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ORIGINAL PAPER

Analytic Approximations to Liénard Nonlinear Oscillators with Modified Energy Balance Method

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Abstract
In this study, a modified He's energy balance method (MEBM) was applied to solve strong nonlinear oscillators characterized by generalizations of Liénard equation. The complete MEBM solution procedure for this oscillator equation is presented. For the illustration of the effectiveness and convenience of the MEBM, different cases of Liénard oscillator with different parameters of a_i and initial conditions were compared with the numerical solution. We found that the solutions obtained show an improvement over the approaches using other two methods considered here. The results show that the MEBM is very convenient and precise, therefore it can be widely applicable in engineering and other sciences.

Keywords Energy balance method (EBM) · Hamiltonian approach · Nonlinear oscillators · Liénard equations

Introduction

The Liénard equations have been extensively studied in many contexts of science and engineering, mainly due to the existence of oscillatory phenomena involving a perturbation model with high non-linearities in many applications in mechanical and electrical systems. The Liénard equation is closely connected with the Rayleigh equation, for example, the Van der Pol equation is an special case of it.

In the field of physics, it has been used for example related to the solitary traveling wave solution of the nonlinear Schrödinger equation [1], or with the development of the Van der Pol equation in the consolidation of oscillators that reproduce irregular heart rate, asystole, certain types of heart block, in an electrical model of the heart considered as a relaxed oscillator [2, 3]. The Duffing equation which is a spatial case of this type of Lienard equations is also widely used and representative in mechanical systems describing the motion of a body subjected to a nonlinear spring power, linear sticky damping and periodic powering [4]. Oscillations of mechanical systems under the action of a periodic external force can be revealed given by using Duffing equation. For example in [5, 6] and therein references, Duffing oscillator application studies are presented to extract the characteristics of the early mechanical failure signal.

The exact solution of this type of nonlinear equations is important in determining the properties and behavior of physical systems. For example, in the case of the Duffing oscillator with softening non-linearity have been reported solutions using perturbation theory for predict instability regions of a damped Duffing oscillator with an external sinusoidal excitation in a parameter space composed of the frequency and amplitude of the excitation, assuming that the system parameters are small [7–10]. Also, chaotic dynamics of a particular non-linear oscillator having Dulling type stiffness, van der Pol damping and dry friction have been examined using averaging technique to obtain informations regarding the bifurcation behavior of the vibrating system [11, 12]. Andrianov and Awrejcewicz [13, 14] developed an asymptotic approach for the analysis of strongly nonlinear dynamic systems and compared the approximated results with those obtained by the fourth-order Runge–Kutta method. However, due to its high non-linearity the possibility of finding exact solutions is quite limited. Apart from the analytic methods that have been developed so far, approximate methods can also be used to find a solution quite close to the numerical or exact value. The general idea is to provide results that are as close as possible to real solutions to be able to study the behavior of the systems with accurate

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approximations. Some of the methods that are available in the literature include the perturbation method [15, 16], the asymmetric homotopy method [17–20], iterative method [21–23], optimal homotopy asypmtotic methods [24–29], the harmonic balance [30–32], the iterative homotopic harmonic method [33], the energy balance method [34–37] and the frequency-Amplitude method (FAM) [38].

In recent years Molla et. al. [39] have presented an modified method to find approximate solutions using the original Energy Balance Method [34] as a basis for constructing a methodology that ensures very accurate approximate solutions that converge with a minimum cost of calculations. This proposed methodology is applicable to resolve highly non-linear antisymmetric oscillators like $\ddot{x}+f(x)=0$ with $f(-x)=-f(x)$. The results obtained by this method prove to be much better than those obtained using other similar methods. Also, the solutions by the method proposed are in agreement with the numerical solution, the general classical EBM and other existing methods explained here.

In the following sections, we turn to the issue of finding very accurate approximate solutions using a generalization of the EBM for antisymmetric systems for a generalized type of Liénard equations. In Sect. 2, we present the method that has been used, detailing its application on the general field of differential equations. In Sect. 3, we present the general system to be solved and some results of the application of the method for some particular cases comparing our results with others used in the literature and also with the corresponding numerical solutions of the system. In Sect. 4, the discussion focuses on the measurement of errors compared with the errors of other methods and determining a great improvement in the solutions. Finally in Sect. 5, some final conclusions are presented.

