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ORIGINAL ARTICLE

On the Negativity of Ricci Curvatures of Complete Conformal Metrics

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Abstract
A version of the singular Yamabe problem in bounded domains yields complete conformal metrics with negative constant scalar curvatures. In this paper, we study whether these metrics have negative Ricci curvatures. Affirmatively, we prove that these metrics indeed have negative Ricci curvatures in bounded convex domains in the Euclidean space. On the other hand, we provide a general construction of domains in compact manifolds and demonstrate that the negativity of Ricci curvatures does not hold if the boundary is close to certain sets of low dimension. The expansion of the Green's function and the positive mass theorem play essential roles in certain cases.

Keywords Negativity of Ricci curvatures · The singular Yamabe problem · Negative sectional curvatures

Mathematics Subject Classification 53C21

1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension n without boundary, for $n \geq 3$, and Γ be a smooth submanifold in M. For $(M, g) = (S^n, g_{S^n})$, Loewner and Nirenberg [15] proved that there exists a complete conformal metric on $S^n \setminus \Gamma$ with a *negative* constant scalar curvature if and only if $\dim(\Gamma) > (n-2)/2$. Aviles

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and McOwen [4] proved a similar result for the general manifold (M, g). As a consequence, we can take the dimension of the submanifold to be $n-1$ and conclude the following result: In any compact Riemannian manifold with boundary, there exists a complete conformal metric with a negative constant scalar curvature. See [4]. For convenience, we always take the constant scalar curvature to be $-n(n-1)$. In this paper, we will study whether Ricci curvatures of such a metric remain negative.

For the case of positive scalar curvatures, the existence and asymptotic behaviors of solutions have been extensively studied over the years. We shall not discuss this case here, but refer to [5, 12, 19–21, 23].

There are several classical results for metrics with negative Ricci curvatures. Gao and Yau [7] proved that there exists a metric of negative Ricci curvature on every compact 3-dimensional manifold without boundary. Lohkamp [16] generalized this to arbitrary dimensions and proved that any manifold of dimension $n \geq 3$ (compact or not) admits a complete metric of negative Ricci curvature. Restricted to conformal metrics, by solving $\det(\mathrm{Ric}) = \text{constant}$ with a precise boundary asymptotics, Guan [8] and Gursky, Streets and Warren [9] proved that there exists a complete conformal metric with negative Ricci curvature on a compact Riemannian manifold with boundary.

In the unit ball in the Euclidean space, the complete conformal metric with scalar curvature $-n(n-1)$ is exactly the Poincaré metric of the unit ball model of the hyperbolic space and has sectional curvatures -1 and Ricci curvatures $-(n-1)$. In particular, it has negative sectional curvatures and Ricci curvatures. A natural question is whether this remains true for the more general case; namely, whether the complete conformal metric with a negative constant scalar curvature in a compact Riemannian manifold with boundary has negative sectional curvatures or negative Ricci curvatures. We point out that a straightforward calculation based on the polyhomogeneous expansion established in [1] and [18] yields that such a metric has sectional curvatures asymptotically equal to -1 near boundary. Our main concern is whether the negativity of the sectional curvatures or Ricci curvatures near boundary can be carried over to the entire domain.

In view of the Poincaré metric in the unit ball model of the hyperbolic space, it is reasonable to expect that the complete conformal metric with a negative constant scalar curvature should have negative sectional curvatures in a domain close to the unit ball in the Euclidean space. We will confirm this in this paper. In fact, we will prove an affirmative result for convex domains in the Euclidean space.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, for $n \geq 3$, and g_Ω be the complete conformal metric in Ω with the constant scalar curvature $-n(n-1)$. Then, g_Ω has negative sectional curvatures in Ω. Moreover, g_Ω has Ricci curvatures strictly less than $-n/2$ in Ω.*

The convexity assumption of the domain Ω is crucial. It allows us to apply a convexity theorem by Kennington [10] directly to conformal factors. Theorem 1.1 does not hold for general bounded domains in $\mathbb{R}^n$. In fact, in certain bounded star-shaped domains, the conformal metrics may have arbitrarily large positive Ricci curvature components. See Example 5.6. Note that bounded convex domains and bounded star-shaped domains have the same topology.

Closely related to the negativity of the Ricci curvatures is whether there is a constant rank theorem for metrics with negative Ricci curvatures, since it is already known that the Ricci curvatures are negative near boundary. Caffarelli, Guan, and Ma [6] proved a constant rank theorem for the σ_k-curvature equations under certain positivity conditions on curvatures. However, their result is not applicable in our case. Our strategy is to connect directly boundary curvatures of domains with the interior curvature tensors of the complete conformal metrics.

We now turn our attention to bounded smooth domains which are sufficiently "far" from the unit ball. According to Aviles and McOwen [4], to have a complete conformal metric with constant negative scalar curvature in $M \setminus \Gamma$, it is required that $\dim(\Gamma) > (n-2)/2$. Closely related is a result proved by Mazzeo and Pacard [19] that there exist complete conformal metrics in $S^n \setminus \Gamma$ with constant *positive* scalar curvatures if $\dim(\Gamma) \leq (n-2)/2$. In view of these results, we can ask what happens to Ricci curvatures of the complete conformal metrics with scalar curvatures fixed at $-n(n-1)$ in domains $\Omega \subset M$ whose $(n-1)$-dimensional boundary is close to a smooth submanifold Γ of dimension $\leq (n-2)/2$. Do Ricci curvatures have mixed signs as $\partial\Omega$ becomes close to a low dimensional set, say a single point?

In this paper, we will construct domains where complete conformal metrics have large positive Ricci curvatures in domains in compact Riemannian manifolds.

Theorem 1.2 *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary and Γ be a disjoint union of finitely many closed smooth embedded submanifolds in M of varying dimensions, between 0 and $(n-2)/2$. Consider the following cases:*

Case 1. Γ *contains a submanifold of dimension j, with $1 \leq j \leq (n-2)/2$.*
Case 2. *If (M, g) is not conformally equivalent to the standard sphere S^n, Γ consists of finitely many points.*
Case 3. *If (M, g) is conformally equivalent to S^n, Γ consists of at least two but only finitely many points.*

Suppose that Ω_i is a sequence of increasing domains with smooth boundary in M which converges to $M \setminus \Gamma$ and that g_i is the complete conformal metric in Ω_i with the constant scalar curvature $-n(n-1)$. Then, for sufficiently large i, g_i has a positive Ricci curvature component somewhere in Ω_i. Moreover, the maximal Ricci curvature in Ω_i diverges to ∞ as $i \to \infty$.

By the convergence of Ω_i to $M \setminus \Gamma$, we mean $\bigcup_{i=1}^{\infty} \Omega_i = M \setminus \Gamma$ and, for any $\varepsilon > 0$, $\partial\Omega_i$ is in the ε-neighborhood of Γ for all large i. By convention, a zero dimensional submanifold is simply a point.

The difference between Case 2 and Case 3 in Theorem 1.2 lies on the number of isolated points when closed smooth embedded submanifolds of positive dimension are absent from Γ. On manifolds conformally equivalent to the standard sphere, the number of the isolated points has to be at least two; while on manifolds not conformally equivalent to the standard sphere, we can allow one point. Theorem 1.2 does not necessarily hold if Γ consists of one point on manifolds conformally equivalent to the standard sphere. See Remark 4.5. Such a difference demonstrates that the background manifolds also play a decisive role in the issue studied in this paper.

As a consequence of Case 2, with Γ consisting of just one point, we have the following rigidity result.

Theorem 1.3 *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary and x_0 be a point in M. Suppose that there exists a sequence Ω_i of increasing domains with smooth boundary in M which converges to $M \backslash \{x_0\}$, such that the complete conformal metric in Ω_i with the constant scalar curvature $-n(n-1)$ has uniformly bounded Ricci curvatures in Ω_i. Then, M is conformally equivalent to the standard sphere S^n.*

The set Γ in Theorem 1.2 resembles that in [19]. The metric g_i in Ω_i as in Theorem 1.2 is assumed to have a *negative* constant scalar curvature, $-n(n-1)$. As Ω_i becomes close to $M \setminus \Gamma$, Ricci curvatures split in sign. Some components become negatively large, while some others positively large.

The proof of Theorem 1.2 relies on a careful analysis of the Ricci curvatures of the complete conformal metrics near boundary. The polyhomogeneous expansion provides correct values near boundary for applications of the maximum principle. The Yamabe invariant of (M, g) plays a crucial role and determines behaviors of the convergence of the conformal factors. Among the three cases listed in Theorem 1.2, Case 2 is the most difficult to prove, especially when the Yamabe invariant is between zero and that of the standard sphere. When Γ consists of one point x_0, we need expansions of Green's functions. If $n = 3, 4, 5$, or M is conformally flat in a neighborhood of x_0, we need to employ the positive mass theorem. If $n \geq 6$ and M is not conformally flat in a neighborhood of x_0, we need to distinguish the two cases $W(x_0) \neq 0$ and $W(x_0) = 0$. Discussions for the latter case is much more complicated than the former case. The proof here seems to resemble the solution of the Yamabe problem, but with one twist. In solving the Yamabe problem, we can choose a point where the Weyl tensor is not zero in the case that M is not conformally flat. In our case, x_0 is a given point and the Weyl tensor can be zero even if M is not conformally flat in a neighborhood of x_0. Different vanishing orders of W at x_0 requires different methods. In fact, we also need to employ the positive mass theorem if the Weyl tensor vanishes at x_0 up to a sufficiently high order. The positive mass theorem has been known to be true if $3 \leq n \leq 7$, or M is locally conformally flat, or M is spin. See [13, 22, 24] and [26]. These conditions might be technical and could be removed according to the recent papers [17] and [25]. Refer to Remark 4.4 on how the positive mass theorem is used in the proof of Theorem 1.2.

The paper is organized as follows. In Sect. 2, we discuss some preliminary identities. In Sect. 3, we study the Ricci curvatures of complete conformal metrics in bounded convex domains in the Euclidean space and prove Theorem 1.1. In Sect. 4, we study the Ricci curvatures of complete conformal metrics in domains in compact manifolds and prove Theorem 1.2. In Sect. 5, we present several examples in the Euclidean space.

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2 Preliminaries

Let (M, g) be a smooth Riemannian manifold of dimension n, for some $n \geq 3$, either compact without boundary or noncompact and complete. Assume $\Omega \subset M$ is a smooth domain, with an $(n-1)$-dimensional boundary. If (M, g) is noncompact, we assume, in addition, that Ω is bounded. We consider the following problem:

$$\Delta_g u - \frac{n-2}{4(n-1)} S_g u = \frac{1}{4} n(n-2) u^{\frac{n+2}{n-2}} \quad \text{in } \Omega, \tag{2.1}$$

$$u = \infty \quad \text{on } \partial\Omega, \tag{2.2}$$

where S_g is the scalar curvature of M. According to Loewner and Nirenberg [15] for $(M, g) = (S^n, g_{S^n})$ and Aviles and McOwen [4] for the general case, (2.1) and (2.2) admits a unique positive solution. We note that $u^{\frac{4}{n-2}} g$ is the complete metric with a constant scalar curvature $-n(n-1)$ on Ω. Andersson, Chruściel and Friedrich [1] and Mazzeo [18] established the polyhomogeneous expansions for the solutions. For the first several terms, we have

$$u = d^{-\frac{n-2}{2}} \left[1 + \frac{n-2}{4(n-1)} H_{\partial\Omega} d + O(d^2) \right],$$

where d is the distance to $\partial\Omega$ and $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ with respect to the interior unit normal vector of $\partial\Omega$. Set

$$v = u^{-\frac{2}{n-2}}. \tag{2.3}$$

Then,

$$v \Delta_g v + \frac{1}{2(n-1)} S_g v^2 = \frac{n}{2} (|\nabla_g v|^2 - 1) \quad \text{in } \Omega, \tag{2.4}$$

$$v = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

Moreover,

$$v = d - \frac{1}{2(n-1)} H_{\partial\Omega} d^2 + O(d^3). \tag{2.6}$$

This implies

$$|\nabla_g v| = 1 \quad \text{on } \partial\Omega. \tag{2.7}$$

We will use this repeatedly later on.

Consider the conformal metric

$$g_\Omega = u^{\frac{4}{n-2}} g = v^{-2} g. \tag{2.8}$$

For a unit vector X of g, vX is a unit vector of g_Ω. Let R_{ij} be the Ricci components of g in a local frame for the metric g and R^Ω_{ij} be the Ricci components of g_Ω in the corresponding frame for the metric g_Ω. Then,

$$R^\Omega_{kl} = v^2 R_{kl} + (n-2)\left[v v_{,kl} - \frac{1}{2} g_{kl} |\nabla_g v|^2 \right] + g_{kl} \left[v \Delta_g v - \frac{n}{2} |\nabla_g v|^2 \right].$$

By (2.4), we have

$$R^\Omega_{kl} = v^2 R_{kl} + (n-2)\left[v v_{,kl} - \frac{1}{2} g_{kl} |\nabla_g v|^2 \right] - g_{kl} \left[\frac{1}{2(n-1)} v^2 S_g + \frac{n}{2} \right], \tag{2.9}$$

or

$$R^\Omega_{kl} = v^2 R_{kl} - \frac{1}{2(n-1)} v^2 g_{kl} S_g + (n-2) v v_{,kl} - \frac{n-2}{2} g_{kl} |\nabla_g v|^2 - \frac{n}{2} g_{kl}. \tag{2.10}$$

We emphasize that (2.9) and (2.10) play important roles in the rest of the paper. By (2.5) and (2.7), we obtain

$$R^\Omega_{kl} = -(n-1) g_{kl} + O(d).$$

In other words, the Ricci curvatures of conformal metrics g_Ω are asymptotically equal to $-(n-1)$ near boundary. We note that this holds in arbitrary smooth domains.

