{(1-t)^2}\left[3B(t) - A(t)\right]. \tag{6.5}$$

By (3.13) and (3.12),

$$3B(t) - A(t) = \frac{1}{1-t}\Psi(t) \quad \text{and} \quad \Psi(t) \geq \int_t^1 \psi(s), \tag{6.6}$$

where

$$\begin{aligned}
\psi(t) &= \int_{\Sigma_t}\left[\frac{3}{4}\left(H - \frac{2|\nabla u|}{1-u}\right)^2 + |\nabla u|^{-2}\left|\nabla_{\Sigma_t}|\nabla u|\right|^2 + \frac{1}{2}|\mathring{\mathbb{II}}|^2 + \frac{1}{2}R\right] \\
&= \frac{1}{2}\int_{\Sigma_t}\left(\frac{|\nabla^2 u - \Phi_u|^2}{|\nabla u|^2} + R\right).
\end{aligned} \tag{6.7}$$

Taking $t_1 = 0$ and letting $t_2 \to 1$, applying Theorem 2.1, we hence have

$$\begin{aligned} 4\pi \mathfrak{m}\mathfrak{c}_\Sigma^{-1} - \mathcal{B}(0) &= \int_0^1 \frac{1}{(1-t)^3} \Psi(t) \\ &\geq \int_0^1 \frac{1}{(1-t)^3} \left(\int_t^1 \psi(s) \right). \end{aligned} \tag{6.8}$$

Integration by parts gives

$$\begin{aligned} &\int_0^1 \frac{1}{(1-t)^3} \left(\int_t^1 \psi(s) \right) \\ &= \frac{1}{2} \left[\lim_{t\to 1} \frac{1}{(1-t)^2} \int_t^1 \psi(s) - \int_0^1 \psi(s) + \int_0^1 \frac{\psi(t)}{(1-t)^2} \right]. \end{aligned} \tag{6.9}$$

We claim

$$\lim_{t\to 1} \frac{1}{(1-t)^2} \int_t^1 \psi(s) = 0. \tag{6.10}$$

This is because, by (2.32),

$$\int_t^1 \int_{\Sigma_s} |\nabla u|^{-2} \big|\nabla_\Sigma |\nabla u|\big|^2 + \frac{1}{2} |\mathring{\mathbb{II}}|^2 = O((1-t)^{1+2\tau}),$$

and, by (2.33),

$$\int_t^1 \int_{\Sigma_s} |R| = o((1-t)^2).$$

Also, by (2.11), (2.12) and Lemma 2.1,

$$\int_t^1 \int_{\Sigma_s} \left(H - \frac{2|\nabla u|}{1-u} \right)^2 = O((1-t)^{1+2\tau}). \tag{6.11}$$

Therefore, (6.10) holds.

Now it follows from (6.8)–(6.10) that

$$\begin{aligned} &4\pi \mathfrak{m}\mathfrak{c}_\Sigma^{-1} - \mathcal{B}(0) \\ &\geq \frac{1}{2} \int_0^1 \left[\frac{1}{(1-t)^2} - 1 \right] \psi(t) \\ &= \frac{1}{4} \int_0^1 \left[\frac{1}{(1-t)^2} - 1 \right] \int_{\Sigma_t} \left(\frac{|\nabla^2 u - \Phi_u|^2}{|\nabla u|^2} + R \right) \\ &= \frac{1}{4} \int_M \left[\frac{1}{(1-u)^2} - 1 \right] \left(\frac{|\nabla^2 u - \Phi_u|^2}{|\nabla u|} + R|\nabla u| \right). \end{aligned} \tag{6.12}$$

This proves (6.2).

Similarly, by (A.17) following Proposition A.1 of Appendix A,

$$[\mathcal{A}(t_2) - \mathcal{B}(t_2)] - [\mathcal{A}(t_1) - \mathcal{B}(t_1)] \geq \int_{t_1}^{t_2} \frac{1}{(1-t)^2} \psi(t). \tag{6.13}$$

Taking $t_1 = 0$, letting $t_2 \to 1$ and applying Theorem 2.1, we have

$$\begin{aligned} & 8\pi \mathfrak{m} \mathfrak{c}_\Sigma^{-1} - (\mathcal{A}(0) - \mathcal{B}(0)) \\ \geq & \ \frac{1}{2} \int_0^1 \frac{1}{(1-t)^2} \int_{\Sigma_t} \left(\frac{|\nabla^2 u - \Phi_u|^2}{|\nabla u|^2} + R \right) \\ = & \ \frac{1}{2} \int_M \frac{1}{(1-u)^2} \left(\frac{|\nabla^2 u - \Phi_u|^2}{|\nabla u|} + R |\nabla u| \right). \end{aligned} \tag{6.14}$$

This together with (6.12) proves (6.3). □

Remark 6.1 If the scalar curvature R is nonnegative, then (6.2) implies (3.16), (6.3) implies (3.15), and (6.14) implies

$$4\pi - \int_\Sigma H |\nabla u| + \int_\Sigma |\nabla u|^2 \leq 8\pi \mathfrak{m} \mathfrak{c}_\Sigma^{-1}. \tag{6.15}$$

For manifolds that are spatial Schwarzschild manifolds near infinity, (6.15) also follows from the work of Agostiniani–Mazzieri–Oronzio [2]. On the other hand, one sees (6.15) is an algebraic consequence of (3.2) and (3.16).

Remark 6.2 If the manifold M in Theorem 6.1 has no boundary, let G be a minimal positive Green's function with a pole at some $p \in M$. Taking $u = 1 - 4\pi G$ in (6.13), and letting $t_2 \to 1$, $t_1 \to -\infty$, one finds

$$\mathfrak{m} \geq \frac{1}{(8\pi)^2} \int_M \frac{1}{G^2} \left(\frac{|\nabla^2 G + \Phi_G|^2}{|\nabla G|} + R |\nabla G| \right). \tag{6.16}$$

Here Φ_G is the $(0, 2)$ tensor given by

$$\Phi_G = \frac{|\nabla G|^2}{G} g - \frac{3 \mathrm{d}G \otimes \mathrm{d}G}{G}.$$

Inequality (6.16) gives the integral version of the proof of the 3-dimensional PMT in [2].

7 Promoting Inequalities via Schwarzschild Models

Inequalities in Sect. 3 are derived via monotone quantities that become constant in Euclidean spaces outside round balls. As a result, they are strict inequalities when evaluated in spatial Schwarzschild manifolds with nonzero mass outside rotationally symmetric spheres.

Inspired by Bray's proof of the Riemannian Penrose inequality [6], in this section we apply results from the previous sections to derive inequalities that become equality in Schwarzschild spaces.

We first outline the idea. Given a tuple (M, g, u) satisfying assumptions in Theorem 3.1 (or equivalently in Theorem 3.2), let v be any other harmonic function on (M, g) with $v \to 1$ at ∞ and $v > 0$ at Σ. The following facts hold:

1. the metric $\bar{g} := v^4 g$ is asymptotically flat, with nonnegative scalar curvature;
2. the function $\bar{u} := v^{-1} u$ is a harmonic function with respect to the metric $\bar{g}$, and satisfies $\bar{u} = 0$ at Σ and $\bar{u} \to 1$ at ∞.

Thus, results from the previous sections are applicable to M with the conformally deformed metric $\bar{g}$ and the $\bar{g}$-harmonic function $\bar{u}$.

To proceed, we compute the quantities involved. Let $\bar{\nabla}$ denote the gradient on $(M, \bar{g})$, let $\bar{H}$ be the mean curvature of Σ in $(M, \bar{g})$ with respect to the ∞-pointing normal. Let $d\sigma$, $d\bar{\sigma}$ denote the surface measure on Σ in (M, g), $(M, \bar{g})$, respectively. As Σ has dimension two, it can be checked

$$\int_\Sigma |\bar{\nabla}\bar{u}|^2_{\bar{g}}\, d\bar{\sigma} = \int_\Sigma |\nabla \bar{u}|^2\, d\sigma. \tag{7.1}$$

(We omitted writing the area and volume measures in previous integrals as there was only one metric g involved therein.) The mean curvature $\bar{H}$ is related to the mean curvature H of Σ in (M, g) via $\bar{H} = v^{-2}(4v^{-1}\partial_\nu v + H)$. Thus,

$$\int_\Sigma \bar{H} |\bar{\nabla}\bar{u}|_{\bar{g}}\, d\bar{\sigma} = \int_\Sigma \big(4v^{-1}\partial_\nu v + H\big) |\nabla \bar{u}|\, d\sigma. \tag{7.2}$$