Energy Balance Method and Modified Version

Considering a general non-linear oscillator in the form:

$$\ddot{x}+f(x)=0, \tag{1}$$

with initial conditions $x(0)=A$ and $\dot{x}(0)=0$, where a dot denotes derivative over the independent variable (time for example) and $f(x)$ is a general non-linear term, it's possible (in major of cases) to find a Hamiltonian conservative equation that describes the solution of the differential Eq. (1)

$$H(x)=\frac{1}{2}\dot{x}^2+F(x)=F(A), \tag{2}$$

where primitive $F(x)=\int f(x)\mathrm{d}x$. The correspondent residual equation for this Hamiltonian is written correspondingly as

$$R(t)=\frac{1}{2}\dot{x}^2+F(x)-F(A)=0. \tag{3}$$

In accordance with the original idea of the Energy Balance Method (EBM) [34], so called He's method, a trial solution is possible to be defined as

$$x_1(t)=A\cos wt, \tag{4}$$

in such a way that when this solution is used in the residual (3), we can conveniently and carefully evaluate this equation by fixing $wt=\pi/4$ to determine a solution for angular frequency. In the general case substituting (4) into (3)

$$R(t)=-\frac{1}{2}A^2w^2\sin^2 wt+F(A\cos wt)-F(A)=0, \tag{5}$$

and

$$R(t)\big|_{wt=\pi/4}\rightarrow w=\frac{2}{A}\sqrt{F(A)-F\left(\frac{\sqrt{2}A}{2}\right)}. \tag{6}$$

In Refs. [39, 40], the authors propose a new method based on the EBM idea. In this case it is possible to define a better solution approximation, which means the generalization

$$x_2(t)=A((1-\alpha)\cos wt+\alpha\cos 3wt). \tag{7}$$

Thus, residual (3) can be written as

$$\begin{aligned} R(t)=&-\frac{1}{2}\left[A^2w^2(1-\alpha)\sin wt+3\alpha\sin 3wt\right]^2\\ &+F(A((1-\alpha)\cos wt+\alpha\cos 3wt))\\ &-F(A)=0. \end{aligned} \tag{8}$$

Note that if we put $wt=\pi/4$, the previous equation reduces to (6). In the general case, residual (8) contains two unknown parameters: angular frequency (w) and α; and another one A fixed by the initial condition for solution x. In order to determine these unknown parameters, two algebraic equations are required. Both can be obtained from the constraint equation:

$$\int_0^{T/4}\frac{R(t)\cos(2k-2)wt}{\sin^2 wt}\mathrm{d}t=0, \tag{9}$$

where $k=1,2$ and the period is $T=2\pi/w$. By solving these two equations simultaneously, we obtain w and α.

In a similar way, it is possible to consider the third approximate solution as in the following form

$$x_3(t)=A((1-\alpha-\beta)\cos wt+\alpha\cos 3wt+\beta\cos 5wt). \tag{10}$$

In this case, the residual function contains three unknown parameters w, α and β. To determine these constants, we need three algebraic equations which are obtained from Eq. (3) for $n=1$, 2 and 3 respectively. Solving these equations simultaneously, we obtain w, α and β. All these relations are valid for $x\geq 0$.

Application on Strongly Non-linear Oscillators

Inspired by the success of the Liénard and Duffing equations [41],

$$\ddot{x} + f_1(y)\dot{x} + f_2(x) = g(t), \tag{11}$$

the following generalized Liénard equations with nonlinear terms of any order were investigated. By considering that the above nonlinear oscillator which is governed by the following differential equation with initial conditions

$$\begin{aligned} \Xi(x,t) &= \ddot{x} + a_0 x + a_1 x^{p+1} + a_2 x^{2p+1} + a_3 x^{3p+1} + \cdots \\ &= \ddot{x} + \sum_{n=0}^{N} a_n x^{np+1} = 0, \end{aligned} \tag{12}$$

with initial conditions $x(0) = A$, $\dot{x}(0) = 0$ and where a_n, for $n = 0, 1, 2, \ldots$, are constant coefficients and $p = 1, 2, 3, \ldots$. A Duffing equation, for example, corresponds to the $N = 1$ and $p = 2$ case of the generalized Liénard equation. For some particular cases, exact solutions of the generalized Liénard equation (12) and their applications have been reported (see, for example refs [42–45]).