If $(M, g) = (\mathbb{R}^n, g_E)$, then (2.1) and (2.2) reduce to

$$\Delta u = \frac{1}{4} n(n-2) u^{\frac{n+2}{n-2}} \quad \text{in } \Omega, \tag{2.11}$$

$$u = \infty \quad \text{on } \partial\Omega. \tag{2.12}$$

In this case, the function v given by (2.3) satisfies

$$v \Delta v = \frac{n}{2} (|\nabla v|^2 - 1). \tag{2.13}$$

Let g_Ω be the metric given by (2.8) with $g = g_E$, i.e., $g_\Omega = v^{-2} g_E$. Denote by R^Ω_{ijij} and R^Ω_{ij} the sectional curvatures and Ricci curvatures of g_Ω in the orthonormal coordinates of g_Ω, respectively. Then, for $i \neq j$,

$$R^\Omega_{ijij} = vv_{ii} + vv_{jj} - |\nabla v|^2, \tag{2.14}$$

and, for any i, j,

$$R^\Omega_{ij} = (n-2)vv_{ij} - \left[\frac{n-2}{2}|\nabla v|^2 + \frac{n}{2}\right]\delta_{ij}. \tag{2.15}$$

Hence, for any $i \neq j$,

$$R^\Omega_{ijij} = -1 + O(d),$$

and, for any i, j,

$$R^\Omega_{ij} = -(n-1)\delta_{ij} + O(d).$$

Note

$$v_i = -\frac{2}{n-2} u^{-\frac{2}{n-2}-1} u_i,$$

and

$$v_{ij} = -\frac{2}{n-2} u^{-\frac{2}{n-2}} \left(\frac{u_{ij}}{u} - \frac{n}{n-2}\frac{u_i u_j}{u^2} \right). \tag{2.16}$$

We can also express R^Ω_{ijij} and R^Ω_{ij} in terms of u.

3 Convex Domains in Euclidean Spaces

In this section, we study Ricci curvatures and sectional curvatures of the complete conformal metrics associated with the Loewner–Nirenberg problem in bounded domains in the Euclidean space. We will prove that the complete conformal metrics in bounded convex domains have negative sectional curvatures. The convexity assumption allows us to apply a convexity theorem by Kennington [10] directly to conformal factors.

Proof of Theorem 1.1 Let u be the solution of (2.11) and (2.12) in Ω and v be given by (2.3). Then, $g = v^{-2} g_E$ is the complete conformal metric in Ω with a constant scalar curvature $-n(n-1)$. Denote by R_{ijij} and R_{ij} the sectional curvatures and Ricci curvatures of g in the orthonormal coordinates of g, given by (2.14) and (2.15), respectively. Here, we suppress Ω from the notations g, R_{ij} and R_{ijij}.

By applying the Laplacian operator to (2.13), we get

$$v\Delta(\Delta v) + (2-n)\nabla v \nabla(\Delta v) = n|\nabla^2 v|^2 - (\Delta v)^2 \geq 0.$$

First, we assume that the boundary of Ω is smooth. By (2.6), we have

$$\Delta v = -H_{\partial\Omega} - \frac{1}{n-1}H_{\partial\Omega} + O(d) = -\frac{n}{n-1}H_{\partial\Omega} + O(d).$$

Since Ω is convex, we have $\Delta v \leq 0$ on $\partial\Omega$. By the strong maximum principle, we obtain $\Delta v < 0$ in Ω. Therefore, $|\nabla v| < 1$ in Ω by (2.13). Next, we apply [10, Theorems 3.1 and 3.2] in Ω and conclude that v is concave. In fact, we write (2.13) as

$$\Delta v = \frac{n(|\nabla v|^2 - 1)}{2v}.$$

Then, we can verify directly the hypothesis (i) of [10, Theorem 3.1] by $|\nabla v| < 1$ and the hypothesis (ii) of [10, Theorem 3.2], since $|\nabla v| < 1$, $v = 0$ on $\partial\Omega$, and ∇v is the inner unit normal vector on $\partial\Omega$.

For general bounded convex domains, we can obtain the concavity of v by approximations.

Since $v_{ii} \leq 0$, by (2.14) and (2.15), we get, for any $i \neq j$,

$$R_{ijij} \leq 0,$$

and, for any i,

$$R_{ii} \leq -\frac{n}{2}.$$

By (2.16), we also have, for any i,

$$\frac{u_{ii}}{u} - \frac{n}{n-2}\frac{u_i^2}{u^2} \geq 0. \tag{3.1}$$

Next, we prove that R_{ijij} does not vanish in Ω for any $i \neq j$. If $R_{ijij} = 0$ at some point $x_0 \in \Omega$ for some $i \neq j$, then

$$\left(\frac{u_{ii}}{u} - \frac{n}{n-2}\frac{u_i^2}{u^2}\right)(x_0) = 0,$$

and

$$\nabla u(x_0) = 0.$$

In fact, by (2.14) and $vv_{ii} \leq 0$, if $R_{ijij}(x_0) = 0$, then $\nabla v(x_0) = 0$ and thus $\nabla u(x_0) = 0$. Hence, $u_{ii}(x_0) = 0$. Applying ∂_i twice to the equation (2.11), we get

$$\Delta u_{ii} = \frac{1}{4}n(n+2)u^{\frac{n+2}{n-2}}\frac{u_{ii}}{u} + \frac{n(n+2)}{n-2}u^{\frac{n+2}{n-2}}\frac{u_i^2}{u^2}.$$

Combining with (3.1), we have

$$\Delta u_{ii} - \frac{1}{4}(n+2)(n+4)u^{\frac{4}{n-2}}u_{ii} = (n+2)u^{\frac{n+2}{n-2}}\left(\frac{n}{n-2}\frac{u_i^2}{u^2} - \frac{u_{ii}}{u}\right) \leq 0.$$

By the strong maximum principle, we have $u_{ii} \equiv 0$ in Ω. On the other hand, by $u_i(x_0) = 0$, we get $u_i \equiv 0$ on $\Omega \cap \{x_0 + te_i \mid t \in \mathbb{R}\}$. Therefore, u is constant on $\Omega \cap \{x_0 + te_i \mid t \in \mathbb{R}\}$. This leads to a contradiction. Therefore, we have, for any $i \neq j$,

$$R_{ijij} < 0.$$

Similarly, we have, for any i,

$$R_{ii} < -\frac{n}{2}.$$

This completes the proof. □

We point out that the upper bounds of sectional curvatures and Ricci curvatures in Theorem 1.1 are given by strict inequalities and, in fact, are optimal. To see this, set

$$D = \left\{(x_1, \dots, x_n) \mid -1 < x_n < 1\right\} \subset \mathbb{R}^n.$$

Let g_D be the complete conformal metric with the constant scalar curvature $-n(n-1)$ in D, and R^D_{ijij} and R^D_{ii} be the sectional curvatures and Ricci curvatures of g_D, respectively. Then, $R^D_{ijij}(0) = 0$, for $i \neq j$, $i, j \neq n$, and $R^D_{ii}(0) = -n/2$, for $i \neq n$. Set $\Omega_R = D \cap B_R$. Then, $R^{\Omega_R}_{ijij}(0) \to 0$, for $i \neq j$, $i, j \neq n$, and $R^{\Omega_R}_{ii}(0) \to -n/2$ for $i \neq n$, as $R \to \infty$.

4 Domains in Compact Manifolds

In this section, we discuss domains in compact Riemannian manifolds without boundary. We construct domains with boundary close to certain sets of low dimension such that the complete conformal metrics with a negative constant scalar curvature have positive Ricci components somewhere. Throughout this section, the Yamabe invariant plays a crucial role. It determines convergence behaviors of conformal factors and, as a consequence, the methods to be employed. In certain cases, we need to employ expansions of the Green's function, and also the positive mass theorem.

Suppose (M, g) is a compact Riemannian manifold of dimension $n \geq 3$ without boundary. The Yamabe invariant of M is given by

$$\lambda(M, [g]) = \inf\left\{ \frac{\int_M \left(|\nabla_g \phi|^2 + \frac{n-2}{4(n-1)}S_g\phi^2\right)dV_g}{\left(\int_M \phi^{\frac{2n}{n-2}}dV_g\right)^{\frac{n-2}{n}}} \,\middle|\, \phi \in C^\infty(M), \phi > 0 \right\}.$$

The conformal Laplacian of (M, g) is given by

$$L_g = -\Delta_g + \frac{n-2}{4(n-1)} S_g.$$

For any function ψ in M, we have

$$L_g(u\psi) = u^{\frac{n+2}{n-2}} L_{u^{\frac{4}{n-2}} g}(\psi).$$

We first prove a convergence result which plays an important role in this section. According to signs of Yamabe invariants, conformal factors exhibit different convergence behaviors. We note that the maximum principle is applicable to the operator L_g if $S_g \geq 0$.

Lemma 4.1 *Suppose (M, g) is a compact Riemannian manifold of dimension $n \geq 3$ without boundary, with a constant scalar curvature S_g, and Γ is a closed smooth submanifold of dimension d in M, $0 \leq d \leq \frac{n-2}{2}$. Suppose Ω_i is a sequence of increasing domains with smooth boundary in M which converges to $M \setminus \Gamma$. Let u_i be the solution of (2.1) and (2.2) in Ω_i. Then, for any positive integer m, if $S_g \geq 0$,*

$$u_i \to 0 \quad \text{in } C^m_{\text{loc}}(M \setminus \Gamma) \text{ as } i \to \infty, \tag{4.1}$$

and, if $S_g < 0$,

$$u_i \to \left(\frac{-S_g}{n(n-1)} \right)^{\frac{n-2}{4}} \quad \text{in } C^m_{\text{loc}}(M \setminus \Gamma) \text{ as } i \to \infty. \tag{4.2}$$

Proof We first consider the case $S_g \geq 0$. By the maximum principle, we have $u_i \geq u_{i+1}$ in Ω_i. It is straightforward to verify, for any m,

$$u_i \to u \quad \text{in } C^m_{\text{loc}}(M \setminus \Gamma) \text{ as } i \to \infty,$$

where u is a nonnegative solution of (2.1) in $M \setminus \Gamma$. By the second part of [4, p. 398], u is bounded. Let $\rho(x)$ be a positive smooth function in $M \setminus \Gamma$ which equals to $\text{dist}(x, \Gamma)$ in a neighborhood of Γ in M. Then,

$$\rho(x)\Delta_g \rho(x) \to n - d - 1 \quad \text{as } x \to \Gamma.$$

Take $\epsilon_0 > 0$ sufficiently small. Then,

$$\Delta_g \rho^{-\frac{n-2}{2}+\epsilon_0} = \left(-\frac{n-2}{2} + \epsilon_0 \right) \rho^{-\frac{n+2}{2}+\epsilon_0} \left(\rho \Delta_g \rho - \left(\frac{n}{2} - \epsilon_0 \right) |\nabla_g \rho|^2 \right).$$

Since $d \leq \frac{n-2}{2}$, we have $\Delta \rho^{-\frac{n-2}{2}+\epsilon_0} < 0$ near Γ. For any $\epsilon > 0$, we can find $\delta < \epsilon$ sufficiently small such that

$$\Delta_g(\delta\rho^{-\frac{n-2}{2}+\epsilon_0} + \epsilon) - S_g(\delta\rho^{-\frac{n-2}{2}+\epsilon_0} + \epsilon) \leq \frac{n-2}{4}(\delta\rho^{-\frac{n-2}{2}+\epsilon_0} + \epsilon)^{\frac{n+2}{n-2}} \quad \text{in } M \setminus \Gamma.$$

By the maximum principle, we have

$$u \leq \delta \rho^{-\frac{n-2}{2}+\epsilon_0} + \epsilon \quad \text{in } M \setminus \Gamma.$$

This implies $u \equiv 0$. In conclusion, we obtain (4.1).