Let $\bar{\mathfrak{m}}$ denote the mass of $(M, \bar{g})$. $\bar{\mathfrak{m}}$ and $\mathfrak{m}$ are related by

$$\bar{\mathfrak{m}} = \mathfrak{m} - 2\mathfrak{c}_v, \tag{7.3}$$

where $\mathfrak{c}_v$ is the constant in the expansion

$$v = 1 - \frac{\mathfrak{c}_v}{|x|} + o(|x|^{-1}),$$

as $x \to \infty$. Since $\bar{u} = v^{-1}u$, $\bar{u}$ satisfies

$$\bar{u} = 1 - \frac{\mathfrak{c}_\Sigma - \mathfrak{c}_v}{|x|} + o(|x|^{-1}),$$

where $\mathfrak{c}_\Sigma > \mathfrak{c}_v$ by the fact $v > u$ and the maximum principle. The capacity of Σ in $(M, \bar{g})$, which we denote by $\bar{\mathfrak{c}}_\Sigma$, is then given by

$$\bar{\mathfrak{c}}_\Sigma = \mathfrak{c}_\Sigma - \mathfrak{c}_v. \tag{7.4}$$

Finally, we note, as $u = 0$ at Σ,

$$|\nabla \bar{u}| = v^{-1} |\nabla u| \quad \text{at } \Sigma. \tag{7.5}$$

We want to seek implications of the inequalities (3.2), (3.16), (3.15) and (6.15), i.e.,

$$4\pi + \int_\Sigma H|\nabla u| \geq 3 \int_\Sigma |\nabla u|^2, \tag{7.6}$$

$$4\pi - \int_\Sigma |\nabla u|^2 \leq 4\pi \mathfrak{m} \, \mathfrak{c}_\Sigma^{-1}, \tag{7.7}$$

$$8\pi - \int_\Sigma H|\nabla u| \leq 12\pi \mathfrak{m} \, \mathfrak{c}_\Sigma^{-1}, \tag{7.8}$$

$$4\pi - \int_\Sigma H|\nabla u| + \int_\Sigma |\nabla u|^2 \leq 8\pi \mathfrak{m} \, \mathfrak{c}_\Sigma^{-1}, \tag{7.9}$$

when they are applied to the conformally deformed triple $(M, \bar{g}, \bar{u})$. As mentioned in Remark 3.6 and Remark 6.15, one knows

$$(7.6) \ + \ (7.7) \ \Longrightarrow \ (7.8) \ \text{ and } \ (7.9).$$

For this reason, we focus on the use of (7.6) and (7.7) below.

Theorem 7.1 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ. Suppose Σ is connected and $H_2(M, \Sigma) = 0$. Let u be the harmonic function such that $u = 0$ at Σ and $u \to 1$ at ∞. If g has nonnegative scalar curvature, then, for any constant $k > 0$,*

$$4\pi + k \int_\Sigma H|\nabla u| \geq k(4 - k) \int_\Sigma |\nabla u|^2. \tag{7.10}$$

Moreover, equality in (7.10) holds for some k if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere, that is, up to isometry,

$$(M, g) = \left(\mathbb{R}^3 \setminus \{|x| < r\}, \left(1 + \frac{\mathfrak{m}}{2|x|}\right)^4 g_E \right),$$

where $r > 0$ is a constant, $g_E = \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j$ is the Euclidean metric, and $\mathfrak{m}$, k, r are related by $\mathfrak{m} = 2r(k - 1)$.

Proof Given any positive harmonic function v on (M, g), let $\bar{g} = v^4 g$ and $\bar{u} = v^{-1} u$. Applying (3.2) in Theorem 3.1 to the triple $(M, \bar{g}, \bar{u})$, we have

$$4\pi + \int_\Sigma \bar{H} |\bar{\nabla} \bar{u}|_{\bar{g}} \, \mathrm{d}\bar{\sigma} \geq 3 \int_\Sigma |\bar{\nabla} \bar{u}|_{\bar{g}}^2 \, \mathrm{d}\bar{\sigma}. \tag{7.11}$$

By (7.1)–(7.5), (7.11) shows

$$4\pi + \int_\Sigma \big(4v^{-1}\partial_\nu v + H\big) v^{-1} |\nabla u| \, \mathrm{d}\sigma \geq 3 \int_\Sigma v^{-2} |\nabla u|^2 \, \mathrm{d}\sigma. \tag{7.12}$$

Given any constant $k > 0$, choose

$$v = u + \frac{1}{k}(1 - u). \tag{7.13}$$

It follows from (7.12) and the fact $\partial_\nu u = |\nabla u|$ at Σ that

$$4\pi + k \int_\Sigma H |\nabla u| \, \mathrm{d}\sigma \geq k(4 - k) \int_\Sigma |\nabla u|^2 \, \mathrm{d}\sigma,$$

which proves (7.10).

The above also shows equality in (7.10) holds for some k if and only if equality in (7.11) holds for the corresponding $(M, \bar{g}, \bar{u})$. By Theorem 3.1, this occurs if and only if $(M, \bar{g})$ is isometric to $\big(\mathbb{R}^3 \backslash B_r, g_E\big)$, where $B_r = \{x \in \mathbb{R}^3 \mid |x| < r\}$ for some constant $r > 0$. In this case,

$$\bar{u} = 1 - \frac{r}{|x|}. \tag{7.14}$$

This combined with (7.13) and the fact $\bar{u} = v^{-1} u$ shows

$$v^{-1} = 1 + \frac{r(k-1)}{|x|}. \tag{7.15}$$

As a result,

$$g = v^{-4} g_E = \left(1 + \frac{r(k-1)}{|x|}\right)^4 \delta_{ij} \, \mathrm{d}x^i \mathrm{d}x^j, \tag{7.16}$$

which is a spatial Schwarzschild metric with mass $\mathfrak{m} = 2r(k-1)$. ◻

Theorem 7.1 implies a sharp bound of $\int_\Sigma |\nabla u|^2$ by the Willmore functional of Σ, with the bound achieved by Schwarzschild spaces outside mean-convex round spheres.

Corollary 7.1 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ. Suppose Σ is connected and $H_2(M, \Sigma) = 0$. Let u be the harmonic function such that $u = 0$ at Σ and $u \to 1$ at ∞. If g has nonnegative scalar curvature, then*

$$\left(\frac{1}{\pi} \int_\Sigma |\nabla u|^2\right)^{\frac{1}{2}} \leq \left(\frac{1}{16\pi} \int_\Sigma H^2\right)^{\frac{1}{2}} + 1. \tag{7.17}$$

Moreover, equality holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative constant mean curvature.

Proof Consider the following quadratic form of k,

$$Q(k) := \alpha(u)\, k^2 + \beta(u)\, k + 4\pi, \tag{7.18}$$

where

$$\alpha(u) = \int_\Sigma |\nabla u|^2, \quad \beta(u) = \int_\Sigma H|\nabla u| - 4\int_\Sigma |\nabla u|^2.$$

We have $Q(0) = 4\pi$, and Theorem 7.1 shows

$$Q(k) \geq 0, \quad \forall k > 0.$$

Thus, by elementary reasons, either

$$\beta(u)^2 - 16\pi\alpha(u) \leq 0 \tag{7.19}$$

or

$$\beta(u)^2 - 16\pi\alpha(u) > 0 \quad \text{and} \quad -\beta(u) + \sqrt{\beta(u)^2 - 16\pi\alpha(u)} < 0. \tag{7.20}$$

The latter case is equivalent to

$$\beta(u) > \sqrt{16\pi\alpha(u)},$$

that is

$$\frac{\int_\Sigma H|\nabla u|}{\left(\int_\Sigma |\nabla u|^2\right)^{\frac{1}{2}}} - 4\left(\int_\Sigma |\nabla u|^2\right)^{\frac{1}{2}} > \sqrt{16\pi}. \tag{7.21}$$

If (7.21) holds, then, by Hölder's inequality,

$$\left(\frac{1}{16\pi}\int_\Sigma H^2\right)^{\frac{1}{2}} > 1 + \left(\frac{1}{\pi}\int_\Sigma |\nabla u|^2\right)^{\frac{1}{2}}. \tag{7.22}$$

If (7.19) holds, then

$$\left|\int_\Sigma H|\nabla u| - 4\int_\Sigma |\nabla u|^2\right| \leq 4\left(\pi\int_\Sigma |\nabla u|^2\right)^{\frac{1}{2}},$$

which in particular implies

$$4\int_\Sigma |\nabla u|^2 \le 4\left(\pi \int_\Sigma |\nabla u|^2\right)^{\frac{1}{2}} + \int_\Sigma H|\nabla u|. \tag{7.23}$$

Combined with Hölder's inequality, this shows

$$\left(\frac{1}{\pi}\int_\Sigma |\nabla u|^2\right)^{\frac{1}{2}} \le 1 + \left(\frac{1}{16\pi}\int_\Sigma H^2\right)^{\frac{1}{2}}. \tag{7.24}$$

Therefore, in either case, we conclude (7.17) holds.

If equality in (7.17) holds, then (7.20) does not hold; (7.23) holds with equality; and $H = c|\nabla u|$ for some constant $c \ge 0$. In particular, this gives

$$-\beta(u) = 4\int_\Sigma |\nabla u|^2 - \int_\Sigma H|\nabla u| = \sqrt{16\pi\alpha(u)} > 0.$$

As a result, $Q(k_0) = 0$ at

$$k_0 = -\frac{\beta(u)}{2\alpha(u)} = 2\left(\frac{1}{\pi}\int_\Sigma |\nabla u|^2\right)^{-\frac{1}{2}} = \frac{2}{1 + \left(\frac{1}{16\pi}\int_\Sigma H^2\right)^{\frac{1}{2}}} > 0.$$

By Theorem 7.1, (M, g) is isometric to a spatial Schwarzschild manifold

$$(M, g) = \left(\mathbb{R}^3 \setminus \{|x| < r\}, \left(1 + \frac{\mathfrak{m}}{2|x|}\right)^4 g_E\right),$$

where $r > 0$ and $\mathfrak{m} = 2r(k_0 - 1)$. As $k_0 \le 2$, the boundary $\{|x| = r\}$ has nonnegative mean curvature in (M, g).

On such an (M, g), direct calculation shows

$$\left(\frac{1}{16\pi}\int_\Sigma H^2\right)^{\frac{1}{2}} = \left|\frac{2}{k} - 1\right| \quad \text{and} \quad \left(\frac{1}{\pi}\int_\Sigma |\nabla u|^2\right)^{\frac{1}{2}} = \frac{2}{k}.$$

As $k \le 2$, equality holds in (7.17). This completes the proof. □

An immediate application of Corollary 7.1 yields a result of Bray and the author [8] on the estimate of the capacity-to-area–radius ratio.

Theorem 7.2 ([8]) *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ. Suppose Σ is connected and $H_2(M, \Sigma) = 0$. If g has nonnegative scalar curvature, then*

$$\frac{2\mathfrak{c}_\Sigma}{r_\Sigma} \le \left(\frac{1}{16\pi}\int_\Sigma H^2\right)^{\frac{1}{2}} + 1. \tag{7.25}$$

Here $\mathfrak{c}_\Sigma$ *is the capacity of* Σ *in* (M, g) *and* $r_\Sigma = \left(\frac{|\Sigma|}{4\pi}\right)^{\frac{1}{2}}$ *is the area–radius of* Σ*. Moreover, equality holds if and only if* (M, g) *is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative constant mean curvature.*

Proof This follows directly from

$$\left(\int_\Sigma |\nabla u|^2\right)^{\frac{1}{2}} \geq \frac{\int_\Sigma |\nabla u|}{|\Sigma|^{\frac{1}{2}}} = \sqrt{\pi}\frac{2\mathfrak{c}_\Sigma}{r_\Sigma}$$

and Corollary 7.1. □

Next, we proceed to find implications of (3.16) in Theorem 3.2.

Theorem 7.3 *Let* (M, g) *be a complete, orientable, asymptotically flat* 3*-manifold with boundary* Σ*. Suppose* Σ *is connected and* $H_2(M, \Sigma) = 0$*. Let* u *be the harmonic function such that* $u = 0$ *at* Σ *and* $u \to 1$ *at* ∞*. If* g *has nonnegative scalar curvature, then*

$$\frac{\mathfrak{m}}{2\mathfrak{c}_\Sigma} \geq 1 - \left(\frac{1}{4\pi}\int_\Sigma |\nabla u|^2 \, d\sigma\right)^{\frac{1}{2}}. \tag{7.26}$$

Moreover, equality holds if and only if (M, g) *is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere.*

Proof Given any positive harmonic function v on (M, g), let $\bar{g} = v^4 g$ and $\bar{u} = v^{-1}u$. Applying (3.16) in Theorem 3.2 to $(M, \bar{g}, \bar{u})$, we have

$$4\pi - \int_\Sigma |\bar{\nabla}\bar{u}|^2_{\bar{g}} \, d\bar{\sigma} \leq 4\pi \bar{\mathfrak{m}}\bar{\mathfrak{c}}_\Sigma^{-1}. \tag{7.27}$$

By (7.1)–(7.5), (7.27) becomes

$$4\pi - \int_\Sigma v^{-2}|\nabla u|^2 \, d\sigma \leq 4\pi \frac{\mathfrak{m} - 2\mathfrak{c}_v}{\mathfrak{c}_\Sigma - \mathfrak{c}_v}. \tag{7.28}$$

Given any constant $k > 0$, choose

$$v = u + \frac{1}{k}(1 - u). \tag{7.29}$$

Then $v = k^{-1}$ at Σ, $\mathfrak{c}_v = (1 - k^{-1})\mathfrak{c}_\Sigma$, and (7.28) shows

$$\frac{\mathfrak{m}}{\mathfrak{c}_\Sigma} \geq 2 - \frac{1}{k} - \frac{k}{4\pi}\int_\Sigma |\nabla u|^2 \, d\sigma. \tag{7.30}$$

Maximizing the right side of (7.30) over all $k > 0$, we have

$$\frac{\mathfrak{m}}{2\mathfrak{c}_\Sigma} \geq 1 - \left(\frac{1}{4\pi} \int_\Sigma |\nabla u|^2 \, d\sigma \right)^{\frac{1}{2}}, \tag{7.31}$$

which proves (7.26).

If equality in (7.26) holds, then equality in (7.27) holds for $v = u + k^{-1}(1 - u)$ with the constant k given by

$$k = \left(\frac{1}{4\pi} \int_\Sigma |\nabla u|^2 \, d\sigma \right)^{-\frac{1}{2}}. \tag{7.32}$$

By Theorem 3.2, $(M, \bar{g})$ is isometric to $\left(\mathbb{R}^3 \setminus B_r, g_E\right)$, where $B_r = \{x \in \mathbb{R}^3 \mid |x| < r\}$ for some $r > 0$, and

$$\bar{u} = 1 - \frac{r}{|x|}. \tag{7.33}$$

This combined with $\bar{u} = v^{-1}u$ and (7.29) shows

$$v^{-1} = 1 + \frac{r(k-1)}{|x|}. \tag{7.34}$$

As a result,

$$g = v^{-4} g_E = \left(1 + \frac{r(k-1)}{|x|} \right)^4 \delta_{ij} \, dx^i dx^j, \tag{7.35}$$

which is a spatial Schwarzschild metric with the mass $\mathfrak{m} = 2r(k-1)$.

On any such an (M, g), direct calculation shows

$$\frac{\mathfrak{m}}{2\mathfrak{c}_\Sigma} = 1 - \frac{1}{k} \quad \text{and} \quad \left(\frac{1}{4\pi} \int_\Sigma |\nabla u|^2 \right)^{\frac{1}{2}} = \frac{1}{k},$$

which verifies equality in (7.26). This completes the proof. □

We now have a succinct lower bound of the mass-to-capacity ratio by the Willmore functional.

Theorem 7.4 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ. Suppose Σ is connected and $H_2(M, \Sigma) = 0$. If g has nonnegative scalar curvature, then*

$$\frac{\mathfrak{m}}{\mathfrak{c}_\Sigma} \geq 1 - \left(\frac{1}{16\pi} \int_\Sigma H^2 \right)^{\frac{1}{2}}. \tag{7.36}$$

Moreover, equality holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative constant mean curvature.

Proof This is a direct consequence of Corollary 7.1 and Theorem 7.3. □

We give a few remarks.

Remark 7.1 Theorem 7.4 gives another way to see the 3-dimensional positive mass theorem. In the context of Proof III in Sect. 4, Theorem 7.4 gives

$$\frac{\mathfrak{m}}{\mathfrak{c}_r} \geq 1 - \left(\frac{1}{16\pi} \int_{\Sigma_r} H^2 \right)^{\frac{1}{2}} = o(1), \quad \text{as } r \to 0,$$

where c_r is the capacity of a small geodesic ball of radius r. Hence, $\mathfrak{m} \geq 0$.

Remark 7.2 Theorem 7.4 improves the result of Bray and the author in [8]. Under an additional assumption of $\int_\Sigma H^2 \leq 16\pi$, in [8] the capacity estimate (7.25) was converted into a Hawking mass estimate

$$\mathfrak{m}_H(\Sigma) \geq \left[1 - \left(\frac{1}{16\pi} \int_\Sigma H^2 \right)^{\frac{1}{2}} \right] \mathfrak{c}_\Sigma$$

and the relation $\mathfrak{m} \geq \mathfrak{m}_H(\Sigma)$ was applied (if Σ is outer-minimizing) to obtain (7.36).

In the current derivation of (7.36), we bound the ratio $\mathfrak{m}\mathfrak{c}_\Sigma^{-1}$ via $\int_\Sigma |\nabla u|^2$ and bound $\int_\Sigma |\nabla u|^2$ via $\int_\Sigma H^2$, hence bypassing the use of $\mathfrak{m}_H(\Sigma)$ in relating $\mathfrak{m}$ and $\mathfrak{c}_\Sigma$.

