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ORIGINAL PAPER

A Modified Newton–Harmonic Balance Approach to Strongly Odd Nonlinear Oscillators

Baisheng Wu[1] · Weijia Liu[1] · Huixiang Zhong[1] · C. W. Lim[2]

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Abstract
Background Since combinations of the Newton's method and the harmonic balance (HB) method require, at each iteration step, calculating the first or the first- and second-order derivatives of the restoring force function, and expanding the function, its first- and second-order derivatives into Fourier series, the procedural costs are high and sometimes difficult to achieve algebraically. It is thus preferable to avoid expensive re-linearization or computation of the second-order derivative.
Purpose A new approach is proposed to construct accurate analytical approximation solutions to strongly nonlinear conservative oscillators with odd nonlinearities.
Methods The approach is based on a combination of a modified Newton method and the HB method. For the modified Newton method, two simplified Newton steps are taken between each Newton step where only one linearization of the restoring force function is required. The resulting equations are solved by applying the HB method appropriately.
Results Using only one modified Newton iteration step may achieve highly accurate analytical approximation solutions to the strongly nonlinear oscillators. Three examples with physical implications are used to illustrate the proposed method.
Conclusion Through the modified Newton iteration step, the multiple cumbersome linearizations of the restoring force function are replaced by only one linearization, and the corresponding governing equations can be properly solved by the HB method. The current work is expected to extend to the study of other nonlinear oscillations.

Keywords Nonlinear oscillation · Analytical approximation · Modified Newton method · Harmonic balance · Odd nonlinearity

Introduction

For nonlinear dynamical systems in various engineering applications, accurate analytical approximate solutions have been constantly in pursuit in research because these solutions are explicitly expressed and they allow direct discussion of the influence of parameters. In this respect, the perturbation methods [1–3] have been well known as some of the most established approaches in the various studies of nonlinear problems. Almost all perturbation methods require the presence of a small parameter in the governing equation. Such a small-parameter assumption greatly restricts the applications of perturbation techniques. The problem becomes most severe for strongly nonlinear problems because no such small parameter does exist. To overcome the shortcomings of classical perturbation methods, new analytical techniques for accurate approximate solutions should be developed.

The harmonic balance (HB) method [1–5] uses a truncated Fourier series to determine analytical approximation solutions. However, the method is difficult to use for constructing higher order accurate analytical approximation solutions because it in turn requires analytical solution of a set of complicated nonlinear algebraic equations. Lau and Cheung [6] improved the HB method and put forward an incremental HB (IHB) method. The latter is a mixture of analytical and numerical techniques. The IHB method, however, is not self-starting, i.e., it is unable to create an initial solution by itself. In addition, the periodic solution is expressed in a truncated Fourier series where coefficients of harmonics can only be obtained by numerical method and

✉ Baisheng Wu
wubs@gdut.edu.cn

1 School of Electro-Mechanical Engineering, Guangdong University of Technology, Guangzhou 510006, People's Republic of China

2 Department of Architecture and Civil Engineering, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, People's Republic of China

the solution is not represented by oscillation amplitude. Its applicability in analytical investigation is rather restricted.

For improving the solution methodology, an analytical approximation technique for strongly nonlinear conservative oscillators with odd nonlinearities was proposed by Wu et al. [7]. The method is based on combining the Newton's method and the HB method. Complexity of the classical HB method was thus significantly simplified. For strongly nonlinear conservative single-degree-of-freedom oscillators with general nonlinearities, Sun and Wu [8] proposed an approach to construct the analytical approximation solutions by introducing two nonlinear oscillators with odd nonlinearities in Wu and Lim [9] and applying the analytical approximation technique in [7]. For generalization and applications of these methods, the readers are referred to some published papers [10–13]. Recently, Wu et al. [14] presented an approach for solving strongly nonlinear conservative symmetric oscillators. This method associates the second-order Newton iteration and the HB method. With only a single iteration, the analytical approximations provide explicit and brief expressions that yield excellent convergent and accurate results. However, the method of Wu et al. [7] requires, at each iteration step, linearizing the restoring force and expanding the function and its first derivative into Fourier series. While in Wu et al. [14] it is necessary to calculate the first- and second-order derivatives of the restoring force function, and expanding the function, its first- and second-order derivatives into Fourier series. The procedural costs are high and sometimes difficult to achieve algebraically. It is thus preferable to avoid expensive re-linearization [7] or computation of the second-order derivative [14], in each iteration and this is exactly what we will achieve in this paper.

For iterative solution of nonlinear algebraic equations in several variables, the modified Newton method was presented where the G-derivative of an n-dimensional function was re-evaluated in every m-steps [15]. The iteration may be considered as the composition of one Newton step with $m-1$ simplified Newton steps. It should be noted that the Newton iteration has a property of quadratic convergence, while the modified Newton iteration exhibits convergence of order $m+1$. It represents a simple way of generating higher order approximations [15].

The modified Newton method has received little attention for determining the periodic solutions of nonlinear oscillators. In this paper, a new efficient alternative, as compared to the published methods [7, 14], is presented to establish accurate analytical approximation solutions to strongly nonlinear conservative single-degree-of-freedom oscillators with odd nonlinearities. This new alternative is based on a modified Newton method and the HB approach. Each modified Newton iteration step may be considered as the composition of one Newton step with two simplified Newton steps without further linearization. The modified Newton iteration equations are established and, subsequently, the HB method is appropriately applied to solve these equations. The classical HB method is, therefore, greatly simplified. The proposed method is self-starting, i.e., it can supply an initial solution by itself. Using only one modified Newton iteration step, we establish analytical, approximate explicit expressions in terms of oscillation amplitude. These expressions are with very high accuracy. Finally, three representative examples with physical implications are used to describe the solution methodology and to show the effectiveness of the new method.

Solution Methodology

Consider a single-degree-of-freedom nonlinear oscillator governed by

$$\frac{d^2x}{dt^2} + g(x) = 0,\ x(0) = A,\ \frac{dx}{dt}(0) = 0, \tag{1}$$

where $g(x)$ is an odd nonlinear function [i.e., $g(-x) = -g(x)$] that satisfies $xg(x) > 0$ for $x \in [-A, A]$, $x \neq 0$. Note that $x = 0$ is the equilibrium position and the system will oscillate in the symmetric interval $[-A, A]$. The corresponding period and periodic solution depend on the oscillation amplitude A.

Introducing the oscillator angular frequency ω, setting $\tau = \omega t$ and defining $x' = dx/d\tau$ and $\Omega = \omega^2$, one can transform Eq. (1) into

$$\Omega x'' + g(x) = 0,\ x(0) = A,\ x'(0) = 0. \tag{2}$$

First, the single-term HB expression

$$x_0(\tau) = A\cos\tau, \tag{3}$$

is used to construct the initial approximation to Eq. (2). Since $g(-x) = -g(x)$, one can expand $g\big(x_0(\tau)\big)$ into the Fourier series as:

$$g\big(x_0(\tau)\big) = \sum_{j=1}^{\infty} \alpha_{2j-1}(A)\cos[(2j-1)\tau],$$
$$\alpha_{2j-1}(A) = \frac{4}{\pi}\int_0^{\pi/2} g\big(x_0(\tau)\big)\cos[(2j-1)\tau]d\tau \tag{4}$$

Note that all the coefficients $\alpha_{2j-1}(A)(j = 1, 2, \cdots)$ are dependent on the oscillation amplitude A. Setting $x(\tau) = x_0(\tau)$ in Eq. (2), using Eqs. (3) and (4), and equating coefficient of $\cos\tau$ to zero result in

$$\alpha_1(A) - A\Omega = 0. \tag{5}$$

The initial approximate frequency ω_0 can then be expressed as:

$$\omega_0(A) = \sqrt{\Omega_0(A)},\ \Omega_0(A) = \frac{\alpha_1(A)}{A}. \tag{6}$$

The initial approximate period and periodic solution are

$$T_0(A) = \frac{2\pi}{\sqrt{\Omega_0(A)}}, x_0(t) = A\cos\tau,\ \tau = \sqrt{\Omega_0(A)}\,t. \tag{7}$$

Next, the modified Newton technique is incorporated with the HB method to solve Eq. (2). The solution to Eq. (2) can be formulated as:

$$x = x_0 + \Delta x_{10},\ \Omega = \Omega_0 + \Delta\Omega_{10}. \tag{8}$$

Substituting Eq. (8) into Eq. (2), expanding in a Taylor's series about x_0 and Ω_0, and neglecting terms of second order and above in Δx_{10} and $\Delta\Omega_{10}$ lead to

$$\begin{aligned}&\Omega_0 x_0'' + g(x_0) + \Omega_0\Delta x_{10}'' + \Delta\Omega_{10}x_0'' + g_x(x_0)\Delta x_{10} = 0,\\ &\Delta x_{10}(0) = 0,\ \Delta x_{10}'(0) = 0,\end{aligned} \tag{9}$$

where $g_x = \mathrm{d}g/\mathrm{d}x$. Note that the correction terms Δx_{10} and $\Delta\Omega_{10}$ are assumed to be smaller than the previously determined ones so that the dynamical equations may be linearized.

Subsequently, the HB method will be used to solve Eq. (9). Since $g(-x) = -g(x)$, $g_x\big(x_0(\tau)\big)$ can be expanded into the following Fourier series:

$$\begin{aligned}&g_x\big(x_0(\tau)\big) = \frac{\beta_0(A)}{2} + \sum_{i=1}^{\infty}\beta_{2i}(A)\cos(2i\tau),\\ &\beta_{2i}(A) = \frac{4}{\pi}\int_0^{\pi/2} g_x\big(x_0(\tau)\big)\cos(2i\tau)\mathrm{d}\tau.\end{aligned} \tag{10}$$

Note that all the coefficients $\beta_{2i}(A)(i = 0, 1, \cdots)$ are related to the oscillation amplitude A. The term $\Delta x_{10}(\tau)$ satisfies the initial condition in Eq. (9) and it can be written as:

$$\Delta x_{10}(\tau) = y_{10}(\cos\tau - \cos 3\tau). \tag{11}$$

Substituting Eqs. (3), (4), (6), (10) and (11) into Eq. (9), expanding the resulting expression into Fourier series and equating the coefficients of $\cos\tau$ and $\cos 3\tau$ to zero, respectively, yield

$$\begin{aligned}&2\alpha_1 - 2A\Omega_0 + \big(\beta_0 - \beta_4 - 2\Omega_0\big)y_{10} - 2A\Delta\Omega_{10} = 0,\\ &\big(\beta_2 + \beta_4 - \beta_0 - \beta_6 + 18\Omega_0\big)y_{10} + 2\alpha_3 = 0.\end{aligned} \tag{12}$$

Solving Eq. (12) for y_{10} and $\Delta\Omega_{10}$ gives

$$\begin{aligned}&y_{10}(A) = -\frac{2A\alpha_3(A)}{A\big[\beta_2(A) + \beta_4(A) - \beta_0(A) - \beta_6(A)\big] + 18\alpha_1(A)},\\ &\Delta\Omega_{10}(A) = -\frac{\alpha_3(A)\big\{A\big[\beta_0(A) - \beta_4(A)\big] - 2\alpha_1(A)\big\}}{A\big\{A\big[\beta_2(A) + \beta_4(A) - \beta_0(A) - \beta_6(A)\big] + 18\alpha_1(A)\big\}}.\end{aligned} \tag{13}$$

It should be verified that $|y_{10}|$ and $\Delta\Omega_{10}$ are smaller than A and Ω_0, respectively, i.e., the corrections have a smaller amplitude than that determined in the previous steps such that the linearization in Eq. (9) is justified.