By considering the first integral, the multiplication of the previous equation by $\dot{x}$ and making by-part integration allow the following conservative Hamiltonian of this equation

$$H(x) = \frac{1}{2}\dot{x}^2 + \sum_{n=0}^{N} \frac{a_n}{np+2} x^{np+2} = \sum_{n=0}^{N} \frac{a_n}{np+2} A^{np+2}, \tag{13}$$

and the residual can be written as

$$R(t) = \frac{1}{2}\dot{x}^2 + \sum_{n=0}^{N} \frac{a_n}{np+2} x^{np+2} - \sum_{n=0}^{N} \frac{a_n}{np+2} A^{np+2} = 0. \tag{14}$$

In this section, we present two examples to illustrate the usefulness and effectiveness of the technique proposed.

Case: $N = 2$ and $p = 2$

In this case the Eq. (12) takes the form

$$\ddot{x} + a_0 x + a_1 x^3 + a_2 x^5, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{15}$$

and the residual is

$$R(t) = \frac{1}{2}\dot{x}^2 + \frac{a_0 x^2}{2} + \frac{a_1 x^4}{4} + \frac{a_2 x^6}{6} - \frac{a_0 A^2}{2} - \frac{a_1 A^4}{4} - \frac{a_2 A^6}{6}. \tag{16}$$

Actually this case is a more general nonlinear system with quintic restoring force [46–49].

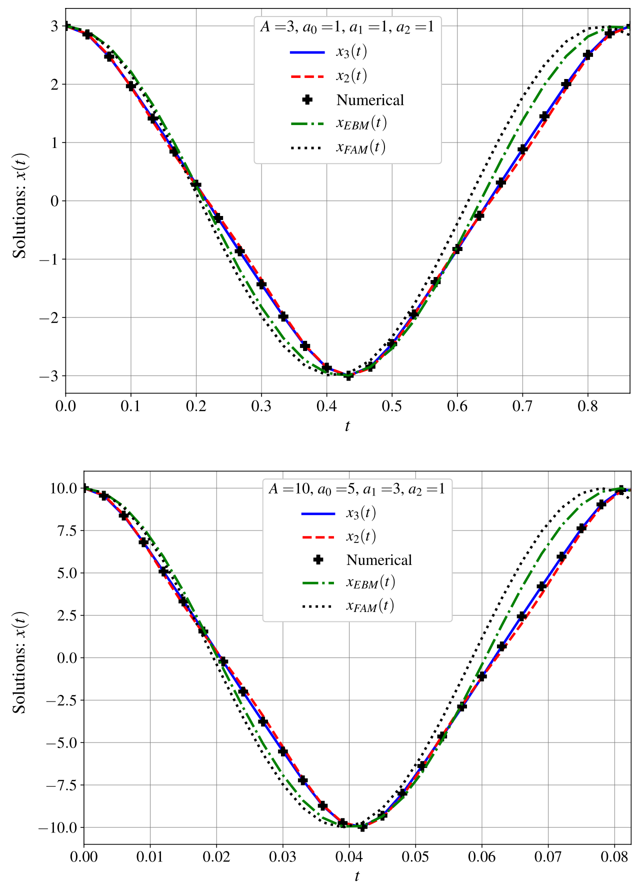

Fig. 1 Comparison of the MEBM solution with the numerical solution for Eq. (15) for some values of the initial conditions and parameters

Table 1 Values for α, β and w for case $N = 2$ and $p = 2$ for some values of parameters and initial conditions

Initial cond.	Params.	$x_2(t)$	$x_3(t)$
$A = 3.0$, $a_0 = 1.0$	α	0.074292425	0.059368243
$a_1 = 1.0$, $a_2 = 1.0$	β	–	0.009239968
	w	7.238439347	7.267885646
$A = 10.0$, $a_0 = 5.0$	α	0.079236901	0.062032774
$a_1 = 3.0$, $a_2 = 1.0$	β	–	0.010399091
	w	75.77535022	76.15868800

The next step is to use the ansatz solution (7) and (10) into the residual (16) and solving the integral in (9) for $n = 1, 2, 3$ respectively, to obtain a system of algebraic equations that can be solved simultaneously by analytic methods to find the values of α, β and w for different values of the initial condition A, despite the strong non-linearities of the equation.

The results of this application are illustrated in the Fig. 1. The values for this solutions are shown in Table 1. The equations for generating these solutions can be seen in A.

