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ORIGINAL ARTICLE

Global Steady Prandtl Expansion over a Moving Boundary III

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Abstract
This is the third paper in a three-part sequence in which we prove that steady, incompressible Navier–Stokes flows posed over the moving boundary, $y=0$, can be decomposed into Euler and Prandtl flows in the inviscid limit globally in $[1,\infty)\times[0,\infty)$, assuming a sufficiently small velocity mismatch. In this paper, we prove existence and uniqueness of solutions to the remainder equation.

1 Introduction

In this paper, we will prove existence and uniqueness in the space Z [the Z norm is recalled below in (1.45)] of solutions to the nonlinear system

$$-\Delta_\epsilon u + S_u(u,v) + P_x = f(u,v), \tag{1.1}$$

$$-\Delta_\epsilon v + S_v(u,v) + \frac{P_y}{\epsilon} = g(u,v), \tag{1.2}$$

$$u_x + v_y = 0, \tag{1.3}$$

with boundary conditions

$$[u,v]|_{\{y=0\}} = [u,v]|_{\{x=1\}} = \lim_{y\to\infty}[u,v] = \lim_{x\to\infty}[u,v] = 0. \tag{1.4}$$

The terms in Eqs. (1.1)–(1.2) are defined:

$$f(u,v) := \epsilon^{-\frac{n}{2}-\gamma}R^{u,n} + \mathcal{N}^u(u,v), \quad g(u,v) := \epsilon^{-\frac{n}{2}-\gamma}R^{v,n} + \mathcal{N}^v(u,v), \tag{1.5}$$

$$S_u(u,v) := u_R u_x + u_{Rx}u + v_R u_y + u_{Ry}v, \quad S_v(u,v) := u_R v_x + v_{Rx}u + v_R v_y + v_{Ry}v, \tag{1.6}$$

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$$\mathcal{N}^u(u,v) := \epsilon^{\frac{n}{2}+\gamma} u u_x + \epsilon^{\frac{n}{2}+\gamma} v u_y, \quad \mathcal{N}^v(u,v) := \epsilon^{\frac{n}{2}+\gamma} u v_x + \epsilon^{\frac{n}{2}+\gamma} v v_y. \tag{1.7}$$

For the convenience of the reader, we recall below the main estimates established in [3] which will be in use throughout this paper.

Theorem 1.1 [3] *Let $n \geq 2 \in \mathbb{N}$. Let δ, ϵ be sufficiently small relative to universal constants, and $\epsilon << \delta$. Let the boundary and in-flow data be prescribed as described in [3]. Then there exist Prandtl profiles $[u^j_p, v^j_p, P^j_p]$ for $j = 1, \dots, n$, Euler profiles $[u^j_e, v^j_e, P^j_e]$ for $j = 1, \dots, n$, and auxiliary pressures $[P^{j,a}_p, P^{j,a}_e]$ for $j = 1, \dots, n$ such that for any $\gamma \in [0, \frac{1}{4})$, $n \geq 2$, and for $\sigma_n = \frac{1}{10000}$, $\kappa > 0$ arbitrarily small, the following remainder estimate holds for any $k \geq 0$:*

$$\epsilon^{-\frac{n}{2}-\gamma} \left| \partial_x^k R^{u,n} + \sqrt{\epsilon} \partial_x^k R^{v,n} \right| \leq C(n,\kappa) \epsilon^{\frac{1}{4}-\gamma-\kappa} x^{-k-\frac{3}{2}+2\sigma_n}, \tag{1.8}$$

$$\epsilon^{-\frac{n}{2}-\gamma} \| \sqrt{\epsilon} \partial_x^k R^{u,n}, \sqrt{\epsilon} \partial_x^k R^{v,n} \|_{L^2_y} \leq C(n,\kappa) \epsilon^{\frac{1}{4}-\gamma-\kappa} x^{-k-\frac{5}{4}+2\sigma_n+\kappa}. \tag{1.9}$$

The following bounds hold on $[u_R, v_R]$ by construction, for any $[k, j, m] \geq 0$, so long as n is sufficiently large relative to m.

$$\| \partial_x^k \partial_y^j v_R^P z^m x^{k+\frac{j}{2}+\frac{1}{2}} \|_{L^\infty} \leq C(k,j,m) \quad \text{if } k \geq 1, \tag{1.10}$$

$$\| \partial_y^j v_R^P z^m x^{\frac{j}{2}+\frac{1}{2}} \|_{L^\infty} \leq C(j,m) \quad \text{if } j \geq 2, \tag{1.11}$$

$$\| \partial_y^j v_R^P z^m x^{\frac{j}{2}+\frac{1}{2}} \|_{L^\infty} \leq \mathcal{O}(\delta; m, j) \quad \text{if } j = 0, 1, \tag{1.12}$$

$$\| \partial_x^k \partial_y^j u_R^P z^m x^{k+\frac{j}{2}} \|_{L^\infty} \leq C(k,j,m) \quad \text{for } k > 1, j \geq 0, \tag{1.13}$$

$$\| \partial_x u_R^P z^m x \|_{L^\infty} \leq \mathcal{O}(\delta; m), \tag{1.14}$$

$$\| \partial_x \partial_y^j u_R^P z^m x \|_{L^\infty} \leq C(m,j) \quad \text{for } j \geq 1, \tag{1.15}$$

$$\| \partial_y^j u_R^{P,n-1} y^j z^m \|_{L^\infty} \leq \mathcal{O}(\delta; m, j) \quad \text{for } 0 \leq j \leq 2, \tag{1.16}$$

$$\| \partial_y^j u_R^{P,n-1} y^j z^m \|_{L^\infty} \leq C(m,j) \quad \text{for } j > 2, \tag{1.17}$$

$$\| \partial_y^j u_p^n y^j x^{\frac{1}{2}-\sigma_n} \|_{L^\infty} \leq C(n,j) \quad \text{for all } j \geq 0, \tag{1.18}$$

$$\|\partial_x^k \partial_Y^j v_R^E x^{k+j+\frac{1}{2}}\|_{L^\infty} \le C(k,j) \quad \text{for } k+j>0, \tag{1.19}$$

$$\|\partial_x^k \partial_Y^j u_R^E x^{k+j+\frac{1}{2}}\|_{L^\infty} \le \sqrt{\epsilon} C(k,j) \quad \text{for } k+j>0, \tag{1.20}$$

$$\|\partial_x^k v_R^E x^{k-\frac{1}{2}} Y\|_{L^\infty} \le C(k,j) \quad \text{for } k \ge 1, \tag{1.21}$$

$$\|\{u_R^E - 1, v_R^E\} x^{\frac{1}{2}}, v_{RY}^E x^{\frac{3}{2}}\|_{L^\infty} \le \mathcal{O}(\delta). \tag{1.22}$$

We also recall briefly the estimate, established in [3], on the remainders, $R^{u,n}, R^{v,n}$ which act as forcing terms:

Lemma 1.2 (Remainder Estimates) *For any* $\gamma \in [0, \frac{1}{4})$, $n \ge 2$, *and for* $\sigma_n = \frac{1}{10000}$, $\kappa > 0$ *arbitrarily small,*

$$\epsilon^{-\frac{n}{2}-\gamma} \left| \partial_x^k R^{u,n} + \sqrt{\epsilon} \partial_x^k R^{v,n} \right| \lesssim C(n,\kappa) \epsilon^{\frac{1}{4}-\gamma-\kappa} x^{-k-\frac{3}{2}+2\sigma_n}, \tag{1.23}$$

$$\epsilon^{-\frac{n}{2}-\gamma} \| \sqrt{\epsilon} \partial_x^k R^{u,n}, \sqrt{\epsilon} \partial_x^k R^{v,n} \|_{L^2_y} \lesssim C(n,\kappa) \epsilon^{\frac{1}{4}-\gamma-\kappa} x^{-k-\frac{5}{4}+2\sigma_n+\kappa}. \tag{1.24}$$

The main result of this paper is:

Theorem 1.3 *For* ϵ, δ *sufficiently small,* $\epsilon << \delta, \kappa > 0$ *small, and* $0 \le \gamma < \frac{1}{4}$, *there exists a unique solution* $[u, v] \in Z(\Omega)$ *to the system* (1.1)–(1.3), (1.4), (1.5) *satisfying the bound:*

$$\|u, v\|_{Z(\Omega)} \lesssim C(u_R, v_R) \epsilon^{\frac{1}{4}-\gamma-\kappa}. \tag{1.25}$$

The main result of the three-paper sequence, Theorem 1.2 of [3], follows immediately from Theorem 1.3. The proof of this theorem proceeds in several steps, which we now outline:

(Step 1) Linear existence of solutions to weighted Stokes system, defined as follows:

$$\Delta_\epsilon^2 \psi + \alpha A(\psi) = F_y - \epsilon G_x \quad \text{on } \Omega^N, \quad F, G \in L^2(\Omega^N), \tag{1.26}$$

$$\psi|_{y=0,N} = \psi_y|_{y=0,N} = 0, \quad \text{and } \psi|_{x=1} = \psi_x|_{x=1} = 0, \tag{1.27}$$

$$\lim_{x \to \infty} [\psi_x, \psi_y] = 0, \tag{1.28}$$

where $\alpha > 0$, and

$$A(\psi) = \big[\psi x^{2m} - \psi_{yy}x^{2m+2} - \partial_x(\psi_x x^{2m+2}) + \psi_{yyyy}x^{2m+4} + \partial_x(\psi_{yyx}x^{2m+4}) + \partial_{xx}(\psi_{xx}x^{2m+4})\big]. \tag{1.29}$$

Here, $m > 0$ is sufficiently large, and can remain temporarily unspecified. The scaled Bilaplacian is defined as $\Delta_\epsilon^2 := \partial_y^4 + \epsilon\partial_y^2\partial_x^2 + \epsilon^2\partial_x^4$. The right-hand sides, F, G, should be thought of as generic elements satisfying $F_y - \epsilon G_x \in H^{-1}$. Upon introducing appropriate function spaces, we define the weak formulation of (1.26)–(1.27) in (2.6). Depicting the weak-solution operator to the above system by S_α^{-1} (see (2.12) for a precise definition), Step 1 amounts to studying the solvability of $S_\alpha\psi = F_y - \epsilon G_x$.

The boundary conditions as $x \to \infty$ in (1.28) are selected in order to be consistent with (1.4). However, due to the terms in $A(\psi)$, the weak solution, $[\psi, u, v]$ exhibits rapid decay as $x \to \infty$.

(Step 2) Linear existence of compact perturbations to S_α. Define the maps:

$$\begin{aligned} T[\psi] :=\ & \partial_y\big[-u_R\psi_{xy} - u_{Rx}\psi_y - (v_R + \epsilon^{\frac{n}{2}+\gamma}\bar{v})\psi_{yy} + u_{Ry}\psi_x\big] \\ & - \epsilon\partial_x\Big[u_R\psi_{xx} - v_{Ry}\psi_y + v_R\psi_{xy} + v_{Ry}\psi_x\Big], \end{aligned} \tag{1.30}$$

$$T_0[\psi] := T[\psi] + \epsilon^{\frac{n}{2}+\gamma}\partial_y[\bar{v}\psi_{yy}], \tag{1.31}$$

$$T_a[\psi] := -u_R\psi_{xy} - u_{Rx}\psi_y - v_R\psi_{yy} + u_{Ry}\psi_x, \tag{1.32}$$

$$T_b[\psi] := u_R\psi_{xx} - v_{Ry}\psi_y + v_R\psi_{xy} + v_{Ry}\psi_x. \tag{1.33}$$

T has a dependence on $\bar{v}$, so to be precise we will sometimes write $T[\psi;\bar{v}]$. When there is no danger of confusion, we simply write $T[\psi]$. The map $T_0[\psi]$ is defined to match the profile terms, $S_u(u, v), S_v(u, v)$ [see the definition in (1.6)], when they are written in terms of the stream function, ψ. We have defined the notation T_a, T_b so that we can write $T_0 = \partial_y T_a - \epsilon\partial_x T_b$. In this step, we are interested in establishing solvability of the system:

$$S_\alpha\psi + T[\psi] = F_y - \epsilon G_x \quad \text{on } \Omega^N, \tag{1.34}$$

$$[\psi = \psi_x]|_{x=1} = [\psi = \psi_y]|_{y=0} = [\psi = \psi_y]|_{y=N} = \lim_{x\to\infty}[\psi_x, \psi_y] = 0. \tag{1.35}$$

The essence of the arguments in this step is that upon applying S_α^{-1} to both sides above, $S_\alpha^{-1}T$ is seen as a compact perturbation of the identity. Despite Ω^N being unbounded in the x-direction, the required compactness arises from the weights, w, present in $A(\psi)$ above in (1.29). The solution of (1.34) is known to decay rapidly as $x \to \infty$, due to the presence of $A(\psi)$. This is captured in estimate (3.67).

(Step 3) Nonlinear existence of auxiliary system: we first invite the reader to refer back to (1.5) for the definitions of f and g. Given this and the definition of T in (1.30), we define:

$$\tilde{f}(\bar{u},\bar{v}) := \epsilon^{-\frac{n}{2}-\gamma} R^{u,n} + \epsilon^{\frac{n}{2}+\gamma} \bar{u}\bar{u}_x, \quad \text{so that} \quad f(u,\bar{u},\bar{v}) = \tilde{f}(\bar{u},\bar{v}) + \epsilon^{\frac{n}{2}+\gamma} \bar{v} u_y. \tag{1.36}$$

The aim of this step is to obtain existence of solutions (which we now index by α and N for clarity) to the nonlinear system:

$$S_\alpha \psi^{\alpha,N} + T[\psi^{\alpha,N}; v^{\alpha,N}] = \tilde{f}_y(u^{\alpha,N}, v^{\alpha,N}) + \epsilon g_x(u^{\alpha,N}, v^{\alpha,N}) \quad \text{on } \Omega^N. \tag{1.37}$$

This existence is obtained in the unit ball of $Z(\Omega^N)$ via Schaefer's fixed point theorem.

(Step 4) Nonlinear existence of solutions to the system (1.1)–(1.3), with f, g as in (1.5): By re-applying the analyses in [4], one obtains the uniform-in-(α, N) estimate: $\|u^{\alpha,N}, v^{\alpha,N}\|_{Z(\Omega^N)} \lesssim \mathcal{O}(\delta)\epsilon^{\frac{1}{4}-\gamma-\kappa}$, which then enables the passage to weak limits in the space $X_1 \cap X_2 \cap X_3$. The weak limit is denoted by $[u, v]$, and is demonstrated to satisfy a weak formulation of system (1.1)–(1.3), see (5.10) for this formulation. Moreover, $[u, v] \in X_1 \cap X_2 \cap X_3$, gives enough regularity to upgrade immediately to a strong solution of (1.1)–(1.3).

***Remark* 1.4** To establish existence, we rely on compactness methods as opposed to applying a contraction mapping. The essential reason for this is seen by examining calculation (4.5) in [4], in which the structure is not preserved under taking differences.

***Remark* 1.5** It is important to establish nonlinear existence of the auxiliary system before establishing nonlinear existence of the system (1.1)–(1.3), as opposed to jumping from linear existence of (1.1)–(1.3) to nonlinear existence because the compactness methods we rely on require the weights from $\alpha A(\psi)$.

(Step 5) Nonlinear uniqueness for solutions to the system (1.1)–(1.3), with f, g as in (1.5): In order to prove uniqueness, we re-apply the estimates from [4] with weights that are weaker by x^{-b}, where $b < 1$, but is arbitrarily close to 1. This step is necessary (with the weaker weight) due again to the calculation in (4.5) from [4], whose structure is destroyed upon considering differences.

1.1 Notations and Norms

We briefly recall the norms introduced in [4] for the convenience of the reader. First define the cut-off functions:

$$\zeta_3(x) = \begin{cases} 0 & \text{for } 1 \le x \le \frac{3}{2}, \\ 1 & \text{for } x \ge 2, \end{cases} \tag{1.38}$$

$$\rho_k(x) = \begin{cases} 0 & \text{for } 1 \le x \le 50 + 50(k-2), \\ 1 & \text{for } x \ge 60 + 50(k-2). \end{cases} \tag{1.39}$$

The energy norms are defined as follows:

$$\|u, v\|_{X_1}^2 := \|u_y\|_{L^2}^2 + \|\{\sqrt{\epsilon}v_x, v_y\}x^{\frac{1}{2}}\|_{L^2}^2 \tag{1.40}$$

$$\|u, v\|_{X_2}^2 := \|u_{xy} \cdot \rho_2 x\|_{L^2}^2 + \|\{\sqrt{\epsilon}v_{xx}, v_{xy}\} \cdot (\rho_2 x)^{\frac{3}{2}}\|_{L^2}^2, \tag{1.41}$$

$$\|u, v\|_{X_3}^2 := \|u_{xxy} \cdot (\rho_3 x)^2\|_{L^2}^2 + \|\{\sqrt{\epsilon}v_{xxx}, v_{xxy}\} \cdot (\rho_3 x)^{\frac{5}{2}}\|_{L^2}^2. \tag{1.42}$$

Definition 1.6 The norms Y_2, Y_3 are strengthenings of X_2, X_3 near the boundary, $x = 1$, and defined through:

$$\|u, v\|_{Y_2}^2 := \|u_{xy}x\|_{L^2}^2 + \|\{\sqrt{\epsilon}v_{xx}, v_{xy}\}x^{\frac{3}{2}}\|_{L^2}^2 + \|u_{yy}\|_{L^2(x \le 2000)}, \tag{1.43}$$

$$\|u, v\|_{Y_3}^2 := \|u_{xxy} \cdot \zeta_3 x^2\|_{L^2}^2 + \|\{\sqrt{\epsilon}v_{xxx}, v_{xxy}\} \cdot \zeta_3 x^{\frac{5}{2}}\|_{L^2}^2. \tag{1.44}$$

Definition 1.7 The norm Z is defined through:

$$\begin{aligned} \|u, v\|_Z := & \|u, v\|_{X_1 \cap X_2 \cap X_3} + \epsilon^{N_2}\|u, v\|_{Y_2} + \epsilon^{N_3}\|u, v\|_{Y_3} + \epsilon^{N_4}\|ux^{\frac{1}{4}}, \sqrt{\epsilon}vx^{\frac{1}{2}}\|_{L^\infty} \\ & + \epsilon^{N_5} \sup_{x \ge 20} \|\sqrt{\epsilon}v_x x^{\frac{3}{2}}, u_x x^{\frac{5}{4}}\|_{L^\infty} + \epsilon^{N_6} \sup_{x \ge 20} \|u_y x^{\frac{1}{2}}\|_{L^2_y} \\ & + \epsilon^{N_7}\Big[\int_{20}^\infty x^4 \|\sqrt{\epsilon}v_{xx}\|_{L^\infty_y}^2 \, dx\Big]^{\frac{1}{2}}. \end{aligned} \tag{1.45}$$

Here, N_i, are some large universal numbers.

The following embedding result from [4] appears repeatedly in the present paper:

Lemma 1.8 *For $\sigma > 0$ arbitrarily small,*

$$\sup_{x \ge 1}\Big[\|\sqrt{\epsilon}\psi x^{-1-\sigma}\|_{L^2_y}^2 + \|ux^{-\sigma}\|_{L^2_y}^2 + \|\sqrt{\epsilon}vx^{-\sigma}\|_{L^2_y}^2\Big] \lesssim C(\sigma)\|u, v\|_{X_1}^2, \tag{1.46}$$

$$\sup_{x \ge 1}\Big[\|u_y x^{\frac{1}{2}}\|_{L^2_y}^2 + \|u_x x\|_{L^2_y}^2\Big] + \sup_{x \ge 20}\|\sqrt{\epsilon}v_x x\|_{L^2_y}^2 \lesssim \|u, v\|_{X_1 \cap Y_2}^2, \tag{1.47}$$

$$\sup_{x\geq 20}\left[\|u_{xy}x^{\frac{3}{2}}\|_{L^2_y}^2+\|\{v_{xy},\sqrt{\epsilon}v_{xx}\}x^2\|_{L^2_y}^2\right]\lesssim\|u,v\|_{Y_2\cap Y_3}^2. \tag{1.48}$$

The constant $C(\sigma)\uparrow\infty$ *as* $\sigma\downarrow 0$. *Finally, for* $[u,v]\in Z$, *we have the following property:*

$$\sup_{x\geq 20}\|\{v_{xxx},u_{xxy}\}x\|_{L^2_y}<\infty, \tag{1.49}$$

2 Invertibility of Weighted Stokes Operator, S_α

In this step, we study the system (1.26)–(1.27). We remind the reader that still, all integrations and all norms are taken over Ω^N unless otherwise specified. There is an abuse of notation here; ψ should be indexed by α and N, but this will not cause any confusion for this step, as we view both α and N as fixed. Our intention of this section is to exhibit solvability of the system (1.26) in the space $Z(\Omega^N)$. Denote by $\chi_1(x)$ a cut-off function satisfying (refer to (1.38) for the definition of ζ_3):

$$\chi_1=1\ \text{ on } x\geq\frac{12}{10},\quad \chi_1=0\ \text{ for } 1\leq x\leq\frac{11}{10}. \tag{2.1}$$

We define higher-order cut-offs similar to (2.1), satisfying the following property: support $(\chi_k)\subset\{\chi_{k-1}=1\}$. Define the following auxiliary norms via:

$$\|\psi\|_{H^2_w}^2:=\int\int\psi^2x^{2m}+|\nabla\psi|^2x^{2m+2}+|\nabla^2\psi|^2x^{2m+4}, \tag{2.2}$$

$$\|\psi\|_{H^3_w}^2:=\|\psi\|_{H^2_w}+\int\int|\nabla^2\psi_x|^2x^{2m+4}, \tag{2.3}$$

$$\|\psi\|_{G^k_{w,B}}^2:=\|\psi\|_{H^k_w}^2+\int\int_B|\partial_y^k\psi|^2\ \text{ for any bounded subset } B\subset\Omega^N,\ k=0,\dots,3, \tag{2.4}$$

$$\|\psi\|_{H^k_w}^2:=\|\psi\|_{H^3_w}^2+\int\int\chi_k^2|\nabla^2\partial_x^{k-2}\psi|^2x^{2m+4}\ \text{ for } k\geq 4. \tag{2.5}$$

We will also call $G^k_{w,\text{loc}}(\Omega^N)$ the space such that $\|\psi\|_{G^k_{w,B}}\leq C(B)$ for all compact subsets B. Define the weak formulation of (1.26) to be:

$$\int\int\nabla_\epsilon^2\psi:\nabla_\epsilon^2\phi+\alpha\Big[\int\int\psi\phi x^{2m}+\int\int\nabla\psi\cdot\nabla\phi x^{2m+2}+\int\int\nabla^2\psi:\nabla^2\phi x^{2m+4}\Big]$$
$$=\langle F_y-\epsilon G_x,\phi\rangle_{H^{-1},H^1}\ \text{for all}\ \phi\in C_0^\infty(\Omega^N),\ \text{where}\ \psi\in H^2_w(\Omega^N). \tag{2.6}$$

Above, ∇^2 is the Hessian matrix, and the inner product between two matrices is given by $A : B = \text{trace}(AB)$. We will need one more norm:

$$\|(F,G)\|_{H_k^{-1}} := \sum_{j=0}^{k} \|\partial_x^j \{F_y - \epsilon G_x\}\|_{H^{-1}}. \tag{2.7}$$

Relevant spaces are defined here:

Definition 2.1 $H^2_w(\Omega^N)$ is defined to be the closure of $C_0^\infty(\Omega^N)$ under the norm $\|\cdot\|_{H^2_w}$. $H^k_w(\Omega^N)$ for $k \geq 3$ consists of the subspace of $H^2_w(\Omega^N)$ whose $H^k_w(\Omega^N)$ norm is finite. Note that $H^3_w(\Omega^N)$ does not contain all of the third derivatives of ψ; it is missing $\partial_y^3 \psi$, which is the reason for the norm, $\|\cdot\|_{G_{w,B}}$.

***Remark* 2.2** There is a distinction between $H^2_w(\Omega^N)$, and $H^k_w(\Omega^N)$ in that:

$$H^2_w(\Omega^N) = \overline{C_0^\infty(\Omega^N)}^{\|\cdot\|_{H^2_w}} \text{ but for } k \geq 3,\ H^3_w(\Omega^N) \not\subset \overline{C_0^\infty(\Omega^N)}^{\|\cdot\|_{H^3_w}}. \tag{2.8}$$

Due to the weights, there is no "$H = W$" theorem generically for $H^k_w(\Omega^N)$.

Lemma 2.3 *For $\psi \in H^2_w(\Omega^N)$, the following boundary conditions are satisfied:*

$$\psi|_{y=0,N} = \psi_y|_{y=0,N} = \psi|_{x=1} = \psi_x|_{x=1} = 0. \tag{2.9}$$

Proof If $\psi \in H^2_w(\Omega^N)$, obtain a sequence $\phi^{(n)}$ such that $\|\phi^{(n)} - \psi\|_{H^2_w} \to 0$. The claim now follows by the standard boundedness properties of the trace operator. □

Lemma 2.4 $H^2_w(\Omega^N)$ *as defined in Definition 2.1 is a Banach space.*

Proof Consider the auxiliary space:

$$H^2_{0,w}(\Omega^N) = \left\{\psi : \nabla\psi, \nabla^2\psi \text{ exist in the weak sense, and } \|\psi\|_{H^2_w(\Omega^N)} < \infty\right\}. \tag{2.10}$$

Through standard arguments, $H^2_{0,w}(\Omega^N)$ is a Banach space. Suppose $\{\psi^{(n)}\}$ is a Cauchy sequence in $H^2_w(\Omega^N)$. Then $\{\psi^{(n)}\}$ is Cauchy in $H^2_{0,w}(\Omega^N)$, and so there exists a limit point ψ such that: $\|\psi - \psi^{(n)}\|_{H^2_w} \xrightarrow{n\to\infty} 0$. As $\psi^{(n)} \in H^2_w(\Omega^N)$, we may find a sequence $\{\phi^{(n)}_m\}_{m\geq 1}$ such that $\|\phi^{(n)}_m - \psi^{(n)}\|_{H^2_w} \xrightarrow{m\to\infty} 0$, where $\phi^{(n)}_m \in C_0^\infty(\Omega^N)$. In particular, define, for each n, by selecting m large enough: $\|\phi^{(n)} - \psi^{(n)}\|_{H^2_w} < 2^{-n}$. Thus, $\|\phi^{(n)} - \psi\|_{H^2_w} \xrightarrow{n\to\infty} 0$, proving that $\psi \in \overline{C_0^\infty}^{\|\cdot\|_{H^2_w}}$. This establishes the desired result.

□

Lemma 2.5 *Endowed with the inner product,*

$$\langle \psi, \varphi \rangle_{H^2_w} := \int\int \psi\varphi x^{2m} + \nabla\psi \cdot \nabla\varphi x^{2m+2} + \nabla^2\psi : \nabla^2\varphi x^{2m+4}, \tag{2.11}$$

H^2_w is a Hilbert Space. The inner product in (2.11) induces the norm defined in (2.2).

Proof One easily verifies the standard axioms of an inner-product for (2.11). Non-degeneracy of (2.11) is obtained via the boundary conditions in (1.27). Completeness is then obtained via Lemma 2.4. □

Definition 2.6 The α-Stokes operator is defined through:

$$S_\alpha \psi = F_y - \epsilon G_x \quad \text{for } \psi \in H^2_w(\Omega^N), \quad F_y - \epsilon G_x \in H^{-1}(\Omega^N), \tag{2.12}$$

and is equivalent to (2.6) holding.

It is our aim to study the invertibility of S_α.

Lemma 2.7 *Given $F_y - \epsilon G_x \in H^{-1}(\Omega^N)$, there exists a unique weak solution $\psi \in H^2_w(\Omega^N)$ satisfying (2.6). Such a weak solution satisfies the energy inequality:*

$$\|\psi\|^2_{H^2_w} \lesssim \frac{1}{\alpha}\|F_y - \epsilon G_x\|^2_{H^{-1}} = \frac{1}{\alpha}\|S_\alpha \psi\|^2_{H^{-1}}. \tag{2.13}$$

Proof Define:

$$\begin{aligned} B[\psi, \phi] := \int\int \nabla^2_\epsilon \psi : \nabla^2_\epsilon \phi + \alpha\Big(\int\int \psi\phi x^{2m} \\ + \int\int \nabla\psi \cdot \nabla\phi x^{2m+2} + \int\int \nabla^2\psi : \nabla^2 \phi x^{2m+4}\Big). \end{aligned} \tag{2.14}$$

It is immediate to see that B is bilinear, bounded, and coercive on $H^2_w(\Omega^N)$. Next, $F_y - \epsilon G_x$ act as bounded linear functionals on $H^2_w(\Omega^N)$ through the pairing: $\langle F_y - \epsilon G_x, \phi\rangle_{H^{-2}_w, H^2_w} := \langle F_y - \epsilon G_x, \phi\rangle_{H^{-1}, H^1}$. This follows from: $|\langle F_y - \epsilon G_x, \phi\rangle_{H^{-1},H^1}| \le \|F_y - \epsilon G_x\|_{H^{-1}}\|\phi\|_{H^2_w}$. The existence of $\psi \in H^2_w(\Omega^N)$, a solution to (2.6) is then a standard application of the Lax–Milgram lemma to the Hilbert space $H^2_w(\Omega^N)$. The energy identity above follows from density of $C^\infty_0(\Omega^N)$ in $H^2_w(\Omega^N)$, which enables us to replace ϕ with ψ in (2.6). □

The above lemma then says that $S^{-1}_\alpha : H^{-1}(\Omega^N) \to H^2_w(\Omega^N)$ is well-defined. Our intention now is to upgrade regularity.

Lemma 2.8 *Given $F_y - \epsilon G_x \in H^{-1}(\Omega)$, the unique weak solution in $H^2_w(\Omega^N)$ guaranteed by Lemma 2.7 is in $H^3_w(\Omega^N)$ and satisfies:*

$$\|\psi\|^2_{H^3_w} \lesssim \frac{1}{\alpha}\|F_y - \epsilon G_x\|^2_{H^{-1}} = \frac{1}{\alpha}\|S_\alpha \psi\|^2_{H^{-1}}. \tag{2.15}$$

Proof As our weak solutions are only in $H^2_w(\Omega^N)$, we must formally use difference quotients within the weak formulation (2.6) to upgrade to $H^3_w(\Omega^N)$. However, we will generate the H^3_w estimate via differentiating (1.26), with the understanding that everything that is done can be formalized through the use of difference quotients in the standard manner. As such, we take ∂_x of the system (1.26), which gives:

$$\Delta_\epsilon^2 \psi_x + \alpha A(\psi_x) + [\partial_x, \alpha A]\psi = \partial_x(F_y - \epsilon G_x), \tag{2.16}$$

where

$$\begin{aligned}[\partial_x, \alpha A]\psi = \alpha \big[&2mx^{2m-1}\psi - (2m+2)x^{2m+1}\psi_{yy} - (2m+2)\partial_x(\psi_x x^{2m+1}) \\ &+ (2m+4)x^{2m+3}\psi_{yyyy} + (2m+4)\partial_x(\psi_{yyx}x^{2m+3}) \\ &+ (2m+4)\partial_{xx}(\psi_{xx}x^{2m+3})\big].\end{aligned} \tag{2.17}$$

Let χ_1 be as above in (2.1). Define the quantities:

$$\tilde{\chi}_1 = 1 - \chi_1, \quad \rho_M(x) = \chi_1(x)\tilde{\chi}_1\left(\frac{x}{M}\right), \text{ which implies } |x^k \partial_x^k \rho_M(x)| \leq 2. \tag{2.18}$$

We now test the above equation, (2.16), against the multiplier $\rho_M \psi_x$. Doing so first gives from the Bilaplacian:

$$\begin{aligned}\int\int \Delta_\epsilon^2 \psi_x \cdot \rho_M \psi_x = \int\int \rho_M \big[\epsilon|\psi_{xxy}|^2 + \epsilon^2|\psi_{xxx}|^2 + |\psi_{xyy}|^2\big] \\ + c_0 \int\int \rho_M''\big[\epsilon|\psi_{xy}|^2 + \epsilon^2|\psi_{xx}|^2\big] + c_1 \int\int \partial_x^4 \rho_M \cdot |\psi_x|^2,\end{aligned} \tag{2.19}$$

for constants c_0, c_1. Next, we have the terms coming from A:

$$\begin{aligned}&\int\int \alpha A(\psi_x) \cdot \rho_M \psi_x \\ &\gtrsim \alpha \int\int \big[\psi_x^2 x^{2m} + \psi_{xy}^2 x^{2m+2} + \psi_{xx}^2 x^{2m+4} + \psi_{xyy}^2 x^{2m+4} \\ &\quad + \psi_{xxy}^2 x^{2m+4} + \psi_{xxx}^2 x^{2m+4}\big]\rho_M - \|\psi\|^2_{H^2_w}.\end{aligned} \tag{2.20}$$

Through a direct integration by parts, the commutator contains lower order terms:

$$\left|\int\int [\partial_x, \alpha A]\psi \cdot \psi_x \rho_M\right| \lesssim \|\psi\|^2_{H^2_w} \lesssim \frac{1}{\alpha}\|F_y - \epsilon G_x\|^2_{H^{-1}}. \tag{2.21}$$

For detailed proofs of calculations (2.20) and (2.21), we refer the reader to (3.93)–(3.100). Finally, on the right-hand side of (2.16), we have:

$$|\langle \partial_x(F_y - \epsilon G_x), \psi_x \rho_M \rangle_{H^{-2},H^2}| \leq \|F_y - \epsilon G_x\|_{H^{-1}} \|\rho_M \psi_{xx}\|_{H^1}. \tag{2.22}$$

We can send $M \to \infty$ so that the weight $\rho_M \uparrow \chi_1$, resulting in

$$\int\int \chi_1 |\nabla^2 \psi_x|^2 x^{2m+4} \lesssim \frac{1}{\alpha} \|F_y - \epsilon G_x\|_{H^{-1}}^2. \tag{2.23}$$

For the region $1 \leq x \leq 20$, and $0 \leq y \leq N$, we apply the standard $\dot{H}^2(\Omega^N)$ estimate for solutions, u^α, v^α Stokes' equation near corners (see [1], Theorems 1 and 2, and Figure 2, P. 562 also in [1] with "C/C" boundary conditions). Formally, fix another cut-off function, $\chi_2(x, y)$ localized near the corner $(1, 0)$ [the identical argument can be given for the other corner, $(1, N)$]. First, by calculation, we have:

$$\Delta_\epsilon^2(\chi_2 \psi) = \chi_2 \Delta_\epsilon^2 \psi + [\Delta_\epsilon^2, \chi]\psi, \tag{2.24}$$

where the expression for the commutator is given explicitly:

$$\begin{aligned}
[\Delta_\epsilon, \chi_2]\psi = {} & 4\partial_y \chi_2 \partial_y^3 \psi + 4\partial_y^3 \chi_2 \partial_y \psi + 6\partial_y^2 \chi_2 \partial_y^2 \psi + 2\epsilon \partial_x^2 \partial_y^2 \chi_2 \psi + 2\epsilon \partial^2 \chi_2 \partial_x^2 \psi \\
& + 4\epsilon \partial_x \partial_y^2 \chi_2 \partial_x \psi + 2\epsilon \partial_x^2 \chi_2 \partial_y^2 \psi + 4\epsilon \partial_x \chi_2 \partial_x \partial_y^2 \psi + 4\epsilon \partial_x^2 \partial_y \chi_2 \partial_y \psi \\
& + 4\epsilon \partial_y \chi_2 \partial_x^2 \partial_y \psi + 8\epsilon \partial_{xy} \chi_2 \partial_{xy} \psi + 6\epsilon^2 \partial_x^2 \chi_2 \partial_x^2 \psi + \epsilon^2 \partial_x^4 \chi_2 \psi \\
& + 4\epsilon^2 \partial_x^3 \chi_2 \partial_x \psi + 4\epsilon^2 \partial_x \chi_2 \partial_x^3 \psi.
\end{aligned} \tag{2.25}$$

The salient feature of (2.25) will be:

$$[\Delta_\epsilon, \chi_2]\psi = o(\chi_2 \partial^3 \psi), \tag{2.26}$$

where this is short-hand notation for containing up to three ψ-derivatives, and localized by χ_2 (or any derivative of χ_2 which is also localized). Localizing (1.26) using χ_2:

$$\|\chi_2 \psi\|_{H^3} \lesssim \|\chi_2(F_y - \epsilon G_x)\|_{H^{-1}} + \|o(\chi_2 \partial^3 \psi)\|_{H^{-1}} \lesssim \|\chi_2(F_y - \epsilon G_x)\|_{H^{-1}}. \tag{2.27}$$

Combining (2.27) and (2.23) gives the desired result. □

Lemma 2.9 *Fix any bounded set $B \subset \Omega^N$. Then we have:*

$$\|\psi\|_{G^3_{w,B}}^2 \lesssim C(B) \frac{1}{\alpha} \|F_y - \epsilon G_x\|_{H^{-1}}^2, \tag{2.28}$$

where the constant $C(B)$ depends on B.

Proof This argument proceeds identically to the calculation from the previous lemma which resulted in (2.27) by simply replacing χ_2 with cut-off functions localized to each interval $x \in [M, M+1]$. The dependence on B in the constant in (2.28) arises from the weights $x^{2m}, x^{2m+2}, x^{2m+4}$ appearing in Eq. (1.26) through $A(\psi)$. □

The above lemmas roughly show that S_α^{-1} gains four derivatives. By repeating this procedure for higher-order x-derivatives, we can upgrade to higher-regularity.

Lemma 2.10 *Given* $(F, G) \in H_2^{-1}$, *the unique weak solution guaranteed by Lemma 2.7 satisfies:*

$$\|\psi\|_{H^5_w}^2 \lesssim \frac{1}{\alpha}\|(F, G)\|_{H_2^{-1}}^2. \tag{2.29}$$

For $(F, G) \in H_2^{-1}$, we can upgrade weak solutions to strong solutions.

Lemma 2.11 *Given* $(F, G) \in H_2^{-1}$, *the unique weak solution guaranteed by Lemma 2.7 is a strong solution of* (1.26).

Proof An integration by parts of the weak formulation (2.6), justified according to the previous lemma, is equivalent to Eq. (1.26) being satisfied pointwise on Ω^N. The boundary conditions at $x = 1, y = 0, y = N$ are satisfied by Lemma 2.3. The boundary condition at $x \to \infty$ comes from the norms, (2.2), which when applying with $k = 5$, imply that up to four derivatives of ψ vanish rapidly at $x \to \infty$. □

3 Compact Perturbations, $S_\alpha \Psi + T[\Psi]$

For this step, we invite the reader to refer back to the specification of $T[\psi]$, given in (1.30), and the system that we will focus on, given in (1.34). Note that $T[\psi]$ contains a loss of three derivatives for ψ. Note also the presence of the term $\epsilon^{\frac{n}{2}+\gamma}\bar{v}$. We will now need some compactness lemmas.

Lemma 3.1 *Fix two weights,* $w_1 = x^{m_1}$, *and* $w_2 = x^{m_2}$, *where* $m_2 > m_1 \geq 0$. *Then, one has the following compact embedding:*

$$H^1_{\mathrm{loc}}(\Omega^N) \cap L^2_{w_2}(\Omega^N) \hookrightarrow\hookrightarrow L^2_{w_1}(\Omega^N). \tag{3.1}$$

Proof Consider a family of functions $\{f^n\}$ defined on Ω^N such that:

$$\sup_n \int\int f_n^2 w_2^2 < \infty, \tag{3.2}$$

and such that $f_n \in H^1_{\mathrm{loc}}(\Omega^N)$, uniformly in n. By taking Sobolev extensions across $\partial\Omega^N$, and subsequently cutting off in the y and negative x directions, we can assume $\{f_n\}$ are defined on $\mathbb{R}^2$, compactly supported in the y direction and negative x direction. Fix any $\delta' > 0$. Since $m_2 > m_1$, there exists a compact set $K = K(\delta')$ such that:

$$\sup_n \|f_n\|_{L^2_{w_1}(K^c)} \le \frac{\delta'}{2}. \tag{3.3}$$

On K, by Rellich compactness, there exists a subsequence (depending on δ') such that

$$\limsup_{j,k\to\infty} \|f_{n_j} - f_{n_k}\|_{L^2(K)} \le \frac{\delta'}{2\times \mathrm{diam}(K)^{m_1}}. \tag{3.4}$$

Then,

$$\limsup_{j,k\to\infty} \|f_{n_j} - f_{n_k}\|_{L^2_{w_1}(K)} \le \frac{\delta'}{2}. \tag{3.5}$$

Combining the above two estimates,

$$\limsup_{j,k\to\infty} \|f_{n_j} - f_{n_k}\|_{L^2_{w_1}} \le \delta'. \tag{3.6}$$

Taking successively $\delta' = 2^{-n}$ and applying a diagonalization argument give the result. □

Lemma 3.2 *Let the weight, x^{2m}, in the expression for $A(\psi)$, Eq. (1.29), be selected for any $m > 0$. Then the map $S_\alpha^{-1}T$ is well-defined and compact $H^2(\Omega^N) \to H^2(\Omega^N)$.*

Proof According to (2.28), this follows from the compactness of $G^3_{w,\mathrm{loc}}(\Omega^N) \hookrightarrow$ $\hookrightarrow H^2(\Omega^N)$, which in turn follows from (3.1). The lemma is proven. □

We are now ready to study system (1.34). The first task is to obtain an energy estimate to the inhomogeneous problem.

Lemma 3.3 *Suppose $\psi \in H^2(\Omega^N)$ is a solution to (1.34), where $(F, G) \in H_2^{-1}$, and $\|\bar{u}, \bar{v}\|_Z \le 1$. Then ψ obeys the following energy estimate:*

$$\|u_y\|^2_{L^2} + \alpha\|\psi\|^2_{H^2_w} \lesssim \mathcal{O}(\delta)\|\sqrt{\epsilon}v_x x^{\frac{1}{2}}, v_y x^{\frac{1}{2}}\|^2_{L^2} + \int\int Fu + \epsilon|G||v|. \tag{3.7}$$

Proof Supposing there exists such a ψ, we would have $T[\psi] \in H^{-1}(\Omega^N)$, and so by (2.15), we know $\psi \in H^3_w(\Omega^N)$. By bootstrapping this regularity, we obtain that:

$$\psi \in H^5_w(\Omega^N). \tag{3.8}$$

We would like to apply the multiplier ψ to Eq. (1.34) in order to repeat the energy estimate from Proposition 3.2 in [4]. Select test functions, $\phi^{(n)} \in C_0^\infty(\Omega^N)$, which satisfy:

$$\|\phi^{(n)} - \psi\|_{H^2_w} \to 0. \tag{3.9}$$

This is possible according to the density of $C_0^\infty(\Omega^N)$ in H^2_w in Definition 2.1. Multiplying (1.34) by $\phi^{(n)}$, then gives on the left-hand side:

$$\int\int (\Delta_\epsilon^2 \psi + T\psi) \cdot \phi^{(n)} + \alpha \int\int A(\psi)\phi^{(n)}. \tag{3.10}$$

First, we shall use (1.31) to write:

$$\begin{aligned}\int\int (\Delta_\epsilon^2 \psi + T[\psi])\phi^{(n)} &= \int\int (\Delta_\epsilon^2 \psi + T_0[\psi])\phi^{(n)} - \int\int \epsilon^{\frac{n}{2}+\gamma} \partial_y (\bar{v} u_y)\phi^{(n)} \\ &= \int\int (\Delta_\epsilon^2 \psi + T_0[\psi])\phi^{(n)} + \int\int \epsilon^{\frac{n}{2}+\gamma} (\bar{v} u_y)\phi_y^{(n)}.\end{aligned} \tag{3.11}$$

According to (3.9), we pass to limits in the following terms:

$$\begin{aligned}\int\int (\Delta_\epsilon^2 \psi + T_0[\psi])\phi^{(n)} &= \int\int \nabla_\epsilon^2 \psi : \nabla_\epsilon^2 \phi^{(n)} + \int\int T_0[\psi]\phi^{(n)} \\ &\xrightarrow{n\to\infty} \int\int |\nabla_\epsilon^2 \psi|^2 + \int\int T_0[\psi]\psi.\end{aligned} \tag{3.12}$$

We have used:

$$\begin{aligned}\left|\int\int (T_0[\psi]\phi^{(n)} - T_0[\psi]\psi)\right| &= \int\int (\partial_y T_a[\psi] - \partial_x T_b[\psi])(\phi^{(n)} - \psi) \\ &\le \|T_a[\psi], T_b[\psi]\|_{L^2} \|\phi^{(n)} - \psi\|_{H^1} \\ &\le \|\psi\|_{H^5_w} \|\phi^{(n)} - \psi\|_{H^1} \xrightarrow{n\to\infty} 0,\end{aligned} \tag{3.13}$$

according to (3.8) and the definition in Eq. (1.31). The integration on the right-hand side of (3.12) produces the lower bound:

$$\left|\lim_{n\to\infty} \int\int (\Delta_\epsilon^2 \psi + T_0[\psi])\phi^{(n)}\right| \gtrsim \|u_y\|_{L^2}^2 - \mathcal{O}(\delta)\|\sqrt{\epsilon} v_x x^{\frac{1}{2}}, v_y x^{\frac{1}{2}}\|_{L^2}^2. \tag{3.14}$$

We may pass to the limit in the final term of (3.11) due to the calculation:

$$\begin{aligned}\left|\int\int \bar{v} u_y (\phi_y^{(n)} - \psi_y)\right| &\le \|\bar{v}\|_{L^\infty} \|u_y\|_{L^2} \|\phi_y^{(n)} - \psi_y\|_{L^2} \\ &\le \epsilon^{-N_4} \|\bar{v}\|_Z \|u_y\|_{L^2} \|\phi_y^{(n)} - \psi_y\|_{L^2} \xrightarrow{n\to\infty} 0.\end{aligned} \tag{3.15}$$

Upon passing to the limit, we integrate by parts:

$$-\int\int \epsilon^{\frac{n}{2}+\gamma}(\bar{v}u_y)\phi_y^{(n)} \xrightarrow{n\to\infty} \int\int \epsilon^{\frac{n}{2}+\gamma}(\bar{v}u_y)u = -\int\int \frac{\epsilon^{\frac{n}{2}+\gamma}}{2}\bar{v}_y u^2. \quad (3.16)$$

From here, we estimate:

$$\left|\int\int \frac{\epsilon^{\frac{n}{2}+\gamma}}{2}\bar{v}_y u^2\right| \leq \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)}\|\bar{u},\bar{v}\|_Z \|v_y x^{\frac{1}{2}}\|_{L^2}^2 \leq \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)}\|v_y x^{\frac{1}{2}}\|_{L^2}^2. \quad (3.17)$$

It remains to treat (3.10), for which we use the compact support of $\phi^{(n)}$ to justify the integration by parts:

$$\int\int \alpha A(\psi)\phi^{(n)} = \int\int (\psi\phi^{(n)}x^{2m} + \nabla\psi\nabla\phi^{(n)}x^{2m+2} + \nabla^2\psi : \nabla^2\phi^{(n)}x^{2m+4}). \quad (3.18)$$

Passing to the limit, according to (3.9):

$$\lim_{n\to\infty}\int\int \alpha A(\psi)\phi^{(n)} = \alpha\int\int (\psi^2 x^{2m} + |\nabla\psi|^2 x^{2m+2} + |\nabla^2\psi|^2 x^{2m+4}). \quad (3.19)$$

On the right-hand side, we have:

$$\int\int F_y\phi^{(n)} = -\int\int F\phi_y^{(n)} \xrightarrow{n\to\infty} \int\int Fu, \quad (3.20)$$

$$\int\int -\epsilon G_x\phi^{(n)} = \int\int \epsilon G\cdot\phi_x^{(n)} \xrightarrow{n\to\infty} \int\int \epsilon G v. \quad (3.21)$$

Consolidating the previous estimates gives the desired estimate, (3.7). □

The task now is to estimate the right-hand side of (3.7) in terms of the left-hand side using the smallness of $\mathcal{O}(\delta)$. We refer the reader to Proposition 3.4 in [4], whose proof we follow closely. We will point out the subtle differences.

Lemma 3.4 *Suppose $\psi \in H^2(\Omega^N)$ is a solution to (1.34), where $(F, G) \in H_2^{-1}$, and $\|\bar{u},\bar{v}\|_Z \leq 1$. Suppose the weight $w = x^{2m}$ from Eq. (1.29) is selected such that m is sufficiently large relative to universal constants. Then ψ obeys:*

$$\|\{\sqrt{\epsilon}v_x, v_y\}x^{\frac{1}{2}}\|_{L^2}^2 \lesssim \|u_y\|_{L^2}^2 + \alpha\|\psi\|_{H_w^2}^2 + \int\int (|F||u_x|x + \epsilon|G||v| + \epsilon|G||v_x|x). \quad (3.22)$$

Proof We apply the multiplier $\psi_x x\chi_{L,\alpha}^2$ to (1.34). Here, χ is a normalized cut-off function equal to 1 on [1, 2] and 0 on [3, ∞), and

$$\chi_{L,\alpha}(x) := \chi\left(\frac{\alpha}{L}x\right), \text{ so that } \partial_x^k \chi_{L,\alpha} = \frac{\alpha^k}{L^k}\chi^{(k)}. \tag{3.23}$$

The necessity of such a cut-off function here is due to the terms arising from $A(\psi)$. The presence of this cut-off function enables us to justify all integrations by parts in the x-direction. For our fixed $\alpha > 0$, we will eventually send $L \to \infty$. Applying the multiplier $\psi_x x\chi_{L,\alpha}^2$ to (1.34), gives on the left-hand side:

$$\begin{aligned} &\int\!\!\int \big(T[\psi] + \Delta_\epsilon^2\psi\big)\psi_x x\chi_{L,\alpha}^2 + \alpha\int\!\!\int A(\psi)\,\psi_x x\chi_{L,\alpha}^2 \\ &= \int\!\!\int \big(T_0[\psi] + \Delta_\epsilon^2\psi + \epsilon^{\frac{n}{2}+\gamma}\partial_y[\bar{v}u_y]\big)\psi_x x\chi_{L,\alpha}^2 \\ &\quad + \alpha\int\!\!\int A(\psi)\psi_x x\chi_{L,\alpha}^2. \end{aligned} \tag{3.24}$$

We will first focus on the first two integrands above. Let us start with the profile terms from $T_0[\psi]$. We will transfer all of the terms to velocity formulation.

$$\begin{aligned} \int\!\!\int \partial_y S_u \cdot vx\chi_{L,\alpha}^2 &= \int\!\!\int S_u u_x x\chi_{L,\alpha}^2 \\ &= \int\!\!\int [u_R u_x + u_{Rx}u + v_R u_y + u_{Ry}v]u_x x\chi_{L,\alpha}^2 \\ &\gtrsim \int\!\!\int u_x^2 x\chi_{L,\alpha}^2 - \left|\int\!\!\int [u_{Rx}u + v_R u_y + u_{Ry}v]u_x x\chi_{L,\alpha}^2\right|. \end{aligned} \tag{3.25}$$

We will treat the three terms on the right-hand side above, using (1.14) and (1.20) starting with:

$$\begin{aligned} \int\!\!\int u_{Rx}uxu_x\chi_{L,\alpha}^2 &= \int\!\!\int \{u_{Rx}^P + u_{Rx}^E\}uxu_x\chi_{L,\alpha}^2 \\ &\le \|yx^{\frac{1}{2}}u_{Rx}^P\|_{L^\infty}\|u_y\|_{L^2}\|v_y x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2} \\ &\quad + \|u_{Rx}^E x^{\frac{3}{2}}\|_{L^\infty}\left\|\frac{u}{x}\chi_{L,\alpha}\right\|_{L^2}\|u_x x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2} \\ &\le \mathcal{O}(\delta)\|u_y\|_{L^2}^2 + \mathcal{O}(\delta)\|u_x x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2}^2 + \frac{\alpha}{L}\|u\|_{L^2}^2 \\ &\le \mathcal{O}(\delta)\|u_y\|_{L^2}^2 + \mathcal{O}(\delta)\|u_x x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2}^2 + \frac{\alpha}{L}\|\psi\|_{H_w^2}^2. \end{aligned} \tag{3.26}$$

Above, we have used the Hardy inequality:

$$\begin{aligned} \left\| \frac{u}{x} \chi_{L,\alpha} \right\|_{L^2} &\lesssim \|\partial_x (u \chi_{L,\alpha})\|_{L^2} \leq \|u_x \chi_{L,\alpha}\|_{L^2} + \frac{\alpha}{L} \|u \chi'_{L,\alpha}\|_{L^2} \\ &\lesssim \|u_x \chi_{L,\alpha}\|_{L^2} + \frac{\alpha}{L} \|\psi\|_{H^2_w}. \end{aligned} \tag{3.27}$$

Next, by (1.12), (1.22), we have:

$$\begin{aligned} \left| \int \int v_R u_y v_y x \chi^2_{L,\alpha} \right| &\leq \|v_R x^{\frac{1}{2}}\|_{L^\infty} \|u_y\|_{L^2} \|v_y x^{\frac{1}{2}} \chi_{L,\alpha}\|_{L^2} \\ &\leq \mathcal{O}(\delta) \|u_y\|_{L^2} \|v_y x^{\frac{1}{2}} \chi_{L,\alpha}\|_{L^2}. \end{aligned} \tag{3.28}$$

Next, by (1.16), (1.18), (1.20) we have:

$$\begin{aligned} \int \int u_{Ry} v u_x x \chi^2_{L,\alpha} &= \int \int \{u^P_{Ry} + \sqrt{\epsilon} u^E_{RY}\} v v_y x \chi^2_{L,\alpha} \\ &\leq \|y u^P_{Ry}\|_{L^\infty} \|v_y x^{\frac{1}{2}} \chi_{L,\alpha}\|^2_{L^2} \\ &\quad + \sqrt{\epsilon} \|u^E_{RY} x^{\frac{3}{2}}\|_{L^\infty} \|v_y x^{\frac{1}{2}} \chi_{L,\alpha}\|_{L^2} \left\| \sqrt{\epsilon} \frac{v}{x} \chi_{L,\alpha} \right\|_{L^2} \\ &\leq \|y u^P_{Ry}\|_{L^\infty} \|v_y x^{\frac{1}{2}} \chi_{L,\alpha}\|^2_{L^2} \\ &\quad + \sqrt{\epsilon} \|u^E_{RY} x^{\frac{3}{2}}\|_{L^\infty} \|v_y x^{\frac{1}{2}} \chi_{L,\alpha}\|_{L^2} \|\sqrt{\epsilon} v_x \chi_{L,\alpha}\|_{L^2} \\ &\quad + \frac{\alpha}{L} \sqrt{\epsilon} \|u^E_{RY} x^{\frac{3}{2}}\|_{L^\infty} \|v_y x^{\frac{1}{2}} \chi_{L,\alpha}\|_{L^2} \|\sqrt{\epsilon} v \chi'_{L,\alpha}\|_{L^2} \\ &\leq \mathcal{O}(\delta) \|v_y x^{\frac{1}{2}} \chi_{L,\alpha}\|^2_{L^2} + \sqrt{\epsilon} \mathcal{O}(\delta) \|\sqrt{\epsilon} v_x \chi_{L,\alpha}\|^2_{L^2} + \mathcal{O}(\delta) \frac{\alpha}{L} \|\psi|^2_{H^2_w}. \end{aligned} \tag{3.29}$$

Summarizing the previous four terms:

$$\begin{aligned} \int \int \partial_y S_u \cdot v x \chi^2_{L,\alpha} &\gtrsim \|v_y x^{\frac{1}{2}} \chi_{L,\alpha}\|^2_{L^2} - \frac{\alpha}{L} \|\psi\|^2_{H^2_w} \\ &\quad - \mathcal{O}(\delta) \|u_y\|^2_{L^2} - \mathcal{O}(\delta) \sqrt{\epsilon} \|\sqrt{\epsilon} v_x \chi_{L,\alpha}\|^2_{L^2}. \end{aligned} \tag{3.30}$$

We will now move to the profile terms from S_v. First,

$$\int \int -\epsilon \partial_x S_v \cdot v x \chi^2_{L,\alpha} = \int \int \epsilon S_v \cdot \left[v_x x \chi^2_{L,\alpha} + v \chi^2_{L,\alpha} + v x \frac{2\alpha}{L} \chi_{L,\alpha} \chi'_{L,\alpha} \right]. \tag{3.31}$$

Referring to definition (1.5), consider the term $u_R v_x$ in S_v, which is the most delicate profile term:

$$\int\int \epsilon u_R v_x^2 x \chi_{L,\alpha}^2 + \int\int \epsilon u_R v_x v \Big[\chi_{L,\alpha} + 2x\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha}\Big]. \tag{3.32}$$

The first term above in (3.32) gives positivity:

$$\int\int \epsilon u_R v_x^2 x \chi_{L,\alpha}^2 \gtrsim \int\int \epsilon v_x^2 x \chi_{L,\alpha}^2. \tag{3.33}$$

We will treat the second term on the right-hand side of (3.32):

$$\int\int \epsilon u_R v v_x \chi_{L,\alpha}^2 = -\int\int \epsilon \frac{v^2}{2}\Big[u_{Rx}\chi_{L,\alpha}^2 + 2u_R\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha}\Big], \tag{3.34}$$

$$\int\int \epsilon u_R v v_x x\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha} = -\int\int \epsilon\frac{v^2}{2}\Big[u_{Rx}x\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha} + u_R\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha} + u_R x\frac{\alpha^2}{L^2}\chi_{L,\alpha}\chi''_{L,\alpha}\Big]. \tag{3.35}$$

The first term on the right-hand side of (3.34) yields:

$$\Big|\int\int \epsilon v^2 u_{Rx}\chi_{L,\alpha}^2\Big| \lesssim \sqrt{\epsilon}\|\chi_{L,\alpha}\{\sqrt{\epsilon}v_x, v_y\}x^{\frac{1}{2}}\|_{L^2}^2 + \frac{\alpha}{L}\|\psi\|_{H^2_w}^2. \tag{3.36}$$

We now estimate:

$$\Big|\int\int \epsilon\frac{u_R}{2}v^2\frac{\alpha}{L}\chi'_{L,\alpha}\Big| \lesssim \int\int \epsilon v^2\frac{\alpha}{L} \lesssim \frac{\alpha}{L}\|\psi\|_{H^2_w}^2. \tag{3.37}$$

This same estimate can be performed for all the terms in (3.35). Consolidating these bounds:

$$\int\int \epsilon v_x^2 x\chi_{L,\alpha}^2 \lesssim (3.32) + \sqrt{\epsilon}\|\chi_{L,\alpha}\{\sqrt{\epsilon}v_x, v_y\}x^{\frac{1}{2}}\|_{L^2}^2 + \frac{\alpha}{L}\|\psi\|_{H^2_w}. \tag{3.38}$$

It remains now to treat the remaining three terms in S_v. The second, third, and fourth terms from S_v are estimated immediately:

$$\epsilon\left|\int\int v_{Rx}uv_x x\chi_{L,\alpha}^2\right| \le \sqrt{\epsilon}\|x^{\frac{3}{2}}v_{Rx}\|_{L^\infty}\|\sqrt{\epsilon}v_x x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2}\|u_x\chi_{L,\alpha}\|_{L^2} + \frac{\alpha}{L}\|\psi\|_{H^2_w}^2. \tag{3.39}$$

$$\epsilon\left|\int\int v_R v_y v_x x\chi_{L,\alpha}^2\right| \le \sqrt{\epsilon}\|v_R\|_{L^\infty}\|v_y x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2}\|\sqrt{\epsilon}v_x x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2} + \frac{\alpha}{L}\|\psi\|_{H^2_w}^2. \tag{3.40}$$

$$\begin{aligned}\epsilon\left|\int\int v_{Ry} v v_x x\chi_{L,\alpha}^2\right| \le{}& \sqrt{\epsilon}\|v_{Ry}^P y\|_{L^\infty}\|v_y x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2}\|\sqrt{\epsilon}v_x x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2}\\ &+ \sqrt{\epsilon}\|v_{RY}^E x^{\frac{3}{2}}\|_{L^\infty}\|\sqrt{\epsilon}v_x x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2}^2 + \frac{\alpha}{L}\|\psi\|_{H^2_w}^2.\end{aligned} \tag{3.41}$$

We now turn back to (3.31), addressing the second term in the bracket for the final three profile terms from S_v:

$$\int\int [v_{Rx}u + v_R v_y + v_{Ry}v]\cdot \epsilon v\chi_{L,\alpha}^2. \tag{3.42}$$

First, through the Hardy inequality and (1.10), (1.19):

$$\begin{aligned}\left|\int\int \epsilon v_{Rx}uv\chi_{L,\alpha}^2\right| &\le \sqrt{\epsilon}\|x^{\frac{3}{2}}v_{Rx}\|_{L^\infty}\left\|\frac{u}{x^{\frac{3}{4}}}\chi_{L,\alpha}\right\|_{L^2}\left\|\sqrt{\epsilon}\frac{v}{x^{\frac{3}{4}}}\chi_{L,\alpha}\right\|_{L^2}\\ &\le \sqrt{\epsilon}\left[\|u_x x^{\frac{1}{4}}\chi_{L,\alpha}\|_{L^2}^2 + \|\sqrt{\epsilon}v_x x^{\frac{1}{4}}\chi_{L,\alpha}\|_{L^2}^2 + \frac{\alpha}{L}\|\{u,\sqrt{\epsilon}vx^{\frac{1}{4}}\chi_{L,\alpha}'\}\|_{L^2}^2\right]\\ &\le \sqrt{\epsilon}\left[\|u_x x^{\frac{1}{4}}\chi_{L,\alpha}\|_{L^2}^2 + \|\sqrt{\epsilon}v_x x^{\frac{1}{4}}\chi_{L,\alpha}\|_{L^2}^2 + \frac{\alpha}{L}\|\psi\|_{H^2_w}^2\right],\end{aligned} \tag{3.43}$$

so long as $w = x^m$ is selected larger than $x^{\frac{1}{4}}$, which is true by the assumption of this lemma. Next, through an integration by parts and (1.12), (1.22):

$$\begin{aligned}\int\int [v_R v_y + v_{Ry}v]\epsilon v\chi_{L,\alpha}^2 = \frac{1}{2}\int\int v_{Ry}v^2\epsilon\chi_{L,\alpha}^2 \le{}& \epsilon\|v_{Ry}^P y^2\|_{L^\infty}\|v_y\chi_{L,\alpha}\|_{L^2}^2\\ &+ \sqrt{\epsilon}\|v_{RY}^E x^{\frac{3}{2}}\|_{L^\infty}\|\sqrt{\epsilon}v_x x^{\frac{1}{4}}\chi_{L,\alpha}\|_{L^2}^2 + \frac{\alpha}{L}\|\psi\|_{H^2_w}^2.\end{aligned} \tag{3.44}$$

The final task for the S_v profile contributions is the third term from (3.31):

$$\left|\int\int \epsilon[v_{Rx}u + v_R v_y + v_{Ry}v]\cdot vx\frac{2\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}'\right| \le \frac{\alpha}{L}\|\psi\|_{H^2_w}^2, \tag{3.45}$$

so long as $w = x^m$ is selected larger than x, which is true by assumption of this lemma. Let us consolidate all of the calculations from S_v:

$$\int\int -\epsilon\partial_x S_v \cdot vx\chi_{L,\alpha}^2 \gtrsim \int\int \epsilon v_x^2 x\chi_{L,\alpha}^2 - \sqrt{\epsilon}\|\chi_{L,\alpha}\{\sqrt{\epsilon}v_x, v_y\}x^{\frac{1}{2}}\|_{L^2}^2 - \frac{\alpha}{L}\|\psi\|_{H_w^2}^2. \tag{3.46}$$

It now remains to come to those terms contributed by $\Delta_\epsilon^2\psi$ into (3.24). Again, we will write these terms in the velocity form. First,

$$\left|\int\int -u_{yy}u_x x\chi_{L,\alpha}^2\right| = \left|-\int\int \frac{u_y^2}{2}\left[\chi_{L,\alpha}^2 + 2x\frac{\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}'\right]\right| \lesssim \|u_y\|_{L^2}^2. \tag{3.47}$$

Next,

$$\begin{aligned}\left|\int\int -\epsilon u_{xx}u_x x\chi_{L,\alpha}^2\right| &= \left|\int\int \epsilon\frac{u_x^2}{2}\left[\chi_{L,\alpha}^2 + 2x\frac{\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}'\right]\right| \\ &\lesssim \epsilon\|u_x\chi_{L,\alpha}\|_{L^2}^2 + \epsilon\frac{\alpha}{L}\|\psi\|_{H_w^2}^2.\end{aligned} \tag{3.48}$$

We now move to the terms from $\Delta_\epsilon v$, starting with:

$$\begin{aligned}\left|\int\int \epsilon^2 v_{xxx}vx\chi_{L,\alpha}^2\right| &= \left|\int\int \epsilon^2 v_{xx}\left[v_x x\chi_{L,\alpha}^2 + v\chi_{L,\alpha}^2 + 2vx\frac{\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}'\right]\right| \\ &\leq \epsilon\|\sqrt{\epsilon}v_x\chi_{L,\alpha}\|_{L^2}^2 + \frac{\alpha}{L}\|\psi\|_{H_w^2}^2.\end{aligned} \tag{3.49}$$

Finally, we have:

$$\begin{aligned}\left|\int\int -\epsilon v_{xyy}vx\chi_{L,\alpha}^2\right| &= \left|\int\int \epsilon v_{xy}v_y x\chi_{L,\alpha}^2\right| \\ &= \left|\int\int \epsilon v_y^2\left[\chi_{L,\alpha}^2 + 2x\frac{\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}\right]\right| \\ &\lesssim \epsilon\|v_y\chi_{L,\alpha}\|_{L^2}^2 + \frac{\alpha}{L}\|\psi\|_{H_w^2}^2.\end{aligned} \tag{3.50}$$

By combining calculations (3.30), (3.46), (3.47)–(3.50), and absorbing relevant terms to the left-hand side below, we have:

$$\|\{\sqrt{\epsilon}v_x, v_y\}x^{\frac{1}{2}}\chi_{L,\alpha}\|_{L^2}^2 \lesssim \|u_y\|_{L^2}^2 + \frac{\alpha}{L}\|\psi\|_{H^2_w}^2 + \alpha\int\int A(\psi)\psi_x x\chi_{L,\alpha}^2 + \int\int \epsilon^{\frac{n}{2}+\gamma}\bar{v}u_y u_x x\chi_{L,\alpha}^2 + \int\int (F_y v x\chi_{L,\alpha}^2 - \epsilon G_x v x\chi_{L,\alpha}^2). \tag{3.51}$$

Via direct integration by parts, which is justified due to the presence of the cut-off function in x, we compute:

$$\left|\alpha\int\int A(\psi)\psi_x x\chi_{L,\alpha}^2\right| \lesssim \alpha\|\psi\|_{H^2_w}^2. \tag{3.52}$$

Let us compute each term in $A(\psi)$ to verify (3.52), referring to the definition in (1.29), starting with:

$$\left|\int\int \alpha\psi x^{2m}vx\chi_{L,\alpha}^2\right| = \left|-\frac{\alpha}{2}\int\int \psi^2\partial_x[x^{2m+1}\chi_{L,\alpha}^2]\right|$$
$$= \left|-\frac{\alpha}{2}\int\int \psi^2\left[Cx^{2m}\chi_{L,\alpha}^2 + 2x^{2m+1}\frac{\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}'\right]\right| \tag{3.53}$$
$$\leq \alpha\|\psi\|_{H^2_w}^2. \tag{3.54}$$

For the second term in (3.53), we have used: $|\frac{\alpha}{L}x\chi_{L,\alpha}| \lesssim 1$. Next, let us turn to:

$$\begin{aligned}\alpha\int\int(-\psi_{yy}x^{2m+2} + \psi_{yyyy}x^{2m+4})vx\chi_{L,\alpha}^2)\\ &= \alpha\int\int(u_y vx^{2m+3}\chi_{L,\alpha}^2 - \alpha u_y u_{xy}x^{2m+5}\chi_{L,\alpha}^2)\\ &= \alpha\int\int[uu_x x^{2m+3}\chi_{L,\alpha}^2 + \alpha u_y^2\partial_x(x^{2m+5}\chi_{L,\alpha}^2]\\ &\lesssim \alpha\int\int(u^2x^{2m+2} + u_y^2x^{2m+4}) \lesssim \alpha\|\psi\|_{H^2_w}^2.\end{aligned} \tag{3.55}$$

Next,

$$\begin{aligned}\alpha\int\int\partial_x(\psi_x x^{2m+2})vx\chi_{L,\alpha}^2 &= \alpha\int\int vx^{2m+2}\partial_x(vx\chi_{L,\alpha}^2)\\ &= \alpha\int\int vx^{2m+2}\left[v_x x\chi_{L,\alpha}^2 + v\chi_{L,\alpha}^2 + 2vx\frac{\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}'\right]\\ &\lesssim \alpha\int\int v^2x^{2m+2} \lesssim \alpha\|\psi\|_{H^2_w}^2.\end{aligned} \tag{3.56}$$

Next,

$$\alpha \int\int \partial_x(\psi_{yyx} x^{2m+4}) v x \chi_{L,\alpha}^2 = \alpha \int\int \partial_x(\psi_{xy} x^{2m+4}) v_y x \chi_{L,\alpha}^2$$
$$= \alpha \int\int \partial_x(u_x x^{2m+4}) u_x x \chi_{L,\alpha}^2$$
$$\lesssim \alpha \int\int u_x^2 x^{2m+4} \lesssim \alpha \|\psi\|_{H^2_w}^2. \tag{3.57}$$

The final term in $A(\psi)$ is:

$$\alpha \int\int \partial_{xx}(\psi_{xx} x^{2m+4}) v x \chi_{L,\alpha}^2 = \alpha \int\int \psi_{xx} x^{2m+4} \partial_{xx}[v x \chi_{L,\alpha}^2]$$
$$= \alpha \int\int v_x x^{2m+4} [v_{xx} x \chi_{L,\alpha}^2 + 2 v_x \partial_x(x \chi_{L,\alpha}) + v \partial_{xx}(x \chi_{L,\alpha})]$$
$$\lesssim \alpha \|\psi\|_{H^2_w}^2. \tag{3.58}$$

This concludes all the terms in $A(\psi)$, according to (1.29). Estimating the next term in (3.51) yields:

$$\left| \int\int \epsilon^{\frac{n}{2}+\gamma} \bar{v} u_y u_x x \chi_{L,\alpha}^2 \right| \leq \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)} \|\bar{u}, \bar{v}\|_Z \|u_y\|_{L^2} \|u_x x^{\frac{1}{2}} \chi_{L,\alpha}\|_{L^2}$$
$$\leq \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)} \|u_x x^{\frac{1}{2}} \chi_{L,\alpha}\|_{L^2}^2 + \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)} \|u_y\|_{L^2}^2. \tag{3.59}$$

Finally, we come to the right-hand side:

$$\int\int [F_y - \epsilon G_x] \cdot v x \chi_{L,\alpha} = \int\int (F u_x x \chi_{L,\alpha} + \epsilon G \partial_x[v x \chi_{L,\alpha}])$$
$$\leq \int\int |F||u_x| x + \left| \int\int \epsilon G \left[v_x x \chi_{L,\alpha} + v \chi_{L,\alpha} + v x \left(\frac{\alpha}{L}\right) \chi_{L,\alpha}' \right] \right|$$
$$\leq \int\int (|F||u_x| x + \epsilon |G||v_x| x + \epsilon |G||v|). \tag{3.60}$$

Inserting the previous few calculations into estimate (3.51) gives:

$$\|\{\sqrt{\epsilon} v_x, v_y\} x^{\frac{1}{2}} \chi_{L,\alpha}\|_{L^2}^2 \lesssim \|u_y\|_{L^2}^2 + \frac{\alpha}{L} \|\psi\|_{H^2_w}^2 + \alpha \|\psi\|_{H^2_w}^2$$
$$+ \int\int (|F||u_x| x + \epsilon |G| [|v_x| x + |v|]). \tag{3.61}$$

We now send $L \to \infty$, and appeal to Monotone Convergence Theorem, as $\chi_{L,\alpha} \uparrow 1$ to establish the desired result. □

Having understood the inhomogeneous problem:

Lemma 3.5 *For* $(F, G) \in H_2^{-1}$*, and* $\|\bar{u}, \bar{v}\|_Z \le 1$*, there exists a unique weak solution* $\psi \in H_w^2(\Omega^N)$ *to the system* (1.34).

Proof We apply S_α^{-1} to both sides of (1.34), which is valid as the right-hand side and therefore the left-hand side is assumed to be in at least $H^{-1}(\Omega^N)$, thereby yielding:

$$\psi + S_\alpha^{-1} T\psi = S_\alpha^{-1}(F_y - \epsilon G_x). \tag{3.62}$$

We will study Eq. (3.62) as an equality in the space $H^2(\Omega^N)$. According to the Fredholm alternative, which is available according to Lemma 3.2, there either exists a unique solution $\psi \in H^2(\Omega^N)$ to the system (3.62), or a non-trivial solution $\psi \in H^2(\Omega^N)$ to:

$$\psi + S_\alpha^{-1} T\psi = 0 \iff S_\alpha \psi = -T\psi. \tag{3.63}$$

Therefore, coupling (3.22) with (3.7), taking $F = G = 0$, we have:

$$\|\sqrt{\epsilon}u_x, u_y\|_{L^2}^2 + \alpha\|\psi\|_{H_w^2}^2 + \|\sqrt{\epsilon}v_x x^{\frac{1}{2}}, v_y x^{\frac{1}{2}}\|_{L^2}^2 \le 0, \tag{3.64}$$

implying $\psi, u, v = 0$. Thus, by the Fredholm alternative, there exists a unique solution $\psi \in H^2(\Omega^N)$ to (3.62). Rearranging (3.62):

$$\psi = S_\alpha^{-1}(F_y - \epsilon G_x - T\psi), \tag{3.65}$$

where $F_y - \epsilon G_x - T\psi \in H^{-1}(\Omega^N)$, and so an application of (2.13) shows that $\psi \in H_w^2(\Omega^N)$. This concludes the proof. □

Lemma 3.6 *Let* ψ *be the unique* $H_w^2(\Omega^N)$ *weak solution from Lemma* 3.5*. Then for* $(F, G) \in H_2^{-1}$, $\psi \in H_w^5(\Omega^N)$.

Proof $T\psi \in H^{-1}(\Omega^N)$, so $S_\alpha \psi = -T\psi + F_y - \epsilon G_x \in H^{-1}(\Omega^N)$, which implies that $\psi \in H_w^3(\Omega^N)$ according to (2.15). Iterating this regularity then gives $\psi \in H_w^5(\Omega^N)$. □

We now introduce more notations, which are more suitable for the velocities:

$$L_{\alpha,\bar{v}}[u, v] = (\tilde{f}, g) \iff S_\alpha \psi + T[\psi; \bar{v}] = \tilde{f}_y - \epsilon g_x. \tag{3.66}$$

Summarizing the established results, we have:

Corollary 3.7 *For $\tilde{f}, g \in H_2^{-1}(\Omega^N), \alpha > 0$, and $\|\bar{u}, \bar{v}\|_{Z(\Omega^N)} \le 1$, the map $L_{\alpha,\bar{v}}[u, v]$ is invertible, where*

$$L_{\alpha,\bar{v}}^{-1} : (\tilde{f}, g) \in H_2^{-1}(\Omega^N) \to [u, v] \in H_w^4(\Omega^N). \tag{3.67}$$

Moreover, the boundary conditions (1.27) *are satisfied by* $[u, v] = L_{\alpha,\bar{v}}^{-1}[\tilde{f}, g]$.

It is now our intention to obtain second- and third-order energy and positivity estimates for our new system (1.34). For this, we will need to understand several calculations. First, we introduce some norms:

$$\|\psi\|_{J^2}^2 := \|\psi\|_{H_w^2}^2, \tag{3.68}$$

$$\begin{aligned} \|\psi\|_{J^{k+2}}^2 := \int\int (|\partial_x^k \psi|^2 (\rho_{k+1})^{2k} x^{2m+2k} + |\nabla \partial_x^k \psi|^2 \rho_{k+1}^{2k} x^{2m+2k+2} \\ + |\nabla^2 \partial_x^k \psi|^2 \rho_{k+1}^{2k} x^{2m+2k+4}) \quad \text{for } k \ge 1. \end{aligned} \tag{3.69}$$

The essential difference between these J^k-norms and the H_w^k norms introduced in (2.2) are the growing weights of x which each application of ∂_x, which mimics the structure of the energy norms, X_k.

Lemma 3.8

$$\int\int A(\partial_x^k \psi) \cdot \partial_x^k \psi x^{2k} \chi_{L,\alpha}^2 \rho_{k+1}^{2k} \gtrsim \|\chi_{L,\alpha} \psi\|_{J^{k+2}}^2 - \sum_{i=0}^{k-1} \|\psi\|_{J^{i+2}}^2. \tag{3.70}$$

Proof Referring to (1.29), the first term is:

$$\int\int |\partial_x^k \psi|^2 x^{2m} x^{2k} \chi_{L,\alpha}^2 \rho_{k+1}^{2k}. \tag{3.71}$$

The next terms, via an integration by parts in y:

$$\int\int -\partial_x^k \psi_{yy} x^{2m+2} \cdot \partial_x^k \psi x^{2k} \rho_{k+1}^{2k} \chi_{L,\alpha}^2 = \int\int |\partial_x^k \psi_y|^2 x^{2m+2k+2} \rho_{k+1}^{2k} \chi_{L,\alpha}^2, \tag{3.72}$$

$$\int\int \partial_x^k \psi_{yyyy} x^{2m+4} \cdot \partial_x^k \psi x^{2k} \rho_{k+1}^{2k} \chi_{L,\alpha}^2 = \int\int |\partial_x^k \psi_{yy}|^2 x^{2m+2k+4} \rho_{k+1}^{2k} \chi_{L,\alpha}^2. \tag{3.73}$$

Next,

$$
\begin{aligned}
&\int\int \partial_x[(\partial_x^k\psi)_{yyx}x^{2m+4}]\cdot\partial_x^k\psi x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}\\
&\quad=\int\int \partial_x^{k+1}\psi_y x^{2m+4}\cdot\partial_x[\partial_x^k\psi_y x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}]\\
&\quad\gtrsim \int\int |\partial_x^{k+1}\psi_y|^2 x^{2m+2k+4}\chi_{L,\alpha}^2\rho_{k+1}^{2k}-\int\int |\partial_x^k\psi_y|^2x^{2m+2k+2}\rho_k^{2(k-1)}\\
&\quad\gtrsim \int\int |\partial_x^{k+1}\psi_y|^2 x^{2m+2k+4}\chi_{L,\alpha}^2\rho_{k+1}^{2k}-\sum_{i=0}^{k-1}\|\psi\|_{J^{i+2}}.
\end{aligned}
\tag{3.74}
$$

Above, we have used the calculation:

$$
\begin{aligned}
\partial_x[x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}]&=2kx^{2k-1}\chi_{L,\alpha}^2\rho_{k+1}^{2k}+x^{2k}\frac{2\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}'\rho_{k+1}^{2k}\\
&\quad+x^{2k}\chi_{L,\alpha}^2 2k\rho_{k+1}^{2k-1}\rho_{k+1}'.
\end{aligned}
\tag{3.75}
$$

For the second term on the right-hand side of (3.75), we estimate: $\frac{2\alpha}{L}x\chi_{L,\alpha}\lesssim 1$. For the third term on the right-hand side, we use that the support of ρ_{k+1}' is localized in x. We also use that: support$(\rho_k)\subset\{\rho_{k-1}=1\}$. Next, we integrate by parts twice in x to obtain:

$$
\begin{aligned}
&\int\int \partial_{xx}[(\partial_x^k\psi)_{xx}x^{2m+4}]\cdot\partial_x^k\psi x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}\\
&\quad=\int\int \partial_x^{k+2}\psi x^{2m+4}\partial_{xx}[\partial_x^k\psi x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}]\\
&\quad=\int\int |\partial_x^{k+2}\psi|^2x^{2m+2k+4}\chi_{L,\alpha}^2\rho_{k+1}^{2k}\\
&\qquad+\int\int \partial_x^{k+2}\psi x^{2m+4}\partial_x^{k+1}\psi\partial_x[x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}]\\
&\qquad+\int\int \partial_x^{k+2}\psi x^{2m+4}\partial_x^{k}\psi\partial_{xx}[x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}].
\end{aligned}
\tag{3.76}
$$

The final two terms on the right-hand side of (3.77) are estimated through further integrations by parts:

$$
\begin{aligned}
&\Big|\int\int \partial_x^{k+2}\psi x^{2m+4}\partial_x^{k+1}\psi\partial_x[x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}]\\
&\qquad+\int\int \partial_x^{k+2}\psi x^{2m+4}\partial_x^{k}\psi\partial_{xx}[x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}]\Big|\\
&\quad\lesssim\|\psi\|_{J^{k+1}}^2.
\end{aligned}
\tag{3.77}
$$

Finally,

$$
\begin{aligned}
& -\int\int \partial_x[(\partial_x^k\psi)_x x^{2m+2}]\cdot\partial_x^k\psi x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k} \\
& \quad = \int\int \partial_x^{k+1}\psi x^{2m+2}\partial_x[\partial_x^k\psi x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}] \\
& \quad = \int\int |\partial_x^{k+1}\psi|^2 x^{2m+2+2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k} \\
& \qquad + \int\int \partial_x^{k+1}\psi x^{2m+2}\partial_x^k\psi\partial_x[x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}] \\
& \quad \gtrsim \int\int |\partial_x^{k+1}\psi|^2 x^{2m+2+2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k} - \|\psi\|_{J^{k+1}}^2.
\end{aligned} \tag{3.78}
$$

Piecing all of the above estimates together yields the desired bound. □

Lemma 3.9

$$
\left|\int\int A(\partial_x^k\psi)\cdot\partial_x^{k+1}\psi x^{2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}\right| \lesssim \sum_{i=0}^k \|\psi\|_{J^{i+2}}^2. \tag{3.79}
$$

Proof Again, referring to definition (1.29), we will proceed term by term, starting with the following, for which we integrate by parts once:

$$
\left|\int\int \partial_x^k\psi x^{2m}\cdot\partial_x^{k+1}\psi x^{2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}\right| \tag{3.80}
$$

$$
= -\frac{1}{2}\int\int |\partial_x^k\psi|^2\partial_x[x^{2m+2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}]. \tag{3.81}
$$

Let us expand the product rule above:

$$
\begin{aligned}
\partial_x[x^{2m+2k+1}\chi_{L,\alpha}^2\rho_k^{2k+1}] = & \, Cx^{2m+2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1} + Cx^{2m+2k+1}\frac{\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}'\rho_{k+1}^{2k+1} \\
& + x^{2m+2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k}\rho_{k+1}' \lesssim x^{2m+2k}\rho_{k+1}^{2k}.
\end{aligned} \tag{3.82}
$$

Thus, the term (3.80) can be controlled via:

$$
|(3.80)| \lesssim \int\int |\partial_x^k\psi|^2 x^{2m+2k}\rho_{k+1}^{2k} \lesssim \sum_{i=0}^k \|\psi\|_{J^{i+2}}^2. \tag{3.83}
$$

The second term in (1.29) is treated via:

$$\begin{aligned}
&\int\int -\partial_x(\partial_x^{k+1}\psi x^{2m+2}) \cdot \partial_x^{k+1}\psi x^{2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1} \\
&\quad = \int\int [-\partial_x^{k+2}\psi x^{2m+2} - C\partial_x^{k+1}\psi x^{2m+1}] \cdot \partial_x^{k+1}\psi x^{2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1} \\
&\quad = \int\int \left(|\partial_x^{k+1}\psi|^2\partial_x[x^{2m+2k+3}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}] - C|\partial_x^{k+1}\psi|^2 x^{2m+2k+2}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}\right) \qquad (3.84) \\
&\quad \lesssim \int\int |\partial_x^{k+1}\psi|^2 x^{2m+2k+2}\rho_{k+1}^{2k} \lesssim \sum_{i=0}^{k} \|\psi\|_{J^{i+2}} \qquad (3.85)
\end{aligned}$$

We have expanded the product in the first term on the right-hand side of (3.84):

$$\begin{aligned}
\partial_x[x^{2m+2k+3}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}] &= Cx^{2m+2k+2}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1} + Cx^{2m+2k+3}\left(\frac{\alpha}{L}\right)\chi_{L,\alpha}\chi_{L,\alpha}'\rho_{k+1}^{2k+1} \\
&\quad + Cx^{2m+2k+3}\chi_{L,\alpha}^2\chi_{k+1}^{2k+1}\rho_{k+1}^{2k}\rho_{k+1}' \\
&\lesssim x^{2m+2k+2}\rho_{k+1}^{2k}.
\end{aligned} \qquad (3.86)$$

Next, we have:

$$\int\int -\partial_x^k\psi_{yy}x^{2m+2} \cdot \partial_x^{k+1}\psi x^{2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1} = \int\int |\partial_x^k\psi_y|^2\partial_x[x^{2m+2k+3}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}]. \qquad (3.87)$$

We will expand the product rule above:

$$\begin{aligned}
&\partial_x[x^{2m+2k+3}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}] \\
&\quad = Cx^{2m+2k+2}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1} + C\frac{\alpha}{L}x^{2m+2k+3}\chi_{L,\alpha}\chi_{L,\alpha}'\rho_{k+1}^{2k+1} \qquad (3.88) \\
&\quad + Cx^{2m+2k+3}\chi_{L,\alpha}^2\rho_{k+1}^{2k}\rho_k' \lesssim x^{2k+2m+2}\rho_{k+1}^{2k}.
\end{aligned}$$

Inserting this above yields: $|(3.87)| \lesssim \sum_{i=0}^{k} \|\psi\|_{J^{i+2}}^2$. Next, after two integrations by parts in y, and one in x:

$$\int\int \partial_x^k\psi_{yyyy}x^{2m+4} \cdot \partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2 = \int\int |\partial_x^k\psi_{yy}|^2\partial_x[x^{2m+2k+5}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2]. \qquad (3.89)$$

Expanding the product rule above yields:

$$\begin{aligned}&\partial_x[x^{2m+2k+5}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2]\\&\quad= Cx^{2m+2k+4}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2 + Cx^{2m+2k+5}\rho_{k+1}^{2k}\rho_{k+1}'\chi_{L,\alpha}^2\\&\qquad+ Cx^{2m+2k+5}\rho_{k+1}^{2k+1}\chi_{L,\alpha}\chi_{L,\alpha}'\frac{\alpha}{L} \lesssim x^{2m+2k+4}\rho_{k+1}^{2k}.\end{aligned} \tag{3.90}$$

Inserting above yields: $|(3.89)| \lesssim \sum_{i=0}^{k} \|\psi\|_{J^{i+2}}^2$. The next term from $A(\psi)$ in definition (1.29) is:

$$\begin{aligned}&\int\int \partial_x[(\partial_x^k\psi)_{yyx}x^{2m+4}]\cdot\partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2\\&\quad= -\int\int \partial_x^{k+1}\psi_y x^{2m+4}\cdot\partial_x[\partial_x^{k+1}\psi_y x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2]\\&\quad= -\int\int \partial_x^{k+1}\psi_y x^{2m+4}\cdot\partial_x^{k+2}\psi_y x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2\\&\qquad-\int\int |\partial_x^{k+1}\psi_y|^2 x^{2m+4}\partial_x[x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2]\\&\quad= \int\int |\partial_x^{k+1}\psi_y|^2\partial_x[x^{2m+2k+5}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2]\\&\qquad-\int\int |\partial_x^{k+1}\psi_y|^2 x^{2m+4}\partial_x[x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2]\\&\quad\lesssim \int\int |\partial_x^{k+1}\psi_y|^2 x^{2m+2k+4}\rho_{k+1}^{2k} \lesssim \sum_{i=0}^{k}\|\psi\|_{J^{i+2}}^2.\end{aligned} \tag{3.91}$$

The final term from $A(\psi)$ in definition (1.29) is:

$$\begin{aligned}&\int\int \partial_{xx}(\partial_x^{k+2}\psi x^{2m+4})\cdot\partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2\\&\quad= -\int\int \partial_x^{k+2}\psi x^{2m+4}\cdot\partial_{xx}[\partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2]\\&\quad= -\int\int \partial_x^{k+2}\psi x^{2m+4}\cdot\partial_x^{k+3}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2\\&\qquad-\int\int |\partial_x^{k+2}\psi|^2 x^{2m+4}\cdot\partial_x[x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2]\\&\qquad-\int\int |\partial_x^{k+1}\psi|^2\partial_{xxx}[x^{2m+2k+5}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2]\\&\quad\lesssim \sum_{i=0}^{k}\|\psi\|_{J^{i+2}}^2.\end{aligned} \tag{3.92}$$

This concludes the proof of the desired estimate, (3.79). □

Lemma 3.10

$$\Big| \int\int [\partial_x^k, A]\psi \cdot \partial_x^k \psi x^{2k} \chi_{L,\alpha}^2 \rho_{k+1}^{2k} \Big| \lesssim \sum_{i=0}^{k-1} \|\psi\|_{J^{i+2}}^2. \tag{3.93}$$

Proof To keep notations simple, we will prove the $k = 1$ case, with the $k \geq 2$ cases following identically. We will proceed term by term from the commutator expression in (2.17). First,

$$\begin{aligned} \int\int \psi x^{2m-1} \cdot \psi_x x^2 \chi_{L,\alpha}^2 \rho_2^2 &= -\frac{1}{2} \int\int |\psi|^2 \partial_x [x^{2m+1} \chi_{L,\alpha}^2 \rho_2^2] \\ &\lesssim \int\int \psi^2 x^{2m} \lesssim \|\psi\|_{J^2}^2. \end{aligned} \tag{3.94}$$

Next,

$$\begin{aligned} \int\int -\psi_{yy} x^{2m+1} \cdot \psi_x x^2 \chi_{L,\alpha}^2 \rho_2^2 &= \int\int \psi_y^2 \partial_x [x^{2m+3} \chi_{L,\alpha}^2 \rho_2^2] \\ &\lesssim \int\int \psi_y^2 x^{2m+2} \lesssim \|\psi\|_{J^2}^2. \end{aligned} \tag{3.95}$$

Next,

$$\int\int -\partial_x(\psi_x x^{2m+1}) \cdot \psi_x x^2 \chi_{L,\alpha}^2 \rho_2^2 \lesssim \int\int \psi_x^2 x^{2m+2} \lesssim \|\psi\|_{J^2}^2. \tag{3.96}$$

We will now move to the higher-order terms, starting with:

$$\begin{aligned} \int\int \psi_{yyyy} x^{2m+3} \cdot \psi_x x^2 \chi_{L,\alpha}^2 \rho_2^2 &= \int\int \psi_{yy} \psi_{yyx} x^{2m+5} \chi_{L,\alpha}^2 \rho_2^2 \\ &= -\frac{1}{2} \int\int |\psi_{yy}|^2 \partial_x [x^{2m+5} \chi_{L,\alpha}^2 \rho_2^2] \lesssim \|\psi\|_{J^2}^2. \end{aligned} \tag{3.97}$$

Next, again integrating by parts several times:

$$\begin{aligned} &\int\int \partial_x [\psi_{yyx} x^{2m+3}] \cdot \psi_x x^2 \chi_{L,\alpha}^2 \rho_2^2 \\ &= \int\int \psi_{xy}^2 \big[x^{2m+3} \partial_x [x^2 \chi_{L,\alpha}^2 \rho_2^2] - \partial_x [x^{2m+5} \chi_{L,\alpha}^2 \rho_2^2] \big] \lesssim \|\psi\|_{J^2}^2. \end{aligned} \tag{3.98}$$

The final term from $A(\psi)$, which after integrating by parts several times in the same way as above,

$$\int\int \partial_{xx}[\psi_{xx}x^{2m+3}] \cdot \psi_x x^2 \chi_{L,\alpha}^2 \rho_2^2 \lesssim \int\int \|\psi\|_{J^2}^2. \tag{3.99}$$

This concludes the proof of (3.93). □

Lemma 3.11

$$\left|\int\int [\partial_x^k, A]\psi \cdot \partial_x^{k+1}\psi x^{2k+1} \chi_{L,\alpha}^2 \rho_{k+1}^{2k+1}\right| \lesssim \sum_{i=0}^{k} \|\psi\|_{J^{i+2}}. \tag{3.100}$$

Proof This estimate proceeds in the same manner as those from (3.93), with the adjustment that the extra derivative in the multiplier from (3.100) is accounted for by the increment in order on the right-hand sides of (3.100) versus (3.93). Indeed, let us take the highest order term from the commutator, $[\partial_x, A]\psi$:

$$\left|\int\int \partial_{xx}(\psi_{xx}x^{2m+3}) \cdot \psi_{xx} x^3 \chi_{L,\alpha}^2 \rho_2^3\right| \lesssim \int\int |\psi_{xxx}|^2 x^{2m+6} \chi_{L,\alpha}^2 \rho_2^3 + \|\psi\|_{J^2}^2. \tag{3.101}$$

The first term on the right-hand side above can be controlled by $\|\psi\|_{J^3}^2$, as can be seen from a comparison to (3.69) with $k = 1$. The remaining terms work identically. □

Using the above calculations, we may repeat the energy and positivity estimates, for $k \geq 1$.