Using Eqs. (3), (6), (8), (11) and (13) produces the analytical approximate period as

$$T_{\mathrm{N1}}(A) = \frac{2\pi}{\sqrt{\Omega_{\mathrm{N1}}(A)}},\ \Omega_{\mathrm{N1}}(A) = \Omega_0(A) + \Delta\Omega_{10}(A), \tag{14}$$

and the periodic solution as

$$x_{\mathrm{N1}}(t) = \big[A + y_{10}(A)\big]\cos\tau - y_{10}(A)\cos 3\tau,\ \tau = \sqrt{\Omega_{\mathrm{N1}}(A)}\,t. \tag{15}$$

These are the approximate solutions from the first Newton step.

Based on the result above, the solution to Eq. (2) can be further expressed as

$$x = x_{\mathrm{N1}} + \Delta x_{20}, \Omega = \Omega_{\mathrm{N1}} + \Delta\Omega_{20}. \tag{16}$$

Substituting Eqs. (16) into (2), expanding in a Taylor series about x_{N1} and Ω_{N1} and neglecting terms of second order and above in Δx_{20} and $\Delta\Omega_{20}$ lead to

$$\begin{aligned}&\Omega_{N1}x_{N1}'' + g\big(x_{N1}\big) + \Omega_{N1}\Delta x_{20}'' + \Delta\Omega_{20}x_{N1}'' + g_x\big(x_{N1}\big)\Delta x_{20} = 0,\\ &\Delta x_{20}(0) = 0,\ \Delta x_{20}'(0) = 0.\end{aligned} \tag{17}$$

To avoid costly and cumbersome computations of the derivative of $g_x\big(x_{\mathrm{N1}}\big)$ and its Fourier series expansion, we replace Ω_{N1}, x_{N1}'' and $g_x\big(x_{\mathrm{N1}}\big)$ in the later three terms in Eq. (17) with Ω_0, x_0'' and $g_x\big(x_0\big)$, respectively, and obtain

$$\begin{aligned}&\Omega_{\mathrm{N1}}x_{\mathrm{N1}}'' + g\big(x_{\mathrm{N1}}\big) + \Omega_0\Delta x_{20}'' + \Delta\Omega_{20}x_0'' + g_x\big(x_0\big)\Delta x_{20} = 0,\\ &\Delta x_{20}(0) = 0,\ \Delta x_{20}'(0) = 0,\end{aligned} \tag{18}$$

which is the governing equation for the first simplified Newton step.

The HB method is again applied to solve Eq. (18) for Δx_{20} and $\Delta\Omega_{20}$. The Fourier series expansion of $g\big(x_{\mathrm{N1}}\big)$ can be expressed as

$$\begin{aligned}&g\big(x_{\mathrm{N1}}(\tau)\big) = \sum_{j=1}^{\infty}\gamma_{2j-1}(A)\cos\big[(2j-1)\tau\big],\\ &\gamma_{2j-1}(A) = \frac{4}{\pi}\int_0^{\pi/2} g\big(x_{\mathrm{N1}}(\tau)\big)\cos\big[(2j-1)\tau\big]\mathrm{d}\tau.\end{aligned} \tag{19}$$

The term $\Delta x_{20}(\tau)$ that satisfies the initial condition in Eq. (18) is set as

$$\Delta x_{20}(\tau) = z_1(\cos\tau - \cos 3\tau) + z_2(\cos 3\tau - \cos 5\tau). \tag{20}$$

Substituting Eqs. (3), (6), (10), (13)-(15), (19) and (20) into Eq. (18), expanding in a Fourier series and equating the coefficients of $\cos\tau$, $\cos 3\tau$ and $\cos 5\tau$ to zero, respectively, generate three relations for unknowns z_1, z_2 and $\Delta\Omega_{20}$ as:

$$\begin{aligned}&2\gamma_1 - 2(A + y_{10})\Omega_{\mathrm{N1}} + \big(\beta_0 - \beta_4 - 2\Omega_0\big)z_1 + \big(\beta_2 - \beta_6\big)z_2\\ &- 2A\Delta\Omega_{20} = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}&2\gamma_3 + 18y_{10}\Omega_{\mathrm{N1}} + \big(\beta_2 + \beta_4 - \beta_0 - \beta_6 + 18\Omega_0\big)\\ &z_1 - \big(\beta_2 + \beta_8 - \beta_0 - \beta_6 + 18\Omega_0\big)z_2 = 0,\end{aligned} \tag{22}$$

$$2\gamma_5 + \big(\beta_4 + \beta_6 - \beta_2 - \beta_8\big)z_1 + \big(\beta_2 + \beta_8 - \beta_0 - \beta_{10} + 50\Omega_0\big)z_2 = 0. \tag{23}$$

The solution to Eqs. (21)–(23) is

$$z_1(A) = \frac{2\left(-\beta_2 - \beta_8 + \beta_0 + \beta_{10} + 50\Omega_0\right)\left(9y_{10}\Omega_{\mathrm{N1}} + \gamma_3\right) + 2\gamma_5\left(\beta_2 + \beta_8 - \beta_0 - \beta_6 + 18\Omega_0\right)}{E}, \tag{24}$$

$$z_2(A) = \frac{-2\left(-\beta_2 + \beta_4 + \beta_6 - \beta_8\right)\left(9y_{10}\Omega_{\mathrm{N1}} + \gamma_3\right) + 2\gamma_5\left(\beta_2 + \beta_4 - \beta_0 - \beta_6 + 18\Omega_0\right)}{E}, \tag{25}$$

$$\Delta\Omega_{20}(A) = \frac{F}{AE}, \tag{26}$$

where

$$\begin{aligned} F(A) = {} & \left(-A\Omega_{\mathrm{N1}} - y_{10}\Omega_{\mathrm{N1}} + \gamma_1\right)E - \left(9y_{10}\Omega_{\mathrm{N1}} + \gamma_3\right)\left(\beta_2 - \beta_6\right)\left(-\beta_2 + \beta_4 + \beta_6 - \beta_8\right) + \\ & \gamma_5\left(\beta_2 - \beta_6\right)\left(\beta_2 + \beta_4 - \beta_0 - \beta_6 + 18\Omega_0\right) + \gamma_5\left(-2\Omega_0 + \beta_0 - \beta_4\right)\left(\beta_2 + \beta_8 - \beta_0 - \beta_6 + 18\Omega_0\right) + \\ & \left(9y_{10}\Omega_{\mathrm{N1}} + \gamma_3\right)\left(-2\Omega_0 + \beta_0 - \beta_4\right)\left(\beta_2 + \beta_8 - \beta_0 - \beta_{10} + 50\Omega_0\right), \\ E(A) = {} & -\left(\beta_2 + \beta_4 - \beta_0 - \beta_6 + 18\Omega_0\right)\left(\beta_2 + \beta_8 - \beta_0 - \beta_{10} + 50\Omega_0\right) + \\ & \left(\beta_2 + \beta_8 - \beta_0 - \beta_6 + 18\Omega_0\right)\left(\beta_2 - \beta_4 - \beta_6 + \beta_8\right). \end{aligned}$$

Consequently, from Eqs. (13)–(16), (20) and (24)–(26), the analytical approximate period and periodic solution for the first simplified Newton step are

$$T_{\mathrm{SN1}}(A) = \frac{2\pi}{\sqrt{\Omega_{\mathrm{SN1}}(A)}}, \Omega_{\mathrm{SN1}}(A) = \Omega_{\mathrm{N1}}(A) + \Delta\Omega_{20}(A), \tag{27}$$

and

$$\begin{aligned} x_{\mathrm{SN1}}(t) = {} & \left[A + y_{10}(A) + z_1(A)\right]\cos\tau + \\ & \left[z_2(A) - y_{10}(A) - z_1(A)\right]\cos 3\tau - z_2(A)\cos 5\tau, \\ \tau = {} & \sqrt{\Omega_{\mathrm{SN1}}(A)}\,t. \end{aligned} \tag{28}$$

The solution to Eq. (2) can be further expressed as:

$$x = x_{\mathrm{SN1}} + \Delta x_{21}, \Omega = \Omega_{\mathrm{SN1}} + \Delta\Omega_{21}. \tag{29}$$

Like the derivation of the first simplified Newton step, we can obtain the governing equation for the second simplified Newton step as follows:

$$\begin{aligned} & \Omega_{\mathrm{SN1}}x''_{\mathrm{SN1}} + g\left(x_{\mathrm{SN1}}\right) + \Omega_0\Delta x''_{21} \\ & + \Delta\Omega_{21}x''_0 + g_x\left(x_0\right)\Delta x_{21} = 0, \\ & \Delta x_{21}(0) = 0,\ \Delta x'_{21}(0) = 0. \end{aligned} \tag{30}$$

The HB method may again be applied to solve Eq. (30) for Δx_{21} and $\Delta\Omega_{21}$. The term $\Delta x_{21}(\tau)$ that satisfies the initial condition in Eq. (30) is set as:

$$\begin{aligned} \Delta x_{21}(\tau) = {} & w_1(\cos\tau - \cos 3\tau) + w_2(\cos 3\tau - \cos 5\tau) \\ & + w_3(\cos 5\tau - \cos 7\tau). \end{aligned} \tag{31}$$

The unknowns Δx_{21} and $\Delta\Omega_{21}$ may be determined using a method similar to that of the first simplified Newton step. The details of solution process are omitted here for brevity. Using Eqs. (27)–(29) and (31) yields the analytical approximate period and periodic solution for the second simplified Newton step:

$$T_{\mathrm{SN2}}(A) = \frac{2\pi}{\sqrt{\Omega_{\mathrm{SN2}}(A)}}, \ \Omega_{\mathrm{SN2}}(A) = \Omega_{\mathrm{SN1}}(A) + \Delta\Omega_{21}(A), \tag{32}$$

and

$$\begin{aligned} x_{\mathrm{SN2}}(t) = {} & \left[A + y_{10}(A) + z_1(A) + w_1(A)\right]\cos\tau + \\ & \left[z_2(A) + w_2(A) - y_{10}(A) - z_1(A) - w_1(A)\right]\cos 3\tau + \\ & \left[w_3(A) - z_2(A) - w_2(A)\right]\cos 5\tau - w_3(A)\cos 7\tau, \\ \tau = {} & \sqrt{\Omega_{\mathrm{SN2}}(A)}\,t. \end{aligned} \tag{33}$$

In fact, many more harmonics can be included in Eqs. (11), (20) and (31) where the initial conditions in Eqs. (9), (17) and (30) can be satisfied, respectively. Finally, the higher order analytical approximations can be constructed using the last approximations x_{SN2} and ω_{SN2} in place of x_0 and ω_0, respectively, and repeating the modified Newton iteration step as before. In the next section, we will use three practical examples to show that using only a single modified Newton iteration step, it is possible to derive highly accurate analytical approximations to the period and periodic solutions of the strongly nonlinear oscillators. Furthermore, these solutions are explicit and expressed in terms of the oscillation amplitude.

Illustrative Examples

In this section, three examples of practical interests are presented to illustrate the solution step, accuracy and effectiveness of the proposed method.

Example 1 The Duffing oscillator.

Consider the following Duffing oscillator with initial conditions ($a \geq 0$)

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + ax + bx^3 = 0, x(0) = A, \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0. \tag{34}$$

For this oscillator, the Fourier series expansions of $g\left(x_0\right)$ and $g_x\left(x_0\right)$ are given in Eqs. (4) and (10), respectively, where

$$\begin{aligned} &\alpha_1(A) = aA + \frac{3bA^3}{4}, \ \alpha_3(A) = \frac{bA^3}{4}, \\ &\beta_0(A) = 2a + 3bA^2, \quad \beta_2(A) = \frac{3bA^2}{2}. \end{aligned} \tag{35}$$

Based on Eqs. (6), (7), (13)–(15) and (35), the analytical approximate periods and periodic solutions for the initial and the first Newton step can be expressed as:

$$T_0(A) = \frac{2\pi}{\sqrt{a + \frac{3}{4}bA^2}}, \tag{36}$$

$$x_0(t) = A\cos\tau, \ \tau = \sqrt{a + \frac{3bA^2}{4}}\, t, \tag{37}$$

and

$$T_{\mathrm{N1}}(A) = \frac{2\pi}{\sqrt{\Omega_{\mathrm{N1}}(A)}}, \ \Omega_{\mathrm{N1}}(A) = \frac{128a^2 + 192abA^2 + 69b^2A^4}{128a + 96bA^2}, \tag{38}$$

$$\begin{aligned} x_{\mathrm{N1}}(t) = &\left(\frac{32aA + 23bA^3}{32a + 24bA^2}\right) \\ &\cos\tau + \left(\frac{bA^3}{32a + 24bA^2}\right)\cos 3\tau, \\ \tau = &\sqrt{\Omega_{\mathrm{N1}}(A)}\, t. \end{aligned} \tag{39}$$

Let $a = 1$ in Eq. (34), Figs. 1 and 2 show the ratios of $y_{10}(A)$, $\Delta\Omega_{10}(A)$ in Eq. (13) to A and $\Omega_0(A)$ in Eq. (6), respectively. Furthermore, we have

$$\lim_{bA^2\to+\infty} \frac{y_1(A)}{A} \approx -0.0416667, \ \lim_{bA^2\to+\infty} \frac{\Delta\Omega_{10}(A)}{\Omega_0(A)} \approx -0.0416667.$$

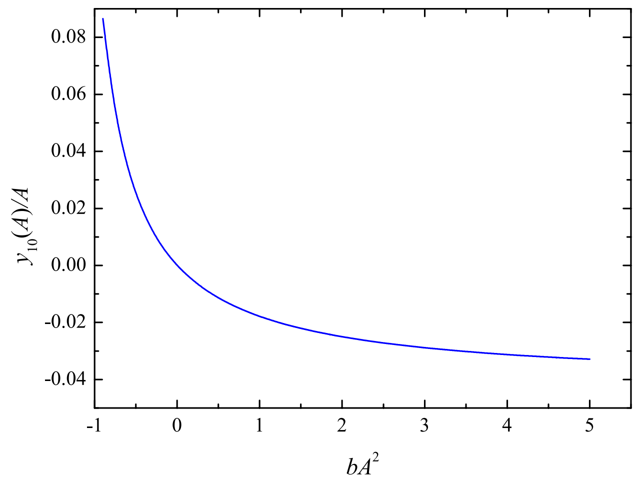

Fig. 1 Variation of ratio $y_{10}(A)/A$ with bA^2 for $a = 1$ in Example 1

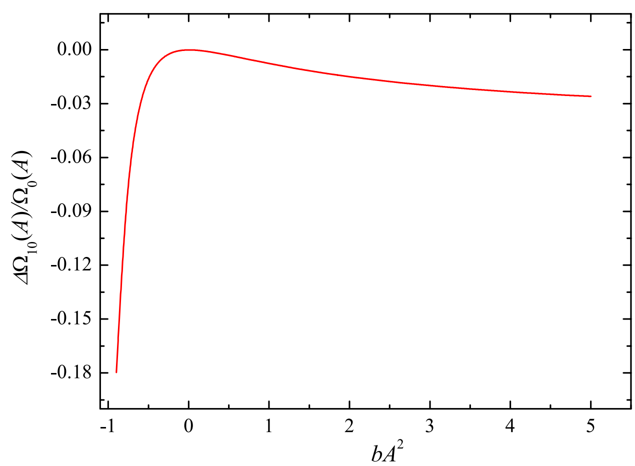

Fig. 2 Variation of ratio $\Delta\Omega_{10}(A)/\Omega_0(A)$ with bA^2 for $a = 1$ in Example 1

From Eq. (11), Figs. 1 and 2 and the two limits above, it can be observed that the correction terms $\Delta x_{10}(A)$ and $\Delta\Omega_{10}(A)$ are smaller than the previously determined ones for $bA^2 \in (-1, +\infty)$ except those value of bA^2 near -1, which confirms the justifiability of the linearization in Eq. (9).

The Fourier series expansion for $g\left(x_{\mathrm{N1}}(\tau)\right)$ is given by Eq. (19) where

$$\gamma_1(A) = \frac{\left(65536Aa^4 + 194560A^3a^3b + 215040A^5a^2b^2 + 104976A^7ab^3 + 19113A^9b^4\right)}{1024\left(4a + 3A^2b\right)^3},$$

$$\gamma_3(A)=\frac{A^3b\left(2304a^3+5184A^2a^2b+3870A^4ab^2+959A^6b^3\right)}{128\left(4a+3A^2b\right)^3},$$

$$\gamma_5(A)=\frac{3A^5b^2\left(32a+23A^2b\right)}{256\left(4a+3A^2b\right)^2}. \tag{40}$$

Based on Eqs. (6), (7), (13)–(15), (24)–(28), (35) and (40), the analytical approximate period and periodic solution for the first simplified Newton step are

$$T_{\mathrm{SN1}}(A)=\frac{2\pi}{\sqrt{\Omega_{\mathrm{SN1}}(A)}}, \tag{41}$$

and

$$x_{\mathrm{SN1}}(t)=L(A)\cos\tau+M(A)\cos3\tau+N(A)\cos5\tau,\ \tau=\sqrt{\Omega_{\mathrm{SN1}}(A)}\,t, \tag{42}$$

where

$$\begin{aligned}
\Omega_{\mathrm{SN1}}(A)&=\frac{128a^2+192A^2ab+69A^4b^2}{128a+96A^2b}-\frac{3A^8b^4\left(2560a^2+3936A^2ab+1513A^4b^2\right)}{2048\left(4a+3A^2b\right)^4\left(32a+23A^2b\right)}\\
L(A)&=\frac{32768Aa^4+96256A^3a^3b+105984A^5a^2b^2+51823A^7ab^3+9491A^9b^4}{16\left(4a+3A^2b\right)^3\left(32a+23A^2b\right)}\\
M(A)&=\frac{A^3b\left(4096a^3+9216A^2a^2b+6984A^4ab^2+1783A^6b^3\right)}{512\left(4a+3A^2b\right)^4}\\
N(A)&=\frac{A^5b\left(4096a^3+9088A^2a^2b+6792A^4ab^2+1711A^6b^3\right)}{512\left(4a+3A^2b\right)^4\left(32a+23A^2b\right)}.
\end{aligned} \tag{43}$$

Finally, using Eqs. (6), (7), (13)–(15), (24)–(28), (30)–(33), (35) and (40) yields the analytical approximate period and periodic solution for the second simplified Newton step as:

$$\begin{aligned}
T_{\mathrm{SN2}}(A)&=\frac{2\pi}{\sqrt{\Omega_{\mathrm{SN2}}(A)}},\\
\Omega_{\mathrm{SN2}}(A)&=\frac{128a^2+192A^2ab+69A^4b^2}{128a+96A^2b}\\
&\quad-\frac{3A^8b^4\left(2560a^2+3936A^2ab+1513A^4b^2\right)}{2048(4a+3A^2b)^4(32a+23A^2b)}+\frac{P_\Omega}{4P_D},
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
x_{\mathrm{SN2}}(t)=&\frac{1}{P_D}\left(P_1\cos\tau+P_2\cos3\tau\right.\\
&\left.+P_3\cos5\tau+P_4\cos7\tau\right),\ \tau=\sqrt{\Omega_{\mathrm{SN2}}(A)}\,t,
\end{aligned} \tag{45}$$

where P_1, P_2, P_3, P_4, P_Ω, P_D are listed in the appendix.

Let $a=1$ in Eq. (34), Table 1 lists the ratios of analytical approximate periods $T_0(A)$, $T_{\mathrm{N1}}(A)$, $T_{SN1}(A)$ and $T_{\mathrm{SN2}}(A)$ in Eqs. (36), (38), (41) and (44), respectively, to the exact period $T_e(A)$ [7]. For comparison, the corresponding ratio of the third approximate period $T_{\mathrm{W3}}(A)$ in Eq. (42) in [7] to $T_e(A)$ is also presented in Table 1. It is observed that Eqs. (41) and (44) give very good approximations to the periods for both small and large values of bA^2. Furthermore, for all $a\ge0$ in Eq. (34), referring to Wu et al. [7], one has

$$\lim_{bA^2\to+\infty}\frac{T_0(A)}{T_e(A)}\approx0.978277,\ \lim_{bA^2\to+\infty}\frac{T_{\mathrm{N1}}(A)}{T_e(A)}\approx0.999318. \tag{46}$$

Further referring to Wu et al. [7], Eqs. (41) and (44) lead to

$$\lim_{bA^2\to+\infty}\frac{T_{\mathrm{SN1}}(A)}{T_\mathrm{e}(A)}=\left(\frac{1}{\pi}\sqrt{\frac{912599}{317952}}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-1/2\sin^2t}}\mathrm{d}t\right)^{-1}\approx1.00015, \tag{47}$$

$$\lim_{bA^2\to+\infty}\frac{T_{\mathrm{SN2}}(A)}{T_\mathrm{e}(A)}=\left(\frac{1}{\pi}\sqrt{\frac{2392173303659990607455}{833239679719213891584}}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-1/2\sin^2t}}\mathrm{d}t\right)^{-1}\approx1.00003, \tag{48}$$

$$\lim_{bA^2\to+\infty}\frac{T_{\mathrm{W3}}}{T_e}=\left(\frac{2}{\pi}\sqrt{\frac{65856986475}{91739270448}}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-1/2\sin^2t}}\mathrm{d}t\right)^{-1}\approx0.999929. \tag{49}$$

Note that the limits in Eqs. (46)–(49) are independent on the value of a. From Table 1 and Eqs. (47) and (48), it is concluded that the proposed approximations to the period converge very fast for both small as well as large oscillation amplitudes.

Table 1 Comparison of approximate periods and exact period for Example 1 ($a = 1$)

bA^2	T_0/T_e	T_{N1}/T_e	T_{SN1}/T_e	T_{SN2}/T_e	T_{W3}/T_e
− 0.9	0.888201	0.980695	0.994290	0.997538	0.996238
− 0.7	0.973202	0.998957	0.999986	0.999985	0.999859
− 0.5	0.992390	0.999917	1.00001	1.00000	0.999990
− 0.3	0.998238	0.999996	1.00000	1.00000	0.999999
− 0.1	0.999863	1.00000	1.00000	1.00000	1.00000
0.5	0.998446	0.999997	1.00000	1.00000	1.00000
1	0.996145	0.999979	1.00000	1.00000	0.999998
5	0.986679	0.999745	1.00006	1.00001	0.999973
10	0.983249	0.999596	1.00009	1.00001	0.999957
50	0.979437	0.999390	1.00013	1.00002	0.999937
100	0.978869	0.999355	1.00014	1.00002	0.999933
500	0.978398	0.999326	1.00014	1.00003	0.999930
1000	0.978338	0.999322	1.00015	1.00003	0.999930
5000	0.978289	0.999319	1.00015	1.00003	0.999930

For $a = 1$, $b = -1$, $A = \sqrt{0.9}$ and $a = 1$, $b = +1$, $A = \sqrt{10}$, the periodic solution $x_e(t)$ obtained using numerical integration to Eq. (34) and the analytical approximate solutions $x_{N1}(t)$, $x_{SN1}(t)$ and $x_{SN2}(t)$ calculated using Eqs. (39), (42) and (45), respectively, as well as their absolute errors are presented in Figs. 3, 4, 5 and 6. The corresponding results of the third approximate periodic solutions x_{W3} in Eq. (43) in Wu et al. [7] are also presented in these figures for comparison purposes. These figures indicate that, for both soft and hard nonlinear oscillators, Eqs. (42) and (45) provide fast-convergent analytical approximate periodic solutions.