We now consider the case $S_g < 0$. We first prove $\Delta_g u_i \geq 0$ in Ω_i, or equivalently

$$u_i \geq \left(\frac{-S_g}{n(n-1)} \right)^{\frac{n-2}{4}} \quad \text{in } \Omega_i. \tag{4.3}$$

If (4.3) is violated somewhere, then u_i must assume its minimum at some point x_0 in the set

$$\left\{ x \in \Omega_i \,:\, \frac{1}{4} n(n-2) u_i^{\frac{n+2}{n-2}} + \frac{n-2}{4(n-1)} S_g u_i < 0 \right\}.$$

On the other hand, we have $\Delta_g u_i(x_0) \geq 0$, which leads to a contradiction. By taking a difference, we have

$$\Delta_g (u_{i+1} - u_i) = c_i (u_{i+1} - u_i) \quad \text{in } \Omega_i,$$

where c_i is a nonnegative function in Ω_i by (4.3). The maximum principle implies $u_{i+1} \leq u_i$ in Ω_i. Then, for any m,

$$u_i \to u \quad \text{in } C^m_{\text{loc}}(M \setminus \Gamma) \text{ as } i \to \infty,$$

where u is a solution of (2.1) in $M \setminus \Gamma$. By (4.3), we have

$$u \geq \left(\frac{-S_g}{n(n-1)} \right)^{\frac{n-2}{4}} \quad \text{in } M \setminus \Gamma. \tag{4.4}$$

For $\epsilon > 0$ sufficiently small, let u_i^ϵ be the solution of

$$\Delta_g u_i^\epsilon = \epsilon \frac{n(n-2)}{4} (u_i^\epsilon)^{\frac{n+2}{n-2}} \quad \text{in } \Omega_i, \tag{4.5}$$

$$u_i^\epsilon = \infty \quad \text{on } \partial \Omega_i. \tag{4.6}$$

The existence of u_i^ϵ can be obtained by the standard method. More specifically, for each integer j, we solve

$$\Delta_g u_i^{\epsilon,j} = \epsilon \frac{n(n-2)}{4} (u_i^{\epsilon,j})^{\frac{n+2}{n-2}} \quad \text{in } \Omega_i, \tag{4.7}$$

$$u_i^{\epsilon,j} = j \quad \text{on } \partial \Omega_i. \tag{4.8}$$

By the maximum principle, we have $u_i^{\epsilon,j} \leq u_i^{\epsilon,k}$ if $j \leq k$. For any $x_0 \in \Omega_i$, choose normal coordinates near x_0. Then, it is easy to check that

$$u_{r,x_0}(x) = \left(\frac{2r}{r^2 - |x|^2}\right)^{\frac{n-2}{2}}$$

is a supersolution of (4.5) when r is sufficiently small, depending on x_0. Hence, by the maximum principle, we have for each point x, $u_i^{\epsilon,j}(x) \leq C(x)$, independent of j. Therefore, by standard estimates, $u_i^{\epsilon,j}$ converges to some u_i^ϵ in $C^m_{\mathrm{loc}}(\Omega_i)$ as $j \to \infty$ for any m, and $u_i^\epsilon \in C^\infty(\Omega_i)$ is a solution of (4.5)–(4.6).

By the same method as in the proof of the case $S_g \geq 0$, we obtain, for any m,

$$u_i^\epsilon \to 0 \quad \text{in } C^m_{\mathrm{loc}}(M \setminus \Gamma) \text{ as } i \to \infty.$$

Next, we can verify

$$\begin{aligned}\Delta_g\left[u_i^\epsilon + \left(\frac{-S_g}{(1-\epsilon)n(n-1)}\right)^{\frac{n-2}{4}}\right] \leq \frac{n-2}{4(n-1)}S_g\left[u_i^\epsilon + \left(\frac{-S_g}{(1-\epsilon)n(n-1)}\right)^{\frac{n-2}{4}}\right] \\ + \frac{n(n-2)}{4}\left[u_i^\epsilon + \left(\frac{-S_g}{(1-\epsilon)n(n-1)}\right)^{\frac{n-2}{4}}\right]^{\frac{n+2}{n-2}}.\end{aligned}$$

To prove this, we simply split the last term according to $1 = \epsilon + (1-\epsilon)$. Then,

$$\begin{aligned}&\frac{n-2}{4(n-1)}S_g\left[u_i^\epsilon + \left(\frac{-S_g}{(1-\epsilon)n(n-1)}\right)^{\frac{n-2}{4}}\right] \\ &\quad + \frac{n(n-2)}{4}\left[u_i^\epsilon + \left(\frac{-S_g}{(1-\epsilon)n(n-1)}\right)^{\frac{n-2}{4}}\right]^{\frac{n+2}{n-2}} - \epsilon\frac{n(n-2)}{4}(u_i^\epsilon)^{\frac{n+2}{n-2}} \\ &\geq \frac{n(n-2)}{4}\left[u_i^\epsilon + \left(\frac{-S_g}{(1-\epsilon)n(n-1)}\right)^{\frac{n-2}{4}}\right] \\ &\quad \cdot \left\{(1-\epsilon)\left[u_i^\epsilon + \left(\frac{-S_g}{(1-\epsilon)n(n-1)}\right)^{\frac{n-2}{4}}\right]^{\frac{4}{n-2}} + \frac{1}{n(n-1)}S_g\right\},\end{aligned}$$

which is nonnegative. By the maximum principle, we have

$$u_i \leq u_i^\epsilon + \left(\frac{-S_g}{(1-\epsilon)n(n-1)}\right)^{\frac{n-2}{4}} \quad \text{in } \Omega_i,$$

where we can verify the boundary condition by the polyhomogeneous expansions of u_i and u_i^ϵ. Therefore, we have

$$u \leq \left(\frac{-S_g}{(1-\epsilon)n(n-1)}\right)^{\frac{n-2}{4}} \quad \text{in } M \setminus \Gamma. \tag{4.9}$$

This holds for any $\epsilon \in (0,1)$. Combining (4.4) and (4.9), we obtain

$$u = \left(\frac{-S_g}{n(n-1)} \right)^{\frac{n-2}{4}}.$$

In conclusion, we obtain (4.2). □

A similar result holds if the scalar curvature has a fixed sign, not necessarily constant.

Now, we study the case that the boundary is close to a closed smooth submanifold of low dimension. The result below holds for all compact manifolds without boundary, but different signs of the Yamabe invariants require different methods, mostly due to the different convergence behaviors as in Lemma 4.1.

Theorem 4.2 *Suppose (M, g) is a compact Riemannian manifold of dimension $n \geq 3$ without boundary and Γ is a closed smooth submanifold of dimension d in M, $1 \leq d \leq \frac{n-2}{2}$. Suppose Ω_i is a sequence of increasing domains with smooth boundary in M which converges to $M \setminus \Gamma$ and g_i is the complete conformal metric in Ω_i with the scalar curvature $-n(n-1)$. Then, for sufficiently large i, g_i has a positive Ricci curvature component somewhere in Ω_i. Moreover, the maximal Ricci curvature of g_i in Ω_i diverges to ∞ as $i \to \infty$.*

Proof Let u_i be the solution of (2.1) and (2.2) in Ω_i and set $v_i = u_i^{-\frac{2}{n-2}}$. Then,

$$g_i = u_i^{\frac{4}{n-2}} g = v_i^{-2} g.$$

By the solution of the Yamabe problem, we can assume the scalar curvature S_g of M is the constant $\lambda(M, [g])$. Since M is compact, we can take $\Lambda > 0$ such that

$$|R_{ij}| \leq \Lambda g_{ij}.$$

We now discuss two cases according to the sign of S_g.

Case 1. We first consider the case $S_g \geq 0$. By Lemma 4.1, for any m,

$$u_i \to 0 \quad \text{in } C^m_{\text{loc}}(M \setminus \Gamma) \text{ as } i \to \infty,$$

and hence

$$v_i \text{ diverges to } \infty \text{ locally uniformly in } M \setminus \Gamma \text{ as } i \to \infty.$$

We now consider two subcases.

Case 1.1. Γ is not totally geodesic. For any $\epsilon > 0$, there exist two points $p, q \in \Gamma$, such that the length of the shortest geodesic σ_{pq} connecting p and q is less than ϵ and $\sigma_{pq} \cap \Gamma = \{p, q\}$. When ϵ is sufficiently small, we can assume q is located in a small neighborhood of p covered by normal coordinates. Without loss of generality, we assume $p = 0$ and $q = Le_n$.

For i large, set $p_i = \widehat{t}_i e_n$ and $q_i = \widetilde{t}_i e_n$, where

$$\widehat{t}_i = \min\{t' \in [0, L/2] \mid te_n \in \Omega_i, \text{ for any } t \in (t', L/2]\},$$
$$\widetilde{t}_i = \max\{t' \in [L/2, L] \mid te_n \in \Omega_i, \text{ for any } t \in [L/2, t')\}.$$

Then, $p_i, q_i \in \partial\Omega_i$. By the convergence of Ω_i to $M \setminus \Gamma$, we have

$$p_i \to p, \quad q_i \to q.$$

By the polyhomogeneous expansions of v_i, we have

$$|\partial_n v_i(p_i)| \le C_i \quad \text{and} \quad |\partial_n v_i(q_i)| \le C_i,$$

where C_i is some positive constant which converges to 1 as $i \to \infty$ and $\epsilon \to 0$.

Since $v_i(Le_n/2) \to \infty$ as $i \to \infty$, for i large, we can take $t_i \in (\widehat{t}_i, \widetilde{t}_i)$ such that, for any $t \in (\widehat{t}_i, \widetilde{t}_i)$,

$$\partial_n v_i(te_n) \le \partial_n v_i(t_i e_n).$$

Then,

$$\partial_n v_i(t_i e_n) > \frac{v_i(\frac{L}{2}e_n) - 0}{\frac{L}{2}} \ge \frac{2}{L} v_i\left(\frac{L}{2}e_n\right),$$

and

$$\partial_{nn} v_i(t_i e_n) = 0.$$

We also have

$$|v_i(t_i e_n)| \le \frac{L}{2} \partial_n v_i(t_i e_n). \tag{4.10}$$

Denote by R^i_{nn} the Ricci curvature of g_i acting on the unit vector $v_i \frac{\partial}{\partial x^n}$ with respect to the metric g_i. By (2.10), we have, at $t_i e_n$,

$$R^i_{nn} \le v_i^2 |R_{nn}| - \frac{n-2}{2}(\partial_n v_i)^2 \le \left[\frac{L^2}{4}|R_{nn}| - \frac{n-2}{2}\right](\partial_n v_i)^2 \to -\infty,$$

if L is sufficiently small. Hence, some component of the Ricci curvature of g_i at the point $t_i e_n$ diverges to ∞ as $i \to \infty$.

Case 1.2. Γ is totally geodesic. Fix a point $x_0 \in \Gamma$ and choose normal coordinates near x_0 such that $x_0 = 0$ and Γ near x_0 is given by $x_i = 0$, $i = 1, \dots, n-d$. Consider the curve σ given by

$$\sigma(t) = (\sqrt{R^2 - t^2} - \sqrt{R^2 - \epsilon^2}, 0, \dots, 0, t) \quad \text{for } t \in [-\epsilon, \epsilon],$$

where R is some sufficiently large constant and ϵ is some sufficiently small constant such that $\sigma \cap \Gamma = \{\sigma(-\epsilon), \sigma(\epsilon)\}$.

For i large, set $p_i = \sigma(\widehat{t}_i)$ and $q_i = \sigma(\widetilde{t}_i)$, where

$$\widehat{t}_i = \min\{t' \in [-\epsilon, 0] \mid \sigma(t) \in \Omega_i, \text{ for any } t \in (t', 0]\},$$
$$\widetilde{t}_i = \max\{t' \in [0, \epsilon] \mid \sigma(t) \in \Omega_i, \text{ for any } t \in [0, t')\}.$$

Then, $p_i, q_i \in \partial\Omega_i$ and

$$p_i \to \sigma(-\epsilon), \quad q_i \to \sigma(\epsilon).$$

By the polyhomogeneous expansion of v_i, we have

$$|\partial_n v_i(\sigma(\widehat{t}_i))| \leq C_i \quad \text{and} \quad |\partial_n v_i(\sigma(\widetilde{t}_i))| \leq C,$$

where C is some positive bounded constant independent of i.