Case: $N = 3$ and $p = 2$

In this case the Eq. (12) takes the form

$$\ddot{x} + a_0 x + a_1 x^3 + a_2 x^5 + a_3 x^7, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{17}$$

and the residual is

$$R(t) = \frac{1}{2}\dot{x}^2 + \frac{a_0 x^2}{2} + \frac{a_1 x^4}{4} + \frac{a_2 x^6}{6} + \frac{a_3 x^8}{8} - \frac{a_0 A^2}{2} - \frac{a_1 A^4}{4} - \frac{a_2 A^6}{6} - \frac{a_3 A^8}{8}. \tag{18}$$

Following a similar procedure, as in the previous case, we calculate the integral (9) using the ansatz solution (7) and (10) to solve the system of corresponding algebraic equations. The results are shown in Fig. 2, and the values for these solutions are presented in Table 2. The equations for generating these solutions can be seen in B.

Table 2 Values for α, β and w for case $N = 3$ and $p = 2$ for some values of parameters and initial conditions

Initial cond.	Params.	$x_2(t)$	$x_3(t)$
$A = 2.0$, $a_0 = 3.0$	α	0.113576546	0.072088263
$a_1 = -7.0$, $a_2 = 2.0$	β	–	0.022105008
$a_3 = 10.0$	w	16.81061055	17.06465940
$A = 7.0$, $a_0 = 2.0$	α	0.111293414	0.071511716
$a_1 = 1.0$, $a_2 = 3.0$	β	–	0.021363164
$a_3 = 5.0$	w	513.5660937	520.8911156

Discussion

In order to compare and measure the effectiveness of our solution, we compare our results with those obtained in previous references and with the corresponding numerical

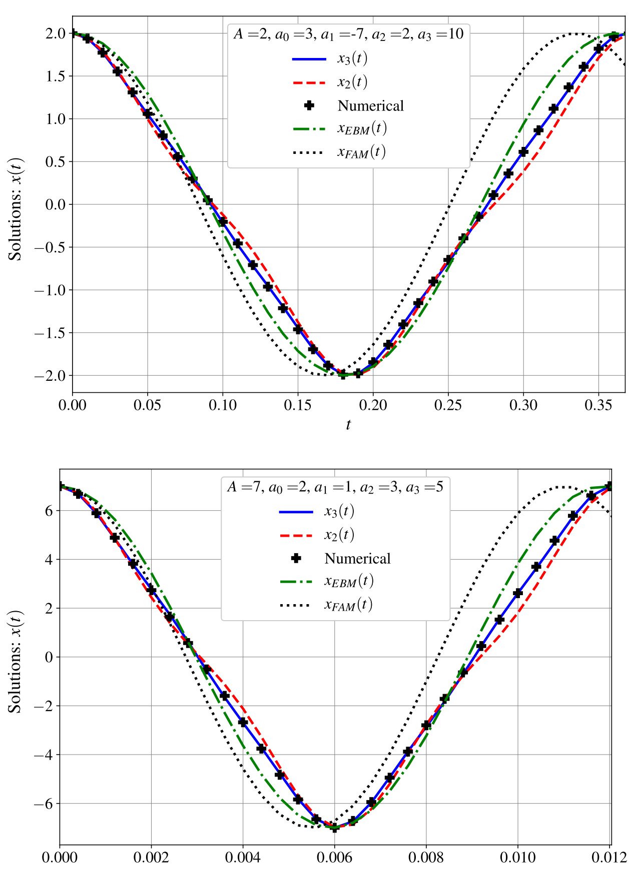

Fig. 2 Comparison of the MEBM solution with the numerical solution for Eq. (17) for some values of the initial conditions and parameters