Example 2 An oscillator with fractional-power restoring force.

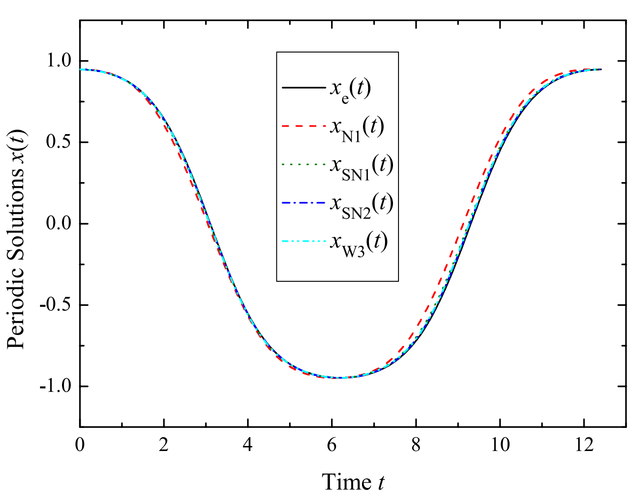

Fig. 3 Comparison of analytical approximations and numerical solution for $a = 1$, $b = -1$ and $A = \sqrt{0.9}$ in Example 1

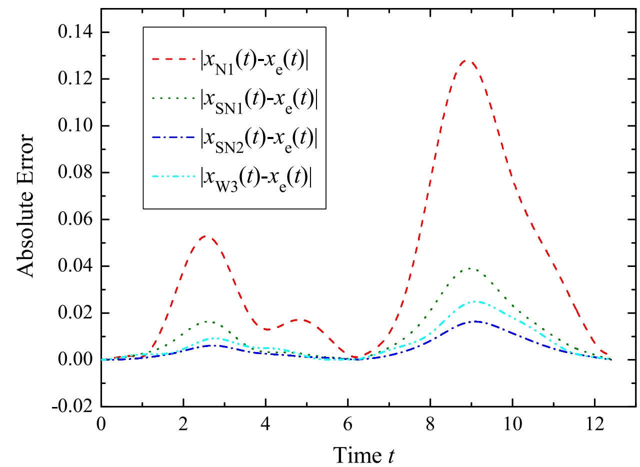

Fig. 4 Absolute error of analytical approximations for $a = 1$, $b = -1$ and $A = \sqrt{0.9}$ in Example 1

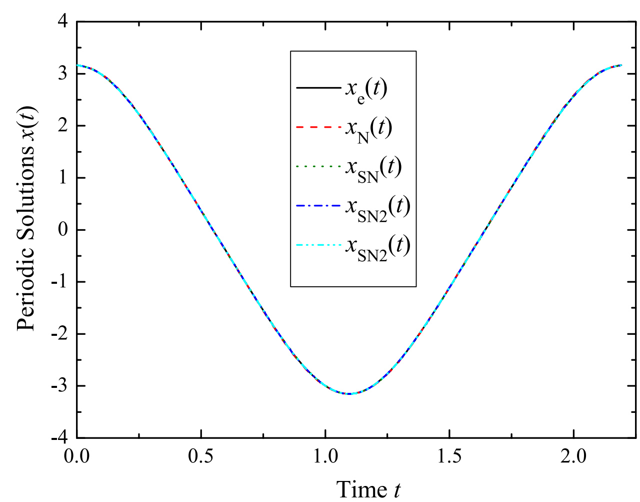

Fig. 5 Comparison of analytical approximations and numerical solution for $a = 1$, $b = 1$ and $A = \sqrt{10}$ in Example 1

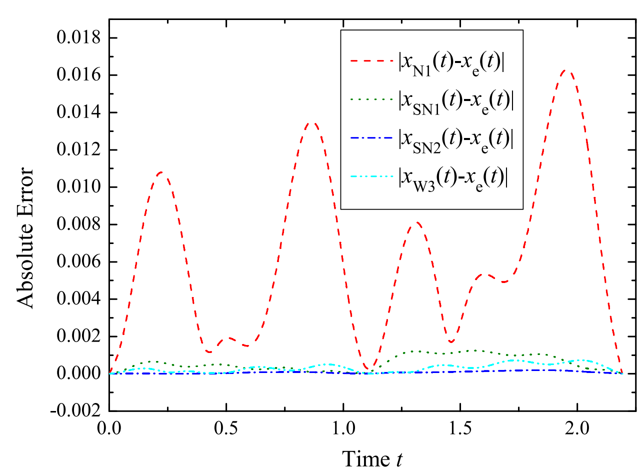

Fig. 6 Absolute error of analytical approximations for $a = 1$, $b = 1$ and $A = \sqrt{10}$ in Example 1

The oscillator with restoring force expressed by a fractional power [16–18] can be represented by the following differential equation with initial conditions:

$$\frac{d^2x}{dt^2} + x^{1/3} = 0,\ x(0) = A,\ \frac{dx}{dt}(0) = 0. \tag{50}$$

For this oscillator, we have $g(x) = x^{1/3}$ and $g_x(x) = \frac{1}{3x^{2/3}}$. The Fourier series expansion of $g(x_0)$ are given in Eq. (4) where

$$\begin{aligned} &\alpha_1(A) = \frac{2A^{1/3}G_1}{\sqrt{\pi}},\ \alpha_3(A) = -\frac{3A^{1/3}G_2}{5\sqrt{\pi}},\ \alpha_5(A) = \frac{3A^{1/3}G_2}{10\sqrt{\pi}},\\ &\alpha_7(A) = -\frac{21A^{1/3}G_2}{110\sqrt{\pi}}, \alpha_9(A) = \frac{3A^{1/3}G_2}{22\sqrt{\pi}},\ \alpha_{11}(A) = -\frac{39A^{1/3}G_2}{374\sqrt{\pi}},\\ &\alpha_{13}(A) = \frac{78A^{1/3}G_2}{935\sqrt{\pi}},\ G_1 = \Gamma\left(\frac{7}{6}\right)\Big/\Gamma\left(\frac{5}{3}\right), G_2 = \Gamma\left(\frac{7}{6}\right)\Big/\Gamma\left(\frac{2}{3}\right). \end{aligned} \tag{51}$$

Based on Eqs. (6), (7) and (51), the initial analytical approximations to the period and periodic solution are

$$\begin{aligned} &T_0(A) = \frac{2\pi}{\sqrt{\Omega_0(A)}}, \Omega_0(A) = 2A^{-2/3}G_1\Big/\sqrt{\pi} \approx 1.15960A^{-2/3},\\ &x_0(t) = A\cos\tau,\ \tau = \sqrt{\Omega_0}t. \end{aligned} \tag{52}$$

Furthermore, we have

$$\begin{aligned} g_x(x_0)\Delta x_{10} &= \frac{y_{10}(\cos\tau - \cos 3\tau)}{3(A\cos\tau)^{2/3}} = \frac{2y_{10}(A\cos\tau)^{1/3}(1-\cos 2\tau)}{3A}\\ &= \frac{2y_{10}}{3A}(\alpha_1\cos\tau + \alpha_3\cos 3\tau + \cdots)(1-\cos 2\tau)\\ &= \beta_1\cos\tau + \beta_3\cos 3\tau + \cdots, \end{aligned} \tag{53}$$

where

$$\beta_1(A) = \frac{(\alpha_1 - \alpha_3)y_{10}}{3A}, \beta_3(A) = -\frac{(\alpha_1 - 2\alpha_3 + \alpha_5)y_{10}}{3A}.$$

Substituting Eqs. (3), (4), (11) and (53) into Eq. (9) and setting the coefficients of $\cos\tau$ and $\cos 3\tau$ to zero yield two linear equations in terms of unknowns y_{10} and $\Delta\Omega_{10}$. These equations can be solved to obtain:

$$\begin{aligned} &y_{10}(A) = \frac{3A\alpha_3(A)}{-26\alpha_1(A) - 2\alpha_3(A) + \alpha_5(A)},\\ &\Delta\Omega_{10}(A) = \frac{\alpha_3\left[2\alpha_1(A) + \alpha_3(A)\right]}{A\left[26\alpha_1(A) + 2\alpha_3(A) - \alpha_5(A)\right]}. \end{aligned} \tag{54}$$

Using Eqs. (6), (51) and (54), we can derive that

$$y_{10}(A)\big/A \approx 0.0235294, \Delta\Omega_{10}(A)\big/\Omega_0(A) \approx -0.0141176,$$

which indicates that the correction terms $\Delta x_{10}(A)$ and $\Delta\Omega_{10}(A)$ are smaller than the previously determined ones for any oscillation amplitude $A > 0$, the linearization in Eq. (9) is thus justified.

Using Eqs. (14), (15), (51), (52) and (54) gives the analytical approximate period and periodic solution for the first Newton step as:

$$\begin{aligned} &T_{N1}(A) = \frac{2\pi}{\sqrt{\Omega_{N1}(A)}},\\ &\Omega_{N1}(A) \approx 1.14323\big/A^{2/3}, \end{aligned} \tag{55}$$

and

$$\begin{aligned} &x_{N1}(t) \approx 1.02353A\cos\tau - 0.0235293A\cos 3\tau,\\ &\quad \tau \approx \left(1.06922\big/A^{1/3}\right)t. \end{aligned} \tag{56}$$

The Fourier series expansion of $g(x_{N1})$ is given by Eq. (19) where

$$\begin{aligned} &\gamma_1(A) \approx 1.17032A^{1/3},\ \gamma_3(A) \approx -0.245242A^{1/3},\\ &\gamma_5(A) \approx 0.119994A^{1/3}. \end{aligned} \tag{57}$$

In addition, we have

$$\begin{aligned} g_x(x_0)\Delta x_{20} &= \frac{z_1(\cos\tau - \cos 3\tau) + z_2(\cos 3\tau - \cos 5\tau)}{3(A\cos\tau)^{2/3}}\\ &= \frac{2}{3A}(A\cos\tau)^{1/3}\left[z_1(1-\cos 2\tau)\right.\\ &\left.+z_2(-1 + 2\cos 2\tau - \cos 4\tau)\right]\\ &= \frac{2}{3A}(\alpha_1\cos\tau + \alpha_3\cos 3\tau + \cdots)\\ &\quad\left[z_1(1-\cos 2\tau) + z_2(-1 + 2\cos 2\tau - \cos 4\tau)\right]\\ &= \eta_1\cos\tau + \eta_3\cos 3\tau + \eta_5\cos 5\tau + \cdots, \end{aligned} \tag{58}$$

where

$$\begin{aligned} &\eta_1(A) = \frac{\alpha_1 - \alpha_3}{3A}z_1 + \frac{\alpha_3 - \alpha_5}{3A}z_2,\\ &\eta_3(A) = \frac{-\alpha_1 + 2\alpha_3 - \alpha_5}{3A}z_1 + \frac{\alpha_1 - 2\alpha_3 + 2\alpha_5 - \alpha_7}{3A}z_2,\\ &\eta_5(A) = \frac{-\alpha_3 + 2\alpha_5 - \alpha_7}{3A}z_1 + \frac{-\alpha_1 + 2\alpha_3 - 2\alpha_5 + 2\alpha_7 - \alpha_9}{3A}z_2. \end{aligned}$$

Substituting Eqs. (3), (19), (20), (51), (57) and (58) into Eq. (18) and setting the coefficients of $\cos\tau$, $\cos 3\tau$ and $\cos 5\tau$

to zero gives three linear equations in terms of z_1, z_2 and $\Delta\Omega_{20}$. These equations can be solved to obtain:

$$z_1(A) \approx -0.00386852A, z_2(A) \approx -0.00421486A,$$
$$\Delta\Omega_{20}(A) \approx 0.00337946/A^{2/3}. \quad (59)$$

Based on Eqs. (27), (28), (51), (54), (55) and (59), the analytical approximation period and periodic solution for the first simplified Newton step are

$$T_{SN1}(A) = 2\pi/\omega_{SN1}(A), \ \ \omega_{SN1}(A) = \sqrt{\Omega_{SN1}(A)},$$
$$\Omega_{SN1}(A) \approx 1.14660/A^{2/3}. \quad (60)$$

and

$$x_{SN1}(t) \approx 1.01966A\cos\tau - 0.0238758A\cos 3\tau$$
$$+ 0.00421486A\cos 5\tau, \ \ \tau \approx 1.07080t. \quad (61)$$

The coefficients of the Fourier series expansion of $g\left(x_{SN1}(\tau)\right)$ are

$$\mu_1(A) \approx 1.16903A^{1/3}, \mu_3(A) \approx -0.245679A^{1/3},$$
$$\mu_5(A) \approx 0.122106A^{1/3}, \mu_7(A) \approx -0.0767870A^{1/3}, \quad (62)$$

and

$$\begin{aligned} &g_x(x_0)\Delta x_{21} \\ &= \frac{w_1(\cos\tau - \cos 3\tau) + w_2(\cos 3\tau - \cos 5\tau) + w_2(\cos 5\tau - \cos 7\tau)}{3(A\cos\tau)^{2/3}} \\ &= \frac{2}{3A}(A\cos\tau)^{1/3}\left[w_1(1-\cos 2\tau) + w_2(-1+2\cos 2\tau - \cos 4\tau)\right. \\ &\quad \left.+ w_3(1 - 2\cos 2\tau + 2\cos 4\tau - \cos 6\tau)\right] \\ &= \delta_1\cos\tau + \delta_3\cos 3\tau + \delta_5\cos 5\tau + \delta_7\cos 7\tau \cdots, \end{aligned} \quad (63)$$

where

$$\delta_1(A) = \frac{\alpha_1 - \alpha_3}{3A}w_1 + \frac{\alpha_3 - \alpha_5}{3A}w_2 + \frac{\alpha_5 - \alpha_7}{3A}w_3,$$

$$\delta_3(A) = \frac{-\alpha_1 + 2\alpha_3 - \alpha_5}{3A}w_1 + \frac{\alpha_1 - 2\alpha_3 + 2\alpha_5 - \alpha_7}{3A}w_2 + \frac{\alpha_3 - 2\alpha_5 + 2\alpha_7 - \alpha_9}{3A}w_3,$$

$$\delta_5(A) = \frac{-\alpha_3 + 2\alpha_5 - \alpha_7}{3A}w_1 + \frac{-\alpha_1 + 2\alpha_3 - 2\alpha_5 + 2\alpha_7 - \alpha_9}{3A}w_2 + \frac{\alpha_1 - 2\alpha_3 + 2\alpha_5 - 2\alpha_7 + 2\alpha_9 - \alpha_{11}}{3A}w_3,$$

$$\delta_7(A) = \frac{-\alpha_5 + 2\alpha_7 - \alpha_9}{3A}w_1 + \frac{-\alpha_3 + 2\alpha_5 - 2\alpha_7 + 2\alpha_9 - \alpha_{11}}{3A}w_2 + \frac{-\alpha_1 + 2\alpha_3 - 2\alpha_5 + 2\alpha_7 - 2\alpha_9 + 2\alpha_{11} - \alpha_{13}}{3A}w_3.$$

Table 2 Ratios of the approximate periods to the exact solution for Example 2

T_0/T_e	T_{N1}/T_e	T_{SN1}/T_e	T_{SN2}/T_e	T_{W3}/T_e
0.994062	1.00115	0.999677	1.00014	0.999692

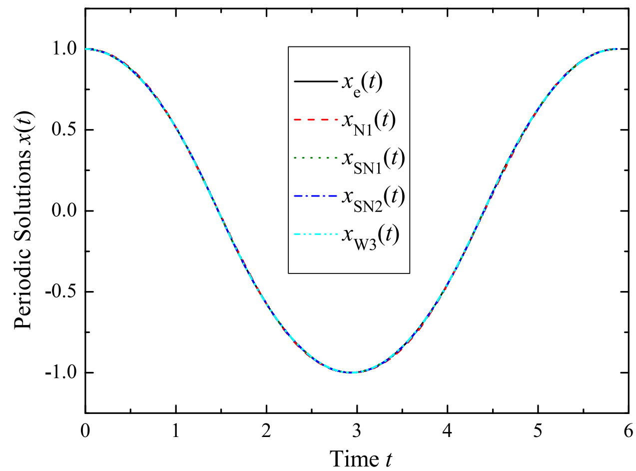

Fig. 7 Comparison of analytical approximations and numerical solution for $A = 1$ in Example 2

Substituting Eqs. (3), (31), (62) and (63) into Eq. (30) and setting the coefficients of $\cos\tau$, $\cos 3\tau$, $\cos 5\tau$ and $\cos 7\tau$ to zero give four linear equations in unknowns w_1, w_2, w_3 and $\Delta\Omega_{21}$. These equations can be solved to yield

$$w_1 \approx 0.00126150A, \ \ w_2 \approx 0.00131078A, \ \ w_2$$
$$\approx 0.00131078A, \ \ w_3 \approx 0.00136572A. \quad (64)$$

Finally, using Eqs. (32), (33), (51), (54), (59) and (64) produces the analytical approximation period and periodic solution for the second simplified Newton step as:

$$T_{SN2}(A) = 2\pi/\omega_{SN2}(A), \ \ \omega_{SN2}(A) = \sqrt{\Omega_{SN2}(A)},$$
$$\Omega_{SN2}(A) \approx 1.14554/A^{2/3}, \quad (65)$$

and

$$x_{SN2}(t) \approx 1.02092A\cos\tau - 0.0238265A\cos 3\tau$$
$$+ 0.00426981A\cos 5\tau - 0.00136572A\cos 7\tau \quad (66)$$
$$\tau \approx 1.07030t.$$

For the exact period $T_e(A)$ of this oscillator, we refer readers to Eq. (72) in Wu et al. [7].

Table 2 shows the ratios of the approximate periods T_0, T_{N1}, T_{SN1}, T_{SN2} to the exact period T_e. For comparison, the ratio of the third approximate period T_{W3} in Eq. (70) in [7]

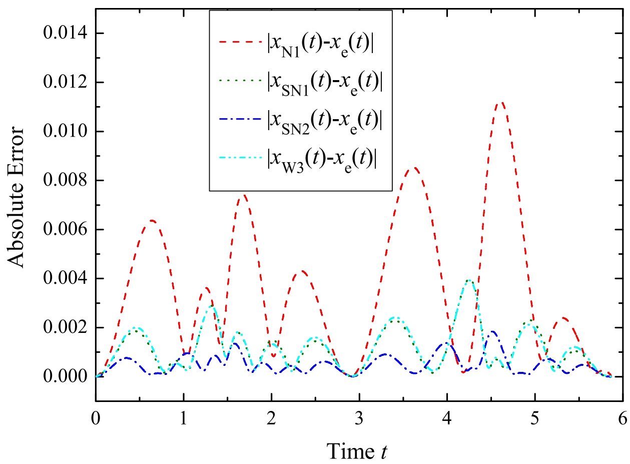

Fig. 8 Absolute errors of analytical approximations for $A = 1$ in Example 2

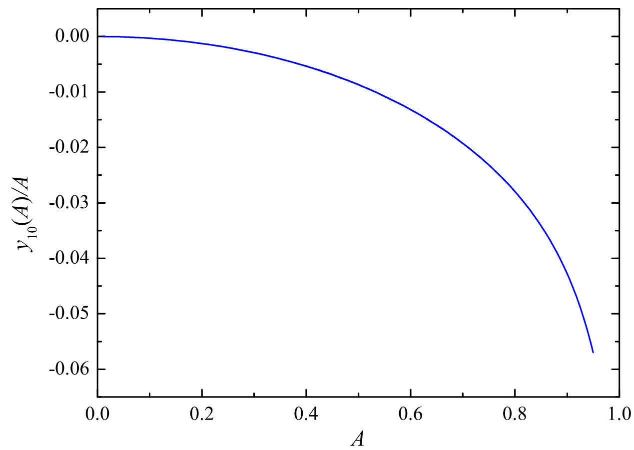

Fig. 9 Variation of ratio $y_{10}(A)/A$ with A in Example 3

to T_e is also given in Table 2. It is clear that Eqs. (60) and (65) provide fast-convergent and excellent approximations to the exact period.

For $A = 1$, the exact periodic solution $x_e(t)$ obtained by numerically integrating Eq. (50) and the approximate periodic solutions $x_{N1}(t)$, $x_{SN1}(t)$ and $x_{SN2}(t)$ computed by Eqs. (56), (61) and (66), respectively, as well as their absolute errors are shown in Figs. 7 and 8. The third approximate periodic solutions $x_{W3}(t)$ in Eq. (71) in [7] are also presented in these figures for comparison. The figure indicates that analytical approximations from the first and second simplified Newton steps well approach the exact periodic solution.

Example 3 A finite extensibility nonlinear oscillator

The non-dimensional equation of motion governing a finite extensibility nonlinear oscillator [19, 20] and the initial conditions are

$$\frac{d^2x}{dt^2} + \frac{x}{1-x^2} = 0,\ x(0) = A,\ \frac{dx}{dt}(0) = 0, \tag{67}$$

where $0 < A < 1$.

Before applying the proposed method to Eq. (67), we rewrite it in a form that gets rid of the fractional term, as

$$\left(1-x^2\right)\frac{d^2x}{dt^2} + x = 0,\ x(0) = A,\ \frac{dx}{dt}(0) = 0. \tag{68}$$

Using a new independent variable $\tau = \omega t$, we can write Eq. (68) as

$$\left(1-x^2\right)\Omega x'' + x = 0,\ x(0) = A,\ x'(0) = 0, \tag{69}$$

where $\Omega = \omega^2$. Substituting Eq. (3) into Eq. (69), expanding the resulting expression into a Fourier series and setting the coefficient of $\cos\tau$ to zero, yield Ω as function of A as:

$$T_0(A) = \frac{2\pi}{\sqrt{\Omega_0(A)}},\ \Omega_0 = \frac{4}{4-3A^2},\ x_0(t) = A\cos\tau,\ \tau = \sqrt{\Omega_0}t. \tag{70}$$

Substituting Eq. (8) into Eq. (69) and neglecting terms of the second order and above in Δx_{10} and $\Delta\Omega_{10}$ lead to

$$\begin{aligned}&\left(1-x_0^2\right)x_0''\Omega_0 + x_0 + \left(1-x_0^2\right)\\&\left(\Delta\Omega_{10}x_0'' + \Omega_0\Delta x_{10}''\right)\\&-2x_0''x_0\Omega_0\Delta x_{10} + \Delta x_{10} = 0,\\&\Delta x_{10}(0) = 0,\ \Delta x_{10}'(0) = 0.\end{aligned} \tag{71}$$

Combining Eqs. (11) and (70) with Eq. (71), expanding in a Fourier series and setting the coefficients of $\cos\tau$ and $\cos 3\tau$ to zero yield two linear equations in terms of unknowns y_{10} and $\Delta\Omega_{10}$. These equations can be solved to obtain:

$$\begin{aligned}y_{10} &= -\frac{4A^3 - 3A^5}{128 - 160A^2 + 43A^4},\ \Delta\Omega_{10}\\&= -\frac{20A^4}{-512 + 1024A^2 - 652A^4 + 129A^6}.\end{aligned} \tag{72}$$

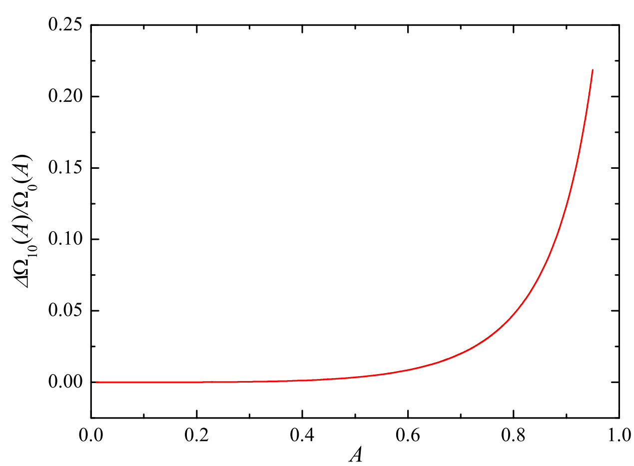

Fig. 10 Variation of ratio $\Delta\Omega_{10}(A)/\Omega_0(A)$ with A in Example 3

For this example, the ratios of $y_{10}(A)$, $\Delta\Omega_{10}(A)$ in Eq. (13) to A and $\Omega_0(A)$ in Eq. (6), respectively, are displayed in Figs. 9 and 10. These figures demonstrate that the correction terms Δx_{10} and $\Delta\Omega_{10}$ are smaller than the previously determined ones for $A \in (0, 1)$ except those values of A near $A = 1$, which verifies the justifiability of the linearization in Eq. (9).

Substituting Eqs. (70) and (72) into Eqs. (14) and (15) gives analytical approximations to the period and periodic solution for the first Newton step as:

$$T_{\mathrm{N1}}(A) = \frac{2\pi}{\sqrt{\Omega_{\mathrm{N1}}(A)}},\ \Omega_{\mathrm{N1}} = \frac{4}{4-3A^2} + \frac{20A^4}{-512+1024A^2-652A^4+129A^6}, \tag{73}$$

and

$$x_{\mathrm{N1}}(t) = \frac{128A-164A^3+46A^5}{128-160A^2+43A^4}\cos\tau + \frac{4A^3-3A^5}{128-160A^2+43A^4}\cos 3\tau,\quad \tau = \sqrt{\Omega_{\mathrm{N1}}(A)}\,t. \tag{74}$$

Substituting Eq. (16) into Eq. (69), following the similar steps for deriving Eqs. (17) and (18), we obtain the governing equation for unknowns Δx_{20} and $\Delta\Omega_{20}$ in the first simplified Newton step as follows:

$$\begin{aligned} &\left(1-x_{\mathrm{N1}}^2\right)x_{\mathrm{N1}}''\Omega_{\mathrm{N1}} + x_{\mathrm{N1}} + \left(1-x_0^2\right)\\ &\left(\Delta\Omega_{20}x_0'' + \Omega_0\Delta x_{20}''\right)\\ &-2x_0''x_0\Omega_0\Delta x_{20} + \Delta x_{20} = 0,\\ &\Delta x_{20}(0) = 0,\ \Delta x_{20}'(0) = 0. \end{aligned} \tag{75}$$

Combining Eqs. (20), (70), (73) and (74) with Eq. (75), expanding in a Fourier series and setting the coefficients of $\cos\tau$, $\cos 3\tau$ and $\cos 5\tau$ to zero gives three linear equations in terms of z_1, z_2 and $\Delta\Omega_{20}$. These equations can be solved to obtain:

$$z_1 = \frac{8K_1(A)}{3H(A)}, z_2 = \frac{K_2(A)}{3H(A)}, \Delta\Omega_{20} = \frac{4K_\Omega(A)}{H(A)}. \tag{76}$$

where

$$\begin{aligned} K_1(A) = {} & 1476395008A^5 - 8476688384A^7\\ & + 21170749440A^9 - 30115528704A^{11}\\ & + 26818781184A^{13} - 15460840448A^{15}\\ & + 5752485184A^{17} - 1327830128A^{19}\\ & + 171976252A^{21} - 9488865A^{23}, \end{aligned}$$

$$\begin{aligned} K_2(A) = {} & 11811160064A^5 - 68014833664A^7\\ & + 170882236416A^9\\ & - 245278703616A^{11} + 221025558528A^{13} - 129193627648A^{15}\\ & + 48740967488A^{17} - 11362056352A^{19} + 1467812432A^{21}\\ & - 78387621A^{23}, \end{aligned}$$

$$\begin{aligned} K_\Omega(A) = {} & -536870912A^6 + 1954545664A^8 - 2900885504A^{10} +\\ & 2281373696A^{12} - 1046603264A^{14} + 294179584A^{16} -\\ & 50428732A^{18} + 4263597A^{20}, \end{aligned}$$

$$\begin{aligned} H(A) = {} & \left(128-160A^2+43A^4\right)^4\\ & \left(-4096+7296A^2-3744A^4+445A^6\right), \end{aligned}$$

Using Eqs. (27), (28) and (76) yields analytical approximations to the period and periodic solution for the first simplified Newton step as:

$$T_{\mathrm{SN1}}(A) = 2\pi / \omega_{\mathrm{SN1}}(A),\ \Omega_{\mathrm{SN1}}(A) = \Omega_{\mathrm{N1}}(A) + \Delta\Omega_{20}(A), \tag{77}$$

and

$$x_{\mathrm{SN1}}(t) = S_1(A)\cos\tau + S_2(A)\cos 3\tau + S_3(A)\cos 5\tau,\ \tau = \sqrt{\Omega_{\mathrm{SN1}}(A)}\,t, \tag{78}$$

where

$S_1(A) = A + y_{10}(A) + z_1(A)$,
$S_2(A) = z_2(A) - y_{10}(A) - z_1(A)$ and $S_3(A) = -z_2(A)$.

Similarly, the governing equation for the second simplified Newton step is

$$\begin{aligned} &\left(1-x_{\mathrm{SN1}}^2\right)x_{\mathrm{SN1}}''\Omega_{\mathrm{SN1}} + x_{\mathrm{SN1}} + \left(1-x_0^2\right)\left(\Delta\Omega_{21}x_0'' + \Omega_0\Delta x_{21}''\right)\\ &-2x_0''x_0\Omega_0\Delta x_{21} + \Delta x_{21} = 0,\\ &\Delta x_{21}(0) = 0,\ \Delta x_{21}'(0) = 0. \end{aligned} \tag{79}$$

Substituting Eqs. (31), (70), (78) into Eq. (79), expanding in a Fourier series and setting the coefficients of $\cos\tau$, $\cos 3\tau$, $\cos 5\tau$ and $\cos 7\tau$ to zero give four linear equations in terms of w_1, w_2, w_3 and $\Delta\Omega_{21}$. These equations can be solved to obtain:

$$\begin{aligned} &w_1 = -\frac{\left(3A^2-4\right)R_1(A)}{12D_{\mathrm{SN2}}(A)},\ w_2 = \frac{\left(3A^2-4\right)R_2(A)}{12D_{\mathrm{SN2}}(A)},\ w_3 = -\frac{\left(3A^2-4\right)R_3(A)}{12D_{\mathrm{SN2}}(A)},\\ &\Delta\Omega_{21} = -\frac{R_\Omega(A)}{AD_{\mathrm{SN2}}(A)}, \end{aligned} \tag{80}$$

where R_1, R_2, R_3, R_Ω, D_{SN2} are listed in the appendix.

Finally, using Eqs. (32), (33), (72), (76) and (80) yields analytical approximations to the period and periodic solution for the second simplified Newton step as:

$$T_{SN2}(A) = 2\pi / \omega_{SN2}(A),\ \Omega_{SN2}(A) = \Omega_{SN1}(A) + \Delta\Omega_{21}(A), \tag{81}$$

and

$$\begin{aligned} x_{SN2}(t) = & \left[A + y_{10}(A) + z_1(A) + w_1(A)\right]\cos\tau + \\ & \left[z_2(A) + w_2(A) - y_{10}(A) - z_1(A) - w_1(A)\right]\cos 3\tau + \\ & \left[w_3(A) - z_2(A) - w_2(A)\right]\cos 5\tau - w_3(A)\cos 7\tau, \\ \tau = & \sqrt{\Omega_{SN2}(A)}\, t. \end{aligned} \tag{82}$$

On the other hand, the exact period is [18]

$$T_e = 4\int_0^A \frac{dx}{\sqrt{\ln\left[\left(1-x^2\right)/\left(1-A^2\right)\right]}}. \tag{83}$$

For comparison, the second-order approximation to the period and periodic solution given by Beléndez et al. [20] using a harmonic balance method (HBM) without linearization are listed as follows:

$$\omega_{B2}(A) = \frac{2}{\sqrt{4 - 3A^2 - 5Ac_1(A) - 30c_1^2(A)}},\ T_{B2}(A) = \frac{2\pi}{\omega_{B2}(A)}, \tag{84}$$

and

$$x_{B2}(t) = A\cos\tau + c_1(\cos 3\tau - \cos\tau),\ \tau = \omega_{B2}\, t, \tag{85}$$

where

$$c_1(A) = \frac{8192A^3 - 12288A^5 + 6512A^7 - 1217A^9}{8\left(32768 - 65536A^2 + 52096A^4 - 19383A^6 + 2811A^8\right)}.$$

The ratios of the approximate periods T_{N1}, T_{SN1}, T_{SN2} and T_{B2} to the exact period T_e are listed in Table 3. Note that for $A < 0.5$, all the period ratios are very close to 1, hence the details are omitted. For $A = 0.9$, numerical solution $x_e(t)$ obtained by integrating Eq. (67) and its analytical approximations $x_{N1}(t)$, $x_{SN1}(t)$, $x_{SN2}(t)$ and $x_{B2}(t)$ computed by Eqs. (74), (78), (82) and (85), respectively, are presented in Figs. 11 and 12. Table 3, Figs. 11 and 12 illustrate clearly again that analytical approximations from the first and second simplified Newton steps yield very accurate period and periodic solution.

Table 3 Comparison of the period and its approximations in Example 3

A	T_{N1}/T_e	T_{SN1}/T_e	T_{SN2}/T_e	T_{B2}/T_e
0.5	1.00003	1.00000	1.00000	0.999948
0.6	1.00015	1.00002	0.999999	0.999839
0.7	1.00062	1.00012	0.999988	0.999624
0.8	1.00274	1.00067	0.999901	0.999707
0.9	1.01544	1.00493	0.999196	1.00599
0.95	1.04815	1.01976	0.999102	1.03014
0.97	1.08794	1.04235	1.00381	1.06346

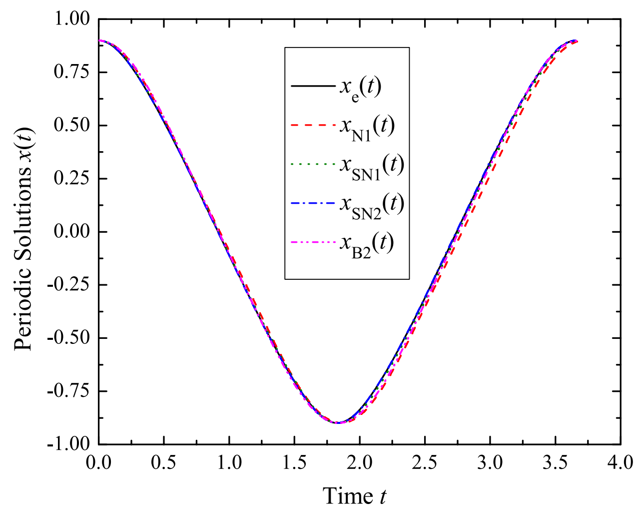

Fig. 11 Comparison of analytical approximations and numerical solution for $A = 0.9$ in Example 3

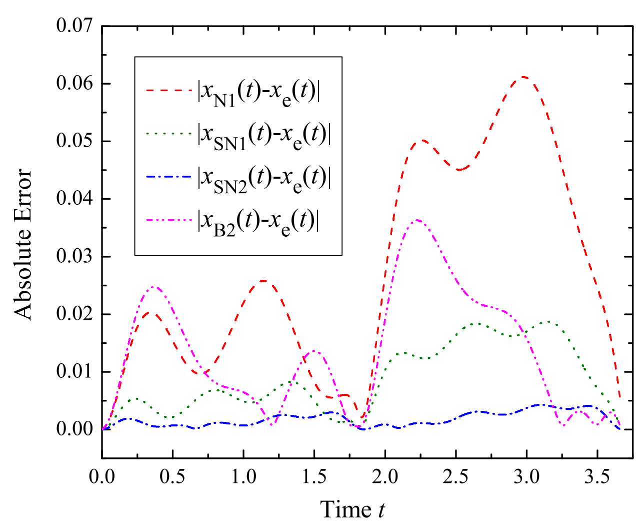

Fig. 12 Absolute error of analytical approximations for $A = 0.9$ in Example 3

Concluding Remarks

A new, efficient and highly accurate alternate approach has been developed for deriving analytical approximate solutions to strongly nonlinear conservative single-degree-of-freedom oscillators with odd nonlinearities. Through the modified Newton iteration step, the multiple cumbersome linearizations of the restoring force function are replaced by only one linearization, and the corresponding governing equations can be properly solved by the harmonic balance method. This improvement is not only accurate, but is generally preferred when it is difficult to obtain the linearizations of the restoring force function, and the corresponding Fourier series expansions. The current work is expected to extend to the study of analytical approximate solutions of general strong nonlinear conservative oscillators, harmonically forced nonlinear oscillators and other nonlinear problems.

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Appendix

The expressions of P_1, P_2, P_3, P_4, P_Ω, P_D are as follows.

$$\begin{aligned}
P_1 = A(&483570327845851669882470 4a^{18} \\
&+ 6445085775820491787652300 8A^2a^{17}b \\
&+ 4056276690460881756304179 20A^4a^{16}b^2 \\
&+ 1601728426736958931994673 152A^6a^{15}b^3 \\
&+ 4446974127479403668997931 008A^8a^{14}b^4 \\
&+ 9218243172947523910489866 240A^{10}a^{13}b^5 \\
&+ 1478584134611140488905254 5024A^{12}a^{12}b^6 \\
&+ 1876352478492245976363827 2000A^{14}a^{11}b^7 \\
&+ 1909768840974522549659710 2592A^{16}a^{10}b^8 \\
&+ 1570650417989894344137087 3856A^{18}a^9b^9 \\
&+ 1046260545932956571534989 7216A^{20}a^8b^{10} \\
&+ 5631582647280319099400355 840A^{22}a^7b^{11} \\
&+ 2431167864190925491283165 184A^{24}a^6b^{12} \\
&+ 8303548112397683603747635 20A^{26}a^5b^{13} \\
&+ 2194413221925636055054254 08A^{28}a^4b^{14} \\
&+ 4329825133629496918117427 2A^{30}a^3b^{15} \\
&+ 6006532208423496497883712 A^{32}a^2b^{16} \\
&+ 522786540947589345532344A^{34}ab^{17} \\
&+ 21485079498804067272519A^{36}b^{18}),
\end{aligned}$$

$$\begin{aligned}
P_2 = &1511157274518286468382 72A^3a^{17}b \\
&+ 1905474875837901843726336A^5a^{16}b^2 \\
&+ 11309108495780967390117888A^7a^{15}b^3 \\
&+ 41958844166223266993668096A^9a^{14}b^4 \\
&+ 108994977581964717830701056A^{11}a^{13}b^5 \\
&+ 210370424231974451810402304A^{13}a^{12}b^6 \\
&+ 312399213313109590649339904A^{15}a^{11}b^7 \\
&+ 364578510827628201919905792A^{17}a^{10}b^8 \\
&+ 338516618651629734938542080A^{19}a^9b^9 \\
&+ 251508624790868463311650816A^{21}a^8b^{10} \\
&+ 149524255540205269240774656A^{23}a^7b^{11} \\
&+ 70726792708396755185565696A^{25}a^6b^{12} \\
&+ 26291888445896461967163392A^{27}a^5b^{13} \\
&+ 7520040035850389352161280A^{29}a^4b^{14} \\
&+ 1598200207932246614587392A^{31}a^3b^{15} \\
&+ 237821684972978253124992A^{33}a^2b^{16} \\
&+ 22124179993691481314304A^{35}ab^{17} \\
&+ 968822190832147158876A^{37}b^{18},
\end{aligned}$$

$$\begin{aligned}
P_3 = &472236648286964521369 6A^5a^{16}b^2 \\
&+ 5600431500778219870617 6A^7a^{15}b^3 \\
&+ 311498637957687129669632A^9a^{14}b^4 \\
&+ 1078614993058995309117440A^{11}a^{13}b^5 \\
&+ 2602451747004176945643520A^{13}a^{12}b^6 \\
&+ 4639338980186234088849408A^{15}a^{11}b^7 \\
&+ 6321057386653510862372864A^{17}a^{10}b^8 \\
&+ 6714526431619256808374272A^{19}a^9b^9 \\
&+ 5619858774906742100721664A^{21}a^8b^{10} \\
&+ 3718495779890683638185984A^{23}a^7b^{11} \\
&+ 1938655501211025962696704A^{25}a^6b^{12} \\
&+ 788025466700714743824384A^{27}a^5b^{13} \\
&+ 244828243490659897270272A^{29}a^4b^{14} \\
&+ 56204195663850721290240A^{31}a^3b^{15} \\
&+ 8991209738990145619840A^{33}a^2b^{16} \\
&+ 895544245281147035584A^{35}ab^{17} \\
&+ 41838930607835396004A^{37}b^{18},
\end{aligned}$$

$$
\begin{aligned}
P_4 = {} & 1475739525896764129 28A^7a^{15}b^3 \\
& + 1639454379550936399872A^9a^{14}b^4 \\
& + 8504309305950292934656A^{11}a^{13}b^5 \\
& + 27323998716602209009664A^{13}a^{12}b^6 \\
& + 60811895858419925516288A^{15}a^{11}b^7 \\
& + 99305437243272100577280A^{17}a^{10}b^8 \\
& + 122919731096436997095424A^{19}a^9b^9 \\
& + 117436631194839132667904A^{21}a^8b^{10} \\
& + 87312998556222355406848A^{23}a^7b^{11} \\
& + 50518231926325423112192A^{25}a^6b^{12} \\
& + 22560699826636333907968A^{27}a^5b^{13} \\
& + 7637023311131482243072A^{29}a^4b^{14} \\
& + 1896881805586013365760A^{31}a^3b^{15} \\
& + 326363394510536538304A^{33}a^2b^{16} \\
& + 34780496959236928136A^{35}ab^{17} \\
& + 1730732174725245369A^{37}b^{18},
\end{aligned}
$$

and

The expressions of R_1, R_2, R_3, R_Ω, D_{SN2} are as follows.

$$
\begin{aligned}
P_\Omega = {} & 3A^8b^4(442721857769029238784a^{15} \\
& + 5102830579389904715776A^2a^{14}b \\
& + 27312638386542166933504A^4a^{13}b^2 \\
& + 90077300039570776129536A^6a^{12}b^3 \\
& + 204746874013667683205120A^8a^{11}b^4 \\
& + 339795427352756527038464A^{10}a^{10}b^5 \\
& + 425361925658199361323008A^{12}a^9b^6 \\
& + 408978801242226020057088A^{14}a^8b^7 \\
& + 304485598886042791837696A^{16}a^7b^8 \\
& + 175505115520599790190592A^{18}a^6b^9 \\
& + 77662337412392581988352A^{20}a^5b^{10} \\
& + 25901296598274638692352A^{22}a^4b^{11} \\
& + 6299629977693939730944A^{24}a^3b^{12} \\
& + 1054263188601994088768A^{26}a^2b^{13} \\
& + 108476685824622319800A^{28}ab^{14} \\
& + 5168358617143777623A^{30}b^{15}),
\end{aligned}
$$

$$
\begin{aligned}
P_{\mathrm{D}} = {} & 134217728\left(4a + 3A^2b\right)^{12}\left(32a + 23A^2b\right)^3 \\
& \left(65536a^3 + 144384A^2a^2b + 105984A^4ab^2 + 25923A^6b^3\right).
\end{aligned}
$$

$$
\begin{aligned}
R_1 = {} & -24576A^2S_1 + 22912A^4S_1 - 3840A^6S_1 \\
& - 98304S_2 + 165376A^2S_2 - 84096A^4S_2 \\
& + 11520A^6S_2 - 32768S_3 + 28672A^2S_3 \\
& + 6704A^4S_3 - 6276A^6S_3 + 24576A^2S_1\Omega_{\mathrm{SN1}} \\
& - 22912A^4S_1\Omega_{\mathrm{SN1}} + 3840A^6S_1\Omega_{\mathrm{SN1}} \\
& - 24576S_1^3\Omega_{\mathrm{SN1}} + 22912A^2S_1^3\Omega_{\mathrm{SN1}} - 3840A^4S_1^3\Omega_{\mathrm{SN1}} \\
& + 884736S_2\Omega_{\mathrm{SN1}} - 1488384A^2S_2\Omega_{\mathrm{SN1}} \\
& + 756864A^4S_2\Omega_{\mathrm{SN1}} - 103680A^6S_2\Omega_{\mathrm{SN1}} \\
& - 630784S1^2S_2\Omega_{\mathrm{SN1}} + 920832A^2S_1^2S_2\Omega_{\mathrm{SN1}} \\
& - 381084A^4S_1^2S_2\Omega_{\mathrm{SN1}} + 35541A^6S_1^2S_2\Omega_{\mathrm{SN1}} \\
& - 233472S_1S_2^2\Omega_{\mathrm{SN1}} + 7296A^2S_1S_2^2\Omega_{\mathrm{SN1}} \\
& + 194180A^4S_1S_2^2\Omega_{\mathrm{SN1}} - 45771A^6S_1S_2^2\Omega_{\mathrm{SN1}} \\
& - 663552S_2^3\Omega_{\mathrm{SN1}} + 1116288A^2S_2^3\Omega_{\mathrm{SN1}} \\
& - 567648A^4S_2^3\Omega_{\mathrm{SN1}} + 77760A^6S_2^3\Omega_{\mathrm{SN1}} \\
& + 819200S_3\Omega_{\mathrm{SN1}} - 716800A^2S_3\Omega_{\mathrm{SN1}} \\
& - 167600A^4S_3\Omega_{\mathrm{SN1}} + 156900A^6S_3\Omega_{\mathrm{SN1}} \\
& - 1216512S_1^2S_3\Omega_{\mathrm{SN1}} + 1651968A^2S_1^2S_3\Omega_{\mathrm{SN1}} \\
& - 555768A^4S_1^2S_3\Omega_{\mathrm{SN1}} + 22194A^6S_1^2S_3\Omega_{\mathrm{SN1}} \\
& - 2007040S_1S_2S_3\Omega_{\mathrm{SN1}} + 2849280A^2S_1S_2S_3\Omega_{\mathrm{SN1}} \\
& - 1274560A^4S_1S_2S_3\Omega_{\mathrm{SN1}} + 210000A^6S_1S_2S_3\Omega_{\mathrm{SN1}} \\
& - 704512S_2^2S_3\Omega_{\mathrm{SN1}} + 352256A^2S_2^2S_3\Omega_{\mathrm{SN1}} \\
& + 390440A^4S_2^2S_3\Omega_{\mathrm{SN1}} - 176214A^6S_2^2S_3\Omega_{\mathrm{SN1}} \\
& - 626688A^2S_1S_2^2\Omega_{\mathrm{SN1}} + 584256A^4S_1S_3^2\Omega_{\mathrm{SN1}} \\
& - 97920A^6S_1S_3^2\Omega_{\mathrm{SN1}} - 3141632S_2S_3^2\Omega_{\mathrm{SN1}} \\
& + 5203328A^2S_2S_3^2\Omega_{\mathrm{SN1}} - 2652640A^4S_2S3_3^2\Omega_{\mathrm{SN1}} \\
& + 403560A^6S_2S_3^2\Omega_{\mathrm{SN1}} - 614400S_3^3\Omega_{\mathrm{SN1}} \\
& + 537600A^2S_3^3\Omega_{\mathrm{SN1}} + 125700A^4S_3^3\Omega_{\mathrm{SN1}} \\
& - 117675A^6S_3^3\Omega_{\mathrm{SN1}}
\end{aligned}
$$

$$
\begin{aligned}
R_2 = {} & 2816A^4S_1 - 1056A^6S_1 + 11264A^2S_2 \\
& - 12672A^4S_2 + 3168A^6S_2 + 32768S_3 \\
& - 53248A^2S_3 + 26368A^4S_3 - 4128A^6S_3 \\
& - 2816A^4S_1\Omega_{\mathrm{SN1}} + 1056A^6S_1\Omega_{\mathrm{SN1}} \\
& + 2816A^2S_1^3\Omega_{\mathrm{SN1}} - 1056A^4S_1^3\Omega_{\mathrm{SN1}} \\
& - 101376A^2S_2\Omega_{\mathrm{SN1}} + 114048A^4S_2\Omega_{\mathrm{SN1}} \\
& - 28512A^6S_2\Omega_{\mathrm{SN1}} + 90112S_1^2S_2\Omega_{\mathrm{SN1}} \\
& - 84480A^2S_1^2S_2\Omega_{\mathrm{SN1}} + 10560A^4S_1^2S_2\Omega_{\mathrm{SN1}} \\
& + 3168A^6S_1^2S_2\Omega_{\mathrm{SN1}} + 233472S_1S_2^2\Omega_{\mathrm{SN1}} \\
& - 350208A^2S_1S_2^2\Omega_{\mathrm{SN1}} + 170620A^4S_1S_2^22^2\Omega_{\mathrm{SN1}} \\
& - 23997A^6S_1S_2^2\Omega_{\mathrm{SN1}} + 76032A^2S_2^3\Omega_{\mathrm{SN1}} \\
& - 85536A^4S_2^3\Omega_{\mathrm{SN1}} + 21384A^6S_2^3\Omega_{\mathrm{SN1}} \\
& - 819200S_3\Omega_{\mathrm{SN1}} + 1331200A^2S_3\Omega_{\mathrm{SN1}} \\
& - 659200A^4S_3\Omega_{\mathrm{SN1}} + 103200A^6S_3\Omega_{\mathrm{SN1}} \\
& + 552960S_1^2S_3\Omega_{\mathrm{SN1}} - 781056A^2S_1^2S_3\Omega_{\mathrm{SN1}} \\
& + 296892A^4S_1^2S_3\Omega_{\mathrm{SN1}} - 26325A^6S_1^2S_3\Omega_{\mathrm{SN1}} \\
& + 286720S_1S_2S_3\Omega_{\mathrm{SN1}} - 161280A^2S_1S_2S_3\Omega_{\mathrm{SN1}} \\
& - 103880A^4S_1S_2S_3\Omega_{\mathrm{SN1}} + 57750A^6S_1S_2S_3\Omega_{\mathrm{SN1}} \\
& + 704512S_2^2S_3\Omega_{\mathrm{SN1}} - 1144832A^2S_2^2S_3\Omega_{\mathrm{SN1}} \\
& + 597184A^4S_2^2S_3\Omega_{\mathrm{SN1}} - 100104A^6S_2^2S_3\Omega_{\mathrm{SN1}} \\
& + 71808A^4S_1S_3^2\Omega_{\mathrm{SN1}} - 26928A^6S_1S_3^2\Omega_{\mathrm{SN1}} \\
& + 241664S_2S_3^2\Omega_{\mathrm{SN1}} + 30208A^2S_2S_3^2\Omega_{\mathrm{SN1}} \\
& - 316004A^4S_2S_3^2\Omega_{\mathrm{SN1}} + 110979A^6S_2S_3^2\Omega_{\mathrm{SN1}} \\
& + 614400S_3^3\Omega_{\mathrm{SN1}} - 998400A^2S_3^3\Omega_{\mathrm{SN1}} \\
& + 494400A^4S_3^3\Omega_{\mathrm{SN1}} - 77400A^6S_3^3\Omega_{\mathrm{SN1}}
\end{aligned}
$$

$$
\begin{aligned}
R_3 = {} & -396A^6S_1 - 1584A^4S_2 + 1188A^6S_2 - 4608A^2S_3 \\
& + 5760A^4S_3 - 1548A^6S_3 + 396A^6S_1\Omega_{\mathrm{SN1}} \\
& - 396A^4S_1^3\Omega_{\mathrm{SN1}} + 14256A^4S_2\Omega_{\mathrm{SN1}} - 10692A^6S_2\Omega_{\mathrm{SN1}} \\
& - 12672A^2S_1^2S_2\Omega_{\mathrm{SN1}} + 7128A^4S_1^2S_2\Omega_{\mathrm{SN1}} \\
& + 1188A^6S_1^2S_2\Omega_{\mathrm{SN1}} - 77824S_1S_2^2\Omega_{\mathrm{SN1}} \\
& + 116736A^2S_1S_2^2\Omega_{\mathrm{SN1}} - 43776A^4S_1S_2^2\Omega_{\mathrm{SN1}} \\
& - 2660A^6S_1S_2^2\Omega_{\mathrm{SN1}} - 10692A^4S_2^3\Omega_{\mathrm{SN1}} \\
& + 8019A^6S_2^3\Omega_{\mathrm{SN1}} + 115200A^2S_3\Omega_{\mathrm{SN1}} \\
& - 144000A^4S_3\Omega_{\mathrm{SN1}} + 38700A^6S_3\Omega_{\mathrm{SN1}} \\
& - 110592S_1^2S_3\Omega_{\mathrm{SN1}} + 134784A^2S_1^2S_3\Omega_{\mathrm{SN1}} \\
& - 34020A^4S_1^2S_3\Omega_{\mathrm{SN1}} - 864A^6S_1^2S_3\Omega_{\mathrm{SN1}} \\
& - 286720S_1S_2S_3\Omega_{\mathrm{SN1}} + 510720A^2S_1S_2S_3\Omega_{\mathrm{SN1}} \\
& - 289800A^4S_1S_2S_3\Omega_{\mathrm{SN1}} + 45010A^6S_1S_2S_3\Omega_{\mathrm{SN1}} \\
& - 99072A^2S_2^2S_3\Omega_{\mathrm{SN1}} + 123840A^4S_2^2S_3\Omega_{\mathrm{SN1}} \\
& - 37539A^6S_2^2S_3\Omega_{\mathrm{SN1}} - 10098A^6S_1S_3^2\Omega_{\mathrm{SN1}} \\
& - 241664S_2S_3^2\Omega_{\mathrm{SN1}} + 430464A^2S_2S_3^2\Omega_{\mathrm{SN1}} \\
& - 267624A^4S_2S_3^2\Omega_{\mathrm{SN1}} + 61301A^6S_2S_3^2\Omega_{\mathrm{SN1}} \\
& - 86400A^2S_3^3\Omega_{\mathrm{SN1}} + 108000A^4S_3^3\Omega_{\mathrm{SN1}} - 29025A^6S_3^3\Omega_{\mathrm{SN1}}
\end{aligned}
$$

$$
\begin{aligned}
R_{\Omega} = & -262144S_1 + 405504A^2S_1 - 166912A^4S_1 + 14100A^6S_1 \\
& - 40960A^2S_2 + 48512A^4S_2 - 10272A^6S_2 \\
& + 16384A^2S_3 - 24576A^4S_3 + 8532A^6S_3 \\
& + 262144S_1\Omega_{\mathrm{SN1}} - 405504A^2S_1\Omega_{\mathrm{SN1}} \\
& + 166912A^4S_1\Omega_{\mathrm{SN1}} - 14100A^6S_1\Omega_{\mathrm{SN1}} \\
& - 196608S_1^3\Omega_{\mathrm{SN1}} + 293888A^2S_1^3\Omega_{\mathrm{SN1}} \\
& - 113056A^4S_1^3\Omega_{\mathrm{SN1}} + 8007A^6S_1^3\Omega_{\mathrm{SN1}} \\
& + 368640A^2S_2\Omega_{\mathrm{SN1}} - 436608A^4S_2\Omega_{\mathrm{SN1}} \\
& + 92448A^6S_2\Omega_{\mathrm{SN1}} - 720896S_1^2S_2\Omega_{\mathrm{SN1}} \\
& + 934912A^2S_1^2S_2\Omega_{\mathrm{SN1}} - 259776A^4S_1^2S_2\Omega_{\mathrm{SN1}} \\
& + 5742A^6S_1^2S_2\Omega_{\mathrm{SN1}} - 2490368S_1S_2^2\Omega_{\mathrm{SN1}} \\
& + 3969024A^2S_1S_2^2\Omega_{\mathrm{SN1}} - 1718816A^4S_1S_2^2\Omega_{\mathrm{SN1}} \\
& + 156180A^6S_1S_2^2\Omega_{\mathrm{SN1}} - 276480A^2S_2^3\Omega_{\mathrm{SN1}} \\
& + 327456A^4S_2^3\Omega_{\mathrm{SN1}} - 69336A^6S_2^3\Omega_{\mathrm{SN1}} \\
& - 409600A^2S_3\Omega_{\mathrm{SN1}} + 614400A^4S_3\Omega_{\mathrm{SN1}} \\
& - 213300A^6S_3\Omega_{\mathrm{SN1}} - 27648A^4S_1^2S_3\Omega_{\mathrm{SN1}} \\
& + 19845A^6S_1^2S_3\Omega_{\mathrm{SN1}} - 4587520S_1S_2S_3\Omega_{\mathrm{SN1}} \\
& + 6522880A^2S_1S_2S_3\Omega_{\mathrm{SN1}} - 2132480A^4S_1S_2S_3\Omega_{\mathrm{SN1}} \\
& - 420A^6S_1S_2S_3\Omega_{\mathrm{SN1}} - 2818048S_2^2S_3\Omega_{\mathrm{SN1}} \\
& + 4711424A^2S_2^2S_3\Omega_{\mathrm{SN1}} - 2322688A^4S_2^2S_3\Omega_{\mathrm{SN1}} \\
& + 335013A^6S_2^2S_3\Omega_{\mathrm{SN1}} - 6684672S_1S_3^2\Omega_{\mathrm{SN1}} \\
& + 10340352A^2S_1S_3^2\Omega_{\mathrm{SN1}} - 4256256A^4S_1S_3^2\Omega_{\mathrm{SN1}} \\
& + 359550A^6S_1S_3^2\Omega_{\mathrm{SN1}} - 1087488A^2S_2S_3^2\Omega_{\mathrm{SN1}} \\
& + 1380128A^4S_2S_3^2\Omega_{\mathrm{SN1}} - 359841A^6S_2S_3^2\Omega_{\mathrm{SN1}} \\
& + 307200A^2S_3^3\Omega_{\mathrm{SN1}} - 460800A^4S_3^3\Omega_{\mathrm{SN1}} + 159975A^6S_3^3\Omega_{\mathrm{SN1}}
\end{aligned}
$$

$$D_{\mathrm{SN2}} = 262144 - 602112A^2 + 460800A^4 - 127156A^6 + 8007A^8$$

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