Consider the single variable function $(v_i \circ \sigma)(t)$. Since $(v_i \circ \sigma)(0) \to \infty$ as $i \to \infty$, for i large, we can take $t_i \in (\widehat{t}_i, \widetilde{t}_i)$ such that, for any $t \in (\widehat{t}_i, \widetilde{t}_i)$,

$$\partial_t (v_i \circ \sigma)(t) \leq \partial_t (v_i \circ \sigma)(t_i).$$

Then,

$$\partial_t (v_i \circ \sigma)(t_i) > \frac{1}{\epsilon}(v_i \circ \sigma)(0),$$

and

$$\partial_{tt} (v_i \circ \sigma)(t_i) = 0. \tag{4.11}$$

We also have

$$|(v_i \circ \sigma)(t_i)| \leq \epsilon \partial_t (v_i \circ \sigma)(t_i). \tag{4.12}$$

Note that

$$\partial_t (v_i \circ \sigma)(t_i) = (\partial_n v_i)(\sigma(t_i)) - \frac{t_i}{\sqrt{R^2 - t_i^2}} (\partial_1 v_i)(\sigma(t_i)).$$

Set

$$\nu_i = \frac{\partial}{\partial x_n} - \frac{t_i}{\sqrt{R^2 - t_i^2}} \frac{\partial}{\partial x_1}.$$

By (4.11), we have

$$(\partial_{\nu_i \nu_i} v_i)(\sigma(t_i)) = \left(-\frac{1}{\sqrt{R^2 - t_i^2}} + \frac{t_i^2}{(R^2 - t_i^2)^{\frac{3}{2}}} \right) \partial_1 v_i(\sigma(t_i)).$$

Hence, $(\partial_{\nu_i \nu_i} v_i)(\sigma(t_i))$ is sufficiently small compared with $(|\nabla v_i|)(\sigma(t_i))$, for R sufficiently large and ϵ sufficiently small. Write $g_{\nu_i \nu_i} = g(\nu_i, \nu_i)$ and denote by $R^i_{\nu_i \nu_i}$ the Ricci curvature of g_i acting on the unit vector $\frac{\nu_i \nu_i}{\sqrt{g_{\nu_i \nu_i}}}$ with respect to the metric g_i.

Similarly as in Case 1.1, we can verify at the point $\sigma(t_i)$, $R^i_{v_i v_i}$ diverges to $-\infty$ as $i \to \infty$. Hence, some component of the Ricci curvature of g_i at the point $\sigma(t_i)$ diverges to ∞ as $i \to \infty$.

Case 2. We now consider the case $S_g < 0$. By Lemma 4.1, for any m,

$$u_i \to \left(\frac{-S_g}{n(n-1)}\right)^{\frac{n-2}{4}} \quad \text{in } C^m_{\text{loc}}(M \setminus \Gamma) \text{ as } i \to \infty,$$

and hence

$$v_i \to \left(\frac{-S_g}{n(n-1)}\right)^{-\frac{1}{2}} \quad \text{in } C^m_{\text{loc}}(M \setminus \Gamma) \text{ as } i \to \infty. \tag{4.13}$$

Fix a point $x_0 \in \Gamma$ and choose normal coordinates in a small neighborhood of x_0 such that $x_0 = 0$ and x_n-axis is a normal geodesic of Γ near x_0. Take $\epsilon > 0$ sufficiently small. For i large, set $p_i = t_i e_n$, where

$$t_i = \min\{t' \in [0, \epsilon] \mid te_n \in \Omega_i, \text{ for any } t \in (t', \epsilon]\}.$$

Then, $p_i \in \partial\Omega_i$ and

$$p_i \to 0.$$

By the polyhomogeneous expansion of v_i, we have

$$|\partial_n v_i(p_i)| \le C_i,$$

where C_i is some positive constant which converges to 1 as $i \to \infty$. By (4.13),

$$\frac{\partial v_i}{\partial x_n}(\epsilon e_n) \to 0 \quad \text{as } i \to \infty.$$

For i large, we take $\widetilde{t}_i \in (t_i, \epsilon)$ such that, for any $t \in (t_i, \epsilon)$,

$$\partial_n v_i(te_n) \le \partial_n v_i(\widetilde{t}_i e_n).$$

Then,

$$\partial_n v_i(\widetilde{t}_i e_n) > \frac{v_i(\epsilon e_n) - 0}{\epsilon - t_i} > \frac{1}{2}\epsilon^{-1}\left(\frac{-S_g}{n(n-1)}\right)^{-\frac{1}{2}},$$

and

$$\partial_{nn} v_i(\widetilde{t}_i e_n) = 0.$$

Denote by R^i_{nn} the Ricci curvature of g_i acting on the unit vector $v_i \frac{\partial}{\partial x^n}$ with respect to the metric g_i. Similarly, by (2.10) at the point $\widetilde{t}_i e_n$, $R^i_{nn} \le -C\epsilon^{-2}$, for all large i, for some positive constant C independent of i and ϵ. By choosing appropriate ϵ, we have the desired result. □

Next, we discuss the case that the boundary is close to a point x_0. The proof of the next result is rather delicate if the Yamabe invariant is positive, in which case expansions of the Green's function play an essential role. We need to employ the positive mass theorem if the manifold has a dimension 3, 4, or 5, or is locally conformally flat. In the case that $n \geq 6$ and M is not conformally flat in a neighborhood of x_0, we need to analyze Weyl tensors and distinguish two cases $W(x_0) \neq 0$ and $W(x_0) = 0$. The proof for the case $W(x_0) = 0$ is quite delicate. It is worth emphasizing that the Weyl tensor can be zero at x_0 even if M is not conformally flat in a neighborhood of x_0. Different vanishing orders of W at x_0 requires different methods. In fact, we also need to employ the positive mass theorem if the Weyl tensor vanishes at x_0 up to a sufficiently high order.

Theorem 4.3 *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary, with $\lambda(M, [g]) < \lambda(S^n, [g_{S^n}])$, where S^n is the sphere with its standard metric g_{S^n}, and let x_0 be a point in M. Suppose that Ω_i is a sequence of increasing domains with smooth boundary in M which converges to $M \setminus \{x_0\}$ and g_i is the complete conformal metric in Ω_i with the scalar curvature $-n(n-1)$. Then, for i sufficiently large, g_i has a positive Ricci curvature component somewhere in Ω_i. Moreover, the maximal Ricci curvature of g_i in Ω_i diverges to ∞ as $i \to \infty$.*

Proof Let u_i be the solution of (2.1) and (2.2) in Ω_i and set $v_i = u_i^{-\frac{2}{n-2}}$. Then,

$$g_i = u_i^{\frac{4}{n-2}} g = v_i^{-2} g.$$

We consider several cases according to the sign of the Yamabe invariant $\lambda(M, [g])$.

Case 1. We first consider $\lambda(M, [g]) < 0$. We point out that the proof of Case 2 of Theorem 4.2 can be adapted to yield the conclusion.

Case 2. Next, we consider $\lambda(M, [g]) = 0$. As in the proof of Theorem 4.2, we assume the scalar curvature of M is 0 and

$$|R_{ij}| \leq \Lambda g_{ij}.$$

Let δ be some small positive constant such that $\Lambda\delta < 1/10$ and there exist normal coordinates in $B_\delta(x_0)$.

Take a sufficiently small $r > 0$ with $r \leq \delta$. Since $\Omega_i \to M \setminus \{x_0\}$, we have $M \setminus B_r(x_0) \subset\subset \Omega_i$ for i large. For such i, by the Harnack inequality, we have

$$\max u_i \leq C \min u_i \quad \text{in } M \setminus B_r(x_0),$$

where C is some positive constant depending only on n, M and r. Then for i sufficiently large, by (4.1), we have

$$|\nabla_g u_i| \leq C\Big(u_i + u_i^{\frac{n+2}{n-2}}\Big) \leq C u_i \quad \text{in } M \setminus B_r(x_0). \tag{4.14}$$

We denote by m_i the minimum of u_i in Ω_i. With the definition of v_i, we have, for i sufficiently large,

$$|\nabla_g v_i| \le C v_i \le C m_i^{-\frac{2}{n-2}} \quad \text{in } M \setminus B_r(x_0), \tag{4.15}$$

where C is some positive constant depending only on n, M and δ. Set

$$A_i = \{x \in \Omega_i \mid u_i(x) < 2m_i\}.$$

Then, for any fixed $r > 0$ with $r \le \delta$ and any i sufficiently large, we have $A_i \cap B_r(x_0) \ne \emptyset$. Otherwise, by the maximum principle, we have

$$u_i(x) \ge 2m_i - C m_i^{\frac{n+2}{n-2}} \quad \text{in } A_i,$$

where C is some constant depending only on n, M and r. Hence,

$$m_i \ge 2m_i - C m_i^{\frac{n+2}{n-2}}.$$

Note that $m_i \to 0$ as $i \to \infty$, which leads to a contradiction.

By $v_i = 0$ on $\partial\Omega_i$, we have, for any fixed $r > 0$ with $r \le \delta$ and for any i sufficiently large,

$$|\nabla_g v_i| \ge \frac{1}{r}(2m_i)^{-\frac{2}{n-2}} \quad \text{somewhere in } \Omega_i \cap B_r(x_0).$$

Therefore, for i sufficiently large, $|\nabla_g v_i|$ must assume its maximum at $p_i \in \Omega_i \cap B_\delta(x_0)$. Write $\nu_i = \frac{\nabla_g v_i}{|\nabla_g v_i|}$ and denote by $R^i_{\nu_i\nu_i}$ the Ricci curvature of g_i acting on the unit vector $\nu_i\nu_i$ with respect to the metric g_i. Then, we can proceed as in the proof of Theorem 4.2 to verify at the point p_i, $R^i_{\nu_i\nu_i}$ diverges to $-\infty$ as $i \to \infty$. Hence, some component of the Ricci curvature of g_i at the point p_i diverges to ∞ as $i \to \infty$.

Case 3. We now consider the case $\lambda(M, [g]) > 0$. In this case, there exists $G_{x_0} \in C^\infty(M \setminus \{x_0\})$, the Green's function for the conformal Laplacian L_g, such that

$$L_g G_{x_0} = (n-2)\omega_{n-1}\delta_{x_0}, \quad G_{x_0} > 0,$$

where ω_{n-1} is the volume of S^{n-1}. Up to a conformal factor, we can assume (M, g) has conformal normal coordinates near x_0. See [13, p. 69] or [24, Chapter 5]. We can perform a conformal blow up at x_0 to obtain an asymptotic flat and scalar flat manifold using G_{x_0}. Specifically, if we define the metric $\widetilde{g} = G_{x_0}^{\frac{4}{n-2}} g$ on $\widetilde{M} = M \setminus \{x_0\}$, then, $(\widetilde{M}, \widetilde{g})$ is an asymptotically flat and scalar flat manifold, and $\widetilde{g}$ has an asymptotic expansion near infinity. See [13, pp. 64–65] or [24, Chapter 5].

Set $\widetilde{u}_i = u_i/G_{x_0}$. Then, $\widetilde{u}_i$ satisfies

$$\Delta_{\widetilde{g}} \widetilde{u}_i = \frac{1}{4} n(n-2) \widetilde{u}_i^{\frac{n+2}{n-2}} \quad \text{in } \Omega_i, \tag{4.16}$$

$$\widetilde{u}_i = \infty \quad \text{on } \partial\Omega_i, \tag{4.17}$$

and for any m,

$$\widetilde{u}_i \to 0 \quad \text{in } C^m_{\text{loc}}(M \setminus \{x_0\}) \text{ as } i \to \infty. \tag{4.18}$$

Fix a point $p_0 \in M \setminus \{x_0\}$. Then, $p_0 \in \Omega_i$, for i sufficiently large. Set $\widetilde{w}_i = \widetilde{u}_i/\widetilde{u}_i(p_0)$. Then, $\widetilde{w}_i(p_0) = 1$, and $\widetilde{w}_i$ satisfies

$$\Delta_{\widetilde{g}}\widetilde{w}_i = \frac{1}{4}n(n-2)u_i(p_0)^{\frac{4}{n-2}}\widetilde{w}_i^{\frac{n+2}{n-2}} \quad \text{in } \Omega_i, \tag{4.19}$$

$$\widetilde{w}_i = \infty \quad \text{on } \partial\Omega_i. \tag{4.20}$$

By interior estimates, there exists a positive function $\widetilde{w} \in \widetilde{M}$ such that, for any m,

$$\widetilde{w}_i \to \widetilde{w} \quad \text{in } C^m_{\text{loc}}(\widetilde{M}) \text{ as } i \to \infty,$$

and

$$\Delta_{\widetilde{g}}\widetilde{w} = 0 \quad \text{in } \widetilde{M}. \tag{4.21}$$

Hence,

$$L_g(G_{x_0}\widetilde{w}) = 0 \quad \text{in } M \setminus \{x_0\}. \tag{4.22}$$

By the expansion of G_{x_0} near x_0 and [14, Proposition 9.1], we conclude that $\widetilde{w}$ converges to some constant as $x \to x_0$. Therefore, $\widetilde{w} \equiv 1$ in $\widetilde{M}$. Hence, for any m,

$$\frac{u_i}{\widetilde{u}_i(p_0)G_{x_0}} \to 1 \quad \text{in } C^m_{\text{loc}}(M \setminus \{x_0\}) \text{ as } i \to \infty. \tag{4.23}$$

In the following, we always discuss in the conformal normal coordinates near x_0. Set

$$v_i = u_i^{-\frac{2}{n-2}} = (\widetilde{u}_i(p_0))^{-\frac{2}{n-2}}\left(\frac{u_i}{\widetilde{u}_i(p_0)G_{x_0}}\right)^{-\frac{2}{n-2}} G_{x_0}^{-\frac{2}{n-2}}.$$

We will fix a direction appropriately, which we call x_1. Denote by R^i_{11} the Ricci curvature of g_i acting on the unit vector $v_i\frac{\partial}{\partial x_1}$ with respect to the metric g_i. To study R^i_{11} given by (2.9), we need to analyze the expansion of $G_{x_0}^{-\frac{2}{n-2}}$. See [13] or [24] for details.