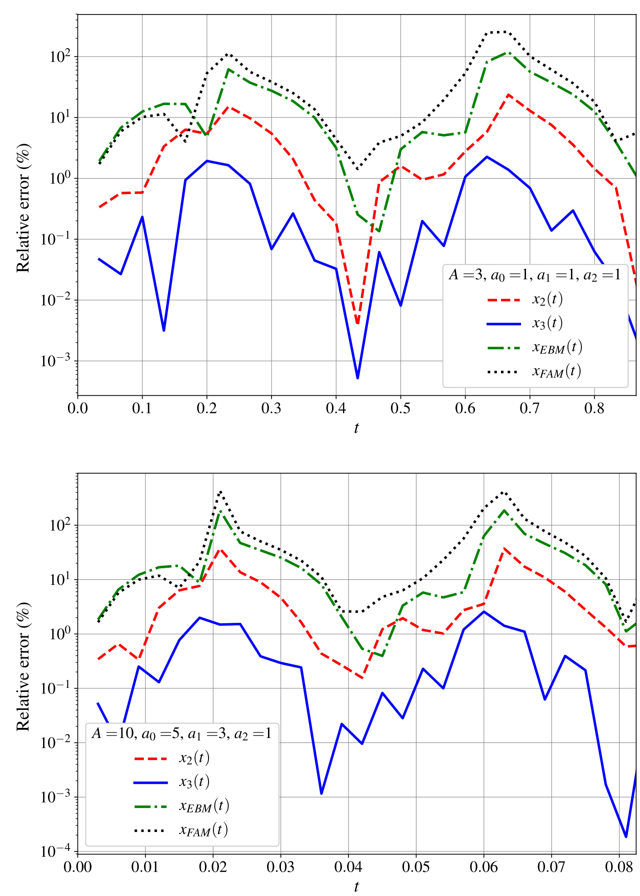

Fig. 3 Comparison of relative error (%) respect to the numerical solution for the MEBM solution, EBM and the FAM [50] for Eq. (15). Vertical axis is in logarithmic scale

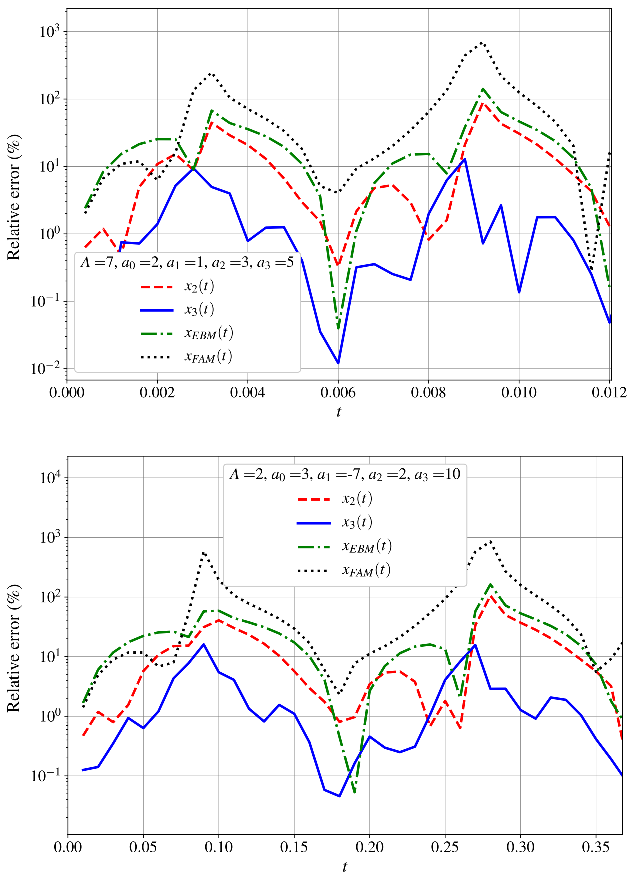

Fig. 4 Comparison of relative error (%) respect to the numerical solution for the MEBM solution, EBM and the FAM [50] for Eq. (17). Vertical axis is in logarithmic scale

solutions. We compare with the values obtained by the EBM and the FAM [50] in Figs. 3 and 4 which for this particular case, using the ansatz function (4) results (for $N = 3$ and $p = 2$)

$$w_{\mathrm{EBM}} = \sqrt{a_0 + \frac{3a_1A^2}{4} + \frac{7a_2A^4}{12} + \frac{15a_3A^6}{32}},$$
$$w_{\mathrm{FAM}} = \sqrt{a_0 + \frac{3a_1A^2}{4} + \frac{5a_2A^4}{8} + \frac{35a_3A^6}{64}}, \tag{19}$$

and the estimation for $N = 3$ and $p = 2$ is the same putting $a_3 = 0$.

The vertical axis in Figs. 3 and 4 is in logarithmic scale to show that the relative errors between all solution considered are at least of three orders of magnitude. This demonstrates that our result improve the performance of the methodology used above. In all the cases studied, it is demonstrated that the solutions found with the MEBM show, in several orders of magnitude, a better behavior than the other referred methods.

For example, for the case of Fig. 1a, we found average values of the relative errors of 0.4102%, compared which result from the EBM and FAM of 18.9376% and 41.134%, respectively (see Fig. 3a); for the case of Fig. 1b, we found average values of the relative errors of 0.4766%, compared with the errors by the EBