Research Article

Nonlinear emergent macroscale PDEs, with error bound, for nonlinear microscale systems

J. E. Bunder[1] · A. J. Roberts[1]

Abstract

Many multiscale physical scenarios have a spatial domain which is large in some dimensions but relatively thin in other dimensions. These scenarios includes homogenization problems where microscale heterogeneity is effectively a 'thin dimension'. In such scenarios, slowly varying, pattern forming, emergent structures typically dominate the large dimensions. Common modelling approximations of the emergent dynamics usually rely on self-consistency arguments or on a nonphysical mathematical limit of an infinite aspect ratio of the large and thin dimensions. Instead, here we extend to nonlinear dynamics a new modelling approach which analyses the dynamics at each cross-section of the domain via a multivariate Taylor series (Roberts and Bunder in IMA J Appl Math 82(5):971–1012, 2017. https://doi.org/10.1093/imamat/hxx021). Centre manifold theory extends the analysis at individual cross-sections to a rigorous global model of the system's emergent dynamics in the large but finite domain. A new remainder term quantifies the error of the nonlinear modelling and is expressed in terms of the interaction between cross-sections and the fast and slow dynamics. We illustrate the rigorous approach by deriving the large-scale nonlinear dynamics of a thin liquid film on a rotating substrate. The approach developed here empowers new mathematical and physical insight and new computational simulations of previously intractable nonlinear multiscale problems.

1 Introduction

Many systems of interest in science and engineering occur in a domain with disparate length scales [10, e.g.]: often a fine structure is modulated on a much larger scale [22, e.g.]. Such disparate scales usually are a major challenge in computational simulations [32, p.14, e.g.]. Two classic examples are Taylor–Couette flow [16, e.g.] and Benard convection [33, e.g.]. Often the fine-scale detail is crucial to the accurate modelling of a multiscale system, but with

multiple length scales a simulation resolving the physical fine scale is not only prohibitively inefficient and severely constrained by memory limitations, it is also an arduous task to analyse simulation data generated at a scale much smaller than the scale of interest. This article further develops a unified mathematical approach and theory to reduce the full set of nonlinear governing equations to a simplified evolution equation, with quantified error, enabling more efficient simulations and analysis. This theoretical methodology should be able to better justify and

✉ J. E. Bunder, judith.bunder@adelaide.edu.au; A. J. Roberts, anthony.roberts@adelaide.edu.au | [1]School of Mathematical Sciences, University of Adelaide, Adelaide, SA, Australia.

illuminate many extant long-wave and homogenisation theories [9, 25, e.g.].

We focus on multiscale systems whose physical domain is 'large' in multiple dimensions, but have a relatively 'thin' cross-section in the other dimensions. As a specific example, Sect. 1.1 takes as prescribed a variant of the integrated boundary layer PDEs for a thin liquid film of Newtonian fluid spreading over a planar rotating surface [7], and rigorously derives a simpler lubrication model of the nonlinear flow of the film [25, 36, §II.K, e.g.]. Appendix A lists computer algebra code for deriving this lubrication model, with the code written to be readily adaptable to a wide range of systems. Thin liquid flows are important in biology, physics, and engineering, as well as in the environment. They may be of common liquids such as water or oil, or of more rheologically complex fluids, and display interesting nonlinear wave patterns [19, 25, 36]. Other examples of systems amenable to our methodology include flood and tsunami modelling [20, 24], pattern formation [9, 35], wave interactions [13], elastic shells [18, 21], and microstructured materials [31].

Section 2 defines the generic nonlinear PDE system to which the methodology applies, and defines the 'large' but 'thin' multiscale domain on which the system evolves. In pattern evolution problems, and in homogenization problems, a 'thin' domain variable is the phase of the underlying small scale pattern [28, §3.3, e.g.]. A first step in the reduction of such PDEs was taken by Mielke [22], but the analysis required solutions to exist for all $\mathbb{R}$-time and for all $\mathbb{R}^m$-space which excludes initial/boundary value problems on finite domains. Here we analyse the dynamics of a general PDE in a thin cross-section of the large, but finite, domain by constructing a multivariate Taylor expansion for the local spatial structures and analysing the evolution of the coefficients. Being the union of local-space-time modelling means the approach is valid everywhere outside of boundary layers [27, e.g.] and initial transients [26, e.g.]. Section 3 details how to capture the emergent behaviour of the system at every chosen cross-section via constructing a set of generalised eigenvectors which span the centre subspace. Based upon these eigenvectors, the system's centre manifold, on which the slow system evolves, is parametrised and a nonlinear PDE derived for the emergent slow evolution. The order of the multivariate Taylor expansion determines the order of accuracy of the derived slow evolution, and Lagrange's Remainder Theorem provides a novel exact expression for the error of this PDE for the emergent slow evolution.

This article significantly extends the approach previously developed to derive the emergent evolution of nonlinear PDE systems on a large one-dimensional physical domain [28] to domains with multiple large dimensions. Based upon the methodology established

for linear systems [30], here we extend the approach to nonlinear PDEs with multiple large dimensions. Our approach is distinctly different to many other methods which derive large-scale models in that here no asymptotic limit is required for the scale separation between the large domain and the thin cross-section [17, 23, e.g.]. Specifically, we have no requirement that a scale separation parameter (often named ϵ) must be asymptotically small; we only require that such a small-large scale separation exists so that we can establish centre and stable subspaces (Assumption 3), and then our approach is valid at finite scale separation.

1.1 Example of a rotating shallow fluid flow

As an example application of some of the results, consider the flow of a shallow layer of fluid on a solid flat rotating substrate, such as in spin coating [25, 36, §II.K, e.g.] or large-scale shallow water waves [11, 15, e.g.]. Let $\vec{x} = (x_1, x_2)$ parametrise location on the rotating substrate, and let the fluid layer have thickness $h(\vec{x}, t)$ and move with depth-averaged horizontal velocity $\vec{v}(\vec{x}, t) = (v_1, v_2)$. We take as given (with its simplified physics) that the (non-dimensional) governing set of PDEs is the nonlinear system

$$\frac{\partial h}{\partial t} = -\nabla \cdot (h\vec{v}), \tag{1a}$$

$$\frac{\partial \vec{v}}{\partial t} = \begin{bmatrix} -b & f \\ -f & -b \end{bmatrix} \vec{v} \tag{1b}$$
$$- (\vec{v} \cdot \nabla)\vec{v} - g\nabla h + \nu\nabla^2\vec{v},$$

where b represents viscous bed drag, f is the Coriolis coefficient, g is the acceleration due to gravity, ν is the kinematic viscosity, and we neglect surface tension. The PDEs (1) are similar to those used by Dellar and Salmon [11, Eq. (79)], but with only one component of the Coriolis force, and the addition of viscous drag and viscosity, and also similar to that used by Hereman [15, Eqs. (22)–(24)], but here with a flat substrate.

For such a shallow fluid flow, the horizontal gradient ∇ of quantities are relatively small [10, e.g.]. Then the flow driven by variations of film thickness, ∇h, is approximately balanced in (1b) by the rotation and the substrate drag, leading to the leading approximate velocity field

$$\vec{v} \approx \frac{-g}{b^2 + f^2} \begin{bmatrix} b & f \\ -f & b \end{bmatrix} \nabla h.$$

Substituting this balance in the conservation of mass equation (1a) derives the single component, 'lubrication', model

$$\frac{\partial h}{\partial t} \approx \frac{gb}{b^2 + f^2} \nabla \cdot (h \nabla h). \tag{2}$$

Having just one component, we can use the PDE model (2) as a simpler description of the shallow fluid dynamics. But PDE (2) is an approximation to the 'original' PDE (1), and so three outstanding questions are: can we find a rigorous error? can the analysis be extended to higher order? and can such an approach apply generally? Our answer to all three questions is yes.

Returning to the original system of PDEs (1) and defining the system field $u(\vec{x}, t) = (h, v_1, v_2)$, we rewrite system (1) as one nonlinear equation, while also grouping terms of like order:

$$\frac{\partial u}{\partial t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -b & f \\ 0 & -f & -b \end{bmatrix} u + \begin{bmatrix} 0 & 0 & 0 \\ -g & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \partial_{x_1} u$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -g & 0 & 0 \end{bmatrix} \partial_{x_2} u \tag{3a}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & v \end{bmatrix} \left[\partial_{x_1}^2 u + \partial_{x_2}^2 u \right] \tag{3b}$$

$$+ \begin{bmatrix} (\partial_{x_1} u)^T \mathfrak{M}_{10,0} u \\ (\partial_{x_1} u)^T \mathfrak{M}_{10,1} u \\ (\partial_{x_1} u)^T \mathfrak{M}_{10,2} u \end{bmatrix} + \begin{bmatrix} (\partial_{x_2} u)^T \mathfrak{M}_{01,0} u \\ (\partial_{x_2} u)^T \mathfrak{M}_{01,1} u \\ (\partial_{x_2} u)^T \mathfrak{M}_{01,2} u \end{bmatrix}, \tag{3c}$$

where matrices

$$\mathfrak{M}_{10,0} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathfrak{M}_{10,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\mathfrak{M}_{10,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$
$$\mathfrak{M}_{01,0} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \mathfrak{M}_{01,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3d}$$
$$\mathfrak{M}_{01,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The system's first two lines (3a) and (3b) are the linear part, whereas the third line (3c) contains the nonlinear quadratic terms which encode inertial acceleration. When the nonlinear terms (3c) are negligible, such as for very viscous fluids with low Reynolds numbers, the method described by Roberts and Bunder [30] derives a slow *linear* PDE

approximation of (3), but the analysis and results of that article do not account for nonlinear effects, such as (3c). Herein we further develop the methodology and theory to account for nonlinear effects.

For the example shallow fluid flow described by (3), the 'large' domain is some 'physical' subset of the $x_1 x_2$-plane, and the 'thin' cross-section is the three-components of $u = (h, v_1, v_2)$. The aim is to capture the long, slow behaviour of the original $u(\vec{x}, t)$ field in a one-component slow field $U(\vec{x}, t)$ (instead of the three components which describe the thin cross-section), and to construct a PDE for $U(\vec{x}, t)$ which is correct to some order N in spatial derivatives, with known error. In general, higher orders N are potentially able to capture more extreme spatial fluctuations but may not be structurally stable, and so we address up to some low–moderate order N. This restricts $U(\vec{x}, t)$ to describing long, relatively gradual, spatial variations of the original field $u(\vec{x}, t)$. Section 3.2 defines the slow field $U(\vec{x}, t)$ in the general case and proves that it describes the behaviour of the original microscale field on the slowly evolving centre manifold (or slow manifold, as is the case in this shallow fluid example).

To determine the nature of the slow field U we first seek to understand the 'slow' and 'fast' evolution of the lowest-order linearisation of the system field u (Assumption 3 elaborates the general case). The linear dynamics of u are dominantly characterised by the lowest order linear term in (3), the term $\mathfrak{L}_{\vec{0}} u$ where here

$$\mathfrak{L}_{\vec{0}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -b & f \\ 0 & -f & -b \end{bmatrix}.$$

The eigenvalues of $\mathfrak{L}_{\vec{0}}$ indicate that u evolves on a one-dimensional slow subspace (one zero eigenvalue) and a two-dimensional stable subspace (two eigenvalues, $-b \pm if$, with negative real part). The stable part of u decays relatively quickly, roughly like e^{-bt}, whereas the slow component of u, namely h, evolves on the one-dimensional slow subspace. In the notation introduced by Assumption 3 and Sect. 3, there exists a slow subspace of dimension $m = 1$ with right and left eigenvectors $V^{\vec{0}} = Z^{\vec{0}} = (1, 0, 0)$, and eigenvalue $A_{\vec{0}} = 0$.

Once the lowest-order linear dynamics of the system field u are known from matrix $\mathfrak{L}_{\vec{0}}$, we construct generalised eigenvectors $\tilde{\mathcal{V}}^{\vec{n}}$, for $|\vec{n}| \leqslant N$, which span the spatially-local slow subspace of the linear system (3a) and (3b) to the specified order of accuracy N. This order of accuracy N is that of a local multivariate Taylor expansion of the field $u(\vec{x}, t)$. The advantages of such a Taylor expansion are that not only does it provide a straightforward way to increase the order of accuracy N, but also Lagrange's Remainder Theorem provides a rigorous error

term. Section 3.1 discusses the general construction of the eigenvectors $\tilde{V}^{\vec{n}}$, which in essence detail spatially-local out-of-equilibrium structures, and then Sect. 3.2 models the full nonlinear system to derive the slow manifold PDE.

Appendix A lists computer algebra Reduce[1] code which applies the general theory of Sects. 2 and 3 to determine the slow PDE for any order N of the Taylor expansion, thus constructing PDEs which describe the slow $U = h$ field evolution of the shallow fluid dynamics to various orders of accuracy. See Sects. 2 and 3 for details that justify (2) for order $N = 2$, and that for order $N = 3$ justify that the slow PDE is the 2D advection-diffusion PDE

$$\frac{\partial h}{\partial t} \approx \sum_{|\vec{\ell}_1|=0}^{3} \sum_{|\vec{\ell}_2|=0}^{|\vec{\ell}_1|} a_{\vec{\ell}_1\vec{\ell}_2}(\partial_{\vec{x}}^{\vec{\ell}_1}h)(\partial_{\vec{x}}^{\vec{\ell}_2}h),$$ (4a)

with multi-indices $\vec{\ell}_{1,2} \in \mathbb{N}_0^2$, symmetry $a_{\vec{\ell}_1\vec{\ell}_2} = a_{\vec{\ell}_2\vec{\ell}_1}$ when $|\vec{\ell}_1| = |\vec{\ell}_2|$, and nonzero constant coefficients[2]

$$a_{(01)(01)} = a_{(10)(10)} = a_{(02)(00)} = a_{(20)(00)} = \frac{bg}{b^2+f^2},$$

$$a_{(03)(01)} = a_{(30)(10)} = a_{(12)(10)} = a_{(21)(01)} = \frac{vg(b^2-f^2)}{(b^2+f^2)^2},$$

$$a_{(03)(10)} = a_{(21)(10)} = -a_{(30)(01)} = -a_{(12)(01)}$$
$$= 2vf\frac{bg}{(b^2+f^2)^2}.$$
(4b)

The four $a_{\vec{\ell}_1\vec{\ell}_2}$ coefficients equal to $bg/(b^2+f^2)$ correspond to the $N = 2$ approximation (2), but the rigorous derivation from Appendix A on the slow subspace empowers a far richer description of the slow dynamics in the x_1x_2-plane. Further, Sect. 3.2 provides an exact expression (44) for the error of slow PDEs such as (4a); Appendix A.3 details how components of this error are constructed for this shallow fluid flow.

Executing the computer algebra code in Appendix A to obtain slow PDEs of higher orders is straightforward. Although the computational time increases rapidly with N, in principle the code is applicable to any order N. For example, the slow PDE for order $N = 4$ is

$$\frac{\partial h}{\partial t} \approx \sum_{|\vec{\ell}_1|=0}^{4} \sum_{|\vec{\ell}_2|=0}^{|\vec{\ell}_1|} a_{\vec{\ell}_1\vec{\ell}_2}(\partial_{\vec{x}}^{\vec{\ell}_1}h)(\partial_{\vec{x}}^{\vec{\ell}_2}h),$$ (5a)

with the same nonzero constant coefficients $a_{\vec{\ell}_1\vec{\ell}_2}$ given in (4b), as well as

$$a_{(40)(00)} = a_{(04)(00)} = \tfrac{1}{2}a_{(22)(00)} = \frac{vg(b^2-f^2)}{(b^2+f^2)^2}.$$ (5b)

For small enough damping, $b < f$, the fourth order hyper-diffusion in (5a) makes the model structurally unstable. As is often necessary, for $b < f$ one would then regularise the model as in the Benjamin, Bona, and Mahony [4] regularised long wave equation. Notwithstanding such practical regularisation, our derived error expression (44) applies and is useful for as long as the spatial gradients in the solutions to (5a) remain small enough.

Figure 1 plots finite difference simulations of the height h for: the original model (1) on a 75×75 grid; the order two slow PDE (2) (lubrication model) on a 15×15 grid; and the order four slow PDE (5) on a 15×15 grid. This coarser resolution of the slow models is appropriate as these PDEs are only accurate for slower variations in space. The example square spatial domain is of width 2π with periodic boundary conditions, and the dimensionless parameters are drag $b = 1$, Coriolis coefficient $f = 5$, gravity $g = 1$ and viscosity $v = 0.5$ (all parameters are nondimensional). We choose a large Coriolis coefficient f as this produces relatively large fourth order coefficients (5b) [although smaller than the second order coefficients shown in the first line of (4b)] and enhances the differences between the order two and order four simulations. For the original model simulation, the initial condition of the height h is a Gaussian peak, wider in the x_1 direction than the x_2 direction, plus a uniformly distributed random component chosen from the interval [0, 0.5] (Fig. 1, top left). The mean of this initial condition in 5×5 blocks provides the initial conditions for both the order two and order four simulations (Fig. 1, top middle and top right). Figure 1 shows that the fluid gradually diffuses across the domain and the two coarse simulations provide reasonable approximations of full model diffusion, although the diffusion of the second order simulation is less accurate.

Figure 2 plots the velocity components of the original model (1) simulation at different times and reveals the dynamics of the flow. All initial velocity components v_1 and v_2 are normally distributed random numbers with mean zero and standard deviation 0.001 (Fig. 2, first row). The system quickly evolves into the expected rotational motion induced by the Coriolis force, with v_1 dominating v_2 due to the asymmetry in the initial fluid height. Although Fig. 2 shows that the velocity plays an important

---

[1] Reduce (http://reduce-algebra.com/) is a free, fast, general purpose, computer algebra system.

[2] Equation (3c) describes two nonlinear terms which, following the derivation in Sect. 3, result in nonlinear terms with constant coefficients $a^j_{\vec{\ell}_1\vec{\ell}_2}$ for $j = 1, 2$. But in this example, since the nonlinear terms are both second order ($P_j = 2$ for $j = 1, 2$), to obtain (4a) we set $a^1_{\vec{\ell}_1\vec{\ell}_2} + a^2_{\vec{\ell}_1\vec{\ell}_2} = a_{\vec{\ell}_1\vec{\ell}_2}$

Fig. 1 Simulations of the fluid height h over a square domain of width 2π with periodic boundary conditions for: (left column) original PDE (1) on a 75×75 grid; (middle column) order two slow PDE (2) on a 15×15 grid; and (right column) order four slow PDE (5) on a 15×15 grid. Plots in the same row are simulated at the same time, and all plots are coloured according to the same colour bar (shown below the row of plots). At any one time, all three plots have the same contour levels, but plots at different times have different contour levels
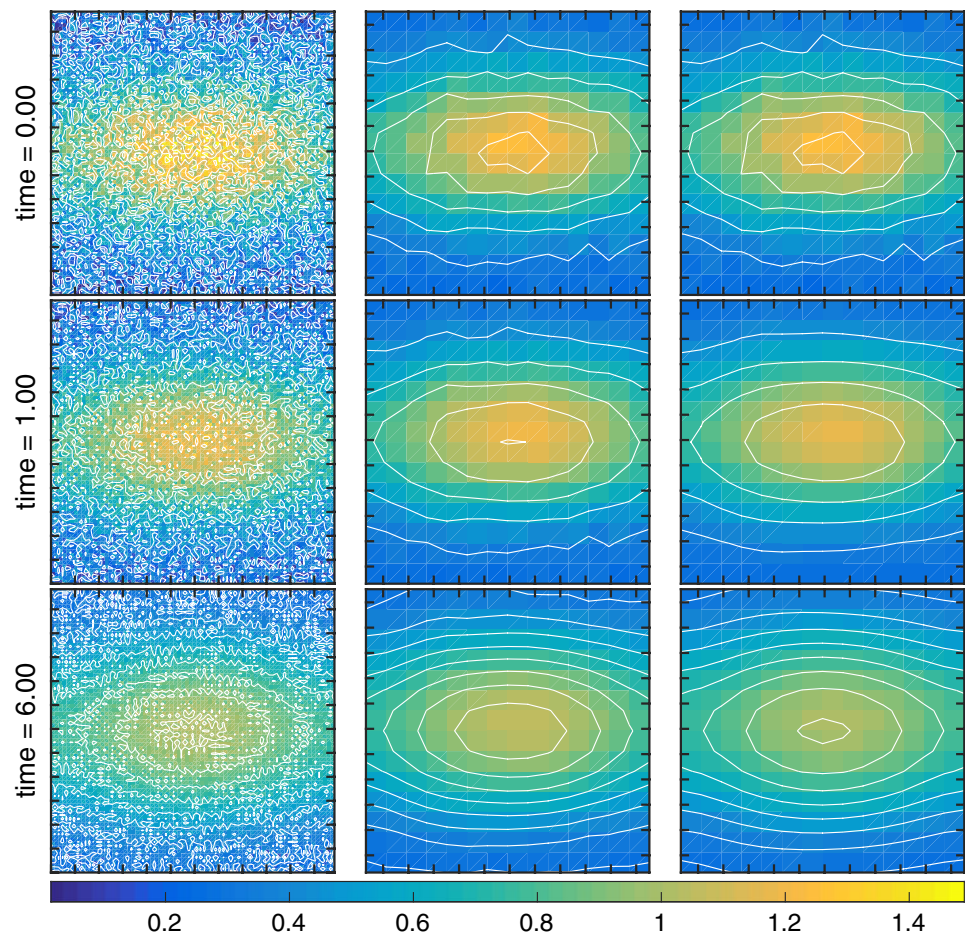
role in explaining the dynamics of this fluid flow model (1), Fig. 1 shows that approximation models such as the fourth order slow PDE (5) accurately predict the slow behaviour of the height h without requiring any simulation of the velocity.

To further explore the accuracy and efficiency of the diffusion in the two coarse approximation simulations of Fig. 1, Fig. 3 plots the fluid height h of the original model (1) averaged over the 5×5 grid in the centre of the domain, and the fluid heights of both the order two slow PDE (2) and the order four slow PDE (5) in the centre of the domain. In general, the fourth order simulation provides a good approximation for the mean height of the full simulation, particularly for times $t \gtrsim 1$, and at $t = 6$ the absolute error of the order four simulation is 3×10^{-3}. In contrast, the second order simulation decays too slowly and produces a mean peak height which is generally larger than the mean peak height of the original PDE, with an absolute error of 5×10^{-2} at $t = 6$. Figure 3 was created using Matlab's ODE solver `ode45`, and for $t \in [0, 6]$ the original model simulation required 4217 time steps, whereas the order two simulation only required 121 and the fourth order required 405, demonstrating that our approximate

PDEs provide a significant improvement in efficiency when restricted to using explicit methods.

The following sections develop theoretical support for the derivation of nonlinear slow PDEs such as (2), (4) and (5), and also derive the novel general exact algebraic expression (44) for the error in such approximate PDEs.

2 Local expansion of general nonlinear dynamics

The local expansion developed here builds on that of Roberts and Bunder [30, §3] by extending it to nonlinear systems. Similarly, Roberts and Bunder [30, §3] generalised the procedure of Roberts [28, §2] from a system with one large dimension and any number of significantly thin dimensions, to a system with some finite number of large dimensions and any number of thin dimensions. For completeness, we here present a full and detailed derivation which simplifies to the derivations of Roberts and Bunder [30, §3] for linear systems and of Roberts [28, §2] for linear systems with a one-dimensional 'large' domain. Thus, necessarily, some of the detailed expressions here reproduce
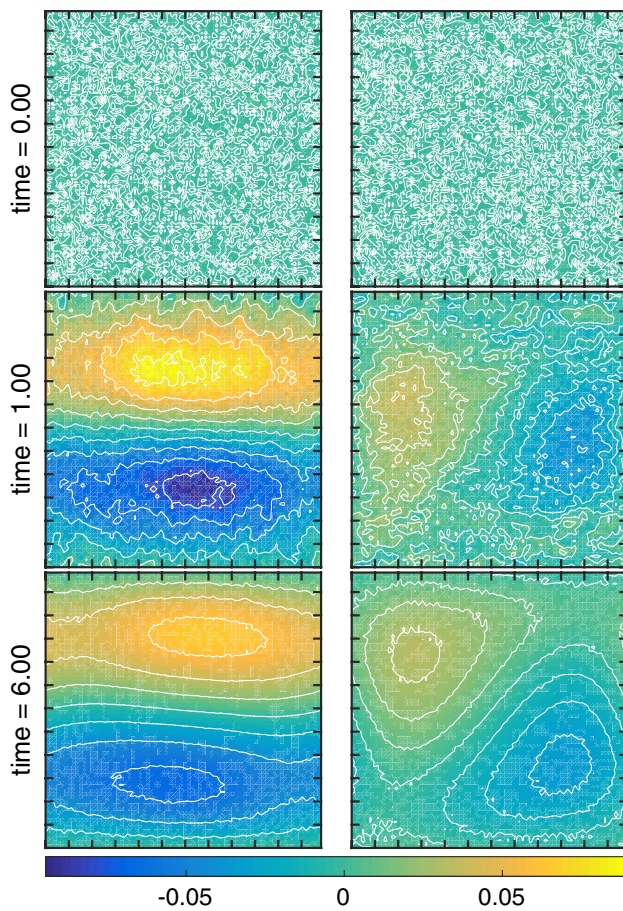
Fig. 2 Simulations of the fluid velocity components (left) v_1 and (right) v_2 over a square domain of width 2π on a 75×75 grid and with periodic boundary conditions. These velocity simulations correspond to the h simulation in the left column of Fig. 1. Plots in the same row are simulated at the same time, and all plots are coloured according to the same colour bar (shown below the row of plots)
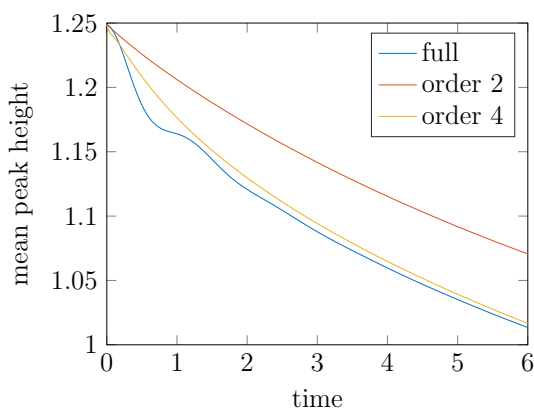

Fig. 3 Mean peak heights of Fig. 1 simulations. For times $t \gtrsim 1$ the order four simulation (Fig. 1, right) provides a good estimation of the expected mean peak height (Fig. 1, left), but the order two simulation (Fig. 1, middle) decays too slowly

that of earlier work as the basis for the new nonlinear theory herein is the previously established linear theory.

Consider some multiscale spatial domain $\mathbb{X} \times \mathbb{Y}$ where $\mathbb{X}$ is some open domain of large macroscale extent in space, and $\mathbb{Y}$ is a 'relatively small' microscale domain (in some Hilbert space). We analyse the dynamics of some field u within the multiscale spatial domain $\mathbb{X} \times \mathbb{Y}$ and determine the emergent behaviour of this field on the macroscale; that is, we aim to derive a description, over some time interval $\mathbb{T}$, of the long, slow u field dynamics on the macroscale domain $\mathbb{X}$ while accounting for the fine details in the microscale domain $\mathbb{Y}$ in an 'averaged', 'homogenised' or 'slaved' sense [28, 30]. As the domain $\mathbb{Y}$ is a small cross-section of the full domain of the system, a description of the large-scale behaviour should not involve fluctuations across $\mathbb{Y}$ as dynamic variables.

We consider the field $u(\vec{x}, y, t)$ in a given Hilbert space $\mathbb{U}$ (finite or infinite dimensional), where $u : \mathbb{X} \times \mathbb{Y} \times \mathbb{T} \to \mathbb{U}$ is a function of M-dimensional position $\vec{x} \in \mathbb{X} \subseteq \mathbb{R}^M$, cross-sectional point $y \in \mathbb{Y}$, and time $t \in \mathbb{T} \subseteq \mathbb{R}$. The derivatives in the large space dimensions $\vec{x}$ are crucial to organising the analysis so we introduce the multivariate (mixed) derivative

$$\partial_{\vec{x}}^{\vec{k}} := \frac{\partial^{|\vec{k}|}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_M^{k_M}}$$

for multi-indices $\vec{k} = (k_1, \ldots, k_M) \in \mathbb{N}_0^M$ (as usual, the set of natural numbers $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$). This multivariate derivative is of order $|\vec{k}| = k_1 + k_2 + \cdots + k_M$. We consider the class of problems where the field $u(\vec{x}, y, t)$ satisfies some specified nonlinear PDE in the form

$$\frac{\partial u}{\partial t} = \mathfrak{L}[u] + f[u] = \sum_{|\vec{k}|=0}^{\infty} \mathfrak{L}_{\vec{k}} \partial_{\vec{x}}^{\vec{k}} u + f[u], \quad (6)$$

where $f[u] : \mathbb{U} \to \mathbb{U}$ is a 'strictly' nonlinear function of field u and its derivatives, the $\mathfrak{L}_{\vec{k}}$ are linear operators (in y), and where the apparently infinite sum over all possible multi-indices $\vec{k}$ in the PDE (6) is finite in applications because we assume that only a finite number of operators $\mathfrak{L}_{\vec{k}}$ are non-zero.

In application to fluid or heat convection the nonlinear term is the quadratic $f[u] = \vec{u} \cdot \vec{\nabla} \vec{u}$, whereas for the Ginzburg–Landau equation $f[u] = u^3$. Consequently we consider nonlinearities that are sums of products of cognate factors.

Assumption 1 The nonlinear function $f[u]$ may be written as, or usefully approximated as, a sum of products of u and its derivatives:

$$f[u] = \sum_j f^j[u], \quad \text{where} \quad f^j[u] = c_j(y) \prod_{i=1}^{P_j} \partial_{\vec{x}}^{\vec{p}_i^j} u(\vec{x}, y, t),$$

$$(7)$$

for P_j the order of each nonlinear term, and for some M-dimensional index $\vec{p}_i^j$. (Sometimes we detail the case when $f[u]$ has only one term in its sum.)

Roberts [28] considered systems such as PDE (6) on the multiscale domain $\mathbb{X} \times \mathbb{Y}$ for *one dimensional* $\mathbb{X} \subset \mathbb{R}$, and by analysing the dynamics of the system at each cross-sectional station $X \in \mathbb{X}$, constructed a reduced PDE for the slowly varying dynamics. The construction relied on a Taylor expansion of the field u to order N, with the expansion made exact by including the Lagrange's Remainder term in the derivation. Analysis of the Taylor coefficients then reveals the slow behaviour of the system near $X \in \mathbb{X}$ within a centre manifold. Then a projection of the u field PDE onto this centre manifold, generalised to the union over all stations $X \in \mathbb{X}$, defines the slow PDE within domain $\mathbb{X}$, and a projection of Lagrange's Remainder determines the error of the PDE. Roberts and Bunder [30] analogously considered linear systems on the multiscale domain $\mathbb{X} \times \mathbb{Y}$, but generalised to M-dimensional $\mathbb{X} \subset \mathbb{R}^M$. Here we further generalise these earlier developments to the class of nonlinear PDEs (6) that have multiple macro-scale dimensions.

2.1 Large-scales modulates local Taylor coefficients

This section adapts the approach of Roberts and Bunder [30] to the additional complication of nonlinear effects $f[u]$. Both the field u and the nonlinearity $f[u]$ are written as Taylor expansions with Lagrange Remainder terms [34, (1.27), e.g.]. These remainder terms ensure the analysis of the dynamics of the system is exact.

First, one chooses a fixed order, denoted by N, for the Taylor series. The analysis begins as in earlier work [30, §3.1], but with subtle differences due to restrictions caused by the nonlinearity that require careful restatement here. For example, the analysis requires some assumptions about the smoothness of u. For $k_{\max}$ denoting the largest magnitude derivative in the linear term of PDE (6), for p_i^j representing all magnitude derivatives in the nonlinear term (7), and for Taylor expansion to order N, the field u must be in differentiability class $C^{N+\max(p_i^j, k_{\max})}$.

At every cross-section station $\vec{X} \in \mathbb{X} \subset \mathbb{R}^M$, every point in the cross-section $y \in \mathbb{Y}$, and every time $t \in \mathbb{T}$, we expand the field u as an Nth order Taylor multinomial about $\vec{x} = \vec{X}$:

$$u(\vec{x}, y, t) = \sum_{|\vec{n}|=0}^{N-1} u^{(\vec{n})}(\vec{X}, y, t) \frac{(\vec{x} - \vec{X})^{\vec{n}}}{\vec{n}!}$$

$$+ \sum_{|\vec{n}|=N} u^{(\vec{n})}(\vec{X}, \vec{x}, y, t) \frac{(\vec{x} - \vec{X})^{\vec{n}}}{\vec{n}!},$$

$$(8a)$$

where we define the multi-index factorial $\vec{n}! := n_1! n_2! \cdots n_M!$, the multi-index magnitude $|\vec{n}| := n_1 + n_2 + \cdots + n_M$, the multi-index power $\vec{x}^{\vec{n}} := x_1^{n_1} x_2^{n_2} \cdots x_M^{n_M}$, and where

- in the first sum, for $|\vec{n}| < N$, we define the coefficients $u^{(\vec{n})} : \mathbb{X} \times \mathbb{Y} \times \mathbb{T} \to \mathbb{U}$ to be

$$u^{(\vec{n})}(\vec{X}, y, t) := \partial_{\vec{x}}^{\vec{n}} u \big|_{\vec{x}=\vec{X}}; \quad (8b)$$

- and in the second sum, for $|\vec{n}| = N$, by Lagrange's Remainder Theorem for multivariate Taylor series, we define the coefficients $u^{(\vec{n})} : \mathbb{X} \times \mathbb{X} \times \mathbb{Y} \times \mathbb{T} \to \mathbb{U}$ to be

$$u^{(\vec{n})}(\vec{X}, \vec{x}, y, t) := N \int_0^1 (1-s)^{N-1} \partial_{\vec{x}}^{\vec{n}} u \big|_{\vec{X}+s(\vec{x}-\vec{X})} \, ds. \quad (8c)$$

The Taylor expansion of the nonlinear term $f[u]$ in PDE (6) is expressed in the same way as for the field u; that is, we do not expand $f[u]$ in a series in u, but instead expand $f[u(\vec{x}, y, t)]$ in a series in $(\vec{x} - \vec{X})$ to order N. As Assumption 1 specifies that $f[u]$ is a sum of a product of linear functions, (7), the smoothness requirements for constructing the Taylor expansion of $f[u]$ are satisfied by u being of class $C^{N+\max(p_i^j, k_{\max})}$. The Nth order Taylor multinomial of $f[u]$ in $\vec{x}$ about $\vec{x} = \vec{X}$ is

$$f[u(\vec{x}, y, t)] = \sum_{|\vec{n}|=0}^{N-1} f^{(\vec{n})}(\vec{X}, y, t) \frac{(\vec{x} - \vec{X})^{\vec{n}}}{\vec{n}!}$$

$$+ \sum_{|\vec{n}|=N} f^{(\vec{n})}(\vec{X}, \vec{x}, y, t) \frac{(\vec{x} - \vec{X})^{\vec{n}}}{\vec{n}!},$$

$$(9a)$$

where we define the coefficients

- for $|\vec{n}| < N$,

$$f^{(\vec{n})}(\vec{X}, y, t) := \partial_{\vec{x}}^{\vec{n}} f[u(\vec{x}, y, t)] \Big|_{\vec{x}=\vec{X}}; \quad (9b)$$

- and, for $|\vec{n}| = N$,

$$f^{(\vec{n})}(\vec{X}, \vec{x}, y, t) := N \int_0^1 (1-s)^{N-1} \partial_{\vec{x}}^{\vec{n}} f[u(\vec{x}, y, t)] \Big|_{\vec{X}+s(\vec{x}-\vec{X})} \, ds. \quad (9c)$$

The Taylor coefficients $f^{(\vec{n})} : \mathbb{U} \to \mathbb{U}$ of the nonlinearity are, in principle, functions of the u field Taylor coefficients $u^{(\vec{k})}$ with $|\vec{k}| \leqslant N$. They may be obtained by substituting the Taylor expansion (8a) of the field u into equations (9b) and

(9c). For example, in the case of nonlinearity $f[u]$ being only one term, a direct substitution of (8a) into the nonlinear form (7) gives

$$
\begin{aligned}
f[u] = c(y)\prod_{i=1}^{P}\partial_{\vec{x}}^{\vec{p}_i}&\left[\sum_{|\vec{n}|=0}^{N-1}u^{(\vec{n})}(\vec{X},y,t)\frac{(\vec{x}-\vec{X})^{\vec{n}}}{\vec{n}!}\right.\\
&\left.+\sum_{|\vec{n}|=N}u^{(\vec{n})}(\vec{X},\vec{x},y,t)\frac{(\vec{x}-\vec{X})^{\vec{n}}}{\vec{n}!}\right]\\
= c(y)\prod_{i=1}^{P}&\left[\sum_{|\vec{n}|=0}^{N-1-|\vec{p}_i|}u^{(\vec{n}+\vec{p}_i)}(\vec{X},y,t)\frac{(\vec{x}-\vec{X})^{(\vec{n}-\vec{p}_i)}}{(\vec{n}-\vec{p}_i)!}\right.\\
&+\sum_{|\vec{n}|=N-|\vec{p}_i|}\sum_{\vec{m}=\vec{0}}^{\vec{p}_i}\\
&\left.\binom{\vec{p}_i}{\vec{m}}\partial_{\vec{x}}^{\vec{m}}u^{(\vec{n}+\vec{p}_i)}(\vec{X},\vec{x},y,t)\frac{(\vec{x}-\vec{X})^{(\vec{n}+\vec{m})}}{(\vec{n}+\vec{m})!}\right],
\end{aligned}
$$
(10)

where, for sums over multi-indices such as $\sum_{\vec{m}=\vec{k}}^{\vec{\ell}}$ we require that $k_i \leqslant m_i \leqslant \ell_i$ for each component $i = 1, 2, \ldots, M$.

As in the linear case [30, Eq. (19)], the multivariate Taylor multinomial (8a) of a field u gives, after some rearrangement, that the $\vec{\ell}$th spatial derivative

$$
\begin{aligned}
\partial_{\vec{x}}^{\vec{\ell}}u = \sum_{|\vec{n}|=0}^{N-|\vec{\ell}|-1}&u^{(\vec{n}+\vec{\ell})}\frac{(\vec{x}-\vec{X})^{\vec{n}}}{\vec{n}!}\\
&+\sum_{|\vec{n}|=N}\sum_{\vec{m}=(\vec{n}-\vec{\ell})^{\oplus}}^{\vec{n}}\binom{\vec{\ell}}{\vec{n}-\vec{m}}\partial_{\vec{x}}^{\vec{m}+\vec{\ell}-\vec{n}}\\
&u^{(\vec{n})}\frac{(\vec{x}-\vec{X})^{\vec{m}}}{\vec{m}!},
\end{aligned}
$$
(11)

where, appearing here and elsewhere in the limits of some sums, $(\vec{k})^{\oplus}$ denotes the multi-index vector with ith component $\max(k_i, 0)$, thus ensuring all multi-index components are non-negative in the sums. Using some details given for the linear case [30], substitute (11) into the nonlinear PDE (6), and after rearrangement we derive that this nonlinear PDE becomes

$$
\begin{aligned}
\sum_{|\vec{n}|=0}^{N-|\vec{\ell}|-1}&\frac{\partial u^{(\vec{n}+\vec{\ell})}}{\partial t}\frac{(\vec{x}-\vec{X})^{\vec{n}}}{\vec{n}!}\\
&+\sum_{|\vec{n}|=N}\sum_{\vec{m}=(\vec{n}-\vec{\ell})^{\oplus}}^{\vec{n}}\binom{\vec{\ell}}{\vec{n}-\vec{m}}\partial_{\vec{x}}^{\vec{m}+\vec{\ell}-\vec{n}}\frac{\partial u^{(\vec{n})}}{\partial t}\frac{(\vec{x}-\vec{X})^{\vec{m}}}{\vec{m}!}\\
= \sum_{|\vec{k}|=0}^{\infty}&\mathfrak{L}_{\vec{k}}\sum_{|\vec{n}|=0}^{N-|\vec{\ell}+\vec{k}|-1}u^{(\vec{n}+\vec{\ell}+\vec{k})}\frac{(\vec{x}-\vec{X})^{\vec{n}}}{\vec{n}!}\\
&+\sum_{|\vec{k}|=0}^{\infty}\mathfrak{L}_{\vec{k}}\sum_{|\vec{n}|=N}\sum_{\vec{m}=(\vec{n}-\vec{\ell}-\vec{k})^{\oplus}}^{\vec{n}}\binom{\vec{\ell}+\vec{k}}{\vec{n}-\vec{m}}\partial_{\vec{x}}^{\vec{m}+\vec{\ell}+\vec{k}-\vec{n}}\\
&u^{(\vec{n})}\frac{(\vec{x}-\vec{X})^{\vec{m}}}{\vec{m}!}\\
&+\partial_{\vec{x}}^{\vec{\ell}}f[u].
\end{aligned}
$$
(12)

As the multivariate Taylor multinomial (8a) is exact, for all stations $\vec{X} \in \mathbb{X}$ and $\vec{x} \in \mathbb{X}$, equation (12) is exact for every $\vec{x} \in \chi(\vec{X})$, where $\chi(\vec{X})$ is an open subset of $\mathbb{X}$ such that for all points $\vec{x} \in \chi(\vec{X})$ the convex combination $\vec{X} + s(\vec{x} - \vec{X}) \in \chi(\vec{X})$ for every $0 \leqslant s \leqslant 1$; this condition ensures that when we take the limit $\vec{x} \to \vec{X}$, $\vec{x}$ will always remain inside $\chi(\vec{X}) \subset \mathbb{X}$ and $(\vec{x} - \vec{X}) \to \vec{0}$.

Now take the limit $\vec{x} \to \vec{X}$ in equation (12) so that all terms with factors of $(\vec{x} - \vec{X})$ vanish. For simplicity, and unless otherwise specified, hereafter $u^{(\vec{n})}$ denotes $u^{(\vec{n})}(\vec{X}, y, t)$ when $|\vec{n}| < N$ and denotes $u^{(\vec{n})}(\vec{X}, \vec{X}, y, t)$ when $|\vec{n}| = N$. Similarly for the nonlinearity: $f^{(\vec{n})}$ denotes $f^{(\vec{n})}(\vec{X}, y, t)$ when $|\vec{n}| < N$ and denotes $f^{(\vec{n})}(\vec{X}, \vec{X}, y, t)$ when $|\vec{n}| = N$. Further, interchange the $\vec{n}$ and $\vec{\ell}$ multi-indices in (12). Then the nonlinear PDE (6) implies that for every station $\vec{X} \in \mathbb{X}$, every point in the cross-section $y \in \mathbb{Y}$, and every time $t \in \mathbb{T}$,

$$
\frac{\partial u^{(\vec{n})}}{\partial t} = \sum_{|\vec{k}|=0}^{N-|\vec{n}|}\mathfrak{L}_{\vec{k}}u^{(\vec{n}+\vec{k})} + f^{(\vec{n})} + r_{\vec{n}},
$$
(13a)

for every $|\vec{n}| \leqslant N$,

where the remainder

$$
r_{\vec{n}} = \sum_{|\vec{k}|\geqslant 1}\sum_{\substack{|\vec{\ell}|=N\\ \vec{\ell}\lneqq\vec{n}+\vec{k}}}\mathfrak{L}_{\vec{k}}\binom{\vec{k}+\vec{n}}{\vec{\ell}}\left[\partial_{\vec{x}}^{\vec{k}+\vec{n}-\vec{\ell}}u^{(\vec{\ell})}(\vec{X},\vec{x},y,t)\right]_{\vec{x}=\vec{X}},
$$
(13b)

where in comparing indices we define that $\lneqq$ means $\leqslant$ for each component, but excluding exact equality of the two multi-indices. The second term on the right-hand side of (12) (when $|\vec{k}| \geqslant 1$) determines the remainder (13b).

Since multi-index $\vec{n} \in \mathbb{N}_0^M$ and $|\vec{n}| \leqslant N$, there are thus $\mathcal{N} := \binom{N+M}{M}$ coupled ODEs (13a).

For every multi-index $\vec{n}, |\vec{n}| \leqslant N$, the $u^{(\vec{n})}$ terms in equation (13a) are evaluated at station $\vec{X}$, but the spatial derivatives of $u^{(\vec{n})}(\vec{X}, \vec{x}, y, t)$ with $|\vec{n}| = N$ that appear in the remainder term $r_{\vec{n}}$ (13b) couple the dynamics at station $\vec{X}$ to dynamics of the system along the line joining fixed station $\vec{X}$ to variable position $\vec{x}$, that is, the dynamics at $\vec{X}$ are coupled to the dynamics at points in the neighbourhood $\chi(\vec{X})$. This dependence of derivatives of $u^{(\vec{n})}(\vec{X}, \vec{x}, y, t)$ on the dynamics at points in $\chi(\vec{X})$ is directly seen from an application of the integral mean value theorem on equation (8c). By this theorem, there exists some $\hat{s} \in (0, 1)$ such that

$$u^{(\vec{n})}(\vec{X}, \vec{x}, y, t) = N \partial_{\vec{x}}^{\vec{n}} u \big|_{\vec{x} + \hat{s}(\vec{x} - \vec{X})} \int_0^1 (1-s)^{N-1} \, ds$$
$$= \partial_{\vec{x}}^{\vec{n}} u \big|_{\vec{x} + \hat{s}(\vec{x} - \vec{X})},$$

and $\vec{X} + \hat{s}(\vec{x} - \vec{X}) \in \chi(\vec{X})$. Spatial derivatives of $u^{(\vec{n})}(\vec{X}, \vec{x}, y, t)$ retain dependence on $\hat{s}$, and thus on the dynamics about $\vec{X}$, even when evaluated at $\vec{x} = \vec{X}$. In contrast, $u^{(\vec{n})}(\vec{X}, \vec{X}, y, t) = \partial_{\vec{x}}^{\vec{n}} u \big|_{\vec{X}}$ is independent of $\hat{s}$. Whereas an $\hat{s} \in (0, 1)$ must exist for each $u^{(\vec{n})}(\vec{X}, \vec{x}, y, t)$, these $\hat{s}$ are generally not determined by users, and so we view gradients of $u^{(\vec{n})}(\vec{X}, \vec{x}, y, t)$ as 'uncertain'. We therefore classify the remainders $r_{\vec{n}}$ as *uncertain forcing* which couple the local dynamics at $\vec{X}$ to the dynamics in its neighbourhood, and thereby to the global dynamics over $\mathbb{X}$.

The nonlinear $f^{(\vec{n})}$ may also contain 'uncertain' gradients of $u^{(\vec{n})}(\vec{X}, \vec{x}, y, t)$, depending on the particular nonlinearity. For example, for the case of a single-term nonlinearity we obtain the last line of equation (10) which contains spatial derivatives up to order $\vec{p}_i$ of $u^{(\vec{n}+\vec{p}_i)}(\vec{X}, \vec{x}, y, t)$ where $|\vec{n} + \vec{p}_i| = N$. So, if at least one $\vec{p}_i > \vec{0}$, the nonlinear term contains uncertain gradients which couple the dynamics at $\vec{X}$ to the dynamics in $\chi(\vec{X})$. Section 2.2 explicitly identifies these uncertain gradients in the nonlinearity $f^{(\vec{n})}$.

2.2 Generating multinomial and PDE

We now pack all the multivariate Taylor coefficients $u^{(\vec{n})}$ together into a generating function (multinomial) in order to handle all the PDEs together, and also connect with established heuristic methodologies. The details here extend those for linear systems [30, §3.2]. For every station $\vec{X} \in \mathbb{X}$ and time $t \in \mathbb{T}$ consider the field u in terms of a local Taylor multinomial (8a) about the cross-section $\vec{x} = \vec{X}$. In terms of the indeterminate $\vec{\xi} \in \mathbb{R}^M$, define the generating multinomial

$$\tilde{u}(\vec{X}, t) := \sum_{|\vec{n}|=0}^{N-1} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} u^{(\vec{n})}(\vec{X}, y, t) + \sum_{|\vec{n}|=N} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} u^{(\vec{n})}(\vec{X}, \vec{X}, y, t), \quad (14)$$

where this generating multinomial $\tilde{u}$, through its range denoted by $\mathbb{U}_N$, is implicitly a function of the indeterminate $\vec{\xi}$ and the cross-sectional variable y. This generating multinomial $\tilde{u} : \mathbb{X} \times \mathbb{T} \to \mathbb{U}_N$ for the vector space $\mathbb{U}_N := \mathbb{U} \otimes_t \mathbb{G}_N$ where $\mathbb{G}_N$ denotes the space of multinomials in $\vec{\xi}$ of degree $\leqslant N$, and where $\otimes_t$ represents the vector space tensor product. The generating operator

$$\mathcal{G} := \left[\sum_{|\vec{n}|=0}^N \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} \partial_{\vec{x}}^{\vec{n}} \right]_{\vec{x}=\vec{X}}, \quad (15)$$

acts to convert the original field $u(\vec{x}, y, t)$ into the generating multinomial $\tilde{u}(\vec{X}, t) = \mathcal{G} u(\vec{x}, y, t)$. The generating operator (15) similarly converts the nonlinear term of PDE (6) into a multinomial in $\vec{\xi}$,

$$\tilde{f}[\tilde{u}] := \mathcal{G} f[u(\vec{x}, y, t)] = \sum_{|\vec{n}|=0}^{N-1} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} f^{(\vec{n})}(\vec{X}, y, t)$$
$$+ \sum_{|\vec{n}|=N} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} f^{(\vec{n})}(\vec{X}, \vec{X}, y, t), \quad (16)$$

with $\tilde{f}[\tilde{u}] : \mathbb{U}_N \to \mathbb{U}_N$ appearing as the nonlinear term in (17) of the next Proposition 2.

We introduce the generating multinomial $\tilde{u}$ because it is more convenient to deal with one multinomial and one PDE than the $\mathcal{N}$ Taylor coefficients $u^{(\vec{n})}$ and the $\mathcal{N}$ differential equations (13) derived in the previous section. Roberts and Bunder [30, §3.2] constructed a similar multinomial $\tilde{u}$ and PDE via the generating operator $\mathcal{G}$; however, here we make new special provisions for the nonlinear term $f[u]$. Although the compact form of multinomial $\tilde{u}$ is useful, a more important property is that the dynamics of $\tilde{u}$ and the original field $u(\vec{x}, y, t)$ are equivalent up to a known difference, as described by Proposition 2.

Proposition 2 (cf. Roberts and Bunder [30, Prop. 2]) *Let $u(\vec{x}, y, t)$ be governed by the specified nonlinear PDE (6). Then the dynamics at every locale $\vec{X} \in \mathbb{X} \subset \mathbb{R}^M$, every $y \in \mathbb{Y}$, and every $t \in \mathbb{T}$, is equivalently governed by the nonlinear PDE*

$$\frac{\partial \tilde{u}}{\partial t} = \tilde{\mathcal{L}} \tilde{u} + \tilde{f}[\tilde{u}] + \tilde{r}[u], \quad (17)$$

for the generating function multinomial $\tilde{u}(\vec{X}, y, t)$ defined in (14), the 'uncertain' forcing $\tilde{r}[u]$ given by (21), the nonlinear $\tilde{f}[\tilde{u}]$ defined by (16), and the operator

$$\tilde{\mathcal{L}} := \sum_{|\vec{k}|=0}^{N} \mathfrak{L}_{\vec{k}} \partial_{\vec{\xi}}^{\vec{k}}. \tag{18}$$

To establish Proposition 2, we first show that the multinomial $\tilde{u}(\vec{X}, y, t)$ (14) satisfies PDE (17). To construct a PDE for $\tilde{u}$, take the time derivative of (14) and replace $\frac{\partial u^{(\vec{n})}}{\partial t}$ using (13a):

$$\frac{\partial \tilde{u}}{\partial t} = \sum_{|\vec{n}|=0}^{N} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} \left[\sum_{|\vec{k}|=0}^{N-|\vec{n}|} \mathfrak{L}_{\vec{k}} u^{(\vec{n}+\vec{k})} \right]$$
$$+ \sum_{|\vec{n}|=0}^{N} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} f^{(\vec{n})} + \sum_{|\vec{n}|=0}^{N} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} r_{\vec{n}} \tag{19}$$
$$= \sum_{|\vec{k}|=0}^{N} \mathfrak{L}_{\vec{k}} \partial_{\vec{\xi}}^{\vec{k}} \tilde{u} + \tilde{f}[\tilde{u}] + \sum_{|\vec{n}|=0}^{N} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} r_{\vec{n}} ,$$

where in the first term the $\vec{n}$ and $\vec{k}$ sums are exchanged, and we then simplify this term using the useful identity

$$\partial_{\vec{\xi}}^{\vec{k}} \tilde{u} = \sum_{|\vec{n}|=0}^{N-|\vec{k}|} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} u^{(\vec{n}+\vec{k})}, \tag{20}$$

obtained from derivatives of the generating multinomial (14) with respect to $\vec{\xi}$. The above PDE (19) is precisely PDE (17) of Proposition 2 with forcing 'remainder'

$$\tilde{r}[u] = \sum_{|\vec{n}|=0}^{N} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} r_{\vec{n}}$$
$$= \sum_{|\vec{k}| \geqslant 1} \mathfrak{L}_{\vec{k}} \sum_{|\vec{n}|=0}^{N} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} \sum_{\substack{|\vec{\ell}| = N \\ \vec{\ell} \lneq \vec{n}+\vec{k}}} \binom{\vec{k}+\vec{n}}{\vec{\ell}} \tag{21}$$
$$\left[\partial_{\vec{x}}^{\vec{k}+\vec{n}-\vec{\ell}} u^{(\vec{\ell})}(\vec{X}, \vec{x}, y, t) \right]_{\vec{x}=\vec{X}},$$

upon using expression (13b) for $r_{\vec{n}}$.

The second task for establishing Proposition 2 is to show that the generating PDE (17) and the original PDE (6) describe the same dynamics at every locale $\vec{X} \in \mathbb{X} \subset \mathbb{R}^M$. We do this by providing a more physical interpretation of the generating operator $\mathcal{G}$ and the generating multinomial $\tilde{u}(\vec{X}, t)$, beyond just a convenient way to pack the Taylor coefficients of $u(\vec{x}, y, t)$.

Consider the Taylor expansion of some general function $g(\vec{x}) \in C^{N+1}$ at $\vec{x} = \vec{X} + \vec{\xi}$ about $\vec{x} = \vec{X}$:

$$[g(\vec{x})]_{\vec{x}=\vec{X}+\vec{\xi}} = g(\vec{X} + \vec{\xi}) = \sum_{|\vec{n}|=0}^{N} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} \partial_{\vec{x}}^{\vec{n}} g(\vec{X}) + R_N(g)$$
$$= \mathcal{G}g(\vec{x}) + \mathcal{O}(|\vec{\xi}|^{N+1}), \tag{22}$$

where $R_N(g)$ is the order N Lagrange remainder term of $g(\vec{X} + \vec{\xi})$ [30]. So, $\mathcal{G}g(\vec{x})$ evaluates $g(\vec{X} + \vec{\xi})$ correct to $\mathcal{O}(|\vec{\xi}|^{N+1})$. Similarly, $\tilde{u}(\vec{X}, t) = \mathcal{G}u(\vec{x}, y, t)$ evaluates $u(\vec{X} + \vec{\xi}, y, t)$ correct to $\mathcal{O}(|\vec{\xi}|^{N+1})$. We interpret $\tilde{u}(\vec{X}, t)$ as the projection of $u(\vec{x}, y, t)$ at $\vec{x} = \vec{X} + \vec{\xi}$ onto the space $\mathbb{U}_N = \mathbb{U} \otimes_t \mathbb{G}_N$, with $\mathcal{O}(|\vec{\xi}|^{N+1})$ interpreted not as an error but as the difference between $u(\vec{X} + \vec{\xi}, y, t)$ and its projection onto $\mathbb{U}_N$ [30]. As $\mathcal{G}$ commutes with the temporal derivative $\mathcal{G}\frac{\partial u(\vec{x}, y, t)}{\partial t} = \frac{\partial \tilde{u}(\vec{X}, t)}{\partial t}$, and $\frac{\partial \tilde{u}(\vec{X}, t)}{\partial t}$ is equivalent to the Taylor expansion of $\frac{\partial u(\vec{x}, y, t)}{\partial t}$ at $\vec{x} = \vec{X} + \vec{\xi}$ correct to $\mathcal{O}(|\vec{\xi}|^{N+1})$. Therefore the generating PDE (17) for multinomial $\tilde{u}(\vec{X}, t)$ is equivalent to the PDE (6) for $u(\vec{x}, y, t)$ evaluated at $\vec{x} = \vec{X} + \vec{\xi}$ correct to $\mathcal{O}(|\vec{\xi}|^{N+1})$. Thus the dynamics of PDE (17) are identical to the dynamics of PDE (6) at every $\vec{x} = \vec{X} \in \mathbb{X}$. This completes the proof of Proposition 2.

The dynamics of the original nonlinear PDE (6) for field u are equivalent to the dynamics of the nonlinear PDE (17) for the N dimensional multinomial $\tilde{u}$ (14); furthermore, the two PDEs are symbolically the same with $u \leftrightarrow \tilde{u}$ and $\vec{x} \leftrightarrow \vec{\xi}$, except for a forcing term. But the advantage of the multinomial form is that the derivatives $\partial_{\vec{\xi}}$ operate only on $\mathbb{G}_N$, that is, multinomials of at most degree N in $\vec{\xi} \in \mathbb{R}^M$, and are thus bounded in $\mathbb{G}_N$. In contrast, the derivatives $\partial_{\vec{x}}$ in the original PDE are potentially unbounded (e.g., for u rapidly oscillating or containing irrational functions). The slowly varying modelling of Sect. 3.2 takes advantage of the near symbolic equivalence between PDE (6) and PDE (17) with $u \leftrightarrow \tilde{u}$ and $\vec{x} \leftrightarrow \vec{\xi}$.

We now expand the nonlinear term (16) of PDE (17) explicitly in terms of generating multinomial $\tilde{u}$ and nonlinear 'uncertain' terms involving gradients of $u^{(\vec{n})}$ with $|\vec{n}| > N$. Section 3.2 makes use of this expansion to simplify the remainder term ρ of the slow PDE, and Appendix A applies the expansion in the construction of the slow PDE for the fluid flow example discussed in Sect. 1.1.

The nonlinear term (16) in the generating PDE (17), expanded according to (7) in Assumption 1 is

$$
\begin{aligned}
\tilde{f}[\tilde{u}] &= \sum_j \sum_{|\vec{n}|=0}^{N} \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!}\left[\partial_{\vec{x}}^{\vec{n}} c_j(y) \prod_{i=1}^{P_j} \partial_{\vec{x}}^{\vec{p}_i^j} u(\vec{x},y,t)\right]_{\vec{x}=\vec{X}} \\
&= \sum_j \sum_{|\vec{n}|=0}^{N} \sum_{\sum_{i=1}^{P_j}\vec{m}_i=\vec{n}} c_j(y) \prod_{i=1}^{P_j} \frac{\vec{\xi}^{\vec{m}_i}}{\vec{m}_i!}\left[\partial_{\vec{x}}^{\vec{m}_i}\partial_{\vec{x}}^{\vec{p}_i^j} u(\vec{x},y,t)\right]_{\vec{x}=\vec{X}} \\
&= \sum_j \sum_{|\vec{n}|=0}^{N} \sum_{\substack{\sum_{i=1}^{P_j}\vec{m}_i=\vec{n} \\ |\vec{m}_i+\vec{p}_i^j|\leqslant N}} c_j(y) \prod_{i=1}^{P_j} \frac{\vec{\xi}^{\vec{m}_i}}{\vec{m}_i!}\left[\partial_{\vec{x}}^{\vec{m}_i+\vec{p}_i^j} u(\vec{x},y,t)\right]_{\vec{x}=\vec{X}} \\
&\quad + \sum_j \sum_{|\vec{n}|=0}^{N} \sum_{\substack{\sum_{i=1}^{P_j}\vec{m}_i=\vec{n} \\ \exists |\vec{m}_i+\vec{p}_i^j|> N}} c_j(y) \prod_{i=1}^{P_j} \frac{\vec{\xi}^{\vec{m}_i}}{\vec{m}_i!}\left[\partial_{\vec{x}}^{\vec{m}_i+\vec{p}_i^j} u(\vec{x},y,t)\right]_{\vec{x}=\vec{X}}.
\end{aligned}
$$

(23)

The components with $|\vec{m}_i+\vec{p}_i^j| > N$ in the last term on the right hand side are 'uncertain', similar to the uncertain forcing $\tilde{r}[u]$ (21), although in the special case where all $\vec{p}_i^j = \vec{0}$, no such uncertain nonlinear terms exist. For the uncertain gradients, consider expansion (11) with $|\vec{\ell}| > N$ evaluated at $\vec{x} = \vec{X}$,

$$
\partial_{\vec{x}}^{\vec{\ell}}[u(\vec{x},y,t)]_{\vec{x}=\vec{X}} = \sum_{|\vec{n}|=N, \vec{n}\lneq \vec{\ell}} \binom{\vec{\ell}}{\vec{n}}\left[\partial_{\vec{x}}^{\vec{\ell}-\vec{n}} u^{(\vec{n})}(\vec{X},\vec{x},y,t)\right]_{\vec{x}=\vec{X}}.
$$

Using this expansion for the uncertain terms, as well as (8b) and (20) and Assumption 1, we rewrite the nonlinear term (23) as

$$
\begin{aligned}
\tilde{f}[\tilde{u}] &= \sum_j \sum_{|\vec{n}|=0}^{N} \sum_{\substack{\sum_{i=1}^{P}\vec{m}_i=\vec{n} \\ |\vec{m}_i+\vec{p}_i^j|\leqslant N}} c_j(y) \prod_{i=1}^{P_j} \frac{\vec{\xi}^{\vec{m}_i}}{\vec{m}_i!}\left[\partial_{\vec{\xi}}^{\vec{m}_i+\vec{p}_i^j}\tilde{u}\right]_{\vec{\xi}=\vec{0}} \\
&\quad + \sum_j \sum_{|\vec{n}|=0}^{N} \sum_{\substack{\sum_{i=1}^{P_j}\vec{m}_i=\vec{n} \\ \exists |\vec{m}_i+\vec{p}_i^j|> N}} c_j(y) \prod_{i=1}^{P_j} \frac{\vec{\xi}^{\vec{m}_i}}{\vec{m}_i!} f_i^j[u,\tilde{u}],
\end{aligned}
$$

(24)

where in the second term $f_i[u,\tilde{u}]$ is either a function of the generating multinomial $\tilde{u}$ or of the uncertain gradients of original field u,

$$
f_i^j[u,\tilde{u}] := \begin{cases} \left[\partial_{\vec{\xi}}^{\vec{m}_i+\vec{p}_i^j}\tilde{u}\right]_{\vec{\xi}=\vec{0}} & \text{for } |\vec{m}_i+\vec{p}_i^j|\leqslant N, \\[2em] \displaystyle\sum_{\substack{|\vec{k}|=N \\ \vec{k}\lneq \vec{m}_i+\vec{p}_i^j}} \binom{\vec{m}_i+\vec{p}_i^j}{\vec{k}}\left[\partial_{\vec{x}}^{\vec{m}_i+\vec{p}_i^j-\vec{k}} u^{(\vec{k})}(\vec{X},\vec{x},y,t)\right]_{\vec{x}=\vec{X}} & \text{for } |\vec{m}_i+\vec{p}_i^j|> N. \end{cases}
$$

Thus in the Taylor expansion (16),

$$
\begin{aligned}
f^{(\vec{n})} &= \vec{n}! \sum_j \sum_{\substack{\sum_{i=1}^{P_j}\vec{m}_i=\vec{n} \\ |\vec{m}_i+\vec{p}_i^j|\leqslant N}} c_j(y) \prod_{i=1}^{P_j} \frac{1}{\vec{m}_i!}\left[\partial_{\vec{\xi}}^{\vec{m}_i+\vec{p}_i^j}\tilde{u}\right]_{\vec{\xi}=\vec{0}} \\
&\quad + \vec{n}! \sum_j \sum_{\substack{\sum_{i=1}^{P_j}\vec{m}_i=\vec{n} \\ \exists |\vec{m}_i+\vec{p}_i^j|> N}} c_j(y) \prod_{i=1}^{P_j} \frac{1}{\vec{m}_i!} f_i^j[u,\tilde{u}],
\end{aligned}
$$

(25)

where the second term contains all uncertain gradients.

3 A slow nonlinear model emerges

Section 3.1 constructs the eigenspace which describes the emergent slow dynamics of the generating PDE (17) by analysing a linearisation of the PDE (17). We then show how this eigenspace and associated eigenvalues capture the slow dynamics of original nonlinear PDE (6). It is possible to determine the slow dynamics of PDE (6) from the eigenspace of the linearised Taylor coefficient PDEs (13a), without introducing the generating multinomial and generating function, but then one must explicitly deal with $\mathcal{N}$ Taylor coefficients and their coupled $\mathcal{N}$ PDEs, as seen in the linear example of Roberts and Bunder [30, §2.2]. Employing the linear eigenspace as a foundation to describe the dynamics of a nonlinear system is justified by centre manifold theory [3, 6, 14, e.g.] which assures us that generically the stability properties of a linear system with centre-stable dynamics persist under nonlinear perturbations and time-dependent forcing. Section 3.2 extends the analysis to the nonlinear PDE (17) to describe the slow dynamics of the nonlinear system, including coupling between the centre and stable subspaces via uncertain terms in both the forcing and the nonlinear term.

The slow dynamics are characterised by a set of generalised eigenvectors $\tilde{\mathcal{V}}^{\vec{n}}$ which span the centre subspace on which the slow dynamics of $\tilde{u}$ evolve. As these generalised eigenvectors are determined from the linearised PDE, they

are the same as those we determined [30, §3.3] for linear PDEs. However, there we constructed the generalised eigenvectors $\tilde{\mathcal{V}}^{\vec{n}}$ via a two-step process, firstly constructing generalised eigenvectors for the linearised version of the original PDE (6) (i.e., for $f[u] = 0$), and then mapping these eigenvectors into $\mathbb{U}_N$ [30, Eq. (37) and Lemma 6]. Here we show how to construct the generalised eigenvectors in $\mathbb{U}_N$ directly from the generating PDE (17).

To analyse the eigenspace and determine the slow dynamics of the linearised generating PDE (17), we apply Assumption 3 which describes the eigenspace of $\mathfrak{L}_{\vec{0}}$, the lowest order operator in $\tilde{\mathcal{L}}$ (as was also assumed in the linear case by Roberts and Bunder [30], and are needed here to clearly define quantities). However, Assumption 3 does not provide *necessary* assumptions for the extraction of a slow model; for example, here we derive the slow model after assuming the Hilbert space $\mathbb{U}$ is a centre-stable subspace, but an analogous derivation is possible when $\mathbb{U}$ is a slow-stable subspace (the shallow fluid example of Sect. 1.1 is on a slow-stable subspace and the code in Appendices A.2 and A.3 permit either slow-stable or centre-stable dynamics).

Assumption 3 We assume the following for the primary case of purely centre-stable dynamics.

(1) The Hilbert space $\mathbb{U}$ is the direct sum of two closed $\mathfrak{L}_{\vec{0}}$-invariant subspaces, $\mathbb{E}_c^0$ and $\mathbb{E}_s^0$, and the corresponding restrictions of $\mathfrak{L}_{\vec{0}}$ generate strongly continuous semigroups [2, 12].

(2) The operator $\mathfrak{L}_{\vec{0}}$ has a discrete spectrum of eigenvalues $\lambda_1, \lambda_2, \ldots$ (repeated according to multiplicity) with corresponding linearly independent (possibly generalised) eigenvectors $v_1^{\vec{0}}, v_2^{\vec{0}}, \ldots$ that are complete ($\mathbb{U} = \mathrm{span}\{v_1^{\vec{0}}, v_2^{\vec{0}}, \ldots\}$).

(3) The first m eigenvalues $\lambda_1, \ldots, \lambda_m$ of $\mathfrak{L}_{\vec{0}}$ all have real part satisfying $|\Re\lambda_j| \leqslant \alpha$ and hence the m-dimensional centre subspace $\mathbb{E}_c^0 = \mathrm{span}\{v_1^{\vec{0}}, \ldots, v_m^{\vec{0}}\}$ [8, Chap. 4, e.g.].

(4) All other eigenvalues $\lambda_{m+1}, \lambda_{m+2}, \ldots$ have real part negative and well separated from the centre eigenvalues, namely $\Re\lambda_j \leqslant -\beta < -N\alpha$ for $j = m+1, m+2, \ldots$, and so the stable subspace $\mathbb{E}_s^0 = \mathrm{span}\{v_{m+1}^{\vec{0}}, v_{m+2}^{\vec{0}}, \ldots\}$. For clarity, say the number of stable eigenvalues is m', so that the stable subspace $\mathbb{E}_s^0$ is m'-dimensional, although the number of stable eigenvalues may be infinite, $m' \to \infty$.

For convenience, Definition 4 packs the m eigenvectors which span the centre subspace $\mathbb{E}_c^0$ of $\mathfrak{L}_{\vec{0}}$ into one matrix $V^{\vec{0}}$, and similarly packs the eigenvalues into the matrix $A_{\vec{0}}$ (being linear concepts, these are the same as for the linear case).

Definition 4 Assumption 3 identifies a subset of m eigenvectors of $\mathfrak{L}_{\vec{0}}$ which span the centre subspace $\mathbb{E}_c^0 \subset \mathbb{U}$.

- With these eigenvectors define
$$V^{\vec{0}} := \begin{bmatrix} v_1^{\vec{0}} & v_2^{\vec{0}} & \cdots & v_m^{\vec{0}} \end{bmatrix} \in \mathbb{U}^{1\times m}.$$

- Since the centre subspace is an invariant space of $\mathfrak{L}_{\vec{0}}$, define complex matrix $A_{\vec{0}} \in \mathbb{C}^{m\times m}$ to be such that $\mathfrak{L}_{\vec{0}} V^{\vec{0}} = V^{\vec{0}} A_{\vec{0}}$ (often $A_{\vec{0}}$ will be in Jordan form, but it is not necessarily so).

- Use $\langle \cdot, \cdot \rangle$ to also denote the inner product on the Hilbert space $\mathbb{U}$, $\langle \cdot, \cdot \rangle : \mathbb{U} \times \mathbb{U} \to \mathbb{C}$, the field of complex numbers. Interpret this inner product when acting on two matrices/vectors with elements in $\mathbb{U}$ as the matrix/vector of the corresponding elementwise inner products. For example, for $Z^{\vec{0}}, V^{\vec{0}} \in \mathbb{U}^{1\times m}$, $\langle Z^{\vec{0}}, V^{\vec{0}} \rangle \in \mathbb{C}^{m\times m}$.

- Define $Z^{\vec{0}} \in \mathbb{U}^{1\times m}$ to have m linearly independent columns which are the m left eigenvectors of $\mathfrak{L}_{\vec{0}}$, ordered such that the jth columns of $V^{\vec{0}}$ and $Z^{\vec{0}}$ have the same eigenvalue and normalised such that $\langle Z^{\vec{0}}, V^{\vec{0}} \rangle = I_m$.

Section 3.1 uses the centre subspace eigenvectors $V^{\vec{0}}$ of $\mathfrak{L}_{\vec{0}}$ to generate a set of generalised eigenvectors of $\tilde{\mathcal{L}}$ which describe the slow dynamics of linear PDE $\partial_t \tilde{u} = \tilde{\mathcal{L}}\tilde{u}$ confined to the centre subspace.

3.1 Generalised eigenvectors span the centre subspace

We invoke Assumption 3 to construct a set of eigenvectors (possibly generalised) which span the centre subspace $\mathbb{E}_c^N \subset \mathbb{U}_N$ of the linear operator $\tilde{\mathcal{L}}$ (18). These eigenvectors capture the slow behaviour of the linear PDE

$$\frac{\partial \tilde{u}}{\partial t} = \tilde{\mathcal{L}}\tilde{u}, \tag{26}$$

which is the linearisation of the generating PDE (17), with neglected forcing.

For $0 < |\vec{n}| \leqslant N$, we construct the generalised eigenvector $\tilde{\mathcal{V}}^{\vec{n}} \in \mathbb{U}^{1\times m} \otimes_t \mathbb{G}_N = \mathbb{U}_N^{1\times m}$ from the following recurrence relations, beginning with $\tilde{\mathcal{V}}^{\vec{0}} = V^{\vec{0}}$,

$$A_{\vec{n}} := \sum_{0 < |\vec{k}|, \vec{k} \leqslant \vec{n}} \left\langle Z^{\vec{0}}, \mathfrak{L}_{\vec{k}} \tilde{\mathcal{V}}^{\vec{n}-\vec{k}} \right\rangle_{\vec{\xi}=\vec{0}}, \tag{27a}$$

$$\mathfrak{L}_{\vec{0}} \tilde{\mathcal{V}}^{\vec{n}} - \tilde{\mathcal{V}}^{\vec{n}} A_{\vec{0}} = -\sum_{0 < |\vec{k}|, \vec{k} \leqslant \vec{n}} \mathfrak{L}_{\vec{k}} \tilde{\mathcal{V}}^{\vec{n}-\vec{k}} + \sum_{0 < |\vec{k}|, \vec{k} \leqslant \vec{n}} \tilde{\mathcal{V}}^{\vec{n}-\vec{k}} A_{\vec{k}}, \tag{27b}$$

$$\left\langle Z^{\vec{0}}, \tilde{\mathcal{V}}^{\vec{n}} \right\rangle = \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!} I_m. \tag{27c}$$

The m rows of all $\tilde{\mathcal{V}}^{\vec{n}}$ with $|\vec{n}| \leqslant N$ form a subset of $\mathbb{U}_N$ with $m\mathcal{N}$ elements. Here we show that these $m\mathcal{N}$ elements are generalised eigenvectors of $\tilde{\mathcal{L}}$ which span the centre subspace $\mathbb{E}_c^N$. To do this we show that the $m\mathcal{N}$ elements are linearly independent and that the generalised eigenvector equation $\tilde{\mathcal{L}}\tilde{\mathcal{V}}^{\vec{n}} - \tilde{\mathcal{V}}^{\vec{n}}A_{\vec{0}}$ only produces linear combinations of $\tilde{\mathcal{V}}^{\vec{k}}$ with $\vec{0} \leqslant \vec{k} < \vec{n}$.

The recurrence relations (27) appear at first sight to be those presented by Roberts and Bunder [30, Eq. (37)], but there is an important difference. The recurrence relations (27) are expressed in terms of the generalised eigenvectors $\tilde{\mathcal{V}}^{\vec{n}} \in \mathbb{U}_N^{1\times m}$, whereas the recurrence relations of Roberts and Bunder [30] are in terms of distinctly different generalised eigenvectors $V^{\vec{n}} \in \mathbb{U}^{1\times m}$, which are later converted into generalised eigenvectors $\tilde{\mathcal{V}}^{\vec{n}}$ [30, Lemma 6]. Here, by working only with the generalised eigenvectors $\tilde{\mathcal{V}}^{\vec{n}}$, our notation, and hence our derivation, is significantly simpler than that of Roberts and Bunder [30].

The inner product (27c) ensures that $\tilde{\mathcal{V}}^{\vec{n}} = \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!}V^{\vec{0}} + \tilde{V}^{\vec{n}}$ for some $\tilde{V}^{\vec{n}} \in \mathbb{U}_N^{1\times m}$ such that $\langle Z^{\vec{0}}, \tilde{V}^{\vec{n}} \rangle = 0_m$ for all $|\vec{n}| > 0$. Further, since the $\frac{\vec{\xi}^{\vec{n}}}{\vec{n}!}V^{\vec{0}}$ part of $\tilde{\mathcal{V}}^{\vec{n}}$ gives zero in the left hand side of (27b), the objective of (27b) is to determine the $\tilde{V}^{\vec{n}}$ part of $\tilde{\mathcal{V}}^{\vec{n}}$. As the right hand side of (27b) is a function of $\tilde{\mathcal{V}}^{\vec{k}}$ with $\vec{0} \leqslant \vec{k} < \vec{n}$, we conclude that $\tilde{V}^{\vec{n}}$ has order of $\vec{\xi}$ no larger than the order of these $\tilde{\mathcal{V}}^{\vec{k}}$. Since we know that $V^{\vec{0}}$ is independent of $\vec{\xi}$, for $|\vec{n}| = 1$ equation (27b) ensures that $\tilde{V}^{\vec{n}}$ is independent of $\vec{\xi}$ and the highest order of $\tilde{\mathcal{V}}^{\vec{n}}$ when $|\vec{n}| = 1$ must be $\vec{\xi}^{\vec{n}}$. By induction we conclude that for any $\vec{n}$, $\tilde{V}^{\vec{n}}$ is of order $\vec{k}$ in $\vec{\xi}$, where $\vec{0} \leqslant \vec{k} < \vec{n}$, and $\tilde{\mathcal{V}}^{\vec{n}}$ is of order $\vec{n}$ in $\vec{\xi}$. Thus $\tilde{\mathcal{V}}^{\vec{n}}$ is an $\vec{n}$th order multinomial in $\mathbb{U}_N^{1\times m}$ and for all $|\vec{n}| \leqslant N$ we have $\mathcal{N}$ linearly independent $\tilde{\mathcal{V}}^{\vec{n}}$.

Now consider the rows of each $\tilde{\mathcal{V}}^{\vec{n}}$. Since

$$\tilde{\mathcal{V}}^{\vec{n}} = \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!}V^{\vec{0}} + \tilde{V}^{\vec{n}} = \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!}\begin{bmatrix} v_1^{\vec{0}} & v_2^{\vec{0}} & \cdots & v_m^{\vec{0}} \end{bmatrix} + \tilde{V}^{\vec{n}},$$

with linearly independent eigenvectors $v_j^{\vec{0}}$ for $j = 1, \ldots, m$, and since $\langle Z^{\vec{0}}, \tilde{V}^{\vec{n}} \rangle = 0_m$, each of the m elements of $\tilde{\mathcal{V}}^{\vec{n}}$ are linearly independent. Therefore, the m elements of all $\tilde{\mathcal{V}}^{\vec{n}}$ with $|\vec{n}| \leqslant N$, form a set of $m\mathcal{N}$ linearly independent elements of $\mathbb{U}_N$.

To show that the rows of $\tilde{\mathcal{V}}^{\vec{n}}$ are generalised eigenvectors of $\tilde{\mathcal{L}}$ in the centre subspace $\mathbb{E}_c^N$, consider

$$\begin{aligned}
\tilde{\mathcal{L}}\tilde{\mathcal{V}}^{\vec{n}} - \tilde{\mathcal{V}}^{\vec{n}}A_{\vec{0}} &= \sum_{|\vec{k}|=0}^{N} \mathfrak{L}_{\vec{k}}\partial_{\vec{\xi}}^{\vec{k}}\tilde{\mathcal{V}}^{\vec{n}} - \tilde{\mathcal{V}}^{\vec{n}}A_{\vec{0}} \\
&= \sum_{\vec{0} \leqslant \vec{k} \leqslant \vec{n}} \mathfrak{L}_{\vec{k}}\partial_{\vec{\xi}}^{\vec{k}}\tilde{\mathcal{V}}^{\vec{n}} - \tilde{\mathcal{V}}^{\vec{n}}A_{\vec{0}} \quad \text{since } \tilde{\mathcal{V}}^{\vec{n}} \text{ is order } \vec{n} \text{ in } \vec{\xi} \\
&= \sum_{\vec{0} \leqslant \vec{k} \leqslant \vec{n}} \mathfrak{L}_{\vec{k}}\tilde{\mathcal{V}}^{\vec{n}-\vec{k}} - \tilde{\mathcal{V}}^{\vec{n}}A_{\vec{0}} \quad \text{from Lemma 5} \\
&= \sum_{\vec{0} < \vec{k} \leqslant \vec{n}} \tilde{\mathcal{V}}^{\vec{n}-\vec{k}}A_{\vec{k}} \quad \text{from rearranging (27b).}
\end{aligned}$$

(28)

The left-hand side only produces $\tilde{\mathcal{V}}^{\vec{k}}$ with $\vec{0} \leqslant \vec{k} < \vec{n}$, and thus the rows of $\tilde{\mathcal{V}}^{\vec{n}}$ are generalised eigenvectors of rank $\vec{n}$ with eigenvalues in matrix $A_{\vec{0}}$. Since the rows of all $\tilde{\mathcal{V}}^{\vec{n}}$ with $|\vec{n}| \leqslant N$ provide $m\mathcal{N}$ linearly independent generalised eigenvectors of $\tilde{\mathcal{L}}$ with eigenvalues contained in $A_{\vec{0}}$, these $m\mathcal{N}$ eigenvectors must span the centre subspace $\mathbb{E}_c^N$.

Lemma 5 *For generalised eigenvector $\tilde{\mathcal{V}}^{\vec{n}}$ constructed from recurrence relations (27), derivatives of $\tilde{\mathcal{V}}^{\vec{n}}$ with respect to $\vec{\xi}$ satisfy $\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}} = \tilde{\mathcal{V}}^{\vec{n}-\vec{m}}$ for $\vec{0} < \vec{m} \leqslant \vec{n}$.*

Since $\tilde{\mathcal{V}}^{\vec{n}} = \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!}V^{\vec{0}} + \tilde{V}^{\vec{n}}$ with $\tilde{V}^{\vec{n}}$ of order less than $\vec{n}$ in $\vec{\xi}$, then $\partial_{\vec{\xi}}^{\vec{n}}\tilde{\mathcal{V}}^{\vec{n}} = \tilde{\mathcal{V}}^{\vec{0}}$, in agreement with Lemma 5 when $\vec{m} = \vec{n}$. However, to prove Lemma 5 we need only prove the $|\vec{m}| = 1$ case for general $\vec{n}$, as the additive property of derivative powers[3] then ensure the lemma is true for every $\vec{0} < \vec{m} \leqslant \vec{n}$.

We prove Lemma 5 by induction. For $|\vec{n}| = 1$, since $\tilde{\mathcal{V}}^{\vec{n}} = \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!}V^{\vec{0}} + \tilde{V}^{\vec{n}}$ with $\tilde{V}^{\vec{n}}$ independent of $\vec{\xi}$, we know that $\partial_{\vec{\xi}}^{\vec{n}}\tilde{\mathcal{V}}^{\vec{n}} = V^{\vec{0}} = \tilde{\mathcal{V}}^{\vec{0}}$, and thus Lemma 5 is true for every $|\vec{n}| = 1$. Now assume that $\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{k}} = \tilde{\mathcal{V}}^{\vec{k}-\vec{m}}$ with $|\vec{m}| = 1$ is true for all $\vec{0} < \vec{k} \leqslant \vec{n}$. Then, for $|\vec{m}| = 1$, replace $\vec{n}$ with $\vec{n} + \vec{m}$ in (27b) and take the $\vec{m}$th derivative with respect to $\vec{\xi}$,

---

[3] That is, $\partial_{\vec{\xi}}^{\vec{k}+\vec{m}}\tilde{\mathcal{V}}^{\vec{n}} = \partial_{\vec{\xi}}^{\vec{k}}\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}}$ for $\tilde{\mathcal{V}}^{\vec{n}}$ a multinomial in $\vec{\xi}$.

$$\mathfrak{L}_{\vec{0}}(\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}}) - (\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}})A_{\vec{0}}$$

$$= -\sum_{0<|\vec{k}|,\vec{k}\leqslant\vec{n}+\vec{m}} \mathfrak{L}_{\vec{k}}(\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}-\vec{k}})$$

$$+ \sum_{0<|\vec{k}|,\vec{k}\leqslant\vec{n}+\vec{m}} (\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}-\vec{k}})A_{\vec{k}}$$

$$= -\sum_{0<|\vec{k}|,\vec{k}\leqslant\vec{n}} \mathfrak{L}_{\vec{k}}(\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}-\vec{k}})$$

$$+ \sum_{0<|\vec{k}|,\vec{k}\leqslant\vec{n}} (\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}-\vec{k}})A_{\vec{k}}$$

$$= -\sum_{0<|\vec{k}|,\vec{k}\leqslant\vec{n}} \mathfrak{L}_{\vec{k}}\tilde{\mathcal{V}}^{\vec{n}-\vec{k}} + \sum_{0<|\vec{k}|,\vec{k}\leqslant\vec{n}} \tilde{\mathcal{V}}^{\vec{n}-\vec{k}}A_{\vec{k}}$$

$$= \mathfrak{L}_{\vec{0}}\tilde{\mathcal{V}}^{\vec{n}} - \tilde{\mathcal{V}}^{\vec{n}}A_{\vec{0}},$$

where in the fourth and fifth lines we recall that the highest $\vec{\xi}$ order of $\tilde{\mathcal{V}}^{\vec{n}+\vec{m}-\vec{k}}$ is $\vec{n}+\vec{m}-\vec{k}$, so to take the $\vec{m}$th derivative we must have $\vec{k}\leqslant\vec{n}$; in the sixth line we apply the assumption that $\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{k}} = \tilde{\mathcal{V}}^{\vec{k}-\vec{m}}$ for all $\vec{0}<\vec{k}\leqslant\vec{n}$; and the seventh line comes from equation (27b). On comparing the first and last lines we see that $\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}} - \tilde{\mathcal{V}}^{\vec{n}} \propto V^{\vec{0}}$, but since

$$\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}} - \tilde{\mathcal{V}}^{\vec{n}} = \partial_{\vec{\xi}}^{\vec{m}}\left(\frac{\vec{\xi}^{(\vec{n}+\vec{m})}}{(\vec{n}+\vec{m})!}V^{\vec{0}} + \tilde{V}^{\vec{n}+\vec{m}}\right) - \frac{\vec{\xi}^{\vec{n}}}{\vec{n}!}V^{\vec{0}} - \tilde{V}^{\vec{n}}$$

$$= \partial_{\vec{\xi}}^{\vec{m}}\tilde{V}^{\vec{n}+\vec{m}} - \tilde{V}^{\vec{n}},$$

and we know $\langle Z^{\vec{0}}, \tilde{V}^{\vec{n}} \rangle = 0$ for every $|\vec{n}| > 0$, then $\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}} = \tilde{\mathcal{V}}^{\vec{n}}$. So, if $\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{k}} = \tilde{\mathcal{V}}^{\vec{k}-\vec{m}}$ with $|\vec{m}| = 1$ is true for every $\vec{0}<\vec{k}\leqslant\vec{n}$ and $1\leqslant|\vec{n}|<N$, then $\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}} = \tilde{\mathcal{V}}^{\vec{n}}$ is also true. Since $\partial_{\vec{\xi}}^{\vec{n}}\tilde{\mathcal{V}}^{\vec{n}} = \tilde{\mathcal{V}}^{\vec{0}}$ is true when $|\vec{n}| = 1$, $\partial_{\vec{\xi}}^{\vec{m}}\tilde{\mathcal{V}}^{\vec{n}+\vec{m}} = \tilde{\mathcal{V}}^{\vec{n}}$ must be true for every $|\vec{n}| > 0$ when $|\vec{m}| = 1$. Finally, because derivative orders are additive, Lemma 5 must be true for every $\vec{0}<\vec{m}\leqslant\vec{n}$.

3.2 Slow field and PDE

In this section we complete our primary aim, which is to model the slow dynamics of the original field $u(\vec{x},y,t)$. To do this, we project $u(\vec{x},y,t)$ onto the centre subspace $\mathbb{E}_c^0$ and define this projection as the slow field $U(\vec{x},t) = \langle Z^{\vec{0}}, u(\vec{x},y,t)\rangle \in \mathbb{C}^m$. The aim of this section is to construct a PDE for $U(\vec{x},t)$ with an exact error term. For the shallow fluid flow example of Sect. 1.1, PDEs of different order are (4a) and (5a), but the error term is new.

The slow field $U(\vec{x},t)$ evaluated at station $\vec{x} = \vec{X}$ is equivalent to $\langle Z^{\vec{0}}, u(\vec{X}+\vec{\xi},y,t)\rangle$ evaluated at $\vec{\xi} = \vec{0}$, and since $\tilde{u}(\vec{X},t) = u(\vec{X}+\vec{\xi},y,t) + \mathcal{O}(|\vec{\xi}|^{N+1})$ (Sect. 2.2 and equation (22)) the slow field is also equivalent to $\langle Z^{\vec{0}}, \tilde{u}(\vec{X},t)\rangle$ evaluated at $\vec{\xi} = \vec{0}$. We expand $\tilde{u}(\vec{X},t)$ in terms of the centre modes $\tilde{\mathcal{V}}^{\vec{n}}$ and the analogous stable modes, and then project this parameterisation onto the the centre subspace $\mathbb{E}_c^0$ to obtain the slow field $U(\vec{X},t)$.

Since we project $\tilde{u}(\vec{X},t)$ onto $\mathbb{E}_c^0$, the stable modes may at first seem superfluous in the expansion of $\tilde{u}(\vec{X},t)$. However, while the stable modes decay exponentially rapidly, resulting in the emergence of the evolution of $u(\vec{x},y,t)$ on the centre subspace, through the nonlinearity these stable modes are not generally negligible and their influence must be accounted for in $U(\vec{X},t)$.

We define $\tilde{\mathcal{W}}^{\vec{n}} \in \mathbb{U}^{1\times m'} \otimes_t \mathbb{G}_N = \mathbb{U}_N^{1\times m'}$ as the generalised eigenvectors which span the stable subspace $\mathbb{E}_s^N \subset \mathbb{U}_N$ of $\tilde{\mathcal{L}}$. The full set of generalised eigenvectors, $\tilde{\mathcal{V}}^{\vec{n}}$ and $\tilde{\mathcal{W}}^{\vec{n}}$, span $\mathbb{U}_N$ of $\tilde{\mathcal{L}}$, fully parameterising the field $\tilde{u}$. Many of the properties of the $\tilde{\mathcal{W}}^{\vec{n}}$ are analogous to those of the $\tilde{\mathcal{V}}^{\vec{n}}$ and can be established by proofs similar to those presented in Sect. 3.1. Therefore, here we only briefly comment on those properties of $\tilde{\mathcal{W}}^{\vec{n}}$ which are required for the analysis of generating multinomial $\tilde{u}$, and ultimately the slow field $U(\vec{x},t)$.

For the lowest order case $\vec{n} = \vec{0}$, the centre subspace eigenvector $\tilde{\mathcal{V}}^{\vec{0}} = V^{\vec{0}}$ satisfies $\mathfrak{L}_{\vec{0}}\tilde{\mathcal{V}}^{\vec{0}} = \tilde{\mathcal{V}}^{\vec{0}}A_{\vec{0}}$ (Definition 4) where matrix $A_{\vec{0}} \in \mathbb{C}^{m\times m}$ has centre eigenvalues $\lambda_1, \ldots, \lambda_m$ (Assumption 3), and similarly there is a stable subspace eigenvector $\tilde{\mathcal{W}}^{\vec{0}} = \begin{bmatrix} v_{m+1}^{\vec{0}} & v_{m+2}^{\vec{0}} & \cdots \end{bmatrix} \in \mathbb{U}^{1\times m'}$ which satisfies $\mathfrak{L}_{\vec{0}}\tilde{\mathcal{W}}^{\vec{0}} = \tilde{\mathcal{W}}^{\vec{0}}B_{\vec{0}}$ for some matrix $B_{\vec{0}} \in \mathbb{C}^{m'\times m'}$ with stable eigenvalues $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_{m'}$ (Recall from Assumption 3 that m' may be infinite). Together, $\tilde{\mathcal{V}}^{\vec{0}}$ and $\tilde{\mathcal{W}}^{\vec{0}}$ span the Hilbert space $\mathbb{U}$ of $\mathfrak{L}_{\vec{0}}$.

For convenience, define matrix $\tilde{\mathcal{V}} = [\tilde{\mathcal{V}}^{\vec{n}}]$ where the columns of $\tilde{\mathcal{V}}$ are the centre subspace eigenvectors $\tilde{\mathcal{V}}^{\vec{n}}$. The ordering of the $\tilde{\mathcal{V}}^{\vec{n}}$ in $\tilde{\mathcal{V}}$ is according to the magnitude $|\vec{n}|$, so that the first column is $\tilde{\mathcal{V}}^{\vec{0}}$. From (28), $\tilde{\mathcal{L}}\tilde{\mathcal{V}}^{\vec{n}} = \sum_{\vec{0}\leqslant\vec{k}\leqslant\vec{n}} \tilde{\mathcal{V}}^{\vec{n}-\vec{k}}A_{\vec{k}}$ and so we define block upper triangular matrix $\mathcal{A}$ such that $\tilde{\mathcal{L}}\tilde{\mathcal{V}} = \tilde{\mathcal{V}}\mathcal{A}$. The upper block triangular matrix $\mathcal{A}$ consists of $\mathcal{N}\times\mathcal{N}$ blocks with 0_m below the diagonal, $A_{\vec{0}} \in \mathbb{C}^{m\times m}$ along the main diagonal, and the $(\vec{k},\vec{n})$ block above the diagonal (that is, for $\vec{n}>\vec{k}$) is $A_{\vec{n}-\vec{k}} \in \mathbb{C}^{m\times m}$. The centre eigenvalues of $\tilde{\mathcal{L}}$ are the eigenvalues of the blocks along the diagonal of $\mathcal{A}$, namely, the m eigenvalues of $A_{\vec{0}}$, $\lambda_1, \ldots, \lambda_m$, repeated $\mathcal{N}$ times.

Similarly to matrix $\mathcal{A}$ which satisfies $\tilde{\mathcal{L}}\tilde{\mathcal{V}} = \tilde{\mathcal{V}}\mathcal{A}$, we define matrix $\mathcal{B}$ such that $\tilde{\mathcal{L}}\tilde{\mathcal{W}} = \tilde{\mathcal{W}}\mathcal{B}$. Analogous to $\mathcal{A}$, $\mathcal{B}$ is upper block triangular with $\mathcal{N}\times\mathcal{N}$ blocks of size $\mathbb{C}^{m'\times m'}$, with block $B_{\vec{0}}$ along the main diagonal, and $B_{\vec{n}-\vec{k}}$ the $(\vec{k},\vec{n})$ block above the diagonal. Thus the stable eigenvalues of $\tilde{\mathcal{L}}$ are the eigenvalues of $\mathcal{B}$ which must be $\lambda_{m+1}, \lambda_{m+2}, \ldots$ repeated $\mathcal{N}$ times. Recall that these stable eigenvalues all

have real part $\leqslant -\beta < -N\alpha$, whereas the magnitude of the real part of the centre eigenvalues are $\leqslant \alpha$ (Assumption 3).

We capture the full centre-stable dynamics of the linear generating PDE (26) on $\mathbb{U}_N$ with

$$\tilde{u}(\vec{X}, t) = \sum_{|\vec{n}|=0}^{N} (\tilde{\mathcal{V}}^{\vec{n}} U_{\vec{n}} + \tilde{\mathcal{W}}^{\vec{n}} S_{\vec{n}}) = \tilde{\mathcal{V}}\mathcal{U} + \tilde{\mathcal{W}}\mathcal{S}, \qquad (29)$$

for parameters $\mathcal{U} = [U_{\vec{n}}]$ and $\mathcal{S} = [S_{\vec{n}}]$ with $U_{\vec{n}} \in \mathbb{C}^m$ and $S_{\vec{n}} \in \mathbb{C}^{m'}$. As $\tilde{r}[\underline{u}], \tilde{f}[\tilde{u}] \in \mathbb{U}_N$ and the generalised eigenvectors, $\tilde{\mathcal{V}}^{\vec{n}}$ and $\tilde{\mathcal{W}}^{\vec{n}}$, span $\mathbb{U}_N$ the forcing and nonlinear terms are uniquely parameterised in terms of these generalised eigenvectors,

$$\tilde{r}[u] = \tilde{\mathcal{V}} r_c(t) + \tilde{\mathcal{W}} r_s(t), \quad \tilde{f}[\tilde{u}] = \tilde{\mathcal{V}} f_c(t) + \tilde{\mathcal{W}} f_s(t), \qquad (30)$$

where $r_c = [r_c^{\vec{n}}]$ and $f_c = [f_c^{\vec{n}}]$, and similarly for r_s and f_s, with $r_c^{\vec{n}}, f_c^{\vec{n}} \in \mathbb{C}^m$ and $r_s^{\vec{n}}, f_s^{\vec{n}} \in \mathbb{C}^{m'}$. We substitute the expansion of $\tilde{u}$ (29) into the nonlinear generating PDE (17) and separate the forcing and nonlinear terms into centre and slow components (30). From Sect. 3.1 we know that the centre subspace eigenvectors $\tilde{\mathcal{V}}^{\vec{n}}$ are linearly independent, and similarly the stable subspace eigenvectors $\tilde{\mathcal{W}}^{\vec{n}}$ are linearly independent. Therefore, we separate the centre and stable components of the PDE to obtain

$$\frac{\partial \mathcal{U}}{\partial t} = \mathcal{A}\mathcal{U} + f_c(t) + r_c(t), \qquad (31a)$$

$$\frac{\partial \mathcal{S}}{\partial t} = \mathcal{B}\mathcal{S} + f_s(t) + r_s(t). \qquad (31b)$$

A general solution of the PDE for the stable parameter $\mathcal{S}$ (31b) is

$$\begin{aligned} \mathcal{S}(t) &= e^{\mathcal{B}t}\mathcal{S}(0) + \int_0^t e^{\mathcal{B}(t-\tau)}[f_s(\tau) + r_s(\tau)]d\tau \\ &= e^{\mathcal{B}t}\mathcal{S}(0) + e^{\mathcal{B}t} \star [f_s(t) + r_s(t)], \end{aligned} \qquad (32)$$

with convolution $h(t) \star g(t) := \int_0^t h(t-\tau)g(\tau)\,d\tau$. As all eigenvalues of $\mathcal{B}$ have real part $\leqslant -\beta < -N\alpha$, for some decay rate $\gamma \in (\alpha, \beta)$,

$$\mathcal{S}(t) = e^{\mathcal{B}t} \star [f_s(t) + r_s(t)] + \mathcal{O}(e^{-\gamma t}). \qquad (33)$$

This solution for $\mathcal{S}(t)$ shows that, after a sufficiently long time, the forcing and nonlinear terms dominate $\mathcal{S}$ through the convolution, thus showing how the forcing and nonlinear terms couple the centre and stable solutions through $f_{s/c}$ and $r_{s/c}$ and why the influence of the stable modes are not negligible.

We now construct a PDE for the slow field $U(\vec{X}, t)$ by considering both its the temporal and spatial derivative in terms of the centre-stable dynamics. Since $U(\vec{X}, t) = \langle Z^{\vec{0}}, \tilde{u}(\vec{X}, t)\rangle_{\vec{\xi}=\vec{0}}$ at station $\vec{x} = \vec{X}$,

$$\begin{aligned} &\frac{\partial U(\vec{X}, t)}{\partial t} \\ &= \left\langle Z^{\vec{0}}, \frac{\partial \tilde{u}(\vec{X}, t)}{\partial t} \right\rangle_{\vec{\xi}=\vec{0}} \\ &= \left\langle Z^{\vec{0}}, \tilde{\mathcal{V}}\frac{\partial \mathcal{U}}{\partial t} \right\rangle_{\vec{\xi}=\vec{0}} + \left\langle Z^{\vec{0}}, \tilde{\mathcal{W}}\frac{\partial \mathcal{S}}{\partial t} \right\rangle_{\vec{\xi}=\vec{0}} \\ &= \langle Z^{\vec{0}}, \tilde{\mathcal{V}}\mathcal{A}\mathcal{U}\rangle_{\vec{\xi}=\vec{0}} + \langle Z^{\vec{0}}, \tilde{\mathcal{V}}f_c(t)\rangle_{\vec{\xi}=\vec{0}} + \langle Z^{\vec{0}}, \tilde{\mathcal{V}}r_c(t)\rangle_{\vec{\xi}=\vec{0}} \\ &\quad + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}\mathcal{B}\mathcal{S}\rangle_{\vec{\xi}=\vec{0}} + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}f_s(t)\rangle_{\vec{\xi}=\vec{0}} \\ &\quad + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}r_s(t)\rangle_{\vec{\xi}=\vec{0}} \quad \text{from (31)} \\ &= \langle Z^{\vec{0}}, \tilde{\mathcal{V}}\rangle_{\vec{\xi}=\vec{0}}\mathcal{A}\mathcal{U} + \langle Z^{\vec{0}}, \tilde{\mathcal{V}}\rangle_{\vec{\xi}=\vec{0}}f_c(t) \\ &\quad + \langle Z^{\vec{0}}, \tilde{\mathcal{V}}\rangle_{\vec{\xi}=\vec{0}}r_c(t) \\ &\quad + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}}f_s(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}}r_s(t) \\ &\quad + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}}\mathcal{B}e^{\mathcal{B}t} \star [f_s(t) + r_s(t)] + \mathcal{O}(e^{-\gamma t}) \quad \text{from (33).} \end{aligned}$$
$$(34)$$

From (27c), the inner product $\langle Z^{\vec{0}}, \tilde{\mathcal{V}}\rangle_{\vec{\xi}=0} = [I_m \; 0_m \; \cdots \; 0_m]$. Also, because of the upper block triangular structure of $\mathcal{A}$, the $\vec{k}$ element of $\mathcal{A}\mathcal{U}$ is

$$[\mathcal{A}\mathcal{U}]_{\vec{k}} = \sum_{\vec{n} \geqslant \vec{k}, |\vec{n}| \leqslant N} \mathcal{A}_{\vec{n}-\vec{k}} U_{\vec{n}} = \sum_{|\vec{n}|=0}^{N-|\vec{k}|} \mathcal{A}_{\vec{n}} U_{\vec{n}+\vec{k}}.$$

Therefore, in the first term on the right hand side of (34) we only retain the $\vec{k} = \vec{0}$ element of $\mathcal{A}\mathcal{U}$, and in the second and third terms we only retain $f_c^{\vec{0}}$ and $r_c^{\vec{0}}$, respectively. So now,

$$\begin{aligned} \frac{\partial U(\vec{X}, t)}{\partial t} &= \sum_{|\vec{n}|=0}^{N} A_{\vec{n}} U_{\vec{n}} + f_c^{\vec{0}}(t) + r_c^{\vec{0}}(t) \\ &\quad + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}}f_s(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}}r_s(t) \\ &\quad + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}}\mathcal{B}e^{\mathcal{B}t} \star [f_s(t) + r_s(t)] + \mathcal{O}(e^{-\gamma t}). \end{aligned}$$
$$(35)$$

Now consider the order $\vec{n}$ spatial derivative of the slow field,

$$\begin{aligned} [\partial_{\vec{x}}^{\vec{n}} U(\vec{x}, t)]_{\vec{x}=\vec{X}} &= \langle Z^{\vec{0}}, u(\vec{x}, y, t)\rangle_{\vec{x}=\vec{X}} \\ &= \langle Z^{\vec{0}}, u^{(\vec{n})}\rangle \\ &= \langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}} \tilde{u}(\vec{X}, t)\rangle_{\vec{\xi}=\vec{0}} \quad \text{from (20)} \\ &= \langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}} \tilde{\mathcal{V}}\rangle_{\vec{\xi}=\vec{0}}\mathcal{U} + \langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}} \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}}\mathcal{S}. \end{aligned}$$
$$(36)$$

From Lemma 5, $\langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}} \tilde{\mathcal{V}}^{\vec{k}}\rangle_{\vec{\xi}=\vec{0}} = \langle Z^{\vec{0}}, \tilde{\mathcal{V}}^{\vec{k}-\vec{n}}\rangle_{\vec{\xi}=\vec{0}}$ which, from (27c), equals the identity I_m if $\vec{k} = \vec{n}$, but zero otherwise. Thus, in the first inner product of (36), only the $U_{\vec{n}}$ element of $\mathcal{U}$ remains. Then, in the second inner product of (36), substitute the solution of the stable parameter $\mathcal{S}$ (33). The spatial derivative is now

$$[\partial_{\vec{x}}^{\vec{n}} U(\vec{x},t)]_{\vec{x}=\vec{X}} = U_{\vec{n}} + \langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}}\tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}} e^{\mathcal{B}t} \star [f_s(t) + r_s(t)]$$
$$+ \mathcal{O}(e^{-\gamma t}). \tag{37}$$

Combining (35) and (37),

$$\frac{\partial U(\vec{X},t)}{\partial t} = \sum_{|\vec{n}|=0}^{N} A_{\vec{n}} \partial_{\vec{x}}^{\vec{n}} U(\vec{X},t)$$

$$+ f_c^{\vec{0}}(t) + r_c^{\vec{0}}(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}} f_s(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}} r_s(t)$$

$$+ \left[\langle Z^{\vec{0}}, \mathcal{W}\rangle \mathcal{B} - \sum_{|\vec{n}|=0}^{N} A_{\vec{n}} \langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}}\tilde{\mathcal{W}}\rangle\right]_{\vec{\xi}=\vec{0}} e^{\mathcal{B}t} \star [f_s(t) + r_s(t)]$$

$$+ \mathcal{O}(e^{-\gamma t}). \tag{38}$$

Whereas this equation symbolically resembles a PDE, it is strictly a differential-integral equation which couples the dynamics at each station $\vec{X}$ via the 'uncertain' gradient terms and the stable parameter $\mathcal{S}(t)$, which is dependent on the history convolution integrals (32). To obtain a slow PDE without this coupling to different stations, such as PDE (4a) used in the shallow fluid flow example, we retain all terms which do not couple to different stations (i.e., no dependence on derivatives $\partial_{\vec{x}} u^{(\vec{n})}$ with $|\vec{n}| = N$ and no dependence on $\mathcal{S}(t)$) and regulate all other terms to a *remainder*. The second last line of (38) contains a convolution, so is part of the remainder, and the two forcing terms $r_c^{\vec{0}}(t)$ and $\langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}} r_s(t)$ are dependent on uncertain gradients, so are also in the remainder. In contrast, the nonlinear terms $f_c^{\vec{0}}(t)$ and $f_s(t)$ contain parts which we want to retain in the slow PDE, as well as terms which should be in the remainder.

where $f_c^{\vec{0}}(\vec{X},t) \in \mathbb{C}^m$ and $f_s^{\vec{0}}(\vec{X},t) \in \mathbb{C}^m$ contain no uncertain terms and no dependence on $\mathcal{S}$ (so are retained in the slow PDE), and where $f_{c,r}^{\vec{0}}(t)$ and $f_{s,r}(t)$ contain all uncertain terms and $\mathcal{S}$ dependent terms. The $f_{c,r}^{\vec{0}}(t)$ and $f_{s,r}(t)$, as well as the convolutions and forcing terms in (38), are not retained in the slow PDE. As the nonlinear function $f[u]$ in the original PDE (6) is a sum of nonlinear terms $f^j[u]$ (Assumption 1), the nonlinear $f_c^{\vec{0}}(\vec{X},t)$ and $f_s(\vec{X},t)$ are also a sum of nonlinear terms indexed by integer j,

$$f_c^{\vec{0}}(\vec{X},t) = \sum_j f_c^{j\vec{0}}(\vec{X},t), \quad f_s^{\vec{0}}(\vec{X},t) = \sum_j f_s^{j\vec{0}}(\vec{X},t), \tag{40}$$

where $f_c^{j\vec{0}}(\vec{X},t) \in \mathbb{C}^m$, and $f_s^{j\vec{0}}(\vec{X},t) \in \mathbb{C}^m$. As $f^j[u]$ is of order P_j in u and its derivatives (Assumption 1), $f_c^{j\vec{0}}$ and $f_s^{j\vec{0}}$ must be of order P_j in $U_{\vec{n}} \in \mathbb{C}^m$ for all $|\vec{n}| \leqslant N$ [4]. So, in general, each $k = 1, \dots, m$ element of $f_c^{j\vec{0}} + f_s^j$ must have the form

$$\sum_{\substack{|\vec{\ell}_1|, \dots, |\vec{\ell}_{P_j}| = 0 \\ |\vec{\ell}_1| \geqslant |\vec{\ell}_2| \geqslant \cdots \geqslant |\vec{\ell}_{P_j}|}}^{N} \vec{a}_{k\vec{\ell}_1\vec{\ell}_2\dots\vec{\ell}_{P_j}}^{jT} U_{\vec{\ell}_1}$$
$$\otimes U_{\vec{\ell}_2} \otimes \cdots \otimes U_{\vec{\ell}_{P_j}}, \tag{41}$$

for some constant vector $\vec{a}_{k\vec{\ell}_1\vec{\ell}_2\dots\vec{\ell}_{P_j}}^j \in \mathbb{C}^{m^{P_j}}$ and where $\otimes$ represents the usual Kronecker product, for which $U_{\vec{\ell}_p} \otimes U_{\vec{\ell}_q} \in \mathbb{C}^{m^2}$ and $U_{\vec{\ell}_1} \otimes U_{\vec{\ell}_2} \otimes \cdots \otimes U_{\vec{\ell}_{P_j}} \in \mathbb{C}^{m^{P_j}}$. [5] On replacing all $U_{\vec{\ell}}$ with spatial derivatives of $\partial_{\vec{x}}^{\vec{\ell}} U$, as shown in (37), the kth coordinate of the m-dimensional nonlinear term retained in the slow PDE must have the form

$$[f_c^{\vec{0}}(\vec{X},t) + f_s(\vec{X},t)]_k$$
$$= \sum_j \sum_{\substack{|\vec{\ell}_1|, \dots, |\vec{\ell}_{P_j}| = 0 \\ |\vec{\ell}_1| \geqslant |\vec{\ell}_2| \geqslant \cdots \geqslant |\vec{\ell}_{P_j}|}}^{N} \vec{a}_{k\vec{\ell}_1\vec{\ell}_2\dots\vec{\ell}_{P_j}}^{jT} (\partial_{\vec{x}}^{\vec{\ell}_1} U) \otimes (\partial_{\vec{x}}^{\vec{\ell}_2} U) \otimes \cdots \otimes (\partial_{\vec{x}}^{\vec{\ell}_{P_j}} U). \tag{42}$$

For specific cases, removing the remainder components from the nonlinear terms in (38) is achieved using (25), as shown in Appendix A.2. Here, for the general case, we show that the nonlinear terms which are retained in the slow PDE must take a particular form. First, separate the nonlinear terms in the second line of (38) into two parts,

$$f_c^{\vec{0}}(t) = f_c^{\vec{0}}(\vec{X},t) + f_{c,r}^{\vec{0}}(t), \quad \langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}} f_s(t) = f_s(\vec{X},t) + f_{s,r}(t), \tag{39}$$

Now, on replacing arbitrary station $\vec{X}$ with $\vec{x} \in \mathbb{X}$, the slow PDE determined from the differential-integral equation (38) is

---

[4] The nonlinear $f_c^{\vec{0}}(t)$ and $\langle Z^{\vec{0}}, \tilde{\mathcal{W}}\rangle_{\vec{\xi}=\vec{0}} f_s(t)$ are sums of nonlinear terms of order P_j in both $U_{\vec{n}}$ and $S_{\vec{n}}$ for all $|\vec{n}| \leqslant N$, but as all $S_{\vec{n}}$ dependence is contained in $f_{c,r}^{\vec{0}}(t)$ and $f_{s,r}(t)$ there is only $U_{\vec{n}}$ dependence in $f_c^{\vec{0}}(\vec{X},t)$ and $f_s(\vec{X},t)$.

[5] For $m = 1$, as in the fluid flow example of Sect. 1.1, the constant vector in (41) reduces to a scalar and the Kronecker products reduce to a multiplication of scalars. For $P_j = 2$ and any value of m, each term in (41) is equivalent to $U_{\vec{\ell}_1}^T C_{k\vec{\ell}_1\vec{\ell}_2}^j U_{\vec{\ell}_2}$ for some constant matrix $C_{k\vec{\ell}_1\vec{\ell}_2}^j \in \mathbb{C}^{m\times m}$.

$$\frac{\partial U(\vec{x},t)}{\partial t} = \sum_{|\vec{n}|=0}^{N} A_{\vec{n}} \partial_{\vec{x}}^{\vec{n}} U(\vec{x},t)$$

$$+ \sum_j \sum_{\substack{|\vec{\ell}_1|,\dots,|\vec{\ell}_{P_j}|=0 \\ |\vec{\ell}_1| \geqslant |\vec{\ell}_2| \geqslant \cdots \geqslant |\vec{\ell}_{P_j}|}}^{N} \vec{d}^{jT}_{\vec{\ell}_1 \vec{\ell}_2 \dots \vec{\ell}_{P_j}} (\partial_{\vec{x}}^{\vec{\ell}_1} U) \otimes (\partial_{\vec{x}}^{\vec{\ell}_2} U) \otimes \cdots \otimes (\partial_{\vec{x}}^{\vec{\ell}_{P_j}} U) + \rho ,$$

(43)

with $\vec{d}^{jT}_{\vec{\ell}_1 \vec{\ell}_2 \dots \vec{\ell}_{P_j}} (\partial_{\vec{x}}^{\vec{\ell}_1} U) \otimes \cdots \otimes (\partial_{\vec{x}}^{\vec{\ell}_{P_j}} U)$ the m-dimensional vector with elements $k = 1, 2, \dots, m$ defined by (42), and with remainder

$$\rho = f^{\vec{0}}_{c,r}(t) + r^{\vec{0}}_c(t) + f_{s,r}(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} r_s(t)$$

$$+ \left[\langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle \mathcal{B} - \sum_{|\vec{n}|=0}^{N} A_{\vec{n}} \langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}} \tilde{\mathcal{W}} \rangle \right]_{\vec{\xi}=\vec{0}} e^{\mathcal{B}t} \star [f_s(t) + r_s(t)]$$

$$+ \mathcal{O}(e^{-\gamma t}) .$$

(44)

Analogous slow PDEs were derived by Robert [28] (equation (22)) and Roberts and Bunder [30] (equation (51)), but without the nonlinear terms.

Simplifications of the remainder ρ (44) are possible when the order N is chosen to be higher than the order of the spatial derivatives in the original PDE (6). The original PDE contains linear operators $\mathfrak{L}_{\vec{k}} \partial_{\vec{x}}^{\vec{k}}$ for $\vec{k}$ satisfying $0 \leqslant |\vec{k}| < \infty$, but in practice there will be an upper limit on $|\vec{k}|$, say k_{max} (often $k_{max} = 1, 2$—the example of Sect. 1.1 has $k_{max} = 1$). Assume that $N > k_{max}$ and consider the uncertain linear terms in ρ:

$$r^{\vec{0}}_c(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} r_s(t) = \langle Z^{\vec{0}}, \tilde{\mathcal{V}} r_c(t) + \tilde{\mathcal{W}} r_s(t) \rangle_{\vec{\xi}=\vec{0}}$$
$$= \langle Z^{\vec{0}}, \tilde{r}[u] \rangle_{\vec{\xi}=\vec{0}} ,$$

Since $Z^{\vec{0}}$ is independent of $\vec{\xi}$, we need only consider $\vec{\xi} = \vec{0}$ in $\tilde{r}[u]$ (21). When $\vec{\xi} = \vec{0}$ the right hand side of (21) requires $|\vec{n}| = 0$, $|\vec{\ell}| = N$ and $\vec{\ell} \lneqq \vec{k}$, but since $N > k_{max} \geqslant |\vec{k}|$ we can never satisfy $|\vec{\ell}| = N$ and $\vec{\ell} \lneqq \vec{k}$. So, when $N > k_{max}$ we have $\tilde{r}[u]_{\vec{\xi}=\vec{0}} = 0$ and $r^{\vec{0}}_c(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} r_s(t) = 0$. Similarly, consider the projection of the nonlinear term

$$f^{\vec{0}}_c(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} f_s(t) = \langle Z^{\vec{0}}, \tilde{\mathcal{V}} f_c(t) + \tilde{\mathcal{W}} f_s(t) \rangle_{\vec{\xi}=\vec{0}}$$
$$= \langle Z^{\vec{0}}, \tilde{f}[\tilde{u}] \rangle_{\vec{\xi}=\vec{0}} ,$$

and then expand $\tilde{f}[\tilde{u}]$ using (24). If N is chosen to be larger than any spatial derivative in the nonlinear term, that is $N > p^j_i$ for every $i = 1, 2, \dots, P_j$ and for every j the number of nonlinear terms, then

$$\tilde{f}[\tilde{u}]_{\vec{\xi}=\vec{0}} = \sum_j c_j(y) \prod_{i=1}^{P_j} \left[\partial_{\vec{\xi}}^{\vec{p}^j_i} \tilde{u} \right]_{\vec{\xi}=\vec{0}}$$

contains no uncertain terms. So, when separating the nonlinear terms according to (30) and (39) $f^{\vec{0}}_{c,r}(t)$ and $f_{s,r}(t)$ contain all $\mathcal{S}$ dependence and any convolution terms, but no uncertain terms.

We have shown that $N > \max(p^j_i, k_{max})$ removes the uncertain terms from $f^{\vec{0}}_{c,r}(t)$ and $f_{s,r}(t)$ and sets $r^{\vec{0}}_c(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} r_s(t) = 0$, but this does not remove all uncertain terms from the remainder ρ (44). Uncertain terms are still present in the remainder because of $f_s(t)$ and $r_s(t)$ which appear in the convolution in the second line of (44).

Section 1.1 presents the example of a shallow fluid flow on a rotating substrate and, with computer algebra code provided in Appendix A, constructs slow PDE of the form given in (43), as shown in equations (4a) and (5a), for $N = 3$ and $N = 4$, respectively. Appendix A.3 calculates parts of the remainder ρ, such as $\langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle \mathcal{B}$ and $A_{\vec{n}} \langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}} \tilde{\mathcal{W}} \rangle$, and shows that, since the order is sufficiently large ($N > \max(p^j_i, k_{max}) = 1$) we have $r^{\vec{0}}_c(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} r_s(t) = 0$. Whereas the Appendix is written to support the example presented in Sect. 1.1, only Appendix A.1 is specific to this example, with the code in Appendices A.2 and A.3 written in a general format so as to be readily adaptable to a large number of systems.

4 Conclusion

This article further develops a general theory to support practical approximations of slow variations in space. This methodology was initially developed by Roberts [28] for linear systems in one dimensional space, and then extended by Roberts and Bunder [30] to linear systems in multi-dimensional space. We here extend theoretical support to nonlinear systems of PDEs in spatial domains that are large in multiple dimensions. In addition, we substantially simplify the derivations of Roberts [28] and Roberts and Bunder [30] by directly constructing the generating PDE (Sect. 3.1), rather than applying an intermediary step which later requires conversion to the generating PDE. As is illustrated by our realistic example concerning the flow of a layer of fluid on rotating substrate, the significant advantages of the theoretical methodology are:

- the approach is readily applicable to a wide range of systems, as illustrated by the general theory provided in Sects. 2 and 3;
- higher order PDEs are obtained in a straightforward manner by increasing the order N of the Taylor expansion; and
- every resulting slow PDE has a well-defined error, with a derived algebraic form which can be bounded in applications.

In the general theory, we make some assumptions about the structure of the nonlinear microscale system and its dynamics. Assumption 1 requires that the nonlinearity in the original microscale PDE should be a sum of products of the unknown field and its derivatives, and Assumption 3 requires centre-stable dynamics. The key requirement for the presented methodology is the persistence of the centre manifold of the linear system (described in Sect. 3.1) when perturbed by non-linearities and time-dependent forcing, thus justifying the importance of the eigenspace of the linearised system to the full nonlinear microscale system (Sect. 3.2). As other invariant manifolds are often similarly persistent, we expect that the methodology is not restricted to the centre-stable dynamics required by Assumption 3. Indeed, we show that other invariant manifolds are possible with the fluid flow example in Sect. 1.1, which has slow-stable dynamics. Furthermore, this fluid flow example has a more complex nonlinear structure than that required by Assumption 1. The Reduce Algebra code presented in Appendix A is designed for this fluid flow example, but is written so as to be adaptable to other systems, including those with different nonlinear structures.

Future research will aim to further generalise the methodology. Of particular interest is stochastic dynamics [1, 29], deriving boundary conditions for the slowly varying model from microscale boundary conditions [22, 27, 33], and non-local operators [i.e., beyond the local operators $\mathfrak{L}_{\vec{k}}\partial_{\vec{x}}^{\vec{k}}$ in (6)] [5].

Compliances with ethical standard

Conflict of interest The authors declare that they have no conflict of interest.

Availability of data and material Not applicable.

Code availability Appendix A provides Reduce Algebra code which derives the emergent slow dynamics of microscale fluid flow equations (1). Matlab code which generates Figs. 1, 2 and 3 is available on request.

Appendix

A: Computer algebra determines the emergent macroscale model

The Reduce Algebra code presented here takes the microscale fluid flow equations (1) for field $u(\vec{x}, y, t)$ and derives the emergent slow macroscale dynamics (4a) or (5a) for the slow field $U(\vec{x}, t)$ on $\vec{x} \in \mathbb{X}$, where domain $\mathbb{X} \subset \mathbb{R}^M$ with $M = 2$ is specified as 'large'. The microscale system (1) decomposes into a stable subspace and an $m = 1$ dimensional slow subspace, with the slow field evolving on this slow subspace $U \in \mathbb{R}^m$.

The Reduce code is applicable to other systems with original microscale field $u \in \mathbb{U} \subset \mathbb{R}^d$ on an $M = 2$ dimensional large domain, and described by PDEs with an order $P = 2$ nonlinear term, provided the dynamics of these systems decompose into a $d - 1$ dimensional stable subspace and an $m = 1$ dimensional centre or slow subspace, thus producing a slow field $U(\vec{x}, t) \in \mathbb{R}$. But this code is readily adaptable to systems with different specifications. Other parameters in the code are easily changed; for example, the dimension d of the field u and the order of the Taylor expansion N are variables in the computer algebra.

We firstly set some printing options and ensure that Reduce will provide us with complex number solutions. We then specify the dimensions of the u field and the desired order of the Taylor expansion N.

```
1 on div; off allfac; on revpri; on complex;
2 d:=3;  % dimension of u field
3 nn:=3; % order of Taylor expansion
```

Appendix A.1 defines the microscale PDE of field $u(\vec{x}, y, t)$, introduced in Sect. 1.1, which describe fluid flow on a rotating substrate. Appendix A.2 is generic code for any system with an $m = 1$ dimensional slow subspace, a 'large' domain of dimension $M = 2$, and nonlinearity of order $P = 2$. This generic code constructs both slow eigenvectors and eigenvalues, $\tilde{\mathcal{V}}^{\vec{n}}$ and $A_{\vec{n}}$, and stable eigenvectors and eigenvalues, $\tilde{\mathcal{W}}^{\vec{n}}$ and $B_{\vec{n}}$, of the matrix operator $\tilde{\mathcal{L}}$ (18) for $|\vec{n}| \leqslant N$. Then the code produces the slow macroscale PDE for the slow field $U(\vec{x}, t) \in \mathbb{R}^m$ with $\vec{x}$ defined on the large domain $\mathbb{X} \subset \mathbb{R}^M$.

A.1: Thin film flow

The provided code allows for a more general PDE than that given in (1), by including the Weber number We to describe surface tension,

$$\frac{\partial h}{\partial t} = -\nabla \cdot (h\vec{v}), \tag{45a}$$

$$\frac{\partial \vec{v}}{\partial t} = \begin{bmatrix} -b & f \\ -f & -b \end{bmatrix} \vec{v} - (\vec{v} \cdot \nabla)\vec{v} - g\nabla h + \nu \nabla^2 \vec{v} + \text{We } \nabla^3 \vec{v}, \tag{45b}$$

although here, for simplicity, we set We $= 0$.

The code below defines matrices $\mathfrak{L}_{\vec{n}}$ and $\mathfrak{M}_{\vec{n}, j}$ for $|\vec{n}| \leqslant N$ for PDEs (1) and (45).

```
 4 we:=0$ % simple example
 5
 6 % linear L matrices
 7 ll00:=mat((0, 0, 0),(0, -b, f), (0, -f, -b));
 8 ll10:=mat((0, 0, 0),(-g, 0, 0), (0, 0, 0));
 9 ll01:=mat((0, 0, 0),(0, 0, 0), (-g, 0, 0));
10 ll20:=nu*mat((0, 0, 0),(0, 1, 0), (0, 0, 1));
11 ll02:=nu*mat((0, 0, 0),(0, 1, 0), (0, 0, 1));
12 ll30:=we*mat((0,0,0),(1,0,0),(0,0,0));
13 ll03:=we*mat((0,0,0),(0,0,0),(1,0,0));
14 ll12:=we*mat((0,0,0),(1,0,0),(0,0,0));
15 ll21:=we*mat((0,0,0),(0,0,0),(1,0,0));
16
17 % nonlinear M matrices
18 mm000:=mat((0,0,0),(0,0,0),(0,0,0));
19 mm001:=mm000;
20 mm002:=mm000;
21 mm100:=mat((0,-1,0),(-1,0,0),(0,0,0));
22 mm101:=mat((0,0,0),(0,-1,0),(0,0,0));
23 mm102:=mat((0,0,0),(0,0,0),(0,-1,0));
24 mm010:=mat((0,0,-1),(0,0,0),(-1,0,0));
25 mm011:=mat((0,0,0),(0,0,-1),(0,0,0));
26 mm012:=mat((0,0,0),(0,0,0),(0,0,-1));
```

Now specify the eigenvalues of $\mathfrak{L}_{\vec{0}}$, thus defining $A_{\vec{0}}$ (Definition 4), the single eigenvalue (since $m = 1$) of the slow subspace, and $B_{\vec{0}}$, the $(d-1) \times (d-1)$ matrix of eigenvalues of the stable subspace (from Definition 4), $m' = d - 1$).

```
27 array aa(nn,nn);
28 aa(0,0):=0$
29 dr:=d-1$
30 array bb(dr-1,dr-1,nn,nn);
31 bb(0,0,0,0):=-b+i*f$
32 bb(1,1,0,0):=-b-i*f$
```

For coding purposes, it is more convenient to define the matrices $\mathfrak{L}_{\vec{n}}$ and $\mathfrak{M}_{\vec{n}, j}$ as arrays.

```
33 kmax:=3$ % highest order spatial derivative
34 array lls(dr,dr,kmax,kmax);
35 for k1:=0:dr do for k2:=0:dr do lls(k1,k2,0,0):=ll00(k1+1,k2+1);
36 for k1:=0:dr do for k2:=0:dr do lls(k1,k2,1,0):=ll10(k1+1,k2+1);
37 for k1:=0:dr do for k2:=0:dr do lls(k1,k2,0,1):=ll01(k1+1,k2+1);
38 for k1:=0:dr do for k2:=0:dr do lls(k1,k2,2,0):=ll20(k1+1,k2+1);
39 for k1:=0:dr do for k2:=0:dr do lls(k1,k2,0,2):=ll02(k1+1,k2+1);
40 for k1:=0:dr do for k2:=0:dr do lls(k1,k2,3,0):=ll30(k1+1,k2+1);
41 for k1:=0:dr do for k2:=0:dr do lls(k1,k2,0,3):=ll03(k1+1,k2+1);
42 for k1:=0:dr do for k2:=0:dr do lls(k1,k2,1,2):=ll12(k1+1,k2+1);
43 for k1:=0:dr do for k2:=0:dr do lls(k1,k2,2,1):=ll21(k1+1,k2+1);
44 array mms(dr,dr,dr,kmax,kmax);
45 for k1:=0:dr do for k2:=0:dr do mms(0,k1,k2,0,0):=mm000(k1+1,k2+1);
46 for k1:=0:dr do for k2:=0:dr do mms(0,k1,k2,1,0):=mm100(k1+1,k2+1);
47 for k1:=0:dr do for k2:=0:dr do mms(0,k1,k2,0,1):=mm010(k1+1,k2+1);
48 for k1:=0:dr do for k2:=0:dr do mms(1,k1,k2,0,0):=mm001(k1+1,k2+1);
49 for k1:=0:dr do for k2:=0:dr do mms(1,k1,k2,1,0):=mm101(k1+1,k2+1);
50 for k1:=0:dr do for k2:=0:dr do mms(1,k1,k2,0,1):=mm011(k1+1,k2+1);
51 for k1:=0:dr do for k2:=0:dr do mms(2,k1,k2,0,0):=mm002(k1+1,k2+1);
52 for k1:=0:dr do for k2:=0:dr do mms(2,k1,k2,1,0):=mm102(k1+1,k2+1);
53 for k1:=0:dr do for k2:=0:dr do mms(2,k1,k2,0,1):=mm012(k1+1,k2+1);
```

A.2: Generalised eigenvectors and slow PDE

The code below derives the centre (or slow) and stable modes for some PDE of the form (6), and constructs the slow PDE projected onto the centre (or slow) subspace (as describe in Sect. 3.2). The code is applicable to any d dimensional microscale field on an $M = 2$ dimensional 'large' domain with a PDE of nonlinear order $P = 2$, where the d dimensional field separates into a $m' = d - 1$ stable subspace and a $m = 1$ dimensional slow or centre

subspace (although v, defined in line 63, is list of at least d dummy variables, so the number of dummy variables in this list should be increased if field u has dimension $d > 7$, and similarly for w defined in line 96, is list of at least $d(d - 1)$ dummy variables).

This code is used to derive the slow PDE of the fluid flow problem described in Sect 3.2, with parameters defined in Appendix A.1

Calculate the centre (or slow) left and right eigenvectors, $Z^{\vec{0}}$ and $V^{\vec{0}}$, of $\mathfrak{L}_{\vec{0}}$ from the nullspaces of $[\mathfrak{L}_{\vec{0}} - A_{\vec{0}}]^T$ and $[\mathfrak{L}_{\vec{0}} - A_{\vec{0}}]$, respectively (Definition 4).

```
54 matrix ii(d,d)$ % identity matrix
55 for j:=1:d do ii(j,j):=1;
56 vvec:=first(nullspace (ll00-ii*aa(0,0)))$ % right eigenvec
57 zvec:=first(nullspace tp(ll00-ii*aa(0,0)))$ % left eigenvec
58 zvec:=zvec/det(tp(zvec)*vvec)$ % rescale
59 array vvt(dr,nn,nn);
60 for j:=0:dr do vvt(j,0,0):=vvec(j+1,1);
61 array zz0(dr);
62 for j:=0:dr do zz0(j):=zvec(j+1,1);
63 v:={v0, v1, v2, v3, v4, v5, v6, v7}$ % dummy variables
```

The recurrence relation (27) is applied to construct the slow eigenvectors $\tilde{\mathcal{V}}^{\vec{n}}$ and eigenvalues $A_{\vec{n}}$ of $\tilde{\mathcal{L}}$ for all $0 < |\vec{n}| \leqslant N$. Once we know all $A_{\vec{n}}$, we know the linear part of the slow PDE (43) (or linear parts of the slow PDEs (4a) and (5a) for the thin film flow example).

Now consider the stable subspace. As was done above for slow subspace, first calculate the left and right eigenvectors associated with the eigenvalues in $B_{\vec{0}}$ of matrix $\mathfrak{L}_{\vec{0}}$, and then apply a recurrence relation analogous to (27) to construct the stable eigenvectors $\tilde{\mathcal{W}}^{\vec{n}}$ and associated matrices of eigenvalues $B_{\vec{n}}$ of $\tilde{\mathcal{L}}$ for all $0 < |\vec{n}| \leqslant N$.

```
64 for p1:=0:nn do for p2:=0:(nn-p1) do if (p1+p2)>0 then <<
65     aa(p1,p2):=(for k1:=0:min(kmax,p1) sum for k2:=0:min(kmax,p2) sum
66         for j1:=0:dr sum for j2:=0:dr sum
67         zz0(j1)*lls(j1,j2,k1,k2)*vvt(j2,p1-k1,p2-k2));
68     aa(p1,p2):=(aa(p1,p2) where {xi1=>0, xi2=>0});
69     eqn:={(for j1:=0:dr sum
70         zz0(j1)*part(v,j1+1))-xi1^p1*xi2^p2
71         /factorial(p1)/factorial(p2)};
72     for j1:=0:dr do <<
73         co:=(for j2:=0:dr sum lls(j1,j2,0,0)*part(v,j2+1) )
74             -part(v,j1+1)*aa(0,0)
75             +(for k1:=0:min(kmax,p1) sum
76             for k2:=0:min(kmax,p2) sum for j2:=0:dr sum
77             lls(j1,j2,k1,k2)*vvt(j2,p1-k1,p2-k2))
78             -(for k1:=0:p1 sum for k2:=0:p2 sum
79             vvt(j1,p1-k1,p2-k2)*aa(k1,k2));
80         eqn:=co.eqn;
81     >>;
82     solv:=solve(eqn,v);
83     for j1:=0:dr do vvt(j1,p1,p2):=sub(solv,part(v,j1+1));
84 >>;
```

```
 85 array wwt(dr,dr-1,nn,nn);
 86 for j1:=0:(dr-1) do <<
 87     wvec:=first(nullspace (ll00-ii*bb(j1,j1,0,0))); % right eigenvec
 88     for j2:=0:dr do wwt(j2,j1,0,0):=wvec(j2+1,1);
 89 >>;
 90 array ww0t(dr-1,dr);
 91 for j1:=0:(dr-1) do <<
 92     wvec:=first(nullspace tp(ll00-ii*bb(j1,j1,0,0))); % left eigenvec
 93     scl:=(for j2:=0:dr sum wvec(j2+1,1)*wwt(j2,j1,0,0));
 94     for j2:=0:dr do ww0t(j1,j2):=wvec(j2+1,1)/scl;
 95 >>;
 96 w:={w0, w1, w2, w3, w4, w5, w6, w7, w8, w9, w10, w11, w12, w13, w14}$
 97 array wtmp(dr,dr-1);
 98 for j1:=0:dr do for j2:=0:(dr-1) do wtmp(j1,j2):=part(w,j1*dr+j2+1);
 99
100 operator dirac;
101 let {dirac(~j1,~j2)=>0 when j1 neq j2, dirac(~j1,~j2)=>1 when j1=j2};
102 for p1:=0:nn do for p2:=0:(nn-p1) do if (p1+p2)>0 then <<
103     for m1:=0:(dr-1) do for m2:=0:(dr-1) do bb(m1,m2,p1,p2):=
104     (for k1:=0:min(kmax,p1) sum for k2:=0:min(kmax,p2) sum
105     for j1:=0:dr sum for j2:=0:dr sum
106     ww0t(m1,j1)*lls(j1,j2,k1,k2)*wwt(j2,m2,p1-k1,p2-k2));
107     for m1:=0:(dr-1) do for m2:=0:(dr-1) do
108     bb(m1,m2,p1,p2):=(bb(m1,m2,p1,p2) where {xi1=>0, xi2=>0});
109     eqn:={};
110     for m1:=0:(dr-1) do for m2:=0:(dr-1) do <<
111         co:=(for j1:=0:dr sum
112             ww0t(m1,j1)*wtmp(j1,m2))-
113             dirac(m1,m2)*xi1^p1*xi2^p2/factorial(p1)/factorial(p2);
114         eqn:=co.eqn;
115     >>;
116     for j1:=0:dr do for j3:=0:(dr-1) do <<
117         co:=(for j2:=0:dr sum lls(j1,j2,0,0)*wtmp(j2,j3))
118             -(for j2:=0:(dr-1) sum wtmp(j1,j2)*bb(j2,j3,0,0))
119             +(for k1:=0:min(kmax,p1) sum
120             for k2:=0:min(kmax,p2) sum for j2:=0:dr sum
121             lls(j1,j2,k1,k2)*wwt(j2,j3,p1-k1,p2-k2))
122             -(for k1:=0:p1 sum for k2:=0:p2 sum for j2:=0:(dr-1) sum
123             wwt(j1,j2,p1-k1,p2-k2)*bb(j2,j3,k1,k2));
124         eqn:=co.eqn;
125     >>;
126     solw:=solve(eqn,w);
127     for j1:=0:dr do for j2:=0:(dr-1) do wwt(j1,j2,p1,p2)
128         :=sub(solw,part(w,j1*dr+j2+1));
129 >>;
```

Construct the nonlinear term $\tilde{f}[\tilde{u}]$ (24) with $\tilde{u} = \mathcal{V}\mathcal{U} + \mathcal{W}\mathcal{S}$ as in (29), where $\mathcal{V} = [V^{\vec{n}}]$, $\mathcal{W} = [W^{\vec{n}}]$, $\mathcal{U} = [U_{\vec{n}}]$, and $\mathcal{S} = [S_{\vec{n}}]$. Here we ignore the uncertain gradient terms (second line of (24)) because such terms are regulated to the remainder ρ (44) of the slow PDE and because

we have chosen N large enough so that these uncertain terms only appear in the remainder's convolution term. Then, compare coefficients of multinomial variable $\vec{\xi}$ to determine the slow and stable parts of the nonlinear term, f_c and f_s, as in (30).

```
130 operator uu;
131 operator ss;
132 operator nonl;
133 operator nonlvw;
134 operator tu;
135 operator fc;
136 operator fs;
137
138 for j:=0:dr do tu(j):=(for k1:=0:nn sum for k2:=0:(nn-k1) sum
139     (vvt(j,k1,k2)*uu(k1,k2)+(for n1:=0:(dr-1) sum
140         wwt(j,n1,k1,k2)*ss(n1,k1,k2))) where {b^2=>bf-f^2});
141 for l:=0:dr do nonl(l):=(for p1:=0:min(kmax,nn) sum
142     for p2:=0:min(kmax,nn) sum
143     for n1:=0:nn sum for n2:=0:(nn-n1) sum
144     for m1:=0:n1 sum for m2:=0:n2 sum
145         xi1^(n1)*xi2^(n2)/(factorial(m1)*factorial(m2)
146         *factorial(n1-m1)*factorial(n2-m2))
147         *(for i:=0:dr sum for j:=0:dr sum
148         sub(xi1=0,xi2=0,df(df(tu(i),xi1,m1+p1),xi2,m2+p2))
149         *mms(l,i,j,p1,p2)*sub(xi1=0,xi2=0,tu(j)) ) );
150 for l:=0:dr do nonlvw(l):=(for k1:=0:nn sum for k2:=0:(nn-k1) sum
151     vvt(l,k1,k2)*fc(k1,k2)+
152     (for j:=0:(dr-1) sum wwt(l,j,k1,k2)*fs(j,k1,k2)  ));
153
154 for l:=0:dr do nonlvw(l):=(for k1:=0:nn sum for k2:=0:(nn-k1) sum
155     vvt(l,k1,k2)*fc(k1,k2)+
156     (for j:=0:(dr-1) sum wwt(l,j,k1,k2)*fs(j,k1,k2)  ));
157 ffs:={}$
158 for k1:=0:nn do for k2:=0:(nn-k1) do ffs:=fc(k1,k2).ffs;
159 for k1:=0:nn do for k2:=0:(nn-k1) do for j:=0:(dr-1) do
160     ffs:=fs(j,k1,k2).ffs;
161 eqn:={}$
162 for l:=0:dr do for k1:=0:nn do for k2:=0:(nn-k1) do
163     eqn:=coeffn(coeffn(nonl(l)-nonlvw(l),xi1,k1),xi2,k2).eqn;
164 eqn:=(eqn where b^2=>bf-f^2)$
165 solf:=solve(eqn,ffs)$
```

For the general slow PDE (43) of $U(\vec{x}, t)$ (or example PDES (4a) and (5a)) we require the projection onto the centre (or slow) subspace $\mathbb{E}_c^0$. In addition, as discussed in Sect 3.2, we regulate all dependence on the stable subspace to a remainder ρ (44), which, for deriving the slow PDE effectively requires us to set parameter $\mathcal{S}$ to zero. In the code below we project nonlinear terms $f_c(\vec{x}, t)$ and $f_s(\vec{x}, t)$ onto $\mathbb{E}_c^0$ and set $\mathcal{S}$ to zero, as defined in equation (39). Then, using $U_{\vec{n}} = \partial_{\vec{x}}^{\vec{n}} U(\vec{x}, t)$, from (37) but neglecting the convolutions [which are also placed in remainder ρ (44)], we rewrite the remaining nonlinear terms $f_c^0(\vec{x}, t) + f_s(\vec{x}, t)$ in the form $a_{\vec{n}, \vec{m}} \partial_{\vec{x}}^{\vec{n}} U(\vec{x}, t) \partial_{\vec{x}}^{\vec{m}} U(\vec{x}, t)$ (42).

A.3: Remainder of the slow PDE

In this section we calculate some terms which appear in the remainder ρ (44) to illustrate how this remainder is constructed for a specific model. We assume that the remainder is evaluated after some relatively long time so that terms $\mathcal{O}(e^{-\gamma t})$ are negligible; as $\gamma \in (\alpha, \beta)$, this ensures that the linear effects of the stable modes are negligible (Assumption 3).

We calculate all inner products $\langle Z^{\vec{0}}, \tilde{\mathcal{W}}^{\vec{n}} \rangle_{\vec{\xi}=\vec{0}}$ and $\langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}} \tilde{\mathcal{W}}^{\vec{k}} \rangle_{\vec{\xi}=\vec{0}}$ and use these to calculate

```
166  fc0u:=(sub(solf,fc(0,0)) where {ss(~j1,~j2,~j3)=>0, bf=>b^2+f^2})$
167  fs0u:=(for k1:=0:nn sum for k2:=0:(nn-k1) sum
168       for j:=0:dr sum for l:=0:(dr-1) sum zz0(j)
169       *wwt(j,l,k1,k2)*sub(solf,fs(l,k1,k2))
170       where {xi1=>0, xi2=>0, ss(~j1,~j2,~j3)=>0, bf=>b^2+f^2})$
171
172  fctmp:=fc0u+fs0u$
173  array a(nn,nn,nn,nn);
174  for k1:=0:nn do for k2:=0:(nn-k1) do
175       for j1:=0:(k1+k2) do for j2:=0:(k1+k2-j1) do <<
176            if (k1+k2=j1+j2 and a(j1,j2,k1,k2)=0) then
177                 a(k1,k2,j1,j2):=coeffn(coeffn(fctmp,uu(j1,j2),1)
178                 ,uu(k1,k2),1)/2
179            else a(k1,k2,j1,j2):=coeffn(coeffn(fctmp,uu(j1,j2),1)
180                 ,uu(k1,k2),1);
181                 fctmp:=fctmp-a(k1,k2,j1,j2)*uu(j1,j2)*uu(k1,k2);
182  >>;
183  for k1:=0:nn do for k2:=0:(nn-k1) do <<
184       a(k1,k2,k1,k2):=coeffn(fctmp,uu(k1,k2),2);
185       fctmp:=fctmp-a(k1,k2,k1,k2)*uu(k1,k2)^2;
186  >>;
```

We now write all $A_{\vec{n}}$ and all elements of $\mathcal{C}$ to define the slow PDE (43) of $U(\vec{x}, t)$ (or example PDES (4a) and (5a)).

```
187  for k1:=0:nn do for k2:=0:(nn-k1) do
188       if aa(k1,k2) neq 0 then write aa(k1,k2):=aa(k1,k2);
189  for k1:=0:nn do for k2:=0:(nn-k1) do
190       for j1:=0:nn do for j2:=0:(nn-j1) do
191       if a(k1,k2,j1,j2) neq 0 then write a(k1,k2,j1,j2)
192            :=a(k1,k2,j1,j2);
```

$$\sum_{|\vec{n}|=0}^{N} A_{\vec{n}} \langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}} \tilde{\mathcal{W}}^{\vec{k}} \rangle_{\vec{\xi}=\vec{0}} \quad \text{and} \quad \left[\langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} \mathcal{B} \right]_{\vec{k}} = \sum_{|\vec{n}|=0}^{N} \langle Z^{\vec{0}}, \tilde{\mathcal{W}}^{\vec{n}} \rangle_{\vec{\xi}=\vec{0}} B_{\vec{k}-\vec{n}} .$$

```
193 array ipzw(dr-1,nn,nn);
194 for k1:=0:nn do for k2:=0:(nn-k1) do
195     for l:=0:(dr-1) do ipzw(l,k1,k2):=(for j:=0:dr sum
196         zz0(j)*wwt(j,l,k1,k2) where {xi1=>0, xi2=>0})$
197
198 array ipzdw(dr-1,nn,nn,nn,nn);
199 array asum(dr-1,nn,nn);
200 for k1:=0:nn do for k2:=0:(nn-k1) do for j1:=0:k1 do for j2:=0:k2 do
201     for l:=0:(dr-1) do ipzdw(l,k1,k2,j1,j2):=
202     (for j:=0:dr sum zz0(j)*df(df(wwt(j,l,k1,k2),xi1,j1),xi2,j2))$
203 for k1:=0:nn do for k2:=0:(nn-k1) do
204     for l:=0:(dr-1) do asum(l,k1,k2):=(for j1:=0:nn sum
205     for j2:=0:(nn-j1) sum aa(j1,j2)*ipzdw(l,k1,k2,j1,j2)
206     where {xi1=>0, xi2=>0})$
207
208 array ipzbb(dr-1,nn,nn);
209 for k1:=0:nn do for k2:=0:(nn-k1) do
210     for l:=0:(dr-1) do <<
211         ipzbb(l,k1,k2):=(for j:=0:(dr-1) sum for j1:=0:k1 sum
212         for j2:=0:k2 sum
213         ipzw(j,j1,j2)*bb(j,l,k1-j1,k2-j2)
214             where {xi1=>0, xi2=>0})$
215         if  ipzbb(l,k1,k2) neq 0 then write {k1,k2};
216 >>;
```

For example, $\langle Z^{\vec{0}}, \tilde{\mathcal{W}}^{(1,2)} \rangle_{\vec{\xi}=\vec{0}}$ is

```
217 mat((ipzw(0,1,2)),(ipzw(1,1,2)));
```

$\sum_{|\vec{n}|=0}^{N} A_{\vec{n}} \langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}} \tilde{\mathcal{W}}^{(1,2)} \rangle_{\vec{\xi}=\vec{0}}$ is

```
218 mat((asum(0,1,2)),(asum(1,1,2)));
```

and $[\langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} \mathcal{B}]_{(10)}$ and $[\langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} \mathcal{B}]_{(01)}$ are

```
219 mat((ipzbb(0,1,0)),(ipzbb(1,1,0)));
220 mat((ipzbb(0,0,1)),(ipzbb(1,0,1)));
```

The last line of (44), in terms of $\mathcal{S}$ [which contains the convolution (33)] is

```
221 convterm:=for k1:=0:nn sum for k2:=0:(nn-k1) sum
222      for l:=0:(dr-1) sum
223      ((for n1:=0:k1 sum for n2:=0:k2 sum for j:=0:(dr-1) sum
224           ipzw(j,n1,n2)*bb(j,l,k1-n1,k2-n2) )
225           -asum(l,k1,k2))*ss(l,k1,k2);
```

Finally we calculate linear uncertain term $r_c^{\vec{0}}(t) + \langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} r_s(t)$ and show that it is zero for our chosen order $N > \max(p_i^j, k_{max}) = 1$. To do this we first construct the forcing $\tilde{r}[u]$ (21) in terms of the uncertain gradient terms, that is, spatial derivatives of $u^{\vec{n}}$ with $|\vec{n}| = N$, and compare this to $\tilde{r}[u] = \tilde{\mathcal{V}} r_c(t) + \tilde{\mathcal{W}} r_s(t)$ (29) to determine all components of $r_c(t)$ and $r_s(t)$.

```
226 operator ux;
227 depend ux, x1, x2;
228 array ru(dr);
229 for k1:=0:kmax do for k2:=0:(kmax-k1) do for j1:=0:dr do
230      for j2:=0:dr do
231      for n1:=0:nn do for n2:=0:(nn-n1) do for l1:=0:nn do
232      if (l1 leq n1+k1 and nn-l1 leq n2+k2 and
233          n1+n2+k1+k2 > nn and k1+k2 > 0) then
234          ru(j1):=ru(j1)+lls(j1,j2,k1,k2)*xi1^n1*xi2^n2/
235          (factorial(n1)*factorial(n2))*factorial(k1+n1)*factorial(k2+n2)
236          /(factorial(l1)*factorial(nn-l1))
237          *df(df(ux(l1,nn-l1,j2),x1,k1+n1-l1),x2,k2+n2-nn+l1);
238
239 operator rvw;
240 operator rc;
241 operator rs;
242 for l:=0:dr do rvw(l):=(for k1:=0:nn sum
243      for k2:=0:(nn-k1) sum
244      vvt(l,k1,k2)*rc(k1,k2)+
245      (for j:=0:(dr-1) sum wwt(l,j,k1,k2)*rs(j,k1,k2)  ));
246 rrs:={}$
247 for k1:=0:nn do for k2:=0:(nn-k1) do rrs:=rc(k1,k2).rrs;
248 for k1:=0:nn do for k2:=0:(nn-k1) do for j:=0:(dr-1) do
249      rrs:=rs(j,k1,k2).rrs;
250 eqn:={}$
251 for l:=0:dr do for k1:=0:nn do for k2:=0:(nn-k1) do
252      eqn:=coeffn(coeffn(ru(l)-rvw(l),xi1,k1),xi2,k2).eqn;
253 eqn:=(eqn where b^2=>bf-f^2)$
254 solr:=solve(eqn,rrs)$
255 solr:=(solr where bf=>b^2+f^2)$
```

In the remainder ρ we only retain the $r_c^{\vec{0}}$ component of the centre subspace forcing $r_c(t)$, and for the stable subspace forcing we calculate $\langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} r_s(t) = \sum_{|\vec{n}|=0}^{N} \langle Z^{\vec{0}}, \tilde{\mathcal{W}}^{\vec{n}} \rangle r_s^{\vec{n}}(t)$. Then we show that the sum $r_c^{\vec{0}} + \langle Z^{\vec{0}}, \tilde{\mathcal{W}} \rangle_{\vec{\xi}=\vec{0}} r_s(t) = 0$.

terms which are included in the slow PDE (43) and those which are in the remainder ρ (44). As the nonlinear calculations in Appendix A.2 are already fairly memory intensive, we do not extend these calculations to deriving the nonlinear remainder terms in ρ.

```
256 rc0:=sub(solr,rc(0,0));
257 rsall:=(for j:=0:(dr-1) sum for k1:=0:nn sum
258     for k2:=0:(nn-k1) sum
259     ipzw(j,k1,k2)*sub(solr,rs(j,k1,k2)));
260 zero:=rc0+rsall;
261
262 end;
```

For example problem (45), after linearisation the slow height field h is not influenced by the fast velocity field $\vec{v}$, but the velocity field is influenced by the height via the gravity term. The consequence of this is that while the slow generalised eigenvectors $\tilde{\mathcal{V}}^{\vec{n}}$ may have a nonzero projection onto the stable subspace, the stable generalised eigenvectors $\tilde{\mathcal{W}}^{\vec{n}}$ do not have a nonzero projection onto the slow subspace,

$$
\tilde{\mathcal{W}}^{\vec{n}} = \begin{bmatrix} 0 & 0 \\ \frac{1}{\vec{n}!}\vec{\xi}^{\vec{n}} & \frac{1}{\vec{n}!}\vec{\xi}^{\vec{n}} \\ \frac{i}{\vec{n}!}\vec{\xi}^{\vec{n}} & -\frac{i}{\vec{n}!}\vec{\xi}^{\vec{n}} \end{bmatrix}, \tag{46}
$$

and thus all inner products $\left[\langle Z^{\vec{0}}, \partial_{\vec{\xi}}^{\vec{n}} \tilde{\mathcal{W}} \rangle \right]_{\vec{\xi}=\vec{0}} = \vec{0}$ (lines 216 to 219 in the above code all produce a three dimensional zero vector) and the remainder ρ (44) is greatly simplified. However, a small change in the example model, such as changing line 8 of the code to `lll0:=mat((0, 1, 0),(-g, 0, 0), (0, 0, 0))`, will produce more complicated stable generalised eigenvectors $\tilde{\mathcal{W}}^{\vec{n}}$ with nonzero projection onto the slow subspace, such as

$$
\tilde{\mathcal{W}}^{(1,0)} = \begin{bmatrix} (-if-b)/(f^2+b^2) & (if-b)/(f^2+b^2) \\ \xi_1 & \xi_1 \\ i\xi_1 & -i\xi_1 \end{bmatrix}, \tag{47}
$$

which will have a nonzero projection onto the slow subspace and produce a more complicated remainder term ρ.

Here we do not calculate the nonlinear terms which appear in the remainder ρ (44), but these terms can be calculated by editing the nonlinear calculations in Appendix A.2 to include the second line of (23). As before, equation (30) separates the centre and stable components of $\tilde{f}[\tilde{u}]$, and (39) separates the nonlinear components into

References

1. Arnold L, Imkeller P (1998) Normal forms for stochastic differential equations. Probab Theory Relat Field 110:559–588. https://doi.org/10.1007/s004400050159
2. Aulbach B, Wanner T (1996) Integral manifolds for Caratheodory type differential equations in Banach spaces. In: Aulbach B, Colonius F (eds) Six lectures on dynamical systems. World Scientific, Singapore, pp 45–119. https://doi.org/10.1142/3012
3. Aulbach B, Wanner T (2000) The Hartman–Grobman theorem for Caratheodorytype differential equations in Banach spaces. In: Nonlinear analysis: theory, methods and applications, pp 40.1–104. ISSN: 0362- 546X. https://doi.org/10.1016/S0362-546X(00)85006-3
4. Benjamin TB, Bona JL, Mahony JJ (1972) Model equations for long waves in nonlinear dispersive systems. Philos Trans R Soc Lond A 272(1220):47–78. https://doi.org/10.1098/rsta.1972.0032
5. Calcagni G, Montobbio M, Nardelli G (2008) Localization of nonlocal theories. Phys Lett B 662(3):285–289. https://doi.org/10.1016/j.physletb.2008.03.024
6. Carr J (1981) Applications of centre manifold theory, vol 35. Sci Appl Math. https://doi.org/10.1007/978-1-4612-5929-9
7. Chang H-C (1994) Wave evolution on a falling film. Annu Rev Fluid Mech 26(1):103–136. https://doi.org/10.1146/annurev.fl.26.010194.000535
8. Chicone C (2006) In: Marsden JE, Sirovich L, Antman SS (eds) Ordinary differential equations with applications. Texts in applied mathematics, vol 34. Springer, New York. https://doi.org/10.1007/0-387-35794-7
9. Cross MC, Hohenberg PC (1993) Pattern formation outside of equilibrium. Rev Mod Phys 65:851–1112. https://doi.org/10.1103/RevModPhys.65.851
10. Davis SH (2017) The importance of being thin. J Eng Math 105(1):3–30. https://doi.org/10.1007/s10665-017-9910-1
11. Dellar PJ, Salmon R (2005) Shallow water equations with a complete Coriolis force and topography. Phys Fluids 17:106601. https://doi.org/10.1063/1.2116747
12. Gallay T (1993) A center-stable manifold theorem for differential equations in Banach spaces. Commun Math Phys 152(2):249–268. https://doi.org/10.1007/BF02098299
13. Griffiths SD, Grimshaw RHJ, Khusnutdinova KR (2006) Modulational instability of two pairs of counter-propagating waves and

energy exchange in a two-component system. Phys D 214(1):1–24. https://doi.org/10.1016/j.physd.2005.11.011

14. Haragus M, Iooss G (2011) Local bifurcations, center manifolds, and normal forms in infinite-dimensional. Dyn Syst. https://doi.org/10.1007/978-0-85729-112-7

15. Hereman W (2009) Shallow Water Waves and Solitary Waves. In: Mathematics of complexity and dynamical systems. Springer, New York, pp. 8112–8125. https://doi.org/10.1007/978-0-387-30440-3_480

16. Iooss G, Adelmeyer M (1992) Topics in bifurcation theory and applications. Advanced series in nonlinear dynamics, vol 3. World Scientific, Singapore

17. Kondic L (2003) Instabilities in gravity driven flow of thin fluid films. SIAM Rev 45(1):95–115. https://doi.org/10.1137/S003614450240135

18. Lall S, Krysl P, Marsden JE (2003) Structure-preserving model reduction for mechanical systems. Phys D 184(1):304–318. https://doi.org/10.1016/S0167-2789(03)00227-6

19. Lam MA, Cummings LJ, Lin T-S, Kondic L (2015) Modeling flow of nematic liquid crystal down an incline. J Eng Math 94(1):97–113. https://doi.org/10.1007/s10665-014-9697-2

20. LeVeque RJ, George DL, Berger MJ (2011) Tsunami modelling with adaptively refined finite volume methods. Acta Numer 20:211–289. https://doi.org/10.1017/S0962492911000043

21. Mielke A (1986) A reduction principle for non-autonomous systems in infinite dimensional spaces. J Differ Equ 65:68–88. https://doi.org/10.1016/0022-0396(86)90042-2

22. Mielke A (1992) Reduction of PDEs on domains with several unbounded directions: a first step towards modulation equations. A Angew Math Phys 43(3):449–470. https://doi.org/10.1007/BF00946240

23. Náraigh LO, Thiffeault J-L (2010) Nonlinear dynamics of phase separation in thin films. Nonlinearity 23(7):1559. https://doi.org/10.1088/0951-7715/23/7/003

24. Noakes CJ, King JR, Riley DS (2006) On the development of rational approximations incorporating inertial effects in coating and rimming flows: a multiple-scales approach. Q J Mech Appl Math 59(2):163–190. https://doi.org/10.1093/qjmam/hbj001

25. Oron A, Davis SH, Bankoff SG (1997) Long-scale evolution of thin liquid films. Rev Mod Phys 69:931–980. https://doi.org/10.1103/RevModPhys.69.931

26. Roberts AJ (1989) Appropriate initial conditions for asymptotic descriptions of the long term evolution of dynamical systems. J Aust Math Soc B 31:48–75. https://doi.org/10.1017/S03342700000006470

27. Roberts AJ (1992) Boundary conditions for approximate differential equations. J Aust Math Soc B 34:54–80. https://doi.org/10.1017/S0334270000007384

28. Roberts AJ (2015) Macroscale, slowly varying, models emerge from the microscale dynamics. IMA J Appl Math 80(5):1492–1518. https://doi.org/10.1093/imamat/hxv004

29. Roberts AJ (2008) Normal form transforms separate slow and fast modes in stochastic dynamical systems. Phys A 387:12–38. https://doi.org/10.1016/j.physa.2007.08.023

30. Roberts AJ, Bunder JE (2017) Slowly varying, macroscale models emerge from microscale dynamics over multiscale domains. IMA J Appl Math 82(5):971–1012. https://doi.org/10.1093/imamat/hxx021

31. Romanazzi P, Bruna M, Howey D (2016) Thermal homogenisation of electrical machine windings applying the multiple-scales method. J Heat Trans 139(1):012101. https://doi.org/10.1115/1.4034337

32. Rüde U, Willcox K, McInnes L, Sterck H (2018) Research and education in computational science and engineering. SIAM Rev 60(3):707–754. https://doi.org/10.1137/16M1096840

33. Segel LA (1969) Distant side walls cause slow amplitude modulation of cellular convection. J Fluid Mech 38:203–224. https://doi.org/10.1017/S0022112069000127

34. Taylor M (2011) Partial differential equations I. Applied mathematical sciences. Springer. https://doi.org/10.1007/978-1-4419-7055-8

35. Westra M-T, Binks DJ, van de Water W (2003) Patterns of Faraday waves. J Fluid Mech 496:1–32. https://doi.org/10.1017/S0022112003005895

36. Wilson SK, Hunt R, Duffy BR (2000) The rate of spreading in spin coating. J Fluid Mech 413:65–88. https://doi.org/10.1017/S0022112000008089