Now we discuss several cases.

Case 3.1. $n = 3, 4, 5$, or M is conformally flat in a neighborhood of x_0. In this case, we have

$$G_{x_0} = r^{2-n} + A + O(r),$$

where A is a constant. Since $\lambda(M, [g]) < \lambda(S^n, [g_{S^n}])$, we have $A > 0$ when $3 \le n \le 7$, or M is locally conformally flat, or M is spin. We also have $A > 0$ when M is just conformally flat in a neighborhood of x_0 under the assumption that the positive mass theorem holds. Then,

$$G_{x_0}^{-\frac{2}{n-2}} = r^2 - \frac{2}{n-2}Ar^n + O(r^{n+1}),$$
$$\partial_r G_{x_0}^{-\frac{2}{n-2}} = 2r - \frac{2n}{n-2}Ar^{n-1} + O(r^n),$$
$$\partial_{rr} G_{x_0}^{-\frac{2}{n-2}} = 2 - \frac{2n(n-1)}{n-2}Ar^{n-2} + O(r^{n-1}),$$

and hence

$$G_{x_0}^{-\frac{2}{n-2}}\partial_{rr}G_{x_0}^{-\frac{2}{n-2}} - \frac{1}{2}\left(\partial_r G_{x_0}^{-\frac{2}{n-2}}\right)^2 = -2(n-1)Ar^n + O(r^{n+1}).$$

For $n = 3, 4, 5$, by [13, p. 61], $R_{11}(x_0) = 0$, $R_{11,1}(x_0) = 0$ and $R_{11,11}(x_0) \le 0$, we have

$$R_{11} \le C|x_1|^3 \quad \text{on the } x_1\text{-axis near } x_0 = 0.$$

We also have $S_g(x_0) = 0$ and $S_{g,1}(x_0) = 0$, and hence

$$|S_g| \le Cx_1^2 \quad \text{on the } x_1\text{-axis near } x_0 = 0.$$

Take any $x_1 > 0$ small. Then, at the point $x_1 e_1$,

$$v_i^2 R_{11} \le C(\widetilde{u}_i(p_0))^{-\frac{4}{n-2}} x_1^7, \tag{4.24}$$

and

$$v_i^2 |S_g| \le C(\widetilde{u}_i(p_0))^{-\frac{4}{n-2}} x_1^6. \tag{4.25}$$

For i large, by (2.9), we have, at the point $x_1 e_1$,

$$R_{11}^i \le (\widetilde{u}_i(p_0))^{-\frac{4}{n-2}}\left[-2(n-1)(n-2)Ax_1^n + Cx_1^6 + o(1)\right], \tag{4.26}$$

where $o(1)$ denotes terms converging to zero as $i \to \infty$, uniformly for small x_1 away from 0. The dominant term in (4.26) is the x_1^n-term, with a negative coefficient. Hence, the expression inside the bracket in (4.26) is strictly less than 0, for a fixed small $x_1 \ne 0$ and i large. Therefore, at the point $x_1 e_1$, R_{11}^i diverges to $-\infty$ as $i \to \infty$. Hence, some component of the Ricci curvature of g_i at the point $x_1 e_1$ diverges to ∞ as $i \to \infty$.

If M is conformally flat in a neighborhood of x_0, then $R_{11} = 0$ and $S_g = 0$ on the x_1-axis and near $x_0 = 0$. The x_1^6-term in (4.26) is absent. Similarly, at the point $x_1 e_1$ for $x_1 > 0$ sufficiently small, R_{11}^i diverges to $-\infty$ as $i \to \infty$.

If we denote by R_{rr}^i the Ricci curvature of g_i acting on the unit vector $v_i \frac{\partial}{\partial r}$ with respect to the metric g_i, then we conclude similarly that R_{rr}^i at x diverges to $-\infty$ as $i \to \infty$, for some x sufficiently close to x_0.

Case 3.2. $n = 6$ and M is not conformally flat in a neighborhood of x_0. In this case,

$$G_{x_0}(x) = r^{2-n} - \frac{n-2}{1152(n-1)}|W(x_0)|^2 \log r - \frac{1}{96} S_{g,ij}(x_0)\frac{x^i x^j}{r^2} + P(x)\log r + \alpha(x),$$

where W is the Weyl tensor, $P(x)$ is a polynomial with $P(0) = 0$, and α is a $C^{2,\mu}$-function. We note that W_{ijkl} is given by

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{n-2}\left(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}\right) \\ &+ \frac{S_g}{(n-1)(n-2)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right). \end{aligned} \tag{4.27}$$

Case 3.2.1. If $W(x_0) \neq 0$, then,

$$\begin{aligned} G_{x_0}^{-\frac{2}{n-2}} &= r^2 + \frac{1}{2880}|W(x_0)|^2 r^6 \log r + O(r^7 \log r), \\ \partial_r G_{x_0}^{-\frac{2}{n-2}} &= 2r + \frac{1}{480}|W(x_0)|^2 r^5 \log r + O(r^5), \\ \partial_{rr} G_{x_0}^{-\frac{2}{n-2}} &= 2 + \frac{1}{96}|W(x_0)|^2 r^4 \log r + O(r^4), \end{aligned}$$

and hence

$$G_{x_0}^{-\frac{2}{n-2}} \partial_{rr} G_{x_0}^{-\frac{2}{n-2}} - \frac{1}{2}\left(\partial_r G_{x_0}^{-\frac{2}{n-2}}\right)^2 = \frac{1}{144}|W(x_0)|^2 r^6 \log r + O(r^6).$$

Take any $x_1 > 0$ small. Then, at the point $x_1 e_1$, (4.24) and (4.25) still hold. For i large, instead of (4.26), we have, at the point $x_1 e_1$,

$$R^i_{11} \leq (\widetilde{u}_i(p_0))^{-1}\left[\frac{1}{36}|W(x_0)|^2 x_1^6 \log x_1 + C x_1^6 + o(1)\right].$$

Similarly as in Case 3.1, at the point $x_1 e_1$ for $x_1 > 0$ sufficiently small, R^i_{11} diverges to $-\infty$ as $i \to \infty$.

Similarly, R^i_{rr} at x diverges to $-\infty$ as $i \to \infty$, for some x sufficiently close to x_0.

Case 3.2.2. We now consider the case $W(x_0) = 0$. By (4.27), we have $R_{ijkl}(x_0) = 0$. Hence, $(\widetilde{M}, \widetilde{g})$ is asymptotically flat of order 3. Using the spherical coordinates, we set

$$\phi_4(\theta) = \frac{1}{96} S_{g,ij}(x_0)\frac{x^i x^j}{r^2},$$

and denote by $r^2 g_2(\theta)$ the degree two part of the Taylor expansion of S_g at x_0. Since

$$\sum_{i=1}^n S_{g,ii}(x_0) = -\frac{1}{6}|W|^2(x_0) = 0,$$

then,

$$\int_{S^{n-1}} g_2(\theta)d\theta = 0.$$

By the positive mass theorem, see [13, pp. 79–80], we have

$$\int_{S^{n-1}} \big(\phi_4(\theta) + \alpha(0)\big)d\theta > 0.$$

Along a radial geodesic $\{(r,\theta) : 0 \le r \le \delta\}$, for a small constant δ, we have

$$\begin{aligned}
G_{x_0}^{-\frac{2}{n-2}} &= r^2 - \frac{1}{2}\big(\phi_4(\theta) + \alpha(0)\big)r^6 + o(r^6),\\
\partial_r G_{x_0}^{-\frac{2}{n-2}} &= 2r - 3\big(\phi_4(\theta) + \alpha(0)\big)r^5 + o(r^5),\\
\partial_{rr} G_{x_0}^{-\frac{2}{n-2}} &= 2 - 15\big(\phi_4(\theta) + \alpha(0)\big)r^4 + o(r^4),
\end{aligned}$$

and hence

$$G_{x_0}^{-\frac{2}{n-2}}\partial_{rr}G_{x_0}^{-\frac{2}{n-2}} - \frac{1}{2}\Big(\partial_r G_{x_0}^{-\frac{2}{n-2}}\Big)^2 = -10\big(\phi_4(\theta) + \alpha(0)\big)r^n + o(r^n).$$

By [13, p. 61], along the radial geodesic $(\cdot,\theta)$, $\partial_r^3 R_{rr}(x_0) = 0$ and $\partial_r^4 R_{rr}(x_0) \le 0$. Therefore, for i large, by (2.9), we have, along the radial geodesic $(\cdot,\theta)$,

$$R^i_{rr}|_{(r,\theta)} \le (\widetilde{u}_i(p_0))^{-1}\Big[-\frac{1}{10}g_2(\theta)r^6 - 40\big(\phi_4(\theta) + \alpha(0)\big)r^6 + o(r^6) + o(1)\Big],$$

where $o(1)$ denotes terms converging to zero as $i \to \infty$, uniformly for small x away from 0. Hence,

$$\int_{S^{n-1}} R^i_{rr}d\theta \le (\widetilde{u}_i(p_0))^{-1}\Big[-40r^6\int_{S^{n-1}}\big(\phi_4(\theta) + \alpha(0)\big)d\theta + o(r^6) + o(1)\Big].$$

Therefore, we can find $\theta_0 \in S^{n-1}$ that R^i_{rr} at (r,θ_0) diverges to $-\infty$ as $i \to \infty$, for some r sufficiently small.

Case 3.3. $n \ge 7$ and M is not conformally flat in a neighborhood of x_0. In this case,

$$G_{x_0}(x) = r^{2-n}\Big[1 + \sum_{i=4}^{n}\psi_i\Big] + c\log r + P(x)\log r + \alpha(x),$$

where ψ_i is a homogeneous polynomial of degree i, c is a constant, $P(x)$ is a polynomial with $P(0) = 0$, and α is a $C^{2,\mu}$-function. We note that $c = 0$ and $P \equiv 0$ if n is odd. Moreover,

$$\psi_4(x) = \frac{n-2}{48(n-1)(n-4)}\left(\frac{r^4}{12(n-6)}|W(x_0)|^2 - S_{g,ij}(x_0)x^ix^jr^2\right),$$

where W is the Weyl tensor.

Case 3.3.1. First, we consider the case $|W(x_0)| \neq 0$. Note that $S_g(x_0) = 0$, $\nabla_g S_g(x_0) = 0$, and

$$\Delta_g S_g(x_0) = -\frac{1}{6}|W(x_0)|^2.$$

Without loss of generality, we assume $S_{g,11}(x_0) < 0$. Take any $x_1 > 0$ small. Then, at the point $x_1 e_1$, (4.24) still holds. Set

$$A = \frac{n-2}{48(n-1)(n-4)}\left[\frac{1}{12(n-6)}|W(x_0)|^2 - S_{g,11}(x_0)\right].$$

Then, on the positive x_1-axis near $x_0 = 0$, we have

$$\begin{aligned} G_{x_0}^{-\frac{2}{n-2}} &= x_1^2 - \frac{2}{n-2}Ax_1^6 + O(x_1^7),\\ \partial_{x_1} G_{x_0}^{-\frac{2}{n-2}} &= 2x_1 - \frac{12}{n-2}Ax_1^5 + O(x_1^6),\\ \partial_{x_1x_1} G_{x_0}^{-\frac{2}{n-2}} &= 2 - \frac{60}{n-2}Ax_1^4 + O(x_1^5), \end{aligned}$$

and

$$\begin{aligned} &(n-2)\left[G_{x_0}^{-\frac{2}{n-2}}\partial_{x_1x_1}G_{x_0}^{-\frac{2}{n-2}} - \frac{1}{2}\left(\partial_{x_1}G_{x_0}^{-\frac{2}{n-2}}\right)^2\right] - \frac{1}{2(n-1)}S_g G_{x_0}^{-\frac{4}{n-2}}\\ &\quad = -40Ax_1^6 - \frac{1}{4(n-1)}S_{g,11}(x_0)x_1^6 + O(x_1^7). \end{aligned}$$

By the definition of A, we obtain

$$\begin{aligned} &(n-2)\left[G_{x_0}^{-\frac{2}{n-2}}\partial_{x_1x_1}G_{x_0}^{-\frac{2}{n-2}} - \frac{1}{2}\left(\partial_{x_1}G_{x_0}^{-\frac{2}{n-2}}\right)^2\right] - \frac{1}{2(n-1)}S_g G_{x_0}^{-\frac{4}{n-2}}\\ &\quad = -\frac{1}{12(n-1)(n-4)}\left[\frac{5(n-2)}{6(n-6)}|W(x_0)|^2 - (7n-8)S_{g,11}(x_0)\right]x_1^6 + O(x_1^7). \end{aligned}$$

For i large, instead of (4.26), we have, at the point $x_1 e_1$,

$$R_{11}^i \le (\widetilde{u}_i(p_0))^{-\frac{4}{n-2}}\left[-Bx_1^6 + Cx_1^7 + o(1)\right],$$

for some positive constant B. Then, we conclude R_{11}^i at the point $x_1 e_1$ diverges to $-\infty$ as $i \to \infty$, for $x_1 > 0$ sufficiently small.

Case 3.3.2. We now consider the case $W(x_0) = 0$. By (4.27), we have $R_{ijkl}(x_0) = 0$. Using the spherical coordinates, we set

$$\psi_i = r^i\phi_i(\theta),$$

and denote by $r^i g_i(\theta)$ the i-th Taylor expansion of S_g at x_0. Let $r^l g_l(\theta)$ be the first nonzero term in the Taylor expansion of S_g at x_0.

Subcase 3.3.2(a). $2 \leq l \leq n-5$. By [13] or [24], we have $\psi_i = 0$, $i = 4, \ldots, l-1$, and

$$\mathcal{L}\psi_{l+2} = -\frac{n-2}{4(n-1)} r^{l+2} g_l(\theta),$$

where

$$\mathcal{L} = -r^2\Delta + 2(n-2)r\partial_r.$$

Here, Δ is the standard Laplacian on the Euclidean space, i.e.,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}.$$

Then, we have

$$(l+2)(n-4-l)\int_{S^{n-1}} \psi_{l+2} = \int_{S^{n-1}} \mathcal{L}\psi_{l+2} = -\frac{n-2}{4(n-1)}\int_{S^{n-1}} r^{l+2} g_l(\theta).$$

Hence,

$$\int_{S^{n-1}} \phi_{l+2} = -\frac{n-2}{4(n-1)(l+2)(n-4-l)}\int_{S^{n-1}} g_l(\theta). \tag{4.28}$$

We also have

$$\begin{aligned}\int_{S^{n-1}} g_l(\theta) &= \frac{r^{2-n-l}}{l}\int_0^r \int_{S^{n-1}} \Delta\left(s^l g_l(\theta)\right)s^{n-1}drd\theta \\ &= \lim_{r\to 0}\frac{r^{2-n-l}}{l}\int_{B_r(x_0)} \Delta_g S_g dV_g \\ &= \lim_{r\to 0}\frac{-r^{2-n-l}}{6l}\int_{B_r(x_0)} |W|^2 dV_g \leq 0.\end{aligned} \tag{4.29}$$

Note that the sign of $\lim_{r\to 0} r^{2-n-l}\int_{B_r(x_0)} |W|^2 dV_g$ is independent of $g \in [g]$. Hence, if for some i with $2 \leq i \leq n-5$,

$$0 < \lim_{r\to 0} r^{2-n-i}\int_{B_r(x_0)} |W|^2 dV_g < \infty,$$

then the i-th Taylor expansion of S_g at x_0 must not be identical to zero.

Note

$$
\begin{aligned}
G_{x_0}^{-\frac{2}{n-2}} &= r^2 - \frac{2}{n-2} r^{l+4}\phi_{l+2} + o(r^{l+4}),\\
\partial_r G_{x_0}^{-\frac{2}{n-2}} &= 2r - \frac{2}{n-2}(l+4) r^{l+3}\phi_{l+2} + o(r^{l+3}),\\
\partial_{rr} G_{x_0}^{-\frac{2}{n-2}} &= 2 - \frac{2}{n-2}(l+4)(l+3) r^{l+2}\phi_{l+2} + o(r^{l+2}),
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&(n-2)\left[G_{x_0}^{-\frac{2}{n-2}}\partial_{x_1x_1}G_{x_0}^{-\frac{2}{n-2}} - \frac{1}{2}\left(\partial_{x_1}G_{x_0}^{-\frac{2}{n-2}}\right)^2\right] - \frac{1}{2(n-1)}S_g G_{x_0}^{-\frac{4}{n-2}}\\
&= \left[-2(l+2)(l+3)\phi_{l+2}(\theta) - \frac{1}{2(n-1)}g_l(\theta)\right] r^{l+4} + o(r^{l+4}).
\end{aligned}
$$

By [13, p. 61], along a radial geodesic $(\cdot, \theta)$, $\partial_r^i R_{rr}(x_0) = 0$, $i = 1, \dots, l-1$, and $\partial_r^l R_{rr}(x_0) \le 0$. Therefore, for i large, by (2.9), we have, along a radial geodesic $(\cdot, \theta)$,

$$
\begin{aligned}
R^i_{rr}|_{(r,\theta)} \le (\widetilde{u}_i(p_0))^{-\frac{4}{n-2}} \Bigg\{ &\left[-2(l+2)(l+3)\phi_{l+2}(\theta) - \frac{1}{2(n-1)}g_l(\theta)\right] r^{l+4}\\
&+ o(r^{l+4}) + o(1)\Bigg\},
\end{aligned}
\tag{4.30}
$$

where $o(1)$ denotes terms converging to zero as $i \to \infty$, uniformly for small x away from 0. By (4.28) and (4.29), we have

$$
\begin{aligned}
&-2(l+2)(l+3)\int_{S^{n-1}} \phi_{l+2}(\theta)d\theta - \frac{1}{2(n-1)}\int_{S^{n-1}} g_l(\theta)d\theta\\
&= \frac{(n-2)(l+3) - (n-4-l)}{2(n-1)(n-4-l)}\int_{S^{n-1}} g_l(\theta)d\theta \le 0.
\end{aligned}
$$

By [13, Lemma 5.3] or [24, Chapter 5],

$$
2(l+2)(l+3)\phi_{l+2} \ne \frac{1}{2(n-1)}g_l.
$$

Hence, we can find $\theta_0 \in S^{n-1}$ such that

$$
-2(l+2)(l+3)\phi_{l+2}(\theta_0) - \frac{1}{2(n-1)}g_l(\theta_0) \le -\epsilon_0,
$$

for some positive constant ϵ_0. Therefore, along the radial geodesic (r, θ_0),

$$
R^i_{rr}|_{(r,\theta_0)} \le (\widetilde{u}_i(p_0))^{-\frac{4}{n-2}}\left[-\epsilon_0 r^{l+4} + o(r^{l+4}) + o(1)\right].
$$

Then, we conclude R^i_{rr} at the point (r, θ_0) diverges to $-\infty$ as $i \to \infty$, for $r > 0$ sufficiently small.

Subcase 3.3.2(b). $l \ge n-4$. When n is even, we have

$$\mathcal{L}(\psi_{n-2} + cr^{n-2}\log r) = \mathcal{L}\psi_{n-2} - (n-2)cr^{n-2} = -\frac{n-2}{4(n-1)}r^{n-2}g_{n-4}(\theta).$$

Then,

$$\begin{aligned}(n-2)cw_{n-1} &= -r^{2-n}\int_{S^{n-1}} \mathcal{L}(\psi_{n-2} + cr^{n-2}\log r) \\ &= \frac{(n-2)r^{6-2n}}{4(n-1)(n-4)}\int_0^r \int_{S^{n-1}} \Delta\big(s^{n-4}g_l(\theta)\big)s^{n-1}drd\theta \\ &= \lim_{r\to 0}\frac{(n-2)r^{6-2n}}{4(n-1)(n-4)}\int_{B_r(x_0)} \Delta_g S_g dV_g \\ &= \lim_{r\to 0}\frac{-(n-2)r^{6-2n}}{24(n-1)(n-4)}\int_{B_r(x_0)} |W|^2 dV_g \le 0.\end{aligned} \tag{4.31}$$

We note

$$\lim_{r\to 0} r^{6-2n}\int_{B_r(x_0)} |W|^2 dV_g = 0,$$

when n is odd, since $\int_{S^{n-1}} g_{n-2}(\theta)d\theta = 0$ when n is odd.

If $c > 0$, we can proceed as in the proof of Case 3.2.1, $n = 6$ and $|W(x_0)| \neq 0$, and conclude that R^i_{rr} at x diverges to $-\infty$ as $i \to \infty$, for some x sufficiently close to x_0.

In general, we first consider the case that there exist a pair $(i,j) \in \{1,\dots,n\}\times\{1,\dots,n\}$ and a constant $k < [\frac{n-4}{2}]$ such that $R_{ij} \neq 0$ and k is the order of the first nonzero term in the Taylor expansion of R_{ij} at x_0.

Without loss of generality, we assume the order of the first nonzero term in the Taylor expansion of some R_{pq} at x_0 is k, $k < [\frac{n-4}{2}]$, and all other R_{ij} vanish up to order k at x_0. Then, by (4.27), all R_{ijkl} vanish up to order k, and hence, all $g_{ij} - \delta_{ij}$ vanish up to order $k+2$. By a rotation, we can assume

$$\frac{\partial^k}{\partial x_1^k}R_{pq}|_{x_0} \neq 0.$$

By [13, p. 61], $(p,q) \neq (1,1)$. If $p \neq 1$ and $q \neq 1$, by a rotation, we can assume $p = q = 2$. Otherwise, we can assume $(p,q) = (1,2)$.

We consider the case $(p,q) = (2,2)$. By the Gauss Lemma, we have

$$x_j = \sum_{i=1}^n g_{ji}x_i.$$

Then, we have, on the x_1-axis near $x_0 = 0$,

$$1 = x_1\frac{\partial}{\partial x_2}g_{12} + g_{22}, \tag{4.32}$$

$$0 = x_1 \frac{\partial}{\partial x_2} g_{11} + g_{12}, \tag{4.33}$$

and

$$0 = x_1 \frac{\partial^2}{\partial x_2^2} g_{22} + 2 \frac{\partial}{\partial x_2} g_{12}.$$

We also have, at $x_0 = 0$,

$$(k+2) \frac{\partial^{k+2}}{\partial x_1^{k+1} \partial x_2} g_{21} + \frac{\partial^{k+2}}{\partial x_1^{k+2}} g_{22} = 0,$$

and

$$(k+1) \frac{\partial^{k+2}}{\partial x_1^{k} \partial^2 x_2} g_{11} + 2 \frac{\partial^{k+2}}{\partial x_1^{k+1} \partial x_2} g_{12} = 0.$$

Hence, we have, at x_0,

$$\begin{aligned} \frac{\partial^k}{\partial x_1^k} R_{1212} &= \frac{1}{2} \left(2 \frac{\partial^{k+2}}{\partial x_1^{k+1} \partial x_2} g_{21} - \frac{\partial^{k+2}}{\partial x_1^{k+2}} g_{22} - \frac{\partial^{k+2}}{\partial x_1^{k} \partial^2 x_2} g_{11} \right) \\ &= -\frac{k+3}{k+1} \frac{\partial^{k+2}}{\partial x_1^{k+2}} g_{22}(x_0). \end{aligned}$$

Therefore, by (4.27), we have

$$\frac{\partial^k}{\partial x_1^k} |_{x_0} R_{22} = (n-2) \frac{\partial^k}{\partial x_1^k} |_{x_0} R_{1212} = -(n-2) \frac{k+3}{k+1} \frac{\partial^{k+2}}{\partial x_1^{k+2}} g_{22}(x_0) < 0. \tag{4.34}$$

Then, for i large, by (2.9), we have, at the point $x_1 e_1$,

$$\begin{aligned} R^i_{22} = (\widetilde{u}_i(p_0))^{-\frac{4}{n-2}} \Bigg[& \frac{1}{k!} \frac{\partial^k}{\partial x_1^k} |_{x_0} R_{22} x_1^{k+4} + (n-2)\big(-2\Gamma^1_{22} x_1^3 - 2(g_{22} - 1) x_1^2 \big) \\ & + O(x_1^{k+5}) + o(1) \Bigg], \end{aligned}$$

where $o(1)$ denotes terms converging to zero as $i \to \infty$, uniformly for small x_1 away from 0. At the point $x_1 e_1$,

$$\Gamma^1_{22} = \frac{1}{2} \left(2 \frac{\partial}{\partial x_2} g_{12} - \frac{\partial}{\partial x_1} g_{22} \right).$$

Combining with (4.32), we get, at the point $x_1 e_1$,

$$\begin{aligned} R^i_{22} &= (\widetilde{u}_i(p_0))^{-\frac{4}{n-2}} \left[\frac{1}{k!} \frac{\partial^k}{\partial x_1^k}|_{x_0} R_{22} x_1^{k+4} + (n-2)\frac{\partial}{\partial x_1} g_{22} x_1^3 + O(x_1^{k+5}) + o(1) \right] \\ &= (\widetilde{u}_i(p_0))^{-\frac{4}{n-2}} \left[-(n-2)\frac{k+2}{(k+1)!} \frac{\partial^{k+2}}{\partial x_1^{k+2}} g_{22}(x_0) x_1^{k+4} + O(x_1^{k+5}) + o(1) \right]. \end{aligned}$$

Then, we conclude R^i_{22} at the point $x_1 e_1$ diverges to $-\infty$ as $i \to \infty$, for $x_1 > 0$ sufficiently small.

If $(p, q) = (1, 2)$, we can argue similarly to conclude that $|R^i_{12}|$ at the point $x_1 e_1$ diverges to ∞ as $i \to \infty$, for $x_1 > 0$ sufficiently small.

We now consider the case that the order of the first nonzero term in the Taylor expansion of all R_{pq} is greater or equal to $[\frac{n-4}{2}]$ at x_0, and

$$\lim_{r\to 0} r^{6-2n} \int_{B_r(x_0)} |W|^2 dV_g = 0.$$

Then, by (4.27), the order of the first nonzero term in the Taylor expansion of R_{ijkl} at x_0 is greater or equal to $[\frac{n-4}{2}]$, and hence, the order of the first nonzero term in the Taylor expansion of $g_{ij} - \delta_{ij}$ at x_0 is greater or equal to $[\frac{n-4}{2}] + 2$. Hence, $(\widetilde{M}, \widetilde{g})$ is asymptotically flat of order $[\frac{n-4}{2}] + 2$. Thus, the ADM-mass of $(\widetilde{M}, \widetilde{g})$ is well defined. By the positive mass theorem, we have

$$\int_{S^{n-1}} \big(\phi_{n-2}(\theta) + \alpha(0)\big) d\theta > 0.$$

Then, we can proceed as in the proof of Case 2.2, $n = 6$ and $|W(x_0)| = 0$, and find $\theta_0 \in S^{n-1}$ such that R^i_{rr} at (r, θ_0) diverges to $-\infty$ as $i \to \infty$, for some r sufficiently small. □

Remark 4.4 We point out that we used the positive mass theorem in the proof of Theorem 4.3 if the Yamabe invariant of $(M, [g])$ is between zero and that of the standard sphere and one of the following conditions holds: (1) M is locally conformally flat, (2) $3 \le n \le 5$, or (3) for $n \ge 6$, the Weyl tensor W at x_0 satisfies

$$\nabla^i |W|^2(x_0) = 0 \quad \text{for any } i = 0, \dots, n-6.$$

Remark 4.5 The blow-up phenomena in Theorem 4.3 are significantly different from those for the case that the underlying manifold is S^n. For example, take $\Omega = S^n \setminus B_r(e_n)$, where $B_r(e_n)$ is a small ball on S^n centered at the north pole. Then, Ω is close to $S^n \setminus \{e_n\}$ and the complete conformal metric g_Ω in Ω with the constant scalar curvature $-n(n-1)$ has a constant sectional curvature -1! This can be verified by the stereographic projection, as the image of $S^n \setminus B_r(e_n)$ under the stereographic projection from the north pole is a ball in $\mathbb{R}^n$ centered at the origin.

We note that Theorem 1.3 follows easily from Theorem 4.3. Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 Let u_i be the solution of (2.1) and (2.2) in Ω_i. Then, $g_i = u_i^{\frac{4}{n-2}} g$.

The proof of Theorem 4.2 can be adapted to prove Case 1, i.e., Γ contains a submanifold of dimension j, with $1 \le j \le \frac{n-2}{2}$.

Next, we consider Case 2, i.e., (M, g) is not conformally equivalent to the standard sphere S^n and Γ consists of finitely many points. If $\lambda(M, [g]) \le 0$, the proofs of Theorem 4.2 and Theorem 4.3 can be adapted to yield the desired conclusion. Hence, we only need to discuss the case $\lambda(M, [g]) > 0$ and Γ consists of finitely many points $\{p_1, \dots, p_k\}$.

Let $G_{p_j} \in C^\infty(M \setminus \{p_j\})$ be the Green's function for the conformal Laplacian L_g with the pole at p_j, $j = 1, \dots, k$, respectively; namely,

$$L_g G_{p_j} = (n-2)\omega_{n-1}\delta_{p_j}, \quad G_{p_j} > 0,$$

where ω_{n-1} is the volume of S^{n-1}. Up to conformal factors, we assume (M, g) has conformal normal coordinates in small neighborhoods of p_i. Consider the metric

$$\widetilde{g} = \left(G_{p_1} + \cdots + G_{p_k}\right)^{\frac{4}{n-2}} g \quad \text{on } \widetilde{M} = M \setminus \{p_1, \dots, p_k\}.$$

Then, $(\widetilde{M}, \widetilde{g})$ is an asymptotically flat and scalar flat manifold, and $\widetilde{g}$ has an asymptotic expansion near infinity.

Set $u_i = (G_{p_1} + \cdots + G_{p_k})\widetilde{u}_i$. Then, $\widetilde{u}_i$ satisfies

$$\begin{aligned} \Delta_{\widetilde{g}}\widetilde{u}_i &= \frac{1}{4}n(n-2)\widetilde{u}_i^{\frac{n+2}{n-2}} \quad \text{in } \Omega_i, \\ \widetilde{u}_i &= \infty \quad \text{on } \partial\Omega_i, \end{aligned}$$

and, for any m,

$$\widetilde{u}_i \to 0 \quad \text{in } C^m_{\text{loc}}(M \setminus \{p_1, \dots, p_k\}) \text{ as } i \to \infty.$$

Fix a point $p_0 \in M \setminus \{p_1, \dots, p_k\}$. Then, for i sufficiently large, $p_0 \in \Omega_i$. Set $\widetilde{w}_i = \widetilde{u}_i / \widetilde{u}_i(p_0)$. Then, $\widetilde{w}_i(p_0) = 1$ and $\widetilde{w}_i$ satisfies

$$\begin{aligned} \Delta_{\widetilde{g}}\widetilde{w}_i &= \frac{1}{4}n(n-2)u_i(p_0)^{\frac{4}{n-2}}\widetilde{w}_i^{\frac{n+2}{n-2}} \quad \text{in } \Omega_i, \\ \widetilde{w}_i &= \infty \quad \text{on } \partial\Omega_i. \end{aligned}$$

By interior estimates, there exists a positive function $\widetilde{w} \in \widetilde{M}$ such that, for any m,

$$\widetilde{w}_i \to \widetilde{w} \quad \text{in } C^m_{\text{loc}}(\widetilde{M}) \text{ as } i \to \infty,$$

and

$$\Delta_{\widetilde{g}}\widetilde{w} = 0 \quad \text{in } \widetilde{M}.$$

Hence,

$$L_g\big((G_{p_1}+\cdots+G_{p_k})\widetilde{w}\big)=0 \quad \text{in } M\setminus\{p_1,\ldots,p_k\}.$$

By the expansions of G_{p_j} near p_j, $j=1,\ldots,k$, respectively, and [14, Proposition 9.1], we conclude that $\widetilde{w}$ converges to some constant α_j as $x\to p_j$. Without loss of generality, we assume

$$\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_k\ge 0.$$

Then, $\alpha_1\ge 1$. By [14, Proposition 9.1], $(G_{p_1}+\cdots+G_{p_k})\widetilde{w}$ can be extended to a C^2-function in a neighborhood of p_j if $\alpha_j=0$.

If some of $\alpha_1,\cdots,\alpha_k$ is zero, we denote by l the first integer in $\{1,\ldots,k\}$ such that $\alpha_l=0$. Otherwise, we set $l=k+1$. We always have $l\ge 2$.

Consider the metric

$$\widehat{g}=(G_{p_1}+\cdots+G_{p_{l-1}})^{\frac{4}{n-2}}g \quad \text{on } \widehat{M}=M\setminus\{p_1,\ldots,p_{l-1}\}.$$

Set $(G_{p_1}+\cdots+G_{p_k})\widetilde{w}=(G_{p_1}+\cdots+G_{p_{l-1}})\widehat{w}$. Then, $\widehat{w}$ satisfies

$$L_g\big((G_{p_1}+\cdots+G_{p_{l-1}})\widehat{w}\big)=0 \quad \text{in } M\setminus\{p_1,\cdots,p_{l-1}\},$$

and

$$\Delta_{\widehat{g}}\widehat{w}=0 \quad \text{in } \widehat{M}.$$

We also have that $\widehat{w}$ converges to α_j as $x\to p_j$, $j=1,\ldots,l-1$. By [14, Proposition 9.1] and the maximum principle, we have, near the point p_{l-1},

$$\widehat{w}=\alpha_{l-1}+C_{l-1}r^{n-2}+O(r^{n-1}),$$

for some nonnegative constant C_{l-1}. Then, the proof follows similarly as that of Theorem 4.3.

Next, we consider Case 3, i.e., (M,g) is conformally equivalent to the standard sphere S^n and Γ consists of at least two but only finitely many points. We can assume $(M,g)=(S^n,g_{S^n})$. By Lemma 4.1, we have, for any m,

$$u_i\to 0 \quad \text{in } C^m_{\text{loc}}(S^n\setminus\Gamma) \text{ as } i\to\infty.$$

Set $v_i=u_i^{-\frac{2}{n-2}}$. Then,

$$v_i \text{ diverges to } \infty \text{ locally uniformly in } S^n\setminus\Gamma \text{ as } i\to\infty.$$

Take two different points $p,q\in\Gamma$ and let σ_{pq} be the shorter geodesic connecting p and q. Up to a conformal transform if necessary, we assume $|\sigma_{pq}|=2\epsilon$, which is less than $\frac{1}{100n}$, and $\sigma_{pq}\cap\Gamma=\{p,q\}$. We parametrize σ_{pq} by its arc length $t\in[0,2\epsilon]$, with p corresponding to $t=0$ and q to $t=2\epsilon$.

For i large, let p_i and q_i be the points parametrized by $\widehat{t}_i$ and $\widetilde{t}_i$, respectively, where

$$\widehat{t}_i = \min\{t' \in [0, \epsilon] \mid te_n \in \Omega_i, \text{ for any } t \in (t', \epsilon]\},$$
$$\widetilde{t}_i = \max\{t' \in [\epsilon, 2\epsilon] \mid t' \in \Omega_i, \text{ for any } t \in [\epsilon, t')\}.$$

Then, $p_i, q_i \in \partial\Omega_i$ and

$$p_i \to p, \quad q_i \to q.$$

For convenience, we denote by $v_i(t)$ the function v_i restricted to the geodesic σ_{pq}. By the polyhomogenous expansion of v_i, we have $|\partial_t v_i(p_i)| \leq 1$ and $|\partial_t v_i(q_i)| \leq 1$.

Since $v_i(\epsilon) \to \infty$, for i large, we take $t_i \in (\widehat{t}_i, \widetilde{t}_i)$ such that, for any $t \in (\widehat{t}_i, \widetilde{t}_i)$,

$$\partial_t v_i(t) \leq \partial_t v_i(t_i).$$

Then,

$$\partial_t v_i(t_i) > \frac{v_i(\epsilon) - 0}{\epsilon - \widehat{t}_i} \geq \frac{v_i(\epsilon)}{\epsilon},$$

and

$$\partial_{tt} v_i(t_i) = 0.$$

Denote by R^i_{tt} the Ricci curvature of g_i acting on the unit vector $v_i \frac{\partial}{\partial t}$ with respect to the metric g_i. Then, we can verify at the point $t_i e$, R^i_{tt} diverges to $-\infty$ as $i \to \infty$. □

5 General Domains in Euclidean Spaces

In this section, we present several examples of smooth bounded domains Ω in the Euclidean space and examine whether the complete conformal metrics g_Ω associated with the Loewner–Nirenberg problem have negative Ricci curvatures. We demonstrate by these examples the complexity of the issue studied in this paper. Topological conditions are not sufficient to determine whether these complete conformal metrics have negative Ricci curvatures.

There are two classes of examples. First, we construct nonconvex smooth domains in which the complete conformal metrics with a constant scalar curvature still have negative Ricci curvatures. Second, we construct bounded smooth domains in which the complete conformal metrics have positive Ricci components at some points.

By the Cartan–Hadamard Theorem, we know that $\pi_i(\Omega) = 0$ for $i \geq 2$ if g_Ω has negative sectional curvatures in Ω, where $\pi_i(\Omega)$ is the i-th homotopy group of Ω. The following example shows that $\pi_1(\Omega) = 0$ is not necessary for g_Ω to have negative sectional curvatures.