Remark 7.3 One may re-write (7.36) as

$$\mathfrak{M}(g) := \frac{\mathfrak{m}}{\mathfrak{c}_\Sigma} + \left(\frac{1}{16\pi} \int_\Sigma H^2 \right)^{\frac{1}{2}} - 1 \geq 0.$$

This gives a nonnegative quantity $\mathfrak{M}(g)$ on asymptotically flat 3-manifolds (M, g) with boundary (under the curvature and topological assumptions). $\mathcal{M}(g)$ vanishes precisely if (M, g) is rotationally symmetric with mean-convex boundary.

8 Manifolds with the Mass-to-Capacity Ratio ≤ 1

In this section, prompted by Theorem 7.4, we consider a class of manifolds satisfying a mass-capacity relation

$$\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1. \tag{8.1}$$

As we will see later in Proposition 8.1, such a class of manifolds includes static metric extensions in the context of the Bartnik quasi-local mass [5].

Theorem 8.1 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ, satisfying a mass-capacity relation*

$$\mathfrak{m}\mathfrak{c}_{\Sigma}^{-1} \leq 1.$$

Let u be the harmonic function with $u = 0$ at Σ and $u \to 1$ near ∞. If Σ is connected, $H_2(M, \Sigma) = 0$, and g has nonnegative scalar curvature, then

$$\frac{1}{4\pi} \int_{\Sigma} H|\nabla u| \geq \left(2 - \mathfrak{m}\mathfrak{c}_{\Sigma}^{-1}\right)\left(1 - \mathfrak{m}\mathfrak{c}_{\Sigma}^{-1}\right). \tag{8.2}$$

Moreover, equality holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative constant mean curvature.

Proof For a regular value $t \in [0, 1)$, if $u^{(t)}$ denotes the harmonic function outside Σ_t with $u^{(t)} = 0$ at Σ_t and $u^{(t)} \to 1$ at ∞, then

$$u^{(t)} = \frac{u - t}{1 - t}. \tag{8.3}$$

As a result, the capacity $\mathfrak{c}_{\Sigma_t}$ of Σ_t is related to that of Σ by

$$\mathfrak{c}_{\Sigma_t} = \frac{\mathfrak{c}_{\Sigma}}{1 - t}. \tag{8.4}$$

Therefore, by Theorem 7.4,

$$\begin{aligned} \left(\frac{1}{16\pi} \int_{\Sigma_t} H^2\right)^{\frac{1}{2}} &\geq 1 - \mathfrak{m}\mathfrak{c}_{\Sigma_t}^{-1} \\ &= 1 - \mathfrak{m}\mathfrak{c}_{\Sigma}^{-1}(1 - t). \end{aligned} \tag{8.5}$$

Under the assumption $\mathfrak{m}\mathfrak{c}_{\Sigma}^{-1} \leq 1$, we have

$$1 - \mathfrak{m}\mathfrak{c}_{\Sigma}^{-1}(1 - t) \geq 0.$$

Hence, (8.5) is equivalent to

$$\frac{1}{16\pi} \int_{\Sigma_t} H^2 \geq \left[1 - \mathfrak{m}\mathfrak{c}_{\Sigma}^{-1}(1 - t)\right]^2. \tag{8.6}$$

To proceed, we return to the basic identity (3.7) in Sect. 3. Given any regular values $t_1 < t_2 < 1$, by (3.7),

$$4\pi(t_2 - t_1) + \int_{\Sigma_{t_1}} H|\nabla u| - \int_{\Sigma_{t_2}} H|\nabla u|$$

$$\geq \int_{t_1}^{t_2} \int_{\Sigma_t} \frac{1}{2}|\mathring{\mathbb{II}}|^2 + |\nabla u|^{-2}\big|\nabla_{\Sigma_t}|\nabla u|\big|^2 + \frac{3}{4}H^2 + \frac{1}{2}R$$

$$\geq \int_{t_1}^{t_2} \frac{3}{4} \int_{\Sigma_t} H^2. \tag{8.7}$$

Thus, it follows from (8.6) and (8.7) that

$$4\pi(t_2 - t_1) + \int_{\Sigma_{t_1}} H|\nabla u| - \int_{\Sigma_{t_2}} H|\nabla u|$$
$$\geq 12\pi \int_{t_1}^{t_2} \left[1 - \mathfrak{mc}_\Sigma^{-1}(1-t)\right]^2.$$

Letting $t_2 \to 1$, by Lemma 2.2, we obtain

$$4\pi(1 - t_1) + \int_{\Sigma_{t_1}} H|\nabla u|$$
$$\geq 12\pi \int_{t_1}^{1} \left[1 - \mathfrak{mc}_\Sigma^{-1}(1-t)\right]^2$$
$$= 12\pi(1-t_1) - 12\pi \mathfrak{mc}_\Sigma^{-1}(1-t_1)^2 + 4\pi(\mathfrak{mc}_\Sigma^{-1})^2(1-t_1)^3,$$

or, equivalently

$$12\pi \mathfrak{mc}_\Sigma^{-1}(1-t_1) - 4\pi(\mathfrak{mc}_\Sigma^{-1})^2(1-t_1)^2$$
$$\geq 8\pi - \frac{1}{1-t_1}\int_{\Sigma_{t_1}} H|\nabla u|. \tag{8.8}$$

In particular, at $t_1 = 0$, we have

$$12\pi \mathfrak{mc}_\Sigma^{-1} - 4\pi(\mathfrak{mc}_\Sigma^{-1})^2 \geq 8\pi - \int_\Sigma H|\nabla u|,$$

which proves (8.2).

If equality in (8.2) holds, then equality in (8.8) holds with $t_1 = 0$. This necessarily implies equality in (8.6) holds for a.e. $t \in [0, 1]$. As a result, at $t = 0$,

$$\frac{1}{16\pi}\int_\Sigma H^2 = \left[1 - \mathfrak{mc}_\Sigma^{-1}\right]^2.$$

Since $1 - \mathfrak{mc}_\Sigma^{-1} \geq 0$, we conclude by Theorem 7.4 that (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative mean curvature.

Suppose $(M, g) = \left(\mathbb{R}^3 \setminus \{|x| < r\}, (1 + \frac{\mathfrak{m}}{2|x|})^4 g_E\right)$ with mean-convex boundary $\{|x| = r\}$, then (8.5)–(8.8) all become equality. Hence, equality in (8.2) holds. This completes the proof. □

Remark 8.1 We compare (8.2) and (3.15). If (M, g) has $\mathfrak{m} = 0$, (8.2) is as the same as (3.15), both of which reduce to $\int_\Sigma H|\nabla u| \geq 8\pi$. For (M, g) with $\mathfrak{m} \neq 0$, (8.2) improves (3.15) by unveiling the quadratic term $4\pi(\mathfrak{m}\mathfrak{c}_\Sigma^{-1})^2$.

Remark 8.2 Condition (8.1) is a global condition on the triple (M, g, Σ). It has a feature of being inheritable to other surfaces enclosing Σ. More precisely,

$$\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1 \implies \mathfrak{m}\mathfrak{c}_S^{-1} \leq 1$$

for any other surfaces S in M enclosing Σ. This follows from the fact $\mathfrak{c}_\Sigma \leq \mathfrak{c}_S$, which is a consequence of the variational characterization of the surface capacity.

Remark 8.3 Theorem 8.1 shows a necessary condition of $\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1$ is $\int_\Sigma H|\nabla u| \geq 0$. Therefore, by Remark 8.2,

$$\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1 \implies \int_S H|\nabla u_S| \geq 0,$$

for any surfaces S enclosing Σ. Here u_S denotes the harmonic function in the exterior of S, with $u_S = 0$ at S and $u_S \to 1$ at ∞.

Manifolds (M, g) satisfying $\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1$ include regions, in a spatial Schwarzschild manifold with positive mass, which are the exterior to a surface enclosing the horizon. That is, if

$$(M_\mathfrak{m}, g_\mathfrak{m}) = \left(\mathbb{R}^3 \setminus \left\{|x| < \frac{1}{2}\mathfrak{m}\right\}, \left(1 + \frac{\mathfrak{m}}{2|x|}\right)^4 g_E\right)$$

with $\mathfrak{m} > 0$ and if $\Sigma \subset M_\mathfrak{m}$ is a closed surface bounding some region D with the horizon $\Sigma_h = \{|x| = \frac{1}{2}\mathfrak{m}\}$, then $(M_\mathfrak{m} \setminus D, g_\mathfrak{m})$ satisfies

$$\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1.$$

This is because of $c_{\Sigma_h} = \mathfrak{m}$ on $(M_\mathfrak{m}, g_\mathfrak{m})$ and $\mathfrak{c}_{\Sigma_h} \leq \mathfrak{c}_\Sigma$.