Lemma 3.12 (($k + 1$)'th order Auxiliary Energy Estimate) *Let* $k = 1, 2$. *Then,*

$$\begin{aligned} &\|\partial_x^k u_y \cdot (\rho_{k+1}x)^k\|_{L^2}^2 + \alpha\|\psi\|_{J^{k+2}}^2 \\ &\lesssim \alpha \sum_{i=0}^{k-1} \|\psi\|_{J^{i+2}}^2 + \mathcal{O}(\delta)\|\partial_x^k\{\sqrt{\epsilon}v_x, v_y\}x^{k+\frac{1}{2}}\rho_{k+1}^{k+\frac{1}{2}}\|_{L^2}^2 + \mathcal{W}_1 + \sum_{i=1}^{k} \mathcal{W}_{i+1}. \end{aligned} \tag{3.102}$$

Proof We apply the operator ∂_x^k to the system (1.34):

$$\Delta_\epsilon \partial_x^k \psi + \partial_x^k T[\psi] + \alpha A(\partial_x^k \psi) + \alpha[\partial_x^k, A]\psi = \partial_x^k\{F_y - \epsilon G_x\}. \tag{3.103}$$

We subsequently apply the multiplier $\partial_x^k \psi x^{2k} \rho_{k+1}^{2k} \chi_{L,\alpha}^2$:

$$\begin{aligned} &\int\int [\Delta_\epsilon \partial_x^k \psi + \partial_x^k T[\psi] + \alpha A(\partial_x^k \psi) + \alpha[\partial_x^k, A]\psi] \cdot \partial_x^k \psi x^{2k} \rho_{k+1}^{2k} \chi_{L,\alpha}^2 \\ &= \int\int [\partial_x^k\{F_y - \epsilon G_x\}] \cdot \partial_x^k \psi x^{2k} \rho_{k+1}^{2k} \chi_{L,\alpha}^2. \end{aligned} \tag{3.104}$$

The desired estimate now follows using similar calculations as in Lemma 3.3. □

Lemma 3.13 ((k + 1)'th order Auxiliary Positivity Estimate)

$$\|\partial_x^k\{\sqrt{\epsilon}v_x, v_y\}x^{k+\frac{1}{2}}\rho_{k+1}^{k+\frac{1}{2}}\|_{L^2}^2 \lesssim \|\partial_x^k u_y \cdot (\rho_{k+1}x)^k\|_{L^2}^2 + \alpha\sum_{i=0}^{k}\|\psi\|_{J^{i+2}}^2 + \mathcal{W}_1 + \sum_{i=1}^{k}\mathcal{W}_{i+1}. \tag{3.105}$$

Proof We apply the multiplier $\partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2$ to the system (3.103):

$$\begin{aligned}\int\int [\Delta_\epsilon\partial_x^k\psi + \partial_x^k T[\psi] + \alpha A(\partial_x^k\psi) + \alpha[\partial_x^k, A]\psi]\cdot\partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2\\ = \int\int [\partial_x^k\{F_y - \epsilon G_x\}]\cdot\partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2.\end{aligned} \tag{3.106}$$

The desired estimate now follows using similar calculations as in Lemma 3.4. □

4 Nonlinear Existence of Auxiliary Systems

For this section, it is necessary to be more precise with notation; we will index solutions by (α, N) and also specify domains over which norms are being taken. We shall also transition our right-hand sides from being generic (F, G) to being the particular right-hand sides of interest, $(\tilde{f}, g)$ as defined in (1.36). Our intention now is to study the map, M^α:

$$\begin{aligned}&M^\alpha[\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}] = [u^{\alpha.N}, v^{\alpha,N}]\\ &\iff L_{\alpha,\bar{v}^{\alpha,N}}[u^{\alpha,N}, v^{\alpha,N}] = \tilde{f}_y(\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}) - \epsilon g_x(\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N})\\ &\iff [u^{\alpha,N}, v^{\alpha,N}] = L_{\alpha,\bar{v}^{\alpha,N}}^{-1}\{\tilde{f}_y(\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}) - \epsilon g_x(\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N})\},\end{aligned} \tag{4.1}$$

which corresponds to the system written in vorticity form:

$$\begin{aligned}\Delta_\epsilon^2\psi^{\alpha,N} + \alpha A(\psi^{\alpha,N}) + T(\psi^{\alpha,N};\bar{v}^{\alpha,N})\\ = \tilde{f}_y(\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}) - \epsilon g_x(\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}) \quad \text{on } \Omega^N.\end{aligned} \tag{4.2}$$

A fixed point of (4.2) corresponds to the desired solution of (1.37).

Lemma 4.1 *Suppose $\|\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}\|_{Z(\Omega^N)} \le 1$. Fix any open set $B \subset \Omega^N$. Let $\alpha > 0$ and $N >> 1$. Solutions $\psi^{\alpha,N}$, or equivalently $[u^{\alpha,N}, v^{\alpha,N}]$, to the system (4.2) satisfy the following estimates, independent of N, where $\omega(N_i)$ is based on universal constants:*

$$\begin{aligned}&\epsilon^{N_0}C(B)\|\psi^{\alpha,N}\|_{H^5(B)} + \|u^{\alpha,N}, v^{\alpha,N}\|_{Z(\Omega^N)}\\ &\lesssim \epsilon^{100} + \|u^{\alpha,N}, v^{\alpha,N}\|_{X_1\cap X_2\cap X_3(\Omega^N)} + \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)}\|\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}\|_{Z(\Omega^N)}^2.\end{aligned} \tag{4.3}$$

The following energy and positivity estimates hold:

$$\alpha \|\psi^{\alpha,N}\|^2_{H^4_w(\Omega^N)} + \|u^{\alpha,N}, v^{\alpha,N}\|^2_{X_1 \cap X_2 \cap X_3(\Omega^N)} \lesssim \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3. \tag{4.4}$$

Finally, one has:

$$\begin{aligned} &\alpha \|\psi^{\alpha,N}\|^2_{H^4_w(\Omega^N)} + \epsilon^{N_0} C(B) \|\psi^{\alpha,N}\|^2_{H^5(B)} + \|u^{\alpha,N}, v^{\alpha,N}\|^2_{Z(\Omega^N)} \\ &\quad \lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa} + \epsilon^{\frac{n}{2}-\omega(N_i)} \|\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}\|^4_{Z(\Omega^N)}. \end{aligned} \tag{4.5}$$

All constants appearing in the above estimates are independent of (α, N).

***Proof of Estimate* (4.3)** This follows by repeating the proofs of elliptic regularity in Subsection 2.1, namely Lemmas 2.11 and 2.13 in [4], to the new system, (4.2). At this point, one repeats the estimates in Subsection 2.2 of [4], which hold independent of any equation. □

***Proof of Estimate* (4.4)** This follows from Lemmas 3.12–3.13, and subsequently comparing $\|\cdot\|_{J^k}$ with $\|\cdot\|_{H^k_w}$. □

***Proof of Estimate* (4.5)** This follows by repeating the proof of Lemma 4.1 of [4]. □

Motivated by (4.5), we define the notation:

$$\|u^{\alpha,N}, v^{\alpha,N}\|_{\mathcal{F}(\Omega^N)} := \alpha \|\psi^{\alpha,N}\|^2_{H^4_w(\Omega^N)} + \|u^{\alpha,N}, v^{\alpha,N}\|^2_{Z(\Omega^N)}. \tag{4.6}$$

Lemma 4.2 (Properties of M^α) *Fix any* $\alpha > 0$ *and any* $N > 0$, *and* $\gamma, \kappa > 0$ *arbitrarily small.*

$$\begin{aligned} &(1)\ M^\alpha : B_Z(1) \subset Z(\Omega^N) \to B_Z(1) \subset Z(\Omega^N), \text{ where } B_Z(1) \text{ is the unit ball in } Z(\Omega^N); \\ &(2)\ M^\alpha \text{ is continuous and compact as an operator on } B_Z(1); \\ &(3)\ \text{There exists a fixed point, } [u^{\alpha,N}, v^{\alpha,N}] = M^\alpha[u^{\alpha,N}, v^{\alpha,N}] \text{ in } B_Z(1); \\ &(4)\ \text{The fixed point satisfies, } \|u^{\alpha,N}, v^{\alpha,N}\|_{Z(\Omega^N)} \lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa}, \text{ independent of } \alpha, N. \end{aligned} \tag{4.7}$$

Proof The outline of this proof is as follows. The map M^α is shown to be well-defined in the appropriate domains and codomains, according to (1) above. Continuity of M^α is investigated by considering differences, and compactness of M^α is obtained using our compactness lemmas above. One then applies a fixed point argument to prove (3) and (4).

(1) Suppose $[\bar{u}, \bar{v}] \in Z(\Omega^N)$. This implies that $(\tilde{f}, g) \in H^{-1}_2$, so by (3.67), the map M^α is well-defined on $Z(\Omega^N)$. Lemma 2.7 and the definition of $H^2_w(\Omega^N)$. Definition 2.1

ensures that $[u^\alpha, v^\alpha]$ are contained in $\overline{C_{0,D}^\infty}^{\|\cdot\|_{X_1}}$. Supposing the pre-images are contained in the unit ball of $Z(\Omega^N)$, $\|\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}\|_{Z(\Omega^N)} \le 1$, one has estimate (4.5), which implies that $M^\alpha(\bar{u}, \bar{v}) \in B_Z(1)$.

(2) To check continuity of the map M^α on $B_Z(1)$, suppose:

$$M^\alpha[\bar{u}_i^{\alpha,N}, \bar{v}_i^{\alpha,N}] = [u_i^{\alpha,N}, v_i^{\alpha,N}] \quad \text{for } i = 1, 2, \tag{4.8}$$

where

$$\|\bar{u}_i^{\alpha,N}, \bar{v}_i^{\alpha,N}\|_Z \le 1. \tag{4.9}$$

Define the notation for the differences,

$$\left[\hat{\bar{\psi}}, \hat{\bar{u}}, \hat{\bar{v}}\right] = [\bar{\psi}_2^{\alpha,N} - \bar{\psi}_1^{\alpha,N}, \bar{u}_2^{\alpha,N} - \bar{u}_1^{\alpha,N}, \bar{v}_2^{\alpha,N} - \bar{v}_1^{\alpha,N}], \tag{4.10}$$

$$[\hat{\psi}, \hat{u}, \hat{v}] = \left[\psi_2^{\alpha,N} - \psi_1^{\alpha,N}, u_2^{\alpha,N} - u_1^{\alpha,N}, v_2^{\alpha,N} - v_1^{\alpha,N}\right]. \tag{4.11}$$

By consulting (4.2), one then obtains the following system satisfied by the differences:

$$\begin{aligned} \Delta_\epsilon^2 \hat{\psi} + \alpha A(\hat{\psi}) + T(\hat{\psi}) = {} & \tilde{f}_y(u_2^{\alpha,N}, \bar{u}_2^{\alpha,N}, \bar{v}_2^{\alpha,N}) - \tilde{f}_y(u_1^{\alpha,N}, \bar{u}_1^{\alpha,N}, \bar{v}_1^{\alpha,N}) \\ & - \epsilon g_x(\bar{u}_2^{\alpha,N}, \bar{v}_2^{\alpha,N}) + \epsilon g_x(\bar{u}_1^{\alpha,N}, \bar{v}_1^{\alpha,N}). \end{aligned} \tag{4.12}$$

We may then repeat the estimates which resulted in (4.3)–(4.5) to obtain:

$$\|\hat{u}, \hat{v}\|_{Z(\Omega^N)}^2 \lesssim \frac{1}{\alpha^2} \|\hat{\bar{u}}, \hat{\bar{v}}\|_{Z(\Omega^N)}^2. \tag{4.13}$$

The only non-trivial calculation when repeating the estimates which resulted in (4.3)–(4.5) is the following:

$$\begin{aligned} & \bar{v}_2^{\alpha,N} u_{2y}^{\alpha,N} - \bar{v}_1^{\alpha,N} u_{1y}^{\alpha,N} \\ & \quad = \bar{v}_2^{\alpha,N} u_{2y}^{\alpha,N} - \bar{v}_2^{\alpha,N} u_{1y}^{\alpha,N} + \bar{v}_2^{\alpha,N} u_{1y}^{\alpha,N} - \bar{v}_1^{\alpha,N} u_{1y}^{\alpha,N} \\ & \quad = \bar{v}_2^{\alpha,N} \hat{u}_y + \hat{\bar{v}} u_{1,y}^{\alpha,N}. \end{aligned} \tag{4.14}$$

A straightforward calculation gives:

$$\int\int \epsilon^{\frac{n}{2}+\gamma}[\bar{v}_2^{\alpha,N}u_{2y}^{\alpha,N} - \bar{v}_1^{\alpha,N}u_{1y}^{\alpha,N}]\hat{u} = \int\int \epsilon^{\frac{n}{2}+\gamma}[\bar{v}_2^{\alpha,N}\hat{u}_y + \hat{\bar{v}}u_{1,y}^{\alpha,N}]\hat{u}$$
$$= -\int\int \frac{\epsilon^{\frac{n}{2}+\gamma}}{2}\hat{u}^2\bar{v}_{2y}^{\alpha,N} + \int\int \epsilon^{\frac{n}{2}+\gamma}\hat{\bar{v}}u_{1,y}^{\alpha,N}\hat{u}. \tag{4.15}$$

For the first term in the right-hand side above, we estimate

$$\left|\int\int \frac{\epsilon^{\frac{n}{2}+\gamma}}{2}\hat{u}^2\bar{v}_{2y}^{\alpha,N}\right| \lesssim \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)}\|\hat{u}\|_{Z(\Omega^N)}^2\|\bar{u}_2^{\alpha,N},\bar{v}_2^{\alpha,N}\|_{Z(\Omega^N)}$$
$$\lesssim \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)}\|\hat{u}\|_{Z(\Omega^N)}^2.$$

This then gets absorbed into the left-hand side of (4.13). For the second term on the right-hand side above, we estimate:

$$\begin{aligned}\left|\int\int \epsilon^{\frac{n}{2}+\gamma}\hat{\bar{v}}u_{1,y}^{\alpha,N}\hat{u}\right| &\le \epsilon^{\frac{n}{2}+\gamma}\|\hat{\bar{v}}x^{\frac{1}{2}}\|_{L^\infty}\|u_{1,y}^{\alpha,N}x^m\|_{L^2}\|\hat{u}x^{-m-\frac{1}{2}}\|_{L^2}\\
&\lesssim \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)}\|\hat{\bar{v}}\|_{Z(\Omega^N)}\|u_1^{\alpha,N}\|_{H_w^2(\Omega^N)}\|\hat{u}\|_{Z(\Omega^N)}\\
&\lesssim \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)}\|\hat{\bar{v}}\|_{Z(\Omega^N)}\frac{1}{\alpha}\|u_1^{\alpha,N}\|_{\mathcal{F}(\Omega^N)}\|\hat{u}\|_{Z(\Omega^N)}\\
&\lesssim \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)}\|\hat{\bar{v}}\|_{Z(\Omega^N)}\frac{1}{\alpha}\|\hat{u}\|_{Z(\Omega^N)}\\
&\lesssim \epsilon^{2(\frac{n}{2}+\gamma-\omega(N_i))}\|\hat{u}\|_{Z(\Omega^N)}^2 + \frac{1}{\alpha^2}\|\hat{\bar{v}}\|_{Z(\Omega^N)}^2,\end{aligned} \tag{4.16}$$

where we have used (4.9) coupled with (4.5) to conclude that: $\|u_1^{\alpha,N}\|_{\mathcal{F}(\Omega^N)} \lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa}$. The weight, x^m, arises from the definition (1.29), and consequently in (2.2). The first term on the right-hand side of (4.16) is absorbed into the left-hand side of (4.13), whereas the second term contributes to the right-hand side of (4.13). All of the remaining calculations which produced (4.5) can be repeated in a similar fashion. Estimate (4.13) then implies the continuity of M^α on $B_Z(1)$. The modulus of continuity of M^α is $\frac{1}{\alpha^2}$, which prevents M^α from being a contraction map. Nevertheless, continuity is retained for all $\alpha > 0$.

We now turn to compactness. According to Lemma 3.2, (4.5) shows that $M^\alpha(B_Z(1))$ is compactly embedded in $B_Z(1)$ so long as m is sufficiently large.

(3) and (4) Consider the family of solutions:

$$[u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N}] = \lambda M^\alpha[u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N}] \quad \text{for } 0 \le \lambda \le 1. \tag{4.17}$$

By (4.1) and linearity of L_α^{-1}, this occurs if and only if

$$[u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N}] = L_\alpha^{-1}\{\lambda \tilde{f}_y(u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N}) - \epsilon\lambda g_x(u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N})\}. \tag{4.18}$$

By repeating the estimates which culminated in (4.5), one sees the uniform in λ bound:

$$\|u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N}\|_{Z(\Omega^N)}^2 \lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa}. \tag{4.19}$$

Thus, Schaefer's fixed point theorem applied to the convex subset $B_Z(1) \subset Z(\Omega^N)$ produces a fixed point, $[u^{\alpha,N}, v^{\alpha,N}] \in B_Z(1)$. The estimate it obeys follows from (4.5). □

5 Nonlinear Existence

We now need to pass to the limit as $\alpha \to 0$ and as $N \to \infty$. The fixed point of the system (4.2), from Lemma 4.2 satisfies the following integral identity for any $\phi \in C_0^\infty(\Omega^N)$:

$$\begin{aligned}
&\int\int_{\Omega^N} \nabla_\epsilon^2 \psi^{N,\alpha} : \nabla_\epsilon^2 \phi + \alpha\Big[\int\int_{\Omega^N} \psi^{N,\alpha}\phi x^{2m} + \int\int_{\Omega^N} \nabla\psi^{N,\alpha}\cdot\nabla\phi x^{2m+2} \\
&\qquad + \int\int_{\Omega^N} \nabla^2\psi^{N,\alpha} : \nabla^2\phi x^{2m+4}\Big] + \int\int_{\Omega^N} (-S_u\cdot\phi_y + \epsilon S_v\cdot\phi_x) \\
&\qquad + \int\int_{\Omega^N} \epsilon^{-\frac{n}{2}-\gamma}\big[R^{u,n}\cdot\phi_y - \epsilon R^{v,n}\cdot\phi_x\big] \\
&\quad = \int\int_{\Omega^N} \epsilon^{\frac{n}{2}+\gamma}\big[-u^{N,\alpha}u_x^{N,\alpha}\phi_y - v^{N,\alpha}u_y^{N,\alpha}\phi_y + \epsilon u^{N,\alpha}v_x^{N,\alpha}\phi_x + \epsilon v^{N,\alpha}v_y^{N,\alpha}\phi_x\big].
\end{aligned} \tag{5.1}$$

First, we shall pass to the limit as $\alpha \to 0$, fixing an N. To do so, we first use (4.7) to obtain a weak subsequential limit point:

$$u^{N,\alpha} \overset{\alpha\to 0}{\rightharpoonup} u^N \quad \text{weakly in } (X_1 \cap X_2 \cap X_3)(\Omega^N). \tag{5.2}$$

It is now our task to pass to the limit in the equation, (5.1), along the subsequence $\alpha \to 0$. Given a test-function, denote by U_ϕ to be the support of ϕ. As U_ϕ is bounded, we have Poincare inequalities available:

$$\alpha\Big|\Big[\int\int_{\Omega^N}\psi^{N,\alpha}\phi x^{2m}+\int\int_{\Omega^N}\nabla\psi^{N,\alpha}\cdot\nabla\phi x^{2m+2}+\int\int_{\Omega^N}\nabla^2\psi^{N,\alpha}:\nabla^2\phi x^{2m+4}\Big]\Big|$$
$$\leq C(\phi)\alpha\big[\|\psi^{N,\alpha}\|_{L^2(U_\phi)}+\|\nabla\psi^{N,\alpha}\|_{L^2(U_\phi)}+\|\nabla^2\psi^{N,\alpha}\|_{L^2(U_\phi)}\big]$$
$$\leq C(\phi)\alpha\|\nabla u^{N,\alpha},\nabla v^{N,\alpha}\|_{L^2(U_\phi)}\leq C(\phi)\alpha\|u^{N,\alpha},v^{N,\alpha}\|_{Z(\Omega^N)}\xrightarrow{\alpha\to 0}0. \tag{5.3}$$

For all of the linear terms, we use the weak convergence in $(X_1\cap X_2\cap X_3)(\Omega^N)$:

$$\lim_{\alpha\to 0}\Big\{\int\int_{\Omega^N}\big[\nabla^2_\epsilon\psi^{\alpha,N}:\nabla^2_\epsilon\phi\big]-\int\int_{\Omega^N}\big[S_u(u^{N,\alpha},v^{N,\alpha})\phi_y+\epsilon S_v(u^{N,\alpha},v^{N,\alpha})\phi_x\big]\Big\}$$
$$=\int\int_{\Omega^N}\big[\nabla^2_\epsilon\psi^{N}:\nabla^2_\epsilon\phi\big]-\int\int_{\Omega^N}\big[S_u(u^{N},v^{N})\phi_y+\epsilon S_v(u^{N},v^{N})\phi_x\big]. \tag{5.4}$$

Finally, we turn to the nonlinear terms for which we integrate by parts:

$$\int\int_{\Omega^N}\big(u^{N,\alpha}u^{N,\alpha}_x\phi_y+v^{N,\alpha}u^{N,\alpha}_y\phi_y\big)=\int\int_{\Omega^N}\big(-|u^{N,\alpha}|^2\phi_{xy}-u^{N,\alpha}v^{N,\alpha}\phi_{yy}\big), \tag{5.5}$$

$$\int\int_{\Omega^N}\big(u^{N,\alpha}v^{N,\alpha}_x\phi_x+v^{N,\alpha}v^{N,\alpha}_y\phi_x\big)=\int\int_{\Omega^N}\big(-|v^{N,\alpha}|^2\phi_{xy}-u^{N,\alpha}v^{N,\alpha}\phi_{xx}\big). \tag{5.6}$$

Fixing a compactly supported ϕ, we can localize the integrations above to U_ϕ. On this set, the weak convergence of $u^{N,\alpha}\overset{X_1\cap X_2\cap X_3}{\rightharpoonup}u^N$ implies strong convergence in L^2. Thus,

$$\Big|\int\int_{U_\phi}\big[|u^{N,\alpha}|^2-u^{N,\alpha}u^N+u^{N,\alpha}u^N-|u^N|^2\big]\phi_{xy}\Big|$$
$$\lesssim\|u^{N,\alpha}-u^N\|_{L^2(U_\phi)}\|u^{N,\alpha}\|_{L^2(U_\phi)}+\|u^N\|_{L^2(U_\phi)}\|u^{N,\alpha}-u^N\|_{L^2(U_\phi)}. \tag{5.7}$$

The right-hand side converges to zero. The same bound works for all of the other nonlinear terms. Thus, the weak limit $[u^N,v^N]$ or equivalently ψ^N satisfies the weak formulation:

$$\int\int_{\Omega^N}\nabla^2_\epsilon\psi^N:\nabla^2_\epsilon\phi-\int\int_{\Omega^N}\big[S_u(u^N,v^N)\phi_y-\epsilon S_v(u^N,v^N)\phi_x\big]+\int\int_{\Omega^N}\epsilon^{-\frac{n}{2}-\gamma}\big[R^{u,n}\phi_y-\epsilon R^{v,n}\phi_x\big]$$
$$=\int\int_{\Omega^N}\epsilon^{\frac{n}{2}+\gamma}\big[-u^Nu^N_x\phi_y-v^Nu^N_y\phi_y+\epsilon u^Nv^N_x\phi_x+\epsilon v^Nv^N_y\phi_x\big]. \tag{5.8}$$

The weak limit $[u^N,v^N]$ must satisfy the bound:

$$\|u^N, v^N\|_{(X_1\cap X_2\cap X_3)(\Omega^N)} \lesssim C(u_R, v_R)\epsilon^{\frac{1}{4}-\gamma-\kappa}, \tag{5.9}$$

independent of N. We may now repeat this exact procedure with the subsequential N limit: denote by $[u, v]$ and ψ the subsequential $(X_1 \cap X_2 \cap X_3)(\Omega)$-weak limit as $N \to \infty$, guaranteed by (5.9). One then passes to the limit in Eq. (5.8) to obtain:

$$\begin{aligned}\int\int_\Omega \nabla_\epsilon^2\psi : \nabla_\epsilon^2\phi - \int\int_\Omega \big[S_u(u,v)\phi_y - \epsilon S_v(u,v)\phi_x\big] + \int\int_\Omega \epsilon^{-\frac{n}{2}-\gamma}\big[R^{u,n}\phi_y - \epsilon R^{v,n}\phi_x\big] \\ = \int\int_\Omega \epsilon^{\frac{n}{2}+\gamma}\big[-uu_x\phi_y - vu_y\phi_y + \epsilon uv_x\phi_x + \epsilon vv_y\phi_x\big],\end{aligned} \tag{5.10}$$

with the limit satisfying:

$$\|u, v\|_{(X_1\cap X_2\cap X_3)(\Omega)} \lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa}. \tag{5.11}$$

We now state the main existence result.

Theorem 5.1 *For ϵ, δ sufficiently small, $\kappa > 0$ small, and $0 \le \gamma < \frac{1}{4}$, there exists a solution to the system* (1.1)–(1.3), (1.4), (1.5) *satisfying:*

$$\|u, v\|_{Z(\Omega)} \lesssim C(u_R, v_R)\epsilon^{\frac{1}{4}-\gamma-\kappa}. \tag{5.12}$$

Proof Estimate (5.11) implies enough regularity to integrate by parts identity (5.1) to:

$$\int\int_\Omega \big[\Delta_\epsilon^2\psi + \partial_y S_u - \epsilon\partial_x S_v - \partial_y f + \epsilon\partial_x g\big]\phi = 0, \tag{5.13}$$

which then implies that the PDE is satisfied pointwise in Ω. The boundary conditions (1.4) are satisfied by elements in $(X_1 \cap X_2 \cap X_3)(\Omega)$. From here, one applies the available embedding theorems for the norm Z which gives the estimate (5.12). A nearly identical proof to Lemma 2.11 in [29] then yields:

$$\sup_{x\le 2\,000} \|u, v\|_{L^\infty_y} + \|u, v\|_{\dot{H}^2(x\le 2\,000)} \lesssim \epsilon^{-M_2}. \tag{5.14}$$

One now bootstraps the estimate in Lemma 2.13, [4] in the identical manner. This gives estimate (5.12). We have verified that $[u, v] \in Z(\Omega)$ satisfies (1.1)–(1.3), (1.4), (1.5). □

6 Uniqueness

In this final section, we prove uniqueness of the solution $[u, v]$ from Theorem 5.1. Suppose there existed two solutions, $[u_1, v_1]$ and $[u_2, v_2]$ to the system in (1.1)–(1.3), (1.4), (1.5). Define:

$$\hat{u} = u_1 - u_2, \quad \hat{v} = v_1 - v_2, \quad \hat{P} = P_1 - P_2. \tag{6.1}$$

Then the new unknowns satisfy:

$$-\Delta_\epsilon \hat{u} + S_u(\hat{u}, \hat{v}) + \hat{P}_x = \hat{f} := \epsilon^{\frac{n}{2}+\gamma}[u_1 u_{1x} - u_2 u_{2x} + v_1 u_{1y} - v_2 u_{2y}], \tag{6.2}$$

$$-\Delta_\epsilon \hat{v} + S_v(\hat{u}, \hat{v}) + \frac{\hat{P}_y}{\epsilon} = \hat{g} := \epsilon^{\frac{n}{2}+\gamma}[u_1 v_{1x} - u_2 v_{2x} + v_1 v_{1y} - v_2 v_{2y}], \tag{6.3}$$

together with the divergence-free condition, $\hat{u}_x + \hat{v}_y = 0$, and also satisfy the boundary conditions:

$$\{\hat{u}, \hat{v}\}|_{\{y=0\}} = \{\hat{u}, \hat{v}\}|_{\{x=1\}} = 0. \tag{6.4}$$

Going to vorticity,

$$\begin{aligned} \partial_y[-\Delta_\epsilon \hat{u} + S_u(\hat{u}, \hat{v})] - \epsilon\partial_x[-\Delta_\epsilon v + S_v(\hat{u}, \hat{v})] = \epsilon^{\frac{n}{2}} \big\{ \partial_y[u_1 u_{1x} \\ - u_2 u_{2x} + v_1 u_{1y} - v_2 u_{2y}] - \epsilon\partial_x[u_1 v_{1x} - u_2 v_{2x} + v_1 v_{1y} - v_2 v_{2y}] \big\}. \end{aligned} \tag{6.5}$$

We shall repeat the basic energy and positivity estimates using a slightly weaker weight. It is convenient to work with the weak formulation, which is given in (5.10). Then, $\hat{u}, \hat{v}$ satisfy the following:

$$\int\int \nabla_\epsilon^2 \hat{\psi} : \nabla_\epsilon^2 \phi + \int\int (\epsilon S_v(\hat{u}, \hat{v}) \cdot \phi_x - S_u(\hat{u}, \hat{v}) \cdot \phi_y) = \int\int \big[-\hat{f}\phi_y + \epsilon\hat{g}\phi_x \big] \tag{6.6}$$

for all $\phi \in C_0^\infty(\Omega)$. We make the notational convention that

$$\int\int := \int\int_\Omega . \tag{6.7}$$

Lemma 6.1 *There exists a $0 < b < 1$, sufficiently close to 0, depending only on universal constants, such that for δ, ϵ sufficiently small and $\epsilon << \delta << b$, the solutions $[\hat{u}, \hat{v}] \in Z$ to the system (6.2)–(6.3) with boundary conditions (6.4) satisfy the following estimate:*

$$b\|\{\hat{u}, \sqrt{\epsilon}\hat{v}\}x^{-b-\frac{1}{2}}\|_{L^2}^2 + \|\hat{u}_y x^{-b}\|_{L^2}^2 \lesssim \mathcal{O}(\delta)\|\{\sqrt{\epsilon}\hat{v}_x, \hat{v}_y\}x^{\frac{1}{2}-b}\|_{L^2}^2 + \mathcal{W}_{1,E,b}, \tag{6.8}$$

where

$$\mathcal{W}_{1,E,b} := \int\int \big(\hat{f}\hat{u}x^{-2b} + \epsilon\hat{g}\hat{v}x^{-2b} - 2b\epsilon\hat{g}\hat{\psi}x^{-2b-1} \big), \tag{6.9}$$

$$\mathcal{W}_{1,P,b} := \int\int \left(\hat{f}\hat{u}_x x^{1-2b} + \epsilon\hat{g}\hat{v}_x x^{1-2b}\right), \tag{6.10}$$

$$\mathcal{W}_{1,b} = \mathcal{W}_{1,E,b} + \mathcal{W}_{1,P,b}. \tag{6.11}$$

Proof The estimate will follow upon applying the multiplier $\hat{\psi} \cdot x^{-2b}$ to the system in (6.5). To work rigorously, we will apply approximate multipliers, and work with the weak formulation given in (6.6). Fix $[\hat{u}^n, \hat{v}^{(n)}, \hat{\psi}^{(n)}] \in C_0^\infty(\Omega)$, such that:

$$[\hat{u}^{(n)}, \hat{v}^{(n)}] \xrightarrow{X_1} [\hat{u}, \hat{v}], \tag{6.12}$$

where X_1 is defined in (1.40) of [4]. Within the notation of (6.6), $\phi = \hat{\psi}^{(n)} x^{-2b}$. The existence of the sequence specified in (6.12) is guaranteed by $[\hat{u}, \hat{v}] \in Z(\Omega)$. That ϕ is compactly supported in (x, y) follows from the representations:

$$\hat{\psi}^{(n)} = -\int_0^y \hat{u}^{(n)} = \int_0^x \hat{v}^{(n)}. \tag{6.13}$$

Let us first treat the second-order terms:

$$\begin{aligned}\int\int \nabla_\epsilon^2\hat{\psi} : \nabla_\epsilon^2\phi &= \int\int \nabla_\epsilon^2\hat{\psi} : \nabla_\epsilon^2\left(\hat{\psi}^{(n)}x^{-2b}\right)\\ &= \int\int \left(\hat{\psi}_{yy}\hat{\psi}_{yy}^{(n)}x^{-2b} + 2\epsilon\hat{\psi}_{xy}\partial_x\left(\hat{\psi}_y^{(n)}x^{-2b}\right) + \epsilon^2\hat{\psi}_{xx}\partial_{xx}\left(\hat{\psi}^{(n)}x^{-2b}\right)\right).\end{aligned} \tag{6.14}$$

The first two terms from (6.14) above are:

$$\begin{aligned}&\int\int (\hat{\psi}_{yy}\hat{\psi}_{yy}^{(n)}x^{-2b} + 2\epsilon\hat{\psi}_{xy}\hat{\psi}_{xy}^{(n)}x^{-2b} + 2\epsilon\hat{\psi}_{xy}\hat{\psi}_y^{(n)}\partial_x x^{-2b})\\ &= \int\int (\hat{u}_y\hat{u}_y^{(n)}x^{-2b} + 2\epsilon\hat{u}_x\hat{u}_x^{(n)}x^{-2b} - 2\epsilon\hat{u}_x\hat{u}^{(n)}\partial_x x^{-2b}).\end{aligned} \tag{6.15}$$

We shall take the limit as $n \to \infty$ above. According to the definition (1.40), the convergence in (6.12) implies:

$$\begin{aligned}&\left|\int\int \hat{u}_y(\hat{u}_y^{(n)} - \hat{u}_y)x^{-2b}\right| + \left|\int\int \hat{u}_x(\hat{u}_x^{(n)} - \hat{u}_x)x^{-2b}\right|\\ &+ \left|\int\int \hat{u}_x(\hat{u}^{(n)} - \hat{u})\partial_x x^{-2b}\right| \xrightarrow{n\to\infty} 0.\end{aligned} \tag{6.16}$$

Expanding the third term from (6.14),

$$\int\int \epsilon^2 \hat{\psi}_{xx}\partial_{xx}\big(\hat{\psi}^{(n)}x^{-2b}\big) = \int\int \epsilon^2 \hat{v}_x \cdot \big[\hat{v}^{(n)}_x x^{-2b} + 2\hat{v}^{(n)}\partial_x x^{-2b} + \hat{\psi}^{(n)}\partial_{xx}x^{-2b}\big]. \tag{6.17}$$

By referring to the definition of X_1 in (1.40) and (6.12), we may pass to the limit:

$$\begin{aligned}
\text{Eq. (6.15)} \xrightarrow{n\to\infty} & \int\int [\hat{u}_y^2 + 2\epsilon\hat{u}_x^2]x^{-2b} - \int\int \epsilon\hat{u}_x\hat{u}\partial_x x^{-2b} \\
&= \int\int \Big([\hat{u}_y^2 + 2\epsilon\hat{u}_x^2]x^{-2b} + b(2b+1)\epsilon\hat{u}^2x^{-2-2b}\Big) + \epsilon b \lim_{M\to\infty}\int_{x=M} \hat{u}^2x^{-1-2b} \\
&= \int\int \Big([\hat{u}_y^2 + 2\epsilon\hat{u}_x^2]x^{-2b} + b(2b+1)\epsilon\hat{u}^2x^{-2-2b}\Big),
\end{aligned} \tag{6.18}$$

and:

$$\text{Eq. (6.17)} \xrightarrow{n\to\infty} \int\int \Big(\epsilon^2\hat{v}_x^2x^{-2b} - 4b\epsilon^2\hat{v}_x\hat{v}x^{-1-2b} + 2b(2b+1)\epsilon^2\hat{v}_x\hat{\psi}x^{-2-2b}\Big). \tag{6.19}$$

Integrating by parts the final two terms above in (6.19), and referring to estimate (1.46),

$$\begin{aligned}
-4b\int\int \epsilon^2\hat{v}_x\hat{v}x^{-1-2b} &= \int\int 2b\epsilon^2\hat{v}^2\partial_x x^{-1-2b} - 2b\lim_{M\to\infty}\int_{x=M}\epsilon^2\hat{v}^2x^{-1-2b} \\
&= -2b(1+2b)\int\int \epsilon^2\hat{v}^2x^{-2-2b},
\end{aligned} \tag{6.20}$$

and similarly, to treat the final term in (6.19), we appeal to the estimates in (1.46):

$$\begin{aligned}
\int\int \epsilon^2\hat{v}_x\hat{\psi}x^{-2-2b} &= -\int\int \epsilon^2\hat{v}^2x^{-2-2b} - \int\int \epsilon^2\hat{v}\hat{\psi}\partial_x x^{-2-2b} \\
&\quad + \lim_{M\to\infty}\int_{x=M}\epsilon^2\hat{v}\hat{\psi}x^{-2-2b} \\
&= -\int\int \epsilon^2\hat{v}^2x^{-2-2b} + \int\int \frac{(2b+3)(2b+2)}{2}\epsilon^2\hat{\psi}^2x^{-4-2b} \\
&\quad + \lim_{M\to\infty}\int_{x=M}\frac{2b+2}{2}\epsilon^2\hat{\psi}^2x^{-3-2b} \\
&= -\int\int \epsilon^2\hat{v}^2x^{-2-2b} + \int\int \frac{(2b+3)(2b+2)}{2}\epsilon^2\hat{\psi}^2x^{-4-2b}.
\end{aligned}$$

Therefore, summarizing the highest order calculation:

$$\begin{aligned}\int\int \nabla_\epsilon^2 \hat{\psi} : \nabla_\epsilon^2(\hat{\psi}^{(n)} x^{-2b}) \gtrsim \int\int [\hat{u}_y^2 x^{-2b} + 2\epsilon \hat{u}_x^2 + \epsilon^2 \hat{v}_x^2] x^{-2b} \\ - \int\int [\epsilon^2 \hat{v}^2 + \epsilon \hat{u}^2] x^{-2-2b} + \epsilon^2 \hat{\psi}^2 x^{-4-2b}\end{aligned} \tag{6.21}$$

$$\gtrsim \int\int \hat{u}_y^2 x^{-2b} - C\int\int \epsilon^2 \hat{v}_x^2 x^{-2b} - C\int\int \epsilon \hat{u}_x^2 x^{-2b}. \tag{6.22}$$

To go from (6.21) to (6.22), we have used the Hardy inequality in the x-direction. We will now address the profile terms arising from $S_u(\hat{u}, \hat{v})$ in the weak formulation (6.6), whose definition has been given in (1.5):

$$\begin{aligned}&- \int\int [u_R \hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R \hat{u}_y] \cdot \partial_y \phi \\ &= - \int\int [u_R \hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R \hat{u}_y] \cdot \partial_y \hat{\psi}^{(n)} x^{-2b} \\ &= \int\int [u_R \hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R \hat{u}_y] \cdot \hat{u}^{(n)} x^{-2b}.\end{aligned} \tag{6.23}$$

We will first pass to the limit in (6.23), using the definition of X_1 in (1.40), which gives:

$$(6.23) \xrightarrow{n\to\infty} \int\int [u_R \hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R \hat{u}_y] \cdot \hat{u} x^{-2b}. \tag{6.24}$$

We proceed to treat each term in (6.24), starting with:

$$\begin{aligned}\int\int u_R \hat{u}_x \hat{u} x^{-2b} &= - \int\int \hat{u}^2 \frac{\partial_x}{2}(u_R x^{-2b}) + \lim_{M\to\infty} \int_{x=M} \hat{u}^2 x^{-2b} \\ &= - \int\int \hat{u}^2 (u_{Rx} x^{-2b} - 2b u_R x^{-2b-1}) \\ &\gtrsim - \|u_{Rx} x\|_{L^\infty} \int\int \hat{u}^2 x^{-2b-1} + 2b \min u_R \int\int \hat{u}^2 x^{-2b-1} \\ &\gtrsim b \int\int \hat{u}^2 x^{-2b-1},\end{aligned} \tag{6.25}$$

according to estimates (1.14), (1.19), so long as δ is taken small relative to b. For the M-limit above, we have used estimate (1.46), which is valid so long as $b > 0$. For the second term in (6.24), we again appeal to estimates (1.14), (1.20):

$$\left| \int u_{Rx} \hat{u}^2 x^{-2b} \right| \lesssim \|u_{Rx} x\|_{L^\infty} \|\hat{u} x^{-2b-\frac{1}{2}}\|_{L^2}^2 \lesssim \mathcal{O}(\delta) \|\hat{u} x^{-b-\frac{1}{2}}\|_{L^2}^2.$$

For the third term, we shall split $u_R = u_R^{n-1,p} + \epsilon^{\frac{n}{2}} u_{pR}^n + u_R^E$. First, we apply estimate (1.16):

$$\begin{aligned} \left| \int\int u_{Ry}^{P,n-1} \hat{v} \hat{u} x^{-2b} \right| &\le \|y^2 x^{-\frac{1}{2}} u_{Ry}^{P,n-1}\|_{L^\infty} \left\| \frac{\hat{u}}{y} x^{-b} \right\|_{L^2} \left\| \frac{\hat{v}}{y} x^{\frac{1}{2}-b} \right\|_{L^2} \\ &\le \mathcal{O}(\delta) \|\hat{u}_y x^{-b}\|_{L^2} \|\hat{v}_y x^{\frac{1}{2}-b}\|_{L^2}. \end{aligned} \tag{6.26}$$

Next, according to estimate (1.18),

$$\begin{aligned} \left| \int\int u_{Ry}^{P,n} \hat{v} \hat{u} x^{-2b} \right| &\le \epsilon^{\frac{n}{2}} \|u_{py}^n y x^{\frac{1}{2}-\sigma_n}\|_{L^\infty} \|\hat{u} x^{-1+\sigma_n-b}\|_{L^2} \left\| \frac{\hat{v}}{y} x^{\frac{1}{2}-b} \right\|_{L^2} \\ &\lesssim \epsilon^{\frac{n}{2}} \|\hat{u}_x x^{\sigma_n-b}\|_{L^2} \|\hat{v}_y x^{\frac{1}{2}-b}\|_{L^2} \lesssim \epsilon^{\frac{n}{2}} \mathcal{O}(\delta) \|\hat{v}_y x^{\frac{1}{2}-b}\|_{L^2}^2. \end{aligned} \tag{6.27}$$

Finally, the Eulerian contribution is handled by an application of (1.20):

$$\begin{aligned} \left| \int\int \sqrt{\epsilon} u_{RY}^E \hat{u} \hat{v} x^{-2b} \right| &\le \sqrt{\epsilon} \|u_{RY}^E x^{\frac{3}{2}}\|_{L^\infty} \left\| \frac{\hat{u}}{x^{\frac{3}{4}+b}} \right\|_{L^2} \left\| \frac{\hat{v}}{x^{\frac{3}{4}+b}} \right\|_{L^2} \\ &\lesssim \sqrt{\epsilon} \|\hat{u}_x x^{\frac{1}{4}-b}\|_{L^2} \|\sqrt{\epsilon} \hat{v}_x x^{\frac{1}{4}-b}\|_{L^2}. \end{aligned} \tag{6.28}$$

The fourth term from (6.23), upon using estimate (1.14) and (1.20), reads:

$$\begin{aligned} \left| \int\int v_R \hat{u}_y \hat{u} x^{-2b} \right| &= \left| \int\int \frac{v_{Ry}}{2} \hat{u}^2 x^{-2b} \right| \\ &\lesssim \|u_{Rx} x\|_{L^\infty} \|\hat{u} x^{-\frac{1}{2}-b}\|_{L^2}^2 \lesssim \mathcal{O}(\delta) \|\hat{u} x^{-\frac{1}{2}-b}\|_{L^2}^2. \end{aligned} \tag{6.29}$$

Summarizing these calculations,

$$\begin{aligned} |(6.24)| &\gtrsim b\|\hat{u} x^{-\frac{1}{2}-b}\|_{L^2}^2 - \mathcal{O}(\delta)\|\hat{u} x^{-\frac{1}{2}-b}\|_{L^2}^2 - \mathcal{O}(\delta)\|\hat{u}_y x^{-b}\|_{L^2}^2 \\ &\quad - \mathcal{O}(\delta)\|\{\sqrt{\epsilon} v_x \hat{v}_y\} x^{\frac{1}{2}-b}\|_{L^2}^2 \\ &\gtrsim b\|\hat{u} x^{-\frac{1}{2}-b}\|_{L^2}^2 - \mathcal{O}(\delta)\|\{\sqrt{\epsilon} v_x, v_y\} x^{\frac{1}{2}-b}\|_{L^2}^2. \end{aligned} \tag{6.30}$$

We have absorbed the $\hat{u}_y$ terms into (6.21), and taken δ sufficiently small relative to b. We shall now address the profile terms from S_v:

$$\int\int \epsilon S_v(\hat{u},\hat{v})\cdot\hat{\phi}_x = \int\int \epsilon\left[u_R\hat{v}_x + v_{Rx}\hat{u} + v_R\hat{v}_y + v_{Ry}\hat{v}\right] \times\left[\hat{v}^{(n)}x^{-2b} - 2b\hat{\psi}^{(n)}x^{-2b-1}\right]. \tag{6.31}$$

We may take $n\to\infty$ above due to the definition of X_1 from (1.40) and (6.12):

$$(6.31)\xrightarrow{n\to\infty}\int\int \epsilon\left[u_R\hat{v}_x + v_{Rx}\hat{u} + v_R\hat{v}_y + v_{Ry}\hat{v}\right]\cdot\left[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\right]. \tag{6.32}$$

We will now proceed to treat each term in (6.32). The first profile term, $u_R v_x$ is the most delicate:

$$\int\int \epsilon u_R\hat{v}_x\left[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\right]. \tag{6.33}$$

First,

$$\begin{aligned}\int\int \epsilon u_R\hat{v}_x\hat{v}x^{-2b} &= -\int\int \epsilon\hat{v}^2\frac{\partial_x}{2}(u_Rx^{-2b}) + \lim_{M\to\infty}\int_{x=M}\frac{\epsilon u_R}{2}\hat{v}^2x^{-2b}\\ &= -\int\int \epsilon\hat{v}^2\frac{u_{Rx}}{2}x^{-2b} + \int\int b\epsilon u_R\hat{v}^2x^{-2b-1}.\end{aligned} \tag{6.34}$$

The M-limit above vanishes due to (1.46). Staying with the term (6.33):

$$\begin{aligned}-2b\int\int \epsilon u_R\hat{v}_x\hat{\psi}x^{-2b-1} &= 2b\int\int \epsilon\hat{v}\partial_x(u_R\hat{\psi}x^{-2b-1})\\ &= \int\int 2b\epsilon u_{Rx}\hat{v}\hat{\psi}x^{-2b-1} + \int\int 2b\epsilon u_R\hat{v}^2x^{-2b-1}\\ &\quad - \int\int 2b(2b+1)\epsilon u_R\hat{\psi}\hat{v}x^{-2b-2}\end{aligned} \tag{6.35}$$

$$\begin{aligned}&= \int\int 2b\epsilon u_{Rx}\hat{v}\hat{\psi}x^{-2b-1} + \int\int 2b\epsilon u_R\hat{v}^2x^{-2b-1}\\ &\quad + \int\int b(2b+1)\epsilon\hat{\psi}^2u_{Rx}x^{-2b-2}\\ &\quad - \int\int b(2b+1)(2b+2)\epsilon u_R\hat{\psi}^2x^{-2b-3}.\end{aligned} \tag{6.36}$$

Combining the positive terms in (6.36) and (6.34), the total positive contribution is $\int\int 3b\epsilon u_R \hat{v}^2 x^{-2b-1}$. For the final term in (6.36), we will now give the estimate:

$$\begin{aligned}\int\int u_R \hat{\psi}^2 x^{-2b-3} &= \int\int u_R \hat{\psi}^2 \frac{-\partial_x}{2b+2} x^{-2b-2} \\ &= \int\int \frac{2}{2b+2} u_R \hat{\psi}\hat{v} x^{-2b-2} + \int\int \frac{u_{Rx}}{2b+2} \hat{\psi}^2 x^{-2b-2} \\ &\leq \left[\frac{1}{2}\|u_R^{\frac{1}{2}}\hat{\psi} x^{-b-\frac{3}{2}}\|_{L^2}^2 + \frac{1}{2}\frac{4}{(2b+2)^2}\|u_R^{\frac{1}{2}}\hat{v} x^{-b-\frac{1}{2}}\|_{L^2}^2\right] \\ &\quad + \frac{\|u_{Rx}x\|_{L^\infty}}{2b+2}\frac{\sup|u_R|}{\inf|u_R|}\int\int u_R \hat{\psi}^2 x^{-2b-3}. \end{aligned} \tag{6.37}$$

By collecting terms and rearranging, we obtain:

$$\left[1 - \frac{1}{2} - \frac{\|u_{Rx}x\|_{L^\infty}}{2b+2}\frac{\sup|u_R|}{\inf|u_R|}\right]\|u_R^{\frac{1}{2}}\hat{\psi} x^{-b-\frac{3}{2}}\|_{L^2}^2 \leq \frac{2}{(2b+2)^2}\|u_R^{\frac{1}{2}}\hat{v} x^{-b-\frac{1}{2}}\|_{L^2}^2. \tag{6.38}$$

This then implies:

$$\|u_R^{\frac{1}{2}}\hat{\psi} x^{-b-\frac{3}{2}}\|_{L^2}^2 \leq \frac{1}{1-\mathcal{O}(\delta)}\frac{4}{(2b+2)^2}\|u_R^{\frac{1}{2}}\hat{v} x^{-b-\frac{1}{2}}\|_{L^2}^2. \tag{6.39}$$

Inserting this into (6.36), one arrives at:

$$\begin{aligned}&\left|\int\int b(2b+1)(2b+2)\epsilon u_R \hat{\psi}^2 x^{-2b-3}\right| \\ &\quad \leq \frac{1}{1-\mathcal{O}(\delta)}\frac{4b(2b+1)(2b+2)}{(2b+2)^2}\int\int \epsilon u_R \hat{v}^2 x^{-1-2b} \\ &\quad \leq \int\int \frac{5b}{2} u_R \epsilon \hat{v}^2 x^{-1-2b}, \end{aligned} \tag{6.40}$$

so long as b is sufficiently close to 0, by the following calculation:

$$\lim_{b\to 0}\frac{(2b+1)(2b+2)}{(2b+2)^2} = \frac{1}{2}. \tag{6.41}$$

Thus, taking b sufficiently small, and recalling the positive contributions from (6.36) and (6.34), we have:

$$3b \int\int \epsilon u_R \hat{v}^2 x^{-2b-1} - \frac{5b}{2} \int\int \epsilon u_R \hat{v}^2 x^{-2b-1} = \frac{b}{2} \int\int \epsilon u_R \hat{v}^2 x^{-2b-1}. \tag{6.42}$$

The remaining terms from (6.34) and (6.36) are then estimated in terms of (6.42) using the smallness of $\mathcal{O}(\delta)$. Summarizing, we have established control over:

$$\int\int \epsilon u_R \hat{v}_x \cdot \left[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\right] \gtrsim \int\int b\epsilon\hat{v}^2 x^{-1-2b} \tag{6.43}$$

for a constant independent of small δ and b. We will now move to the second term from (6.31), for which we recall estimates (1.10) and (1.19):

$$\begin{aligned} \left|\int\int \epsilon v_{Rx}\hat{u} \cdot \left[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\right]\right| &\leq \sqrt{\epsilon}\|v_{Rx}x^{\frac{3}{2}}\|_{L^\infty}\left\|\frac{\hat{u}}{x^{\frac{3}{4}-b}}\right\|_{L^2}\left\|\sqrt{\epsilon}\frac{\hat{v}}{x^{\frac{3}{4}-b}}\right\|_{L^2} \\ &\leq \sqrt{\epsilon}\|\hat{u}_x x^{\frac{1}{4}-b}\|_{L^2}\|\sqrt{\epsilon}\hat{v}_x x^{\frac{1}{4}-b}\|_{L^2}. \end{aligned} \tag{6.44}$$

For the third term from (6.31), we use Young's inequality and estimates (1.12), (1.22):

$$\begin{aligned} &\left|\int\int \epsilon v_R \hat{v}_y\left[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\right]\right| \\ &\leq \|v_R x^{\frac{1}{2}}\|_{L^\infty}\left[\|\hat{v}_y x^{\frac{1}{2}-b}\|_{L^2}^2 + \|\sqrt{\epsilon}\hat{v}x^{-b-\frac{1}{2}}\|_{L^2}^2 + \|\sqrt{\epsilon}\hat{\psi}x^{-b-\frac{3}{2}}\|_{L^2}^2\right] \\ &\leq \mathcal{O}(\delta)\left[\|\hat{v}_y x^{\frac{1}{2}-b}\|_{L^2}^2 + \|\sqrt{\epsilon}\hat{v}x^{-b-\frac{1}{2}}\|_{L^2}^2 + \|\sqrt{\epsilon}\hat{\psi}x^{-b-\frac{3}{2}}\|_{L^2}^2\right]. \end{aligned} \tag{6.45}$$

For the final term from (6.31), we use Young's inequality and estimates (1.12), (1.22):

$$\begin{aligned} &\left|\int\int \epsilon v_{Ry}\hat{v} \cdot \left[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\right]\right| \\ &\lesssim \|v_{Ry}x\|_{L^\infty}\left[\|\sqrt{\epsilon}\hat{v}x^{-b-\frac{1}{2}}\|_{L^2}^2 + b\|\sqrt{\epsilon}\hat{\psi}x^{-b-\frac{3}{2}}\|_{L^2}^2\right]. \end{aligned} \tag{6.46}$$

Summarizing these last few terms, we obtain:

$$|(6.32)| \gtrsim \int\int b\epsilon\hat{v}x^{-1-2b} + \mathcal{O}(\delta)\left[\|\{\hat{u}, \sqrt{\epsilon}v\}x^{\frac{1}{2}-b}\|_{L^2}^2 + \|\hat{v}_y x^{\frac{1}{2}-b}\|_{L^2}^2\right]. \tag{6.47}$$

The final task is to turn to the right-hand side. Reading from (6.6), and (6.2)–(6.3):

$$\int\int (\hat{f} \cdot \phi_y + \epsilon \hat{g} \cdot \phi_x) = \int\int (\hat{f} \cdot \hat{u}^{(n)} x^{-2b} + \epsilon \hat{g} \cdot [\hat{v}^{(n)} x^{-2b} + \hat{\psi}^{(n)} \partial_x x^{-2b}]) \xrightarrow{n \to \infty} \int\int (\hat{f} \cdot \hat{u}^{(n)} x^{-2b} + \epsilon \hat{g} \cdot [\hat{v} x^{-2b} + \hat{\psi} \partial_x x^{-2b}]), \tag{6.48}$$

where we have passed to the limit using again the definition of X_1 from (1.40). Combining (6.21), (6.30), (6.47), and (6.48), one obtains the desired result, estimate (6.8). □

We now repeat the positivity estimate, with a correspondingly weaker weight in order to close the above energy estimate. We refer the reader to Proposition 3.4 in [4] for a comparison.

Lemma 6.2 *Fix any* $0 < b < 1$*. Let* δ, ϵ *be sufficiently small relative to universal constants, and* $\epsilon << \delta$*. Then for* $[\hat{u}, \hat{v}] \in Z$ *solutions to* (6.2)–(6.3) *with boundary conditions* (6.4) *satisfy the following estimate:*

$$\|\{\hat{u}_x, \sqrt{\epsilon}\hat{v}_x\} x^{\frac{1}{2}-b}\|_{L^2}^2 \lesssim \|\hat{u}_y x^{-b}\|_{L^2}^2 + \|\{\sqrt{\epsilon}\hat{v}, \hat{u}\} x^{-\frac{1}{2}-b}\|_{L^2}^2 + \mathcal{W}_{1,P,b}. \tag{6.49}$$

Proof The estimate will follow upon applying the multiplier $\hat{v} x^{1-2b}$ to the system (6.5). In order to proceed formally, we must start with the weak formulation given in (6.6), and select the test function:

$$\phi = \hat{v}^{(n)} x^{1-2b}, \quad [\hat{u}^{(n)}, \hat{v}^{(n)}] \xrightarrow{X_1} [\hat{u}, \hat{v}], \tag{6.50}$$

where X_1 is defined in (1.40). Turning to the weak formulation in (6.6), we will first expand the second-order terms:

$$\begin{aligned} \int\int \nabla_\epsilon^2 \hat{\psi} : \nabla_\epsilon^2 \phi &= \int\int \nabla_\epsilon^2 \hat{\psi} : \nabla_\epsilon^2 \hat{v}^{(n)} x^{1-2b} \\ &= \int\int \Big(\hat{\psi}_{yy} \hat{v}_{yy}^{(n)} x^{1-2b} + 2\epsilon \hat{\psi}_{xy} \partial_x \big(\hat{v}_y^{(n)} x^{1-2b} \big) + \epsilon^2 \hat{\psi}_{xx} \partial_{xx} \big(\hat{v}^{(n)} x^{1-2b} \big) \Big) \\ &= \int\int \Big(-\hat{u}_y \hat{v}_{yy}^{(n)} x^{1-2b} + 2\epsilon \hat{v}_y \partial_x \big(\hat{v}_y^{(n)} x^{1-2b} \big) + \epsilon^2 \hat{v}_x \partial_{xx} \big(\hat{v}^{(n)} x^{1-2b} \big) \Big). \end{aligned} \tag{6.51}$$

We first arrive at the first two terms on the right-hand side of (6.51):

$$\begin{aligned} &\int\int \big(-\hat{u}_y \hat{v}_{yy}^{(n)} x^{1-2b} - 2\epsilon \hat{u}_x \hat{v}_{xy}^{(n)} x^{1-2b} - 2\epsilon \hat{u}_x \hat{v}_y^{(n)} x^{-2b} \big) \\ &= \int\int \big(-\hat{u}_y^{(n)} \partial_x [\hat{u}_y x^{1-2b}] - 2\epsilon \hat{u}_x^{(n)} \partial_x [\hat{u}_x x^{1-2b}] - 2\epsilon \hat{u}_x \hat{v}_y^{(n)} x^{-2b} \big). \end{aligned} \tag{6.52}$$

Referring to the definition of X_1 in (1.40), according to (6.50), we may pass to the limit as $n \to \infty$, and appeal to the estimates in (1.46) and (1.47), to obtain:

$$\begin{aligned}(6.52)\xrightarrow{n\to\infty} & \int\int \big(-\hat{u}_y \partial_x[\hat{u}_y x^{1-2b}] - 2\epsilon \hat{u}_x \partial_x[\hat{u}_x x^{1-2b}] - 2\epsilon \hat{u}_x \hat{v}_y x^{-2b} \big) \\ &= \int\int \left[-\frac{(1-2b)}{2} \hat{u}_y^2 x^{-2b} + (1+2b)\epsilon \hat{u}_x^2 x^{-2b} \right] \\ &\quad + \lim_{M\to\infty} \int_{x=M} \left[\frac{1}{2} \hat{u}_y^2 x^{1-2b} - \epsilon \hat{u}_x^2 x^{1-2b} \right] \\ &= \int\int \left[-\frac{(1-2b)}{2} \hat{u}_y^2 x^{-2b} + (1+2b)\epsilon \hat{u}_x^2 x^{-2b} \right]. \end{aligned} \tag{6.53}$$

Again referring to the definition in (1.40), the third term from (6.51) is treated by:

$$\begin{aligned} \int\int \epsilon^2 \hat{v}_x \partial_{xx}(\hat{v}^{(n)} x^{1-2b}) &= -\int\int \epsilon^2 \hat{v}_{xx} \partial_x (\hat{v}^{(n)} x^{1-2b}) \\ &\xrightarrow{n\to\infty} -\int\int \epsilon^2 \hat{v}_{xx} \partial_x (\hat{v} x^{1-2b}) \\ &= \int\int -\epsilon^2 \hat{v}_{xx} \hat{v}_x x^{1-2b} - \int\int \epsilon^2 \hat{v}_{xx} \hat{v}(1-2b) x^{-2b}. \end{aligned} \tag{6.54}$$

Integrating by parts the first term on the right-hand side of (6.54), and appealing to estimate (1.47):

$$\begin{aligned} \int\int -\epsilon^2 \hat{v}_{xx} \hat{v}_x x^{1-2b} &= \int\int \epsilon^2 \frac{1-2b}{2} |\hat{v}_x|^2 x^{-2b} - \lim_{M\to\infty} \int_{x=M} \frac{\epsilon^2}{2} \hat{v}_x^2 x^{1-2b} \\ &= \int\int \epsilon^2 \frac{1-2b}{2} |\hat{v}_x|^2 x^{-2b}. \end{aligned} \tag{6.55}$$

Integrating by parts the second term on the right-hand side of (6.54), and again appealing to estimates (1.46)–(1.48) for the M-limit below:

$$
\begin{aligned}
&- \int\int \epsilon^2 \hat{v}_{xx}\hat{v}(1-2b)x^{-2b} \\
&\quad = \int\int \epsilon^2(1-2b)\hat{v}_x \partial_x[\hat{v}x^{-2b}] + \lim_{M\to\infty}\int_{x=M} \epsilon^2(1-2b)\hat{v}_x\hat{v}x^{-2b} \\
&\quad = \int\int \epsilon^2(1-2b)\hat{v}_x^2 x^{-2b} - \int\int \epsilon^2 2b(1-2b)\hat{v}_x\hat{v}x^{-2b-1} \\
&\quad = \int\int \epsilon^2(1-2b)\hat{v}_x^2 x^{-2b} + \int\int \epsilon^2 b(1-2b)\hat{v}^2\partial_x x^{-2b-1} \\
&\qquad - \lim_{M\to\infty}\int_{x=M} \epsilon^2 b(1-2b)\hat{v}^2 x^{-2b-1} \\
&\quad = \int\int \epsilon^2(1-2b)\hat{v}_x^2 x^{-2b} - \int\int \epsilon^2 b(1-2b)(2b+1)\hat{v}^2 x^{-2b-2}.
\end{aligned}
\tag{6.56}
$$

Combining the above estimates:

$$
(6.54) = \int\int \left[\frac{3}{2}(1-2b)\epsilon^2\hat{v}_x^2 x^{-2b} - b(2b+1)(1-2b)\epsilon^2\hat{v}^2 x^{-2b-2}\right]. \tag{6.57}
$$

Hence, summarizing (6.52)–(6.57):

$$
\left|\lim_{n\to\infty}\int\int \nabla_\epsilon^2\hat{\psi} : \nabla_\epsilon^2\phi\right| = \left|\int\int \nabla_\epsilon^2\hat{\psi} : \nabla_\epsilon^2(\hat{v}x^{1-2b})\right| \lesssim \int\int [\epsilon^2\hat{v}_x^2 + \epsilon\hat{u}_x^2 + \hat{u}_y^2]x^{-2b}. \tag{6.58}
$$

We will now turn to the profile terms from S_u, which upon consultation with (6.6), the definition in (1.40), and (6.50), read:

$$
\begin{aligned}
&\int\int \left[u_R\hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R\hat{u}_y\right]\cdot \hat{u}_x^{(n)}x^{1-2b} \\
&\quad \xrightarrow{n\to\infty} \int\int \left[u_R\hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R\hat{u}_y\right]\cdot \hat{u}_x x^{1-2b}.
\end{aligned}
\tag{6.59}
$$

We now turn our attention to (6.59). The first term yields the desired positivity:

$$
\int\int u_R\hat{u}_x^2 x^{1-2b} \gtrsim \min u_R \int\int \hat{u}_x^2 x^{1-2b}. \tag{6.60}
$$

Next, by (1.14), (1.20):

$$
\begin{aligned}
\left|\int\int u_{Rx}\hat{u}\hat{u}_x x^{1-2b}\right| &\le \|u_{Rx}x\|_{L^\infty}\|\hat{u}_x x^{\frac{1}{2}-b}\|_{L^2}\|\hat{u}x^{-\frac{1}{2}-b}\|_{L^2} \\
&\le \mathcal{O}(\delta)\|\hat{u}_x x^{\frac{1}{2}-b}\|_{L^2}^2.
\end{aligned}
\tag{6.61}
$$

Next, we shall split $u_R = u_R^P + u_R^E$, and use estimate (1.16) and (1.18) for (6.62) below and (1.20) for (6.63) below:

$$\left| \int\int u_{Ry}^P \hat{v} \hat{u}_x x^{1-2b} \right| \leq \|y u_{Ry}^P\|_{L^\infty} \|\hat{v}_y x^{\frac{1}{2}-b}\|_{L^2}^2, \tag{6.62}$$

$$\begin{aligned} \left| \int\int \sqrt{\epsilon} u_{RY}^E \hat{v} \hat{u}_x x^{1-2b} \right| &\leq \|u_{RY}^E x^{\frac{3}{2}}\|_{L^\infty} \|\sqrt{\epsilon} \hat{v} x^{-\frac{1}{2}-b}\|_{L^2} \|\hat{u}_x x^{\frac{1}{2}-b}\|_{L^2} \\ &\lesssim \sqrt{\epsilon} \Big[\|\sqrt{\epsilon} \hat{v} x^{-\frac{1}{2}-b}\|_{L^2}^2 + \|\hat{u}_x x^{\frac{1}{2}-b}\|_{L^2}^2 \Big]. \end{aligned} \tag{6.63}$$

For the fourth term from (6.59), by estimates (1.12) and (1.22):

$$\begin{aligned} \left| \int\int v_R \hat{u}_y \hat{u}_x x^{1-2b} \right| &\leq \|v_R x^{\frac{1}{2}}\|_{L^\infty} \|\hat{u}_y x^{-b}\|_{L^2} \|\hat{u}_x x^{\frac{1}{2}-b}\|_{L^2} \\ &\leq \mathcal{O}(\delta) \|\hat{u}_y x^{-b}\|_{L^2} \|\hat{u}_x x^{\frac{1}{2}-b}\|_{L^2}. \end{aligned} \tag{6.64}$$

Summarizing the last four calculations:

$$|(6.59)| \gtrsim \int\int \hat{u}_x^2 x^{1-2b} - \mathcal{O}(\delta) \Big[\|\hat{u} x^{-\frac{1}{2}-b}\|_{L^2}^2 + \|\hat{u}_y x^{-b}\|_{L^2}^2 + \|\sqrt{\epsilon} \hat{v} x^{-\frac{1}{2}-b}\|_{L^2}^2 \Big]. \tag{6.65}$$

The final three terms appearing on the right-hand side above all appear on the right-hand side of estimate (6.49). Turning now to the profile terms, from S_v, for which we read (6.6) with $\phi = \hat{v}^{(n)} x^{1-2b}$, appeal to (1.40) and (6.50), giving ultimately:

$$\begin{aligned} &\int\int \epsilon \big[u_R \hat{v}_x + v_{Rx} \hat{u} + v_R \hat{v}_y + v_{Ry} \hat{v} \big] \cdot \partial_x [\hat{v}^{(n)} x^{1-2b}] \\ &\xrightarrow{n \to \infty} \int\int \epsilon \big[u_R \hat{v}_x + v_{Rx} \hat{u} + v_R \hat{v}_y + v_{Ry} \hat{v} \big] \cdot \partial_x [\hat{v} x^{1-2b}]. \end{aligned} \tag{6.66}$$

We will treat each term in (6.66). For the first term from (6.66):

$$\begin{aligned} &\int\int \epsilon u_R \hat{v}_x \big(\hat{v}_x x^{1-2b} + (1-2b) \hat{v} x^{-2b} \big) \\ &= \int\int (\epsilon u_R \hat{v}_x^2 x^{1-2b} + b(1-2b) u_R \epsilon \hat{v}^2 x^{-1-2b}) - \int\int \frac{1-2b}{2} \epsilon u_{Rx} \hat{v}^2 x^{-2b} \\ &\gtrsim \int\int \epsilon \hat{v}_x^2 x^{1-2b} + b \int\int \epsilon \hat{v}^2 x^{-1-2b}. \end{aligned} \tag{6.67}$$

Above we have used (1.14) and (1.20). For the second term, we integrate by parts:

$$\begin{aligned}\int\int \epsilon v_{Rx}\hat{u}\partial_x[\hat{v}x^{1-2b}] &= \int\int -\epsilon\partial_x\big(v_{Rx}\hat{u}\big)\cdot\hat{v}x^{1-2b} + \lim_{M\to\infty}\int_{x=M} v_{Rx}\hat{u}\hat{v}x^{1-2b} \\ &= \int\int (-\epsilon v_{Rxx}\hat{u}\hat{v}x^{1-2b} - \epsilon v_{Rx}\hat{v}\hat{u}_x x^{1-2b}) \\ &\le \sqrt{\epsilon}\|v_{Rx}x^{\frac{3}{2}}, v_{Rxx}x^{\frac{5}{2}}\|_{L^\infty}\Big[\|\hat{u}x^{-\frac{1}{2}-b}\|_{L^2}^2 \\ &\quad + \|\hat{u}_x x^{\frac{1}{2}-b}\|_{L^2}^2 + \|\sqrt{\epsilon}\hat{v}x^{-\frac{1}{2}-b}\|_{L^2}^2\Big]. \end{aligned} \tag{6.68}$$

The above M-limit vanishes according to estimates (1.46), and we have used estimates (1.10) and (1.19). For the third term, we recall estimates (1.12), (1.22):

$$\begin{aligned}&\int\int (\epsilon v_R\hat{v}_y\hat{v}_x x^{1-2b} + c_0\epsilon v_R\hat{v}_y\hat{v}x^{-2b}) \\ &\quad\le \sqrt{\epsilon}\|v_R x^{\frac{1}{2}}\|_{L^\infty}\Big[\|\hat{v}_y x^{\frac{1}{2}-b}\|_{L^2}^2 + \|\sqrt{\epsilon}\hat{v}_x x^{\frac{1}{2}-b}\|_{L^2}^2 + \|\sqrt{\epsilon}\hat{v}x^{-\frac{1}{2}-b}\|_{L^2}^2\Big].\end{aligned} \tag{6.69}$$

For the fourth term, we integrate by parts and appeal to (1.46), (1.10)–(1.12), and (1.19):

$$\begin{aligned}\int\int \epsilon v_{Ry}\hat{v}\cdot\partial_x[\hat{v}x^{1-2b}] &= -\int\int \epsilon\partial_x[v_{Ry}\hat{v}]\cdot\hat{v}x^{1-2b} + \lim_{M\to\infty}\int_{x=M}\epsilon v_{Ry}\hat{v}^2x^{1-2b} \\ &= \int\int (-\epsilon v_{Rxy}\hat{v}^2x^{1-2b} - \epsilon v_{Ry}\hat{v}_x\hat{v}x^{1-2b}) \\ &\le \|v_{Ry}x, v_{Rxy}x^2\|_{L^\infty}\Big[\|\sqrt{\epsilon}\hat{v}_x x^{\frac{1}{2}-b}\|_{L^2}^2 + \|\sqrt{\epsilon}\hat{v}x^{-\frac{1}{2}-b}\|_{L^2}^2\Big].\end{aligned} \tag{6.70}$$