Example 5.1 Set

$$\Omega_r = \left\{ (x_1, \dots, x_n) \,\middle|\, \left(x_1 - \frac{x_1}{\sqrt{x_1^2 + x_n^2}} \right)^2 + \sum_{i=2}^{n-1} x_i^2 + \left(x_n - \frac{x_n}{\sqrt{x_1^2 + x_n^2}} \right)^2 < r^2 \right\} \subset \mathbb{R}^n,$$

where $0 < r < 1/100$. Then, $\pi_1(\Omega_r) = \mathbb{Z}$ and $\pi_i(\Omega) = 0$ for $i \geq 2$. We claim that g_{Ω_r} has negative sectional curvatures in Ω_r, if r is sufficiently small. By symmetry, it suffices to show g_{Ω_r} has negative sectional curvatures in

$$\left\{ (x_1, \dots, x_n) \,|\, (x_1 - 1)^2 + \sum_{i=2}^{n-1} x_i^2 < r^2, x_n = 0 \right\} \subset \mathbb{R}^n.$$

Note that Ω_r is transformed to

$$\widetilde{\Omega}_r = \left\{ y \in \mathbb{R}^n \,\middle|\, \left(y_1 + \frac{1}{r} - \frac{r y_1 + 1}{r\sqrt{(r y_1 + 1)^2 + (r y_n)^2}} \right)^2 + \sum_{i=2}^{n-1} y_i^2 + \left(y_n - \frac{y_n}{\sqrt{(r y_1 + 1)^2 + (r y_n)^2}} \right)^2 < 1 \right\},$$

under the transform

$$y_1 = \frac{x_1 - 1}{r},\ y_2 = \frac{x_2}{r},\ \dots,\ y_n = \frac{x_n}{r}.$$

Then, $\{\widetilde{\Omega}_r\}$ converges in C^k, for any integer $k \geq 1$, to

$$\Omega_0 = \{(x_1, \dots, x_n) \,|\, x_1^2 + \cdots + x_{n-1}^2 < 1\} \subset \mathbb{R}^n$$

in any compact sets in $\mathbb{R}^n$. Let u^0 be the positive solution of (2.11)–(2.12) for $\Omega = \Omega_0$. Then, u^0 has the form

$$u^0(x_1, \dots, x^n) = u^0(x_1, \dots, x_{n-1}, 0).$$

By the same method as in the proof of Theorem 1.1, we can prove $(u^0)^{\frac{4}{n-2}}|dx|^2$ has negative sectional curvatures in Ω_0. Then, the polyhomogeneous expansions for u_r imply that $g_{\widetilde{\Omega}_r}$ has negative sectional curvatures in

$$\{(x_1, \dots, x_n) \,|\, 1 - \delta < x_1^2 + \cdots + x_{n-1}^2 < 1, x_n = 0\},$$

for some small $\delta > 0$, independent of r. It is straightforward to prove that $g_{\widetilde{\Omega}_r}$ has negative sectional curvatures in

$$\{(x_1, \dots, x_n) \,|\, x_1^2 + \cdots + x_{n-1}^2 \leq 1 - \delta, x_n = 0\},$$

since $\{\widetilde{\Omega}_r\}$ converges to Ω_0 in any compact sets in $\mathbb{R}^n$.

In the next example, we construct a bounded smooth domain $\Omega \subset \mathbb{R}^n$ which is diffeomorphic to the unit ball and cannot be conformally transformed to a bounded convex domain such that the complete conformal metric g_Ω with a negative scalar curvature possesses negative sectional curvatures in Ω.

Example 5.2 Let σ be a U-shaped smooth curve in $\mathbb{R}^n$ with two endpoints p and q. Let Ω^r, $0 < r < 1/100$, be a family of tubular domains with smooth boundaries satisfying the following conditions:

(A1) $\Omega^{r_2} \subseteq \Omega^{r_1}$ if $r_2 < r_1$;
(A2) $\bigcap \Omega^r = \sigma$;
(A3) For a fixed point $x_0 \in \sigma \setminus \{p, q\}$, $\{x \in \mathbb{R}^n \mid rx + x_0 \in \Omega^r\}$ converges in C^k, for any integer $k \geq 1$, to a smooth domain which is equal to $\{(x_1, \dots, x_n) \mid x_1^2 + \cdots + x_{n-1}^2 < 1\} \subset \mathbb{R}^n$ up to a Euclidean transformation in any compact set in $\mathbb{R}^n$ as $r \to 0$;
(A4) At the point p or q, $\{x \in \mathbb{R}^n \mid rx + x_0 \in \Omega^r\}$ converges in C^k, for any integer $k \geq 1$, to a smooth domain which is equal to Ω^0 up to a Euclidean transformation in any compact set in $\mathbb{R}^n$ as $r \to 0$, where Ω^0 is a smooth convex domain which coincides $\{(x_1, \dots, x_n) \mid x_1^2 + \cdots + x_{n-1}^2 < 1\}$ when $x_n \geq 0$, and coincides $\{(x_1, \dots, x_n) \mid -\sqrt{x_1^2 + \cdots + x_{n-1}^2} < x_n < 0\}$ when $-1 < x_n < 1/3$.

Then, Ω^r cannot be conformally transformed to bounded convex domains if r is sufficiently small. Otherwise, there would exist an arc σ_r with endpoints p and q such that $\sigma_r \subseteq \Omega_r$. Let w be the positive solution of (2.11)–(2.12) for $\Omega = \Omega^0$. Then by approximation and using the same method as in the proof of Theorem 1.1, we can prove $w^{-\frac{2}{n-2}}$ is concave in Ω^0 and $w^{\frac{4}{n-2}}|dx|^2$ has negative sectional curvatures in Ω^0. Arguing as in Example 5.1, we can show that the sectional curvatures in Ω^0 is close to the sectional curvatures in Ω_0 when x_n is sufficiently large. Hence, the sectional curvatures in Ω^0 are bounded above by a negative constant. Then, arguing as in Example 5.1 again, we obtain that g_{Ω^r} has negative sectional curvatures in Ω^r when r is sufficiently small.

Example 5.3 For $R > r > 0$, consider the annular region $\Omega_{R,r} = B_R \setminus B_r$ in $\mathbb{R}^n$. Let $g_{\Omega_{R,r}}$ be the complete conformal metric with the constant scalar curvature $-n(n-1)$ in $\Omega_{R,r}$. Arguing as in Example 5.1, we can prove that the maximum Ricci curvature component of $g_{\Omega_{R,r}}$ is close to $-n/2$ as R and r tend to 1. Hence, $g_{\Omega_{R,r}}$ has negative Ricci curvatures in $\Omega_{R,r}$ as R is sufficiently close to r.

In the rest of this section, we construct bounded domains in which the complete conformal metrics have positive Ricci components at some points. The most straightforward way to do this is to combine Theorem 1.2 for the case $(M, g) = (S^n, g_{S^n})$ and the stereographic projections.

We identify $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \{0\}$ and write $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then,

$$S^n = \{(x, x_{n+1}) : |x|^2 + x_{n+1}^2 = 1\}.$$

Consider the transform $T : \mathbb{R}^n \to S^n$ given by

$$T(x) = \left(\frac{2x}{1+|x|^2}, \frac{|x|^2-1}{1+|x|^2} \right).$$

Then, T is the inverse transform of the stereographic projection which lifts $\mathbb{R}^n \times \{0\}$ to S^n.

Proposition 5.4 *Let Γ be a set in S^n as in Theorem 1.2, containing the north pole. Suppose $\widetilde{\Omega}_i$ is a sequence of increasing smooth domains in S^n which converges to $S^n \setminus \Gamma$, with $\partial\widetilde{\Omega}_i$ not containing the north pole, and set $\Omega_i = T^{-1}(\widetilde{\Omega}_i)$. Assume g_i is the complete conformal metric in Ω_i with the constant scalar curvature $-n(n-1)$. Then, for sufficiently large i, g_i has a positive Ricci curvature component somewhere in Ω_i. Moreover, the maximal Ricci curvature of g_i in Ω_i diverges to ∞ as $i \to \infty$.*

Proposition 5.4 follows easily from Theorem 1.2 for the case $(M, g) = (S^n, g_{S^n})$.

We point out that notations in Proposition 5.4 is slightly different from those in Theorem 1.2. In Proposition 5.4, $\widetilde{\Omega}_i$ is a domain in S_n and Ω_i is a domain in $\mathbb{R}^n$. We also note that Ω_i is a bounded domain in $\mathbb{R}^n$ if the north pole is not in the closure of $\widetilde{\Omega}_i$.

Example 5.5 Let $\{p_1, \ldots, p_k\}$ be a collection of finitely many points in $\mathbb{R}^n$, with $k \geq 1$, and set $\Omega_{R,r} = B_R(0) \setminus \bigcup_{i=1}^k B_r(p_i)$. Then, for R sufficiently large and r sufficiently small, the complete conformal metric in $\Omega_{R,r}$ with the constant scalar curvature $-n(n-1)$ has a positive Ricci curvature component somewhere. Note that the corresponding Γ in S^n is given by $\Gamma = \{e_{n+1}, T(p_1), \ldots, T(p_k)\}$, which consists of at least two points. If $k = 1$ and $p_1 = 0$, then $\Omega_{R,r}$ is the annular region as in Example 5.3. Combining with Example 5.3 for a fixed constant r, we conclude that the maximum Ricci curvature component of $g_{\Omega_{R,r}}$ tends to $-n/2$ as $R \to r$ and tends to ∞ as $R \to \infty$.

Next, we construct bounded star-shaped domains in which the complete conformal metrics have positive Ricci components somewhere.

Example 5.6 For $n \geq 4$, set

$$\gamma = \{(0, \ldots, 0, x_n) \mid |x_n| \geq 1\} \subset \mathbb{R}^n.$$

Let Ω_i be a sequence of increasing bounded smooth domains in $\mathbb{R}^n$, star-shaped with respect to the origin, which converges to $\mathbb{R}^n \setminus \gamma$. Then, for i sufficiently large, the complete conformal metric in Ω_i with the constant scalar curvature $-n(n-1)$ has a positive Ricci curvature component somewhere. Note that the corresponding Γ in S^n is given by the equator in the x_n-x_{n+1} plane minus the image under T of the segment $(-1, 1)$ on x_n-axis. Hence, the dimension of Γ is 1. This is the reason we require $n \geq 4$. We point out that domains in this example are diffeomorphic to balls.

References

1. Andersson, L., Chruściel, P., Friedrich, H.: On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations. Comm. Math. Phys. **149**, 587–612 (1992)
2. Aviles, P.: A study of the singularities of solutions of a class of nonlinear elliptic partial differential equations. Comm. Partial Differ. Equ. **7**, 609–643 (1982)
3. Aviles, P., McOwen, R.C.: Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds. J. Differ. Geom. **27**, 225–239 (1988)
4. Aviles, P., McOwen, R.C.: Complete conformal metrics with negative scalar curvature in compact Riemannian manifolds. Duke Math. J. **56**, 395–398 (1988)
5. Caffarelli, L., Gidas, B., Spruck, J.: Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. **42**, 271–297 (1989)
6. Caffarelli, L., Guan, P., Ma, X.: A constant rank theorem for solutions of fully nonlinear elliptic equations. Comm. Pure Appl. Math. **60**, 1769–1791 (2007)
7. Gao, L., Yau, S.-T.: The existence of negatively Ricci curved metrics on three-manifolds. Invent. Math. **85**, 637–652 (1986)
8. Guan, B.: Complete conformal metrics of negative Ricci curvature on compact manifolds with boundary, Int. Math. Res. Not. **2008**, Article ID rnn 105, 25 pp. (2008)
9. Gursky, M., Streets, J., Warren, M.: Existence of complete conformal metrics of negative Ricci curvature on manifolds with boundary. Calc. Var. P. D. E. **41**, 21–43 (2011)
10. Kennington, A.: Power concavity and boundary value problems. Indiana Univ. Math. J. **34**, 687–704 (1985)
11. Kichenassamy, S.: Boundary behavior in the Loewner–Nirenberg problem. J. Funct. Anal. **222**, 98–113 (2005)
12. Korevaar, N., Mazzeo, R., Pacard, F., Schoen, R.: Refined asymptotics for constant scalar curvature metrics with isolated singularities. Invent. Math. **135**, 233–272 (1999)
13. Lee, J., Parker, T.: The Yamabe problem. Bull. Am. Math. Soc. (N.S.) **17**, 37–91 (1987)
14. Li, Y., Zhu, M.: Yamabe type equations on three-dimensional Riemannian manifolds. Comm. Contemp. Math. **1**, 1–50 (1999)
15. Loewner, C., Nirenberg, L.: Partial differential equations invariant under conformal or projective transformations, In: Contributions to Analysis, 245–272. Academic Press, New York (1974)
16. Lohkamp, J.: Metrics of negative Ricci curvature. Ann. Math. **140**, 655–683 (1994)
17. Lohkamp, J.: The higher dimensional positive mass theorem II. arXiv: 1612.07505 (2016)
18. Mazzeo, R.: Regularity for the singular Yamabe problem. Indiana Univ. Math. J. **40**, 1277–1299 (1991)
19. Mazzeo, R., Pacard, F.: Constant scalar curvature metrics with isolated singularities. Duke Math. J. **99**, 353–418 (1999)
20. Mazzeo, R., Pollack, D., Uhlenbeck, K.: Moduli spaces of singular Yamabe metrics. J. Am. Math. Soc. **9**, 303–344 (1996)
21. Schoen, R.: The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation. Comm. Pure Appl. Math. **41**, 317–392 (1988)
22. Schoen, R., Yau, S.-T.: On the proof of the positive mass conjecture in general relativity. Comm. Math. Phys. **65**, 45–76 (1979)
23. Schoen, R., Yau, S.-T.: Conformally flat manifolds, Kleinian groups and scalar curvature. Invent. Math. **92**, 47–71 (1988)
24. Schoen, R., Yau, S.-T.: Lectures on Differential Geometry. International Press, Cambridge, MA (1994)
25. Schoen, R., Yau, S.-T.: Positive scalar curvature and minimal hypersurface singularities. arXiv: 1704.05490 (2017)
26. Witten, E.: A new proof of the positive energy theorem. Comm. Math. Phys. **80**, 381–402 (1981)