To put the next corollary of Theorem 8.1 in context, we mention a few additional facts on $(M_\mathfrak{m}, g_\mathfrak{m})$. Let $\Sigma_r = \{|x| = r\} \subset M_\mathfrak{m}$. The mean curvature H_r of Σ_r equals

$$H_r = k^{-2}(2k^{-1} - 1)2r^{-1},$$

where $k \in (1, 2]$ is the constant determined by $\mathfrak{m} = 2r(k - 1)$. The product $\mathfrak{m}H_r$ satisfies

$$\mathfrak{m}H_r = 2k^{-1}(2k^{-1} - 1)2(1 - k^{-1}).$$

The capacity $\mathfrak{c}_r$ of Σ_r is given by

$$\mathfrak{c}_r = r + \frac{\mathfrak{m}}{2}, \quad \text{and hence } \mathfrak{m}\mathfrak{c}_r^{-1} = 2(1 - k^{-1}).$$

As a result, $\mathfrak{m}H_r$ and $\mathfrak{m}\mathfrak{c}_r^{-1}$ are related by

$$\mathfrak{m}H_r = (2 - \mathfrak{m}\mathfrak{c}_r^{-1})(1 - \mathfrak{m}\mathfrak{c}_r^{-1})\mathfrak{m}\mathfrak{c}_r^{-1}. \tag{8.9}$$

As a function of r, calculation shows

$$\max_{\frac{1}{2}\mathfrak{m} \le r < \infty} \mathfrak{m}H_r = \frac{2}{3\sqrt{3}}, \tag{8.10}$$

where this maximum is achieved uniquely at

$$r_p = \left(1 + \frac{\sqrt{3}}{2}\right)\mathfrak{m}, \quad \text{satisfying} \quad \left(1 + \frac{\mathfrak{m}}{2r_p}\right)^2 r_p = 3\mathfrak{m}. \tag{8.11}$$

The sphere $\{|x| = r_p\}$ is often known as the photon sphere in $(M_\mathfrak{m}, g_\mathfrak{m})$. The mass-to-capacity ratio at Σ_{r_p} is given by

$$\mathfrak{m}\mathfrak{c}_{r_p}^{-1} = 1 - \frac{1}{\sqrt{3}}. \tag{8.12}$$

The following corollary gives a partial classification or comparison result for manifolds with $\mathfrak{m}\mathfrak{c}_\Sigma \le 1$, depending on the maximum of $\mathfrak{m}H$ at the boundary.

Corollary 8.1 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ, with the mass-to-capacity ratio satisfying*

$$0 < \mathfrak{m}\mathfrak{c}_\Sigma^{-1} \le 1.$$

Suppose Σ is connected, $H_2(M, \Sigma) = 0$, and g has nonnegative scalar curvature. Then

(i) *either Σ has vanishing mean curvature, in which case (M, g) must be isometric to a spatial Schwarzschild manifold outside the horizon;*
(ii) *or $H_{\max} = \max_\Sigma H > 0$ and one of the following holds:*

 (a) $\mathfrak{m}H_{\max} < \frac{2}{3\sqrt{3}}$ *and*

$$\mathfrak{c}_\Sigma \le \mathfrak{c}_{r_1} \quad or \quad \mathfrak{c}_\Sigma \ge \mathfrak{c}_{r_2}.$$

Here $\mathfrak{c}_{r_i}$ is the capacity of the sphere $\Sigma_{r_i} = \{|x| = r_i\}$, $i = 1, 2$, in the spatial Schwarzschild manifold

$$(M_\mathfrak{m}, g_\mathfrak{m}) = \left(\mathbb{R}^3 \setminus \left\{|x| < \frac{1}{2}\mathfrak{m}\right\}, \left(1 + \frac{\mathfrak{m}}{2|x|}\right)^4 g_E\right)$$

which has the same mass as (M, g)*, and the constants* r_1, r_2 *are chosen so that*

$$H_{r_1} = H_{r_2} = H_{\max} \quad \text{and} \quad \frac{1}{2}\mathfrak{m} < r_1 < \left(1 + \frac{1}{2}\sqrt{3}\right)\mathfrak{m} < r_2, \tag{8.13}$$

where H_{r_i} *is the mean curvature of* Σ_{r_i} *in* $(M_{\mathfrak{m}}, g_{\mathfrak{m}})$*. Moreover,* $\mathfrak{c}_\Sigma = \mathfrak{c}_{r_i}$ *for an* r_i *if and only if* (M, g) *is isometric to* $(M_{\mathfrak{m}}, g_{\mathfrak{m}})$ *outside* Σ_{r_i}*;*

(b) $\mathfrak{m}H_{\max} \geq \frac{2}{3\sqrt{3}}$ *and equality holds if and only if* (M, g) *is isometric to the spatial Schwarzschild manifold* $(M_{\mathfrak{m}}, g_{\mathfrak{m}})$ *outside the photon sphere* $\left\{|x| = (1 + \frac{1}{2}\sqrt{3})\mathfrak{m}\right\}$.

Proof Let $q = \mathfrak{m}\mathfrak{c}_\Sigma^{-1} \in (0, 1]$. By Theorem 8.1,

$$\frac{1}{4\pi}\int_\Sigma H|\nabla u| \geq (2-q)(1-q). \tag{8.14}$$

In particular, $\int_\Sigma H|\nabla u| \geq 0$. As $|\nabla u| > 0$ along Σ, we have $H_{\max} \geq 0$, and $H_{\max} = 0$ if and only if $H = 0$.

If $H = 0$, Theorem 7.4 shows $q \geq 1$. Hence, $q = 1$, and therefore $q = 1 - \left(\frac{1}{16\pi}\int_\Sigma H^2\right)^{\frac{1}{2}}$. By the equality case of Theorem 7.4, (M, g) is isometric to a spatial Schwarzschild manifold outside the horizon.

In what follows, we suppose $H_{\max} > 0$. Since $\int_\Sigma |\nabla u| = 4\pi\mathfrak{c}_\Sigma$, (8.14) implies

$$H_{\max}\,\mathfrak{c}_\Sigma \geq (2-q)(1-q). \tag{8.15}$$

As $\mathfrak{m} > 0$, this gives

$$\mathfrak{m}\,H_{\max} \geq (2-q)(1-q)\,q. \tag{8.16}$$

As a result, either

$$\mathfrak{m}\,H_{\max} \geq \frac{2}{3\sqrt{3}} = \max_{x\in[0,1]}(2-x)(1-x)x, \tag{8.17}$$

or

$$0 < \mathfrak{m}\,H_{\max} < \frac{2}{3\sqrt{3}}. \tag{8.18}$$

If (8.17) holds with equality, then

$$H_{\max}\mathfrak{c}_\Sigma = \frac{1}{4\pi}\int_\Sigma H|\nabla u| = (2-q)(1-q),$$

with $q = 1 - \frac{1}{\sqrt{3}}$. By Theorem 8.1 and the fact (8.10)–(8.12), (M, g) is isometric to a spatial Schwarzschild manifold with the photon sphere boundary.

Next, we suppose (8.18) holds. Let r_i, $i = 1, 2$, be the constants given in (8.13). It follows from (8.9) and (8.16) that

$$\begin{aligned} &(2 - \mathfrak{m}\mathfrak{c}_{r_i}^{-1})(1 - \mathfrak{m}\mathfrak{c}_{r_i}^{-1})\mathfrak{m}\mathfrak{c}_{r_i}^{-1} \\ &= \mathfrak{m}H_{r_i} = \mathfrak{m}H_{\max} \\ &\geq (2 - q)(1 - q)q. \end{aligned}$$

Analyzing the function $f(x) = (2-x)(1-x)x$ and using the assumption $0 < q \leq 1$, we conclude

$$q \leq \mathfrak{m}\mathfrak{c}_{r_2}^{-1} \quad \text{or} \quad q \geq \mathfrak{m}\mathfrak{c}_{r_1}^{-1},$$

or equivalently

$$\mathfrak{c}_{r_2} \leq \mathfrak{c}_\Sigma \quad \text{or} \quad \mathfrak{c}_{r_1} \geq \mathfrak{c}_\Sigma . \tag{8.19}$$

If $\mathfrak{c}_\Sigma = \mathfrak{c}_{r_i}$ for an r_i, then

$$H_{\max}\mathfrak{c}_\Sigma = \frac{1}{4\pi}\int_\Sigma H|\nabla u| = (2 - q)\,(1 - q)\,,$$

with $q = \mathfrak{m}\mathfrak{c}_{r_i}^{-1}$. By Theorem 8.1, (M, g) is isometric to a spatial Schwarzschild manifold with boundary $\{|x| = r_i\}$. This completes the proof. □

Remark 8.4 Corollary 8.1 can be applied to manifolds (M, g) with CMC boundary, i.e., Σ has constant mean curvature. In this case, it might be interesting to understand the supremum of $\mathfrak{m}H$ over such manifolds.

Remark 8.5 Corollary 8.1 (i) shows $(M_\mathfrak{m}, g_\mathfrak{m})$, Schwarzschild manifolds outside the horizon, are the unique manifolds with the given topological and curvature assumptions, satisfying $\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1$ and the boundary being minimal.

On the other hand, we note $(M_\mathfrak{m}, g_\mathfrak{m})$ is not characterized by the condition $\mathfrak{m}\mathfrak{c}_\Sigma^{-1} = 1$. To get other examples with $\mathfrak{m}\mathfrak{c}_\Sigma^{-1} = 1$, we can first return to the proof of Theorem 8.1. Suppose we start with an arbitrary (M, g) with boundary Σ satisfying $\mathfrak{c}_\Sigma < \mathfrak{m}$. Then, by (8.4), we may consider the value t_0 determined by

$$\mathfrak{m} = \frac{\mathfrak{c}_\Sigma}{1 - t_0}. \tag{8.20}$$

If t_0 is a regular value of u, then the exterior region of Σ_{t_0} in (M, g) satisfies $\mathfrak{m}\mathfrak{c}_{\Sigma_{t_0}}^{-1} = 1$. (Otherwise, one can still consider regular values t greater than and close to t_0. For these t, the exterior region of Σ_t satisfies $\mathfrak{m}\mathfrak{c}_{\Sigma_t}^{-1} < 1$ and $\mathfrak{m}\mathfrak{c}_{\Sigma_t}^{-1}$ can be made arbitrarily close to 1.) To get a precise example, we may consider a rotationally symmetric, asymptotically flat metric g on $\mathbb{R}^3$ so that g has nonnegative scalar curvature, positive mass, and $(\mathbb{R}^3, g)$ has no closed minimal surfaces; see [20, Section 2.4] for this class of metrics. By (4.7), there is a small, rotationally symmetric sphere Σ_r in $(\mathbb{R}^3, g)$ with

$\mathfrak{c}_\Sigma < \mathfrak{m}$. Let u be the capacitary function on the exterior of Σ_r, then u is rotationally symmetric, and the level set $\{u = t\}$ is a rotationally symmetric sphere for each $t \in (0, 1)$. The region exterior to $\{u = t_0\}$ then satisfies $\mathfrak{m}\mathfrak{c}_{\Sigma_{t_0}}^{-1} = 1$; moreover, it is not $(M_\mathfrak{m}, g_\mathfrak{m})$ as $\{u = t_0\}$ is not minimal.

Next, we mention some other implications of (8.1) which are corollaries of Theorems 7.2 and 7.3.

Corollary 8.2 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ, satisfying*

$$\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1.$$

Suppose Σ is connected, $H_2(M, \Sigma) = 0$, and g has nonnegative scalar curvature. Then

(i) $\mathfrak{m} \leq \frac{r_\Sigma}{2}\left[1 + \left(\frac{1}{16\pi}\int_\Sigma H^2\right)^{\frac{1}{2}}\right]$, *where r_Σ is the area–radius of Σ; and*
(ii) $\frac{1}{\pi}\int_\Sigma |\nabla u|^2 \geq 1$, *where u is the harmonic function on (M, g) with $u = 0$ at Σ and $u \to 1$ near ∞.*

Moreover, equality holds in either inequality if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside the horizon.

Proof Inequalities in (i), (ii) follow from (7.25) in Theorem 7.2, (7.26) in Theorem 7.3, respectively. The rigidity part follows from the rigidity conclusion in Theorem 7.2 or Theorem 7.3, together with the extra information $\mathfrak{m} = \mathfrak{c}_\Sigma$. □

Remark 8.6 Heuristically, (ii) of Corollary 8.2 suggests the condition

$$\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1$$

may rule out manifolds having long cylindrical neighborhoods shielding the boundary. The following is a simple example. Suppose Σ is a sphere or a torus and γ is a metric of nonnegative Gauss curvature on Σ. Given a constant $L > 0$, consider the product manifold

$$(P, g_P) = (\Sigma \times [0, L], \gamma + \mathrm{d}t^2).$$

If (M, g) contains a neighborhood U of ∂M so that (U, g) is isometric to (P, g_P) with $\partial M = \Sigma \times \{0\}$, then one can consider the harmonic function v on (P, g_P) with $v = 0$ on $\Sigma \times \{0\}$ and $v = 1$ on $\Sigma \times \{L\}$. By the maximum principle, $|\nabla u| \leq |\nabla v|$. Hence

$$4r_\Sigma^2 L^{-2} = \frac{1}{\pi}\int_\Sigma |\nabla v|^2 \geq \frac{1}{\pi}\int_\Sigma |\nabla u|^2, \tag{8.21}$$

where $4\pi r_\Sigma^2$ is the area of (Σ, γ). Therefore, if (M, g) satisfies condition (8.1), then (ii) of Corollary 8.2 shows $L \leq 2r_\Sigma$.

One can always construct manifolds satisfying (8.1) by cutting off a compact set in a given asymptotically flat (M, g). For instance, if a triple (M, g, Σ) has $\mathfrak{m} > \mathfrak{c}_\Sigma$, then, by (8.4), the exterior of Σ_t in (M, g) satisfies $\mathfrak{m} \leq \mathfrak{c}_{\Sigma_t}$ for any regular $t \geq 1 - \mathfrak{c}_\Sigma \mathfrak{m}^{-1}$.

The complement of a finite domain enclosing the Schwarzschild horizon in $(M_\mathfrak{m}, g_\mathfrak{m})$ is an example of a static extension in the context of Bartnik's quasi-local mass [5]. Here an asymptotically flat (M, g) is called static (see [12] for instance) if there is a nontrivial function N on (M, g), referred as a static potential, such that

$$\begin{cases} N \,\mathrm{Ric} \ = \nabla^2 N, \\ \Delta N = 0, \end{cases} \tag{8.22}$$

where Ric denotes the Ricci curvature of g. These spaces necessarily have zero scalar curvature.

The next proposition, among other things, shows an asymptotically flat manifold with boundary, admitting a positive static potential, satisfies (8.1).

Proposition 8.1 *Let (M, g) be a complete, orientable, asymptotically flat 3-manifold with boundary Σ. Suppose there is a static potential N that is positive in the interior of (M, g). Then*

$$\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1.$$

If Σ is connected and $H_2(M, \Sigma) = 0$, then

(i) $\mathfrak{m} \leq \frac{r_\Sigma}{2}\left[1 + \left(\frac{1}{16\pi}\int_\Sigma H^2\right)^{\frac{1}{2}}\right]$, *where r_Σ is the area-radius of Σ; and*
(ii) *any closed, regular level set $S_t = N^{-1}(t)$ is connected and enclosing Σ; if N is normalized so that $N \to 1$ at ∞, then, along S_t,*

$$N^2 \leq \frac{1}{16\pi}\int_{S_t} H^2,$$
$$1 - N^2 \leq \left(\frac{1}{\pi}\int_{S_t} |\nabla N|^2\right)^{\frac{1}{2}} \leq (1 - N)\left[1 + \left(\frac{1}{16\pi}\int_{S_t} H^2\right)^{\frac{1}{2}}\right],$$

and

$$N(1 - N)(1 + N) \leq \frac{1}{4\pi}\int_{S_t} H|\nabla N|.$$

Moreover, equality holds in any of these inequalities if and only if the exterior of S_t in (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative constant mean curvature.

Proof The static system (8.22) and the assumption $N > 0$ near ∞ imply that, upon multiplying N by a constant,

$$N = 1 - \frac{\mathfrak{m}}{|x|} + o(|x|^{-1}),$$

where $\mathfrak{m}$ is the mass of (M, g) (see [9, 24] for instance). If in addition $N \geq 0$ at Σ, then $\mathfrak{m} \leq \mathfrak{c}_\Sigma$ by the maximum principle.

The mass estimate in (i) follows from (i) of Corollary 8.2.

Suppose S_t is a closed, regular level set of N. By the topological assumption on M and the fact N is harmonic, S_t only has one connected component and it encloses Σ. Let E_t denote the exterior of S_t in M. The inequalities in (ii), with the rigidity conclusions, follow from applying Theorem 7.4, Corollary 7.1, Theorem 7.3, and Theorem 8.1, respectively, to

$$u^{(t)} = \frac{N - t}{1 - t}$$

on (E_t, g) and using the fact $\mathfrak{c}_{S_t} = \dfrac{\mathfrak{m}}{1 - t}$. □

In the context of the Bartnik mass [5], asymptotically flat extensions are often assumed to have no closed minimal surface enclosing the boundary to prevent the infimum of the mass over all extensions from being trivially zero. We note here, if an asymptotically flat 3-manifold (M, g) with boundary Σ satisfies the mass-to-capacity relation $\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1$, then necessarily there are no closed minimal surfaces enclosing Σ. This is because, if such a minimal surface S exists, then $\mathfrak{m} \geq \mathfrak{c}_S$ by the result of Bray [6]. On the other hand, $\mathfrak{c}_S > \mathfrak{c}_\Sigma$. Hence, $\mathfrak{m} > \mathfrak{c}_\Sigma$, violating (8.1).

Considering this and Proposition 8.1, we think manifolds satisfying condition (8.1) are worthy of further study. The mass-to-capacity ratio condition $\mathfrak{m}\mathfrak{c}_\Sigma^{-1} \leq 1$ may serve as an alternative to the no-minimal-surface or outer-minimizing conditions in the formulation of the Bartnik mass.

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Declarations

Conflict of Interest The author declares no conflict of interest.

Appendix A: Regularization and Integration

In this appendix, we give the regularization arguments that can be used to verify the monotonicity of $\Psi(t)$, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ in Sect. 3.

Lemma A.1 *Let u be a harmonic function on a compact Riemannian manifold (Ω, g) with boundary $\partial\Omega$. Suppose $\max_\Omega u < 1$. Then*

$$\int_{\partial\Omega} \frac{|\nabla u|}{1 - u} \frac{\partial u}{\partial \zeta} = \int_\Omega \frac{|\nabla u|^3}{(1 - u)^2} + \int_{\{\nabla u \neq 0\} \subset \Omega} \frac{\nabla^2 u(\nabla u, \nabla u)}{(1 - u)|\nabla u|} \tag{A.1}$$

and

$$\int_{\partial\Omega} \frac{|\nabla u|}{(1-u)^3} \frac{\partial u}{\partial \zeta} = \int_\Omega \frac{3|\nabla u|^3}{(1-u)^4} + \int_{\{\nabla u \neq 0\} \subset \Omega} \frac{\nabla^2 u(\nabla u, \nabla u)}{(1-u)^3 |\nabla u|}. \tag{A.2}$$

Here ζ *denotes the unit normal to* $\partial\Omega$ *pointing out of* Ω.

Proof Given any constant $\epsilon > 0$, one has

$$\operatorname{div}\left(\frac{\sqrt{|\nabla u|^2+\epsilon}}{1-u}\nabla u\right) = \frac{\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^2}|\nabla u|^2 + \frac{1}{1-u}\frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2+\epsilon}}.$$

Therefore,

$$\int_{\partial\Omega} \frac{\sqrt{|\nabla u|^2+\epsilon}}{1-u}\frac{\partial u}{\partial \zeta} = \int_\Omega \frac{\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^2}|\nabla u|^2 + \int_\Omega \frac{1}{1-u}\frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2+\epsilon}}. \tag{A.3}$$

For the third term in (A.3), one notes

$$\begin{aligned}\int_\Omega \frac{1}{1-u}\frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2+\epsilon}} &= \int_{\{\nabla u \neq 0\}\subset\Omega} \frac{1}{1-u}\frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2+\epsilon}} \\ &= \int_{\{\nabla u \neq 0\}\subset\Omega} \frac{1}{1-u}\,\nabla^2 u\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)\frac{|\nabla u|^2}{\sqrt{|\nabla u|^2+\epsilon}}.\end{aligned}$$

Thus, taking $\epsilon \to 0$ in (A.3) proves (A.1).

Similarly, to show (A.2), one has

$$\operatorname{div}\left[\frac{\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^3}\nabla u\right] = \frac{3\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^4}|\nabla u|^2 + \frac{1}{(1-u)^3}\frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2+\epsilon}}.$$

Consequently,

$$\int_{\partial\Omega} \frac{\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^3}\frac{\partial u}{\partial \zeta} = \int_\Omega \frac{3\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^4}|\nabla u|^2 + \int_\Omega \frac{1}{(1-u)^3}\frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2+\epsilon}}. \tag{A.4}$$

The third term above satisfies

$$\begin{aligned}&\int_\Omega \frac{1}{(1-u)^3}\frac{\nabla^2 u(\nabla u, \nabla u)}{\sqrt{|\nabla u|^2+\epsilon}} \\ &\quad= \int_{\{\nabla u \neq 0\}\subset\Omega} \frac{1}{(1-u)^3}\,\nabla^2 u\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)\frac{|\nabla u|^2}{\sqrt{|\nabla u|^2+\epsilon}}.\end{aligned}$$

Thus, (A.2) follows by taking $\epsilon \to 0$ in (A.4). □

Lemma A.2 *Let u be a harmonic function on a compact, orientable, Riemannian 3-manifold (Ω, g) with boundary $\partial\Omega$. Suppose $\max_\Omega u < 1$ and u equals a constant on each connected component of $\partial\Omega$. Then*

$$\int_{\partial\Omega} \frac{H|\nabla u|}{(1-u)^2} \leq \int_{t_1}^{t_2} \frac{1}{(1-t)^2} \left\{ \int_{\Sigma_t} \left[\frac{2H|\nabla u|}{1-t} - \frac{1}{2} \left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + R \right) \right] + 2\pi\chi(\Sigma_t) \right\}. \tag{A.5}$$

Here the mean curvature H of $\partial\Omega$ is taken with respect to the unit normal ζ pointing out of Ω, the mean curvature H of a regular level set Σ_t is taken with respect to $|\nabla u|^{-1}\nabla u$, $\chi(\Sigma_t)$ is the Euler characteristic of Σ_t, $t_1 = \min_\Omega u$, and $t_2 = \max_\Omega u$.

Proof For any constant $\epsilon > 0$, one has

$$\operatorname{div}\left[\frac{\nabla\sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} \right] = \frac{\Delta\sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} + \frac{2\,\nabla^2 u(\nabla u, \nabla u)}{(1-u)^3\sqrt{|\nabla u|^2 + \epsilon}}.$$

Therefore,

$$\int_{\partial\Omega} \frac{\partial_\zeta\sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} = \int_\Omega \frac{\Delta\sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} + \int_\Omega \frac{2\,\nabla^2 u(\nabla u, \nabla u)}{(1-u)^3\sqrt{|\nabla u|^2 + \epsilon}}. \tag{A.6}$$

As u is constant on each connected component of $\partial\Omega$, direct calculations gives

$$\partial_\zeta\sqrt{|\nabla u|^2 + \epsilon} = -\frac{|\nabla u|^2}{\sqrt{|\nabla u|^2 + \epsilon}} H.$$

(See Lemma 2.1 in [16] for instance.) Thus,

$$\lim_{\epsilon\to 0} \int_{\partial\Omega} \frac{\partial_\zeta\sqrt{|\nabla u|^2 + \epsilon}}{(1-u)^2} = -\int_{\partial\Omega} \frac{H|\nabla u|}{(1-u)^2}. \tag{A.7}$$

As in the proof of the previous lemma, taking $\epsilon \to 0$ in the third term in (A.6) gives

$$\begin{aligned} \lim_{\epsilon\to 0} \int_\Omega \frac{2\,\nabla^2 u(\nabla u, \nabla u)}{(1-u)^3\sqrt{|\nabla u|^2 + \epsilon}} &= \int_{\{\nabla u \neq 0\}\subset\Omega} \frac{2\,\nabla^2 u(\nabla u, \nabla u)}{(1-u)^3|\nabla u|} \\ &= -\int_{t_1}^{t_2} \frac{2}{(1-t)^3} \int_{\Sigma_t} H|\nabla u|, \end{aligned} \tag{A.8}$$

where the second equation follows from the coarea formula and (3.5).

To deal with the second term in (A.6), we follow an argument of Stern [30]. Let $\mathcal{C}$ denote the set of critical values of u in $[t_1, t_2]$. Let W denote an open set of $[t_1, t_2]$ such that W contains $\mathcal{C}$. Let D be the complement of W in $[t_1, t_2]$.

On $u^{-1}(D)$, $\frac{\Delta\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^2|\nabla u|}$ is integrable. By coarea formula,

$$\int_{u^{-1}(D)} \frac{\Delta\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^2} = \int_D \int_{\Sigma_t} \frac{\Delta\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^2|\nabla u|}.$$

Along Σ_t which a regular level set of u, by equation (14) in [30],

$$\Delta\sqrt{|\nabla u|^2+\epsilon} \geq \frac{1}{2\sqrt{|\nabla u|^2+\epsilon}}\left[|\nabla^2 u|^2 + (R - 2K_{\Sigma_t})|\nabla u|^2\right], \tag{A.9}$$

where K_{Σ_t} is the Gauss curvature of Σ_t. Thus,

$$\int_{u^{-1}(D)} \frac{\Delta\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^2} \geq \int_D \int_{\Sigma_t} \frac{1}{(1-u)^2|\nabla u|} \frac{\left[|\nabla^2 u|^2 + (R - 2K_{\Sigma_t})|\nabla u|^2\right]}{2\sqrt{|\nabla u|^2+\epsilon}}. \tag{A.10}$$

With W fixed, letting $\epsilon \to 0$ in (A.10) gives

$$\begin{aligned} \liminf_{\epsilon\to 0} \int_{u^{-1}(D)} \frac{\Delta\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^2} &\geq \int_D \int_{\Sigma_t} \frac{\left[|\nabla^2 u|^2 + (R - 2K_{\Sigma_t})|\nabla u|^2\right]}{(1-u)^2\, 2|\nabla u|^2} \\ &= \int_D \frac{1}{(1-t)^2}\left[\int_{\Sigma_t} \frac{1}{2}\left(|\nabla u|^{-2}|\nabla^2 u|^2 + R\right)\right. \\ &\quad \left. -2\pi\chi(\Sigma_t)\right], \end{aligned} \tag{A.11}$$

where one also used the Gauss–Bonnet theorem.

To estimate the integral on $u^{-1}(W)$, one notes

$$\begin{aligned} \Delta\sqrt{|\nabla u|^2+\epsilon} &= \frac{1}{\sqrt{|\nabla u|^2+\epsilon}}\left[|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) - \frac{1}{4(|\nabla u|^2+\epsilon)}\left|\nabla|\nabla u|^2\right|^2\right] \\ &\geq \frac{1}{\sqrt{|\nabla u|^2+\epsilon}}\mathrm{Ric}(\nabla u, \nabla u). \end{aligned}$$

This implies

$$\begin{aligned} \int_{u^{-1}(W)} \frac{\Delta\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^2} &\geq -\max_{\Omega}|\mathrm{Ric}| \int_{u^{-1}(W)} \frac{|\nabla u|}{(1-u)^2} \\ &= -\max_{\Omega}|\mathrm{Ric}| \int_W \int_{\Sigma_t} \frac{1}{(1-u)^2}. \end{aligned} \tag{A.12}$$

It follows from (A.11) and (A.12) that

$$\lim_{\epsilon\to 0}\int_\Omega \frac{\Delta\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^2} \geq \int_D \frac{1}{(1-t)^2}\left[\int_{\Sigma_t}\frac{1}{2}\left(|\nabla u|^{-2}|\nabla^2 u|^2+R\right)-2\pi\chi(\Sigma_t)\right] - \max_\Omega |\mathrm{Ric}| \int_W \int_{\Sigma_t} \frac{1}{(1-u)^2}.$$

As $\int_\Omega \frac{|\nabla u|}{(1-u)^2} < \infty$, by choosing the measure of W to be arbitrarily small, one has

$$\lim_{\epsilon\to 0}\int_\Omega \frac{\Delta\sqrt{|\nabla u|^2+\epsilon}}{(1-u)^2} \geq \int_{t_1}^{t_2} \frac{1}{(1-t)^2}\left[\int_{\Sigma_t}\frac{1}{2}\left(|\nabla u|^{-2}|\nabla^2 u|^2+R\right)-2\pi\chi(\Sigma_t)\right]. \tag{A.13}$$

The lemma now follows from (A.6), (A.7), (A.8) and (A.13). □

Remark A.1 (A.5) may be viewed as a weighted version of the identity (3.4).

Proposition A.1 *Let (Ω, g) be a connected, compact, orientable, Riemannian 3-manifold with boundary $\partial\Omega$. Suppose $\partial\Omega$ is the disjoint union of two nonempty pieces S_1 and S_2. Let u be a harmonic function on (Ω, g) such that $u = c_i$ on S_i, $i = 1, 2$, where c_1, c_2 are constants with $c_1 < c_2 < 1$. For regular values t, let*

$$A(t) = 8\pi - \frac{1}{(1-t)}\int_{\Sigma_t} H|\nabla u|, \quad \mathcal{A}(t) = \frac{A(t)}{1-t},$$
$$B(t) = 4\pi - \frac{1}{(1-t)^2}\int_{\Sigma_t} |\nabla u|^2, \quad \mathcal{B}(t) = \frac{B(t)}{1-t}.$$

Then, for any two regular values $t_1 < t_2$,

$$\mathcal{B}(t_2) - \mathcal{B}(t_1) = \int_{t_1}^{t_2} \frac{1}{(1-t)^2}\left[3B(t) - A(t)\right], \tag{A.14}$$

and

$$\mathcal{A}(t_2) - \mathcal{A}(t_1) \geq \int_{t_1}^{t_2} \frac{1}{(1-t)^2}\left[3B(t) - A(t) + 2\pi\left(2-\chi(\Sigma_t)\right) + \psi(t)\right], \tag{A.15}$$

where

$$\psi(t) = \int_{\Sigma_t}\left[\frac{3}{4}\left(H - \frac{2|\nabla u|}{1-u}\right)^2 + |\nabla u|^{-2}\left|\nabla_{\Sigma_t}|\nabla u|\right|^2 + \frac{1}{2}|\mathring{\mathrm{II}}|^2 + \frac{1}{2}R\right].$$

As a result, if

- *(M, g) is a complete, orientable, asymptotically flat 3-manifold with connected boundary Σ and $H_2(M, \Sigma) = 0$;*
- *u is the harmonic function on (M, g) with $u = 0$ at Σ and $u \to 1$ at ∞; and*
- *g has nonnegative scalar curvature,*

then Σ_t is connected, $3B(t) - A(t) \geq 0$ by (3.3), and consequently,

$$\mathcal{B}(t_2) - \mathcal{B}(t_1) \geq 0 \quad \textit{and} \quad \mathcal{A}(t_2) - \mathcal{A}(t_1) \geq 0.$$

Proof Applying (A.2) in Lemma A.1 to $\Omega_{[t_1,t_2]} = \{x \mid t_1 \leq u(x) \leq t_2\}$, one has

$$\begin{aligned}\int_{\Sigma_{t_2}} \frac{|\nabla u|^2}{(1-u)^3} - \int_{\Sigma_{t_1}} \frac{|\nabla u|^2}{(1-u)^3} &= \int_{\Omega_{[t_1,t_2]}} \frac{3|\nabla u|^3}{(1-u)^4} + \int_{\{\nabla u \neq 0\} \subset \Omega_{[t_1,t_2]}} \frac{\nabla^2 u(\nabla u, \nabla u)}{(1-u)^3 |\nabla u|} \\ &= \int_{t_1}^{t_2} \int_{\Sigma_t} \left[\frac{3|\nabla u|^2}{(1-t)^4} - \frac{H|\nabla u|}{(1-t)^3} \right],\end{aligned}$$

This, combined with $\frac{1}{1-t_2} - \frac{1}{1-t_1} = \int_{t_1}^{t_2} \frac{1}{(1-t)^2}$, shows

$$\begin{aligned}\mathcal{B}(t_2) - \mathcal{B}(t_1) &= \int_{t_1}^{t_2} \left[\frac{4\pi}{(1-t)^2} + \int_{\Sigma_t} \frac{H|\nabla u|}{(1-t)^3} - \int_{\Sigma_t} \frac{3|\nabla u|^2}{(1-t)^4} \right] \\ &= \int_{t_1}^{t_2} \frac{1}{(1-t)^2} \left[3B(t) - A(t) \right],\end{aligned} \tag{A.16}$$

which proves (A.14).

Similarly, applying Lemma A.2 to u on $\Omega_{[t_1,t_2]}$ and using (3.6), one has

$$\begin{aligned}&\frac{1}{(1-t_2)^2} \int_{\Sigma_{t_2}} H|\nabla u| - \frac{1}{(1-t_1)^2} \int_{\Sigma_{t_1}} H|\nabla u| \\ &\leq \int_{t_1}^{t_2} \frac{1}{(1-t)^2} \left[2\pi\chi(\Sigma_t) + \int_{\Sigma_t} \frac{2H|\nabla u|}{1-t} \right. \\ &\quad \left. - \int_{\Sigma_t} \frac{3}{4} H^2 - \int_{\Sigma_t} \left(|\nabla u|^{-2} \left| \nabla_{\Sigma_t} |\nabla u| \right|^2 + \frac{1}{2} |\mathring{\mathbb{I}}|^2 + \frac{1}{2} R \right) \right] \\ &= \int_{t_1}^{t_2} \frac{1}{(1-t)^2} \left[2\pi\chi(\Sigma_t) - \psi(t) - \frac{1}{1-t} \int_{\Sigma_t} H|\nabla u| + \frac{3}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right].\end{aligned}$$

Therefore,

$$\mathcal{A}(t_2) - \mathcal{A}(t_1) \geq \int_{t_1}^{t_2} \frac{1}{(1-t)^2} \left[3B(t) - A(t) + 2\pi \left(2 - \chi(\Sigma_t) \right) + \psi(t) \right],$$

which proves (A.15). □

Remark A.2 As a corollary of (A.14) and (A.15),

$$[\mathcal{A}(t_2) - \mathcal{B}(t_2)] - [\mathcal{A}(t_1) - \mathcal{B}(t_1)] \geq \int_{t_1}^{t_2} \frac{1}{(1-t)^2} \big[2\pi \big(2 - \chi(\Sigma_t) \big) + \psi(t) \big]. \tag{A.17}$$

This corresponds to the monotonicity of $F(t) = \mathcal{A}(t) - \mathcal{B}(t)$ in [2].

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