Summarizing these four terms,

$$\begin{aligned}|(6.66)| \gtrsim &\int\int \epsilon\hat{v}_x^2x^{1-2b} + b\int\int \epsilon\hat{v}^2x^{-1-2b} \\ &- \mathcal{O}(\delta)\Big[\|\hat{u}x^{-\frac{1}{2}-b}, \hat{u}_x x^{\frac{1}{2}-b}\|_{L^2}^2 + \|\sqrt{\epsilon}\hat{v}x^{-\frac{1}{2}-b}, \sqrt{\epsilon}\hat{v}_x x^{\frac{1}{2}-b}\|_{L^2}^2\Big].\end{aligned} \tag{6.71}$$

On the right-hand side, appealing again to (6.50), and the definitions of $\hat{f}, \hat{g}$ in (6.2)–(6.3), one obtains:

$$\int\int\Big(\hat{f}\hat{u}_x^{(n)}x^{1-2b}+\hat{g}\big[\hat{v}_x^{(n)}x^{1-2b}+(1-2b)\hat{v}^{(n)}x^{-2b}\big]\Big) \\ \xrightarrow{n\to\infty}\int\int\Big(\hat{f}\hat{u}_x x^{1-2b}+\hat{g}\big[\hat{v}_x x^{1-2b}+(1-2b)\hat{v}x^{-2b}\big]\Big). \tag{6.72}$$

Placing the above estimates together yields the estimate (6.49). □

We will now introduce some notations involving the weaker weight of x^{-b}. The reader should recall the definitions of the cutoff functions introduced in (1.38)–(1.39). The energy norms are defined as follows:

$$\|u,v\|_{X_{1,b}}^2 := \|u_y x^{-b}\|_{L^2}^2 + \|\{\sqrt{\epsilon}v_x, v_y\}x^{\frac{1}{2}-b}\|_{L^2}^2 \tag{6.73}$$

$$\|u,v\|_{X_{2,b}}^2 := \|u_{xy}\cdot\rho_2 x^{1-b}\|_{L^2}^2 + \|\{\sqrt{\epsilon}v_{xx}, v_{xy}\}\cdot\rho_2^{\frac{3}{2}}x^{\frac{3}{2}-b}\|_{L^2}^2, \tag{6.74}$$

$$\|u,v\|_{X_{3,b}}^2 := \|u_{xxy}\cdot\rho_3^2 x^{2-b}\|_{L^2}^2 + \|\{\sqrt{\epsilon}v_{xxx}, v_{xxy}\}\cdot\rho_3^{\frac{5}{2}}x^{\frac{5}{2}-b}\|_{L^2}^2. \tag{6.75}$$

Definition 6.3 The norms $Y_{2,b}, Y_{3,b}$ are strengthenings of $X_{2,b}, X_{3,b}$ near the boundary, $x=1$, and defined through:

$$\|u,v\|_{Y_{2,b}}^2 := \|u_{xy}x^{1-b}\|_{L^2}^2 + \|\{\sqrt{\epsilon}v_{xx}, v_{xy}\}x^{\frac{3}{2}-b}\|_{L^2}^2 + \|u_{yy}\|_{L^2(x\le 2000)}, \tag{6.76}$$

$$\|u,v\|_{Y_{3,b}}^2 := \|u_{xxy}\cdot\zeta_3 x^{2-b}\|_{L^2}^2 + \|\{\sqrt{\epsilon}v_{xxx}, v_{xxy}\}\cdot\zeta_3 x^{\frac{5}{2}-b}\|_{L^2}^2. \tag{6.77}$$

Definition 6.4 The norm Z_b is defined through:

$$\begin{aligned}\|u,v\|_{Z_b} := {} & \|u,v\|_{X_{1,b}\cap X_{2,b}\cap X_{3,b}} + \epsilon^{N_2}\|u,v\|_{Y_{2,b}} + \epsilon^{N_3}\|u,v\|_{Y_{3,b}} \\ & + \epsilon^{N_4}\|ux^{\frac{1}{4}-b}, \sqrt{\epsilon}vx^{\frac{1}{2}-b}\|_{L^\infty} + \epsilon^{N_5}\sup_{x\ge 20}\|\sqrt{\epsilon}v_x x^{\frac{3}{2}-b}, u_x x^{\frac{5}{4}-b}\|_{L^\infty} \\ & + \epsilon^{N_6}\sup_{x\ge 20}\|u_y x^{\frac{1}{2}-b}\|_{L^2_y} + \epsilon^{N_7}\left[\int_{20}^\infty x^{4-b}\|\sqrt{\epsilon}v_{xx}\|_{L^\infty_y}^2\,dx\right]^{\frac{1}{2}}.\end{aligned} \tag{6.78}$$

Next, we record the second- and third-order versions of the energy and positivity estimates, which mimic Propositions 3.8, 3.13, 3.16 and 3.18 in [4]. We will omit most details, and record only those differences which arise.

Lemma 6.5 (Second-Order Energy Estimate) *Fix any $0 < b < 1$. Let δ, ϵ be sufficiently small relative to universal constants, and $\epsilon << \delta$. Then for $[\hat{u}, \hat{v}] \in Z$ solutions to* (6.2)–(6.3):

$$\|\hat{u}_{xy}\rho_2 x^{1-b}\|_{L^2}^2 \lesssim \mathcal{O}(\delta)\|\{\sqrt{\epsilon}\hat{v}_{xx}, \hat{v}_{xy}\}\rho_2^{\frac{3}{2}} x^{\frac{3}{2}-b}\|_{L^2}^2 + \|\hat{u}, \hat{v}\|_{X_{1,b}}^2 + \mathcal{W}_{1,b} + \mathcal{W}_{2,E,b}, \tag{6.79}$$

where [*recall the definition of ρ_2 from* (1.39)]:

$$\mathcal{W}_{2,E,b} := \int\int \hat{f}_x \hat{u}_x \rho_2^2 x^{2-2b} + \int\int \epsilon \hat{g}_x \hat{v}_x \rho_2^2 x^{2-2b}, \tag{6.80}$$

$$\mathcal{W}_{2,P,b} = \int\int \hat{f}_x \hat{u}_{xx} \rho_2^3 x^{3-2b} + \int\int \epsilon \hat{g}_x \hat{v}_{xx} \rho_2^3 x^{3-2b}, \tag{6.81}$$

$$\mathcal{W}_{2,b} := \mathcal{W}_{2,E,b} + \mathcal{W}_{2,P,b}. \tag{6.82}$$

Proof Differentiating the weak formulation gives:

$$\begin{aligned}\int\int \nabla_\epsilon^2 \hat{\psi}_x : \nabla_\epsilon^2 \phi - \int\int \partial_x S_u(\hat{u}, \hat{v}) \cdot \phi_y + \int\int \epsilon \partial_x S_v(\hat{u}, \hat{v}) \cdot \phi_x \\ = \int\int \epsilon^{\frac{n}{2}+\gamma}\big[-\partial_x \hat{f} \phi_y + \epsilon \partial_x \hat{g} \phi_x\big].\end{aligned} \tag{6.83}$$

For the second-order energy estimate, we select $\phi = \rho_2^2 \hat{v}^{(n)} x^{2-2b}$, where:

$$[\hat{u}^{(n)}, \hat{v}^{(n)}] \in C_{0,D}^\infty, \quad [\hat{u}^{(n)}, \hat{v}^{(n)}] \xrightarrow{X_1} [\hat{u}, \hat{v}]. \tag{6.84}$$

Let us turn to the highest order terms:

$$\begin{aligned}\int\int \nabla_\epsilon^2 \hat{\psi}_x : \nabla_\epsilon^2 \phi &= \int\int \nabla_\epsilon^2 \hat{v} : \nabla_\epsilon^2 \rho_2^2 \hat{v}^{(n)} x^{2-2b} \\ &= \int\int (\hat{v}_{yy} \rho_2^2 \hat{v}_{yy}^{(n)} x^{2-2b} + 2\epsilon \hat{v}_{xy} \partial_x [\hat{v}_y^{(n)} \rho_2^2 x^{2-2b}] + \epsilon^2 \hat{v}_{xx} \partial_{xx} [\rho_2^2 \hat{v}^{(n)} x^{2-2b}]) \\ &= \int\int (\hat{u}_{xy} \rho_2^2 \hat{u}_{xy}^{(n)} x^{2-2b} + 2\epsilon \hat{v}_{xy} \partial_x [\hat{v}_y^{(n)} \rho_2^2 x^{2-2b}] + \epsilon^2 \hat{v}_{xx} \partial_{xx} [\rho_2^2 \hat{v}^{(n)} x^{2-2b}]) \\ &= \int\int (-\partial_x [\hat{u}_{xy} \rho_2^2 x^{2-2b}] \hat{u}_y^{(n)} - 2\epsilon \hat{v}_{xxy} \hat{v}_y^{(n)} \rho_2^2 x^{2-2b} - \epsilon^2 \hat{v}_{xxx} \partial_x [\rho_2^2 \hat{v}^{(n)} x^{2-2b}]).\end{aligned} \tag{6.85}$$

One now checks according to the definition (1.40), that (6.84) suffices to pass to the limit in the above identity, which upon integrating by parts in x yields:

$$(6.85) \xrightarrow{n\to\infty} \int\int (-\partial_x[\hat{u}_{xy}\rho_2^2 x^{2-2b}]\hat{u}_y - 2\epsilon\hat{v}_{xxy}\hat{v}_y\rho_2^2 x^{2-2b} - \epsilon^2\hat{v}_{xxx}\partial_x[\rho_2^2\hat{v}x^{2-2b}])$$
$$= \int\int [\hat{u}_{xy}^2 + \epsilon\hat{u}_{xx}^2 + \epsilon^2\hat{v}_{xx}^2]\rho_2^2 x^{2-2b} + J,$$

where $|J| = |\iint (c_0\epsilon^2\hat{v}_x^2\partial_{xx}(\rho_2^2 x^{2-2b}) + c_1\epsilon^2\hat{v}^2\partial_x^4(\rho_2^2 x^{2-2b}))| \lesssim \|u, v\|_{X_{1,b}}^2$. From here, repeating the calculations in Proposition 3.8 of [4] gives the desired result, where the required integrations by parts are justified upon using that $b > 0$, combined with the estimates in (1.46)–(1.48). These justifications are analogous to those in Lemma 6.1, and so we omit the details. □

Lemma 6.6 (Second-Order Positivity Estimate) *Fix any* $0 < b < 1$. *Let* δ, ϵ *be sufficiently small relative to universal constants, and* $\epsilon << \delta$. *Then for* $[\hat{u}, \hat{v}] \in Z$ *solutions to* (6.2)–(6.3):

$$\|\{\sqrt{\epsilon}\hat{v}_{xx}, \hat{v}_{xy}\}\rho_2^{\frac{3}{2}} x^{\frac{3}{2}-b}\|_{L^2}^2 \lesssim \|\hat{u}_{xy}\rho_2 x^{1-b}\|_{L^2}^2 + \|\hat{u}, \hat{v}\|_{X_{1,b}}^2 + \mathcal{W}_{1,b} + \mathcal{W}_{2,b}. \quad (6.86)$$

Proof We start again with the weak formulation in (6.83). Fix a large $0 < L < \infty$. We then make the selection: $\phi = \hat{v}_x^{(n)} \cdot \rho_2^3 x_L^{3-2b}$, where the weight x_L is defined via: $x_L := (a_L * \phi_L)\chi(\frac{x}{10L})$. Define the domain: $\Omega_L := \{x : 3 < x < 50L + 100\}$, so that $\hat{v}_x \cdot \rho_2^3 x_L^{3-2b} = 0$ on Ω_L^C. The sequence $\hat{v}^{(n)}$ is selected according to:

$$[\hat{u}_x^{(n)}, \hat{v}_x^{(n)}] \in C_{0,D}^\infty(\Omega_L), \quad [\hat{u}_x^{(n)}, \hat{v}_x^{(n)}] \xrightarrow{H^1(\Omega_L)} [\hat{u}_x, \hat{v}_x]. \quad (6.87)$$

The existence of such a sequence is guaranteed due to the standard Sobolev space theory, because we are now in the un-weighted setting. It is now straightforward to repeat all estimates in Proposition 3.13 of [4] using the test function ϕ. Upon doing so, we pass to the limit first as $n \to \infty$, and then as $L \to \infty$ to obtain the desired estimate. □

Lemma 6.7 (Third-Order Energy Estimate) *Fix any* $0 < b < 1$. *Let* δ, ϵ *be sufficiently small relative to universal constants, and* $\epsilon << \delta$. *Then for* $[\hat{u}, \hat{v}] \in Z$ *solutions to* (6.2)–(6.3):

$$\|\hat{u}_{xxy}\rho_3^2 x^{2-b}\|_{L^2}^2 \lesssim \mathcal{O}(\delta)\|\{\sqrt{\epsilon}\hat{v}_{xxx}, \hat{v}_{xxy}\}\rho_3^{\frac{5}{2}} x^{\frac{5}{2}-b}\|_{L^2}^2$$
$$+ \|\hat{u}, \hat{v}\|_{X_{1,b}\cap X_{2,b}}^2 + \sum_{i=1}^{2} \mathcal{W}_{i,b} + W_{3,E,b}, \quad (6.88)$$

where

$$\mathcal{W}_{3,E,b} := \int\int \hat{f}_{xx}\hat{u}_{xx}\rho_3^4 x^{4-2b} + \int\int \epsilon\hat{g}_{xx}\hat{v}_{xx}\rho_3^4 x^{4-2b}, \quad (6.89)$$

$$\mathcal{W}_{3,P,b} := \int\int \hat{f}_{xx}\hat{u}_{xxx}\rho_3^5 x^{5-2b} + \int\int \epsilon\hat{g}_{xx}\hat{v}_{xxx}\rho_3^5 x^{5-2b}, \tag{6.90}$$

$$\mathcal{W}_{3,b} := W_{3,E,b} + W_{3,P,b}. \tag{6.91}$$

Proof The first step is to differentiate the weak formulation (6.83) yet again, which formally takes place using difference quotients, yielding:

$$\begin{aligned}\int\int \nabla_\epsilon^2 \psi_{xx} : \nabla_\epsilon^2 \phi - \int\int \partial_{xx} S_u(\hat{u},\hat{v}) \cdot \phi_y + \int\int \epsilon \partial_{xx} S_v(\hat{u},\hat{v}) \cdot \phi_x \\ = \int\int \epsilon^{\frac{n}{2}+\gamma}\big[-\partial_{xx}\hat{f}\phi_y + \epsilon\partial_{xx}\hat{g}\phi_x\big].\end{aligned} \tag{6.92}$$

Fix any L large, finite. The selection of test function is now $\phi := \hat{v}_x^{(n)}\rho_3^4 x_L^{4-2b}$, where the sequence:

$$[\hat{u}_{xx}^{(n)}, \hat{v}_{xx}^{(n)}] \in C_{0,D}^\infty(\Omega_L), \quad [\hat{u}_{xx}^{(n)}, \hat{v}_{xx}^{(n)}] \xrightarrow{H^1(\Omega_L)} [\hat{u}_{xx}, \hat{v}_{xx}]. \tag{6.93}$$

From here, repeating the estimates given in Proposition 3.16 of [4], and sending $n \to \infty$ and then $L \to \infty$ gives the desired result. □

Lemma 6.8 (Third-Order Positivity Estimate) *Fix any* $0 < b < 1$. *Let* δ, ϵ *be sufficiently small relative to universal constants, and* $\epsilon << \delta$. *Then for* $[\hat{u}, \hat{v}] \in Z$ *solutions to* (6.2)–(6.3):

$$\|\{\sqrt{\epsilon}\hat{v}_{xxx}, \hat{v}_{xxy}\}\rho_3^{\frac{5}{2}} x^{\frac{5}{2}-b}\|_{L^2}^2 \lesssim \|\hat{u}_{xxy}\rho_3^2 x^{2-b}\|_{L^2}^2 + \|\hat{u},\hat{v}\|_{X_{1,b}\cap X_{2,b}}^2 + \sum_{i=1}^3 \mathcal{W}_{i,b}. \tag{6.94}$$

Proof Again, fix any L large, finite. The selection of the test function is now $\phi := \hat{v}_{xx}^{(n)}\rho_3^5 x_L^{5-2b}$, where the sequence $[\hat{u}^{(n)}, \hat{v}^{(n)}]$ is selected according to:

$$[\hat{u}_{xx}^{(n)}, \hat{v}_{xx}^{(n)}] \in C_{0,D}^\infty(\Omega_L), \quad [\hat{u}_{xx}^{(n)}, \hat{v}_{xx}^{(n)}] \xrightarrow{H^1(\Omega_L)} [\hat{u}_{xx}, \hat{v}_{xx}]. \tag{6.95}$$

From here, repeating the estimates in Proposition 3.18 of [4], and sending $n \to \infty$ and then $L \to \infty$ gives the desired result. □

Piecing together the above set of estimates,

Proposition 6.9 *Let* δ, ϵ *be sufficiently small relative to universal constants, and* $\epsilon << \delta << b$. *Then for* $[\hat{u}, \hat{v}] \in Z$ *solutions to* (6.2)–(6.3):

$$\|\hat{u},\hat{v}\|_{X_{1,b}\cap X_{2,b}\cap X_{3,b}}^2 \lesssim \mathcal{W}_{1,b} + \mathcal{W}_{2,b} + \mathcal{W}_{3,b}, \tag{6.96}$$

where $\mathcal{W}_{i,b}$ *have been defined in* (6.11), (6.82), (6.91).

By repeating the analysis in Section 2 of [4], one has:

Lemma 6.10 *Let δ, ϵ be sufficiently small relative to universal constants, and $\epsilon << \delta << b$. Then for $[\hat{u}, \hat{v}] \in Z$ solutions to (6.2)–(6.3):*

$$\|\hat{u}, \hat{v}\|_{Z_b}^2 \lesssim \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)} \|\hat{u}, \hat{v}\|_{Z_b}^4 + \|\hat{u}, \hat{v}\|_{X_{1,b} \cap X_{2,b} \cap X_{3,b}}^2. \tag{6.97}$$

Due to (6.96), we will now turn to estimating $\mathcal{W}_{i,b}$

Lemma 6.11 *Let $\mathcal{W}_{1,b}, \mathcal{W}_{2,b}, \mathcal{W}_{3,b}$ be as in* (3.10), (3.79) *and* (3.216) *in* [4]. *Then:*

$$|\mathcal{W}_{1,b} + \mathcal{W}_{2,b} + \mathcal{W}_{3,b}| \lesssim C(b) \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)} \|\hat{u}, \hat{v}\|_{Z_b}^2, \tag{6.98}$$

where $C(b) \uparrow \infty$ as $b \downarrow 0$.

Proof We will work with the expression:

$$\begin{aligned} \hat{f} &= \epsilon^{\frac{n}{2}+\gamma} \Big[u^{(1)} u_x^{(1)} - u^{(2)} u_x^{(2)} + v^{(1)} u_y^{(1)} - v^{(2)} u_y^{(2)} \Big] \\ &= \epsilon^{\frac{n}{2}+\gamma} \Big[\hat{u} u_x^{(1)} + u^{(2)} \hat{u}_x + \hat{v} u_y^{(1)} + v^{(2)} \hat{u}_y \Big], \end{aligned} \tag{6.99}$$

$$\begin{aligned} \hat{g} &= \epsilon^{\frac{n}{2}+\gamma} \Big[u^{(1)} v_x^{(1)} - u^{(2)} v_x^{(2)} + v^{(1)} v_y^{(1)} - v^{(2)} v_y^{(2)} \Big] \\ &= \epsilon^{\frac{n}{2}+\gamma} \Big[\hat{u} v_x^{(1)} + u^{(2)} \hat{v}_x + \hat{v} v_y^{(1)} + v^{(2)} \hat{v}_y \Big]. \end{aligned} \tag{6.100}$$

Concerning $\mathcal{W}_{1,b}$, let us bring particular attention to the following term from $\int\int |\hat{f}| \cdot |\hat{u}| x^{-2b}$:

$$\begin{aligned} &\int\int \epsilon^{\frac{n}{2}+\gamma} [\hat{v} u_y^{(1)} + v^{(2)} \hat{u}_y] \cdot |\hat{u}| x^{-2b} \\ &\le \epsilon^{\frac{n}{2}+\gamma} \|\hat{v} x^{\frac{1}{2}-b}\|_{L^\infty} \|u_y^{(1)}\|_{L^2} \|\hat{u} x^{-\frac{1}{2}-b}\|_{L^2} \\ &\quad + \epsilon^{\frac{n}{2}+\gamma} \|v^{(2)} x^{\frac{1}{2}}\|_{L^\infty} \|\hat{u}_y x^{-b}\|_{L^2} \|\hat{u} x^{-\frac{1}{2}-b}\|_{L^2} \\ &\le \epsilon^{\frac{n}{2}+\gamma} \Big[\|\hat{v} x^{\frac{1}{2}-b}\|_{L^\infty} \|u_y^{(1)}\|_{L^2} \|\hat{u}_x x^{\frac{1}{2}-b}\|_{L^2} \\ &\quad + \|v^{(2)} x^{\frac{1}{2}}\|_{L^\infty} \|\hat{u}_y x^{-b}\|_{L^2} \|\hat{u}_x x^{\frac{1}{2}-b}\|_{L^2} \Big] \\ &\le C(b) \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)} \|u^{(i)}, v^{(i)}\|_Z \|\hat{u}, \hat{v}\|_{Z_b}^2. \end{aligned} \tag{6.101}$$

The above term requires the weight of $x^{-2b}, b > 0$, in order to apply the Hardy inequality. Indeed, this was not required for the existence proof (see calculation (4.5) in [4]), because the structure of $vu_y \cdot u$ enabled us to integrate by parts, unlike in the present situation. The remaining terms in $\mathcal{W}_{1,b}$, and all terms in $\mathcal{W}_{2,b}, \mathcal{W}_{3,b}$ are treated nearly identically to Lemma 4.1 of [4], and so we omit repeating those calculations. □

Corollary 6.12 *Fix $0 < b < 1$ sufficiently small, relative to universal constants. Suppose ϵ, δ are sufficiently small, such that $\epsilon << \delta << b$. Then $\hat{u}, \hat{v} = 0$.*

Proof Combining estimate (6.98) and (6.97) with estimate (5.12) yields:

$$\|\hat{u}, \hat{v}\|_{Z_b}^2 \lesssim C(b)\epsilon^{\frac{n}{2}+\gamma-\omega(N_i)}\|\hat{u}, \hat{v}\|_{Z_b}^2. \tag{6.102}$$

For ϵ sufficiently small, this then implies $\|\hat{u}, \hat{v}\|_{Z_b} = 0$. Upon consultation with the norm Z_b, and (6.4), this implies that $\hat{u}, \hat{v} = 0$. □

***Remark* 6.13** We have controlled the second- and third-order energy norms, (6.74)–(6.75) in order to treat the term $\int\int \hat{v}u_y^{(1)}|\hat{u}|x^{-2b}$, which appears in (6.101). This term forces us to control $\|\hat{v}x^{\frac{1}{2}-b}\|_{L^\infty}$. One cannot get around placing this term in L^∞ (for instance by integrating by parts from $u_y^{(1)}$) because this produces suboptimal decay rates, according to (2.92)–(2.93) in [4].

This then immediately establishes Theorem 1.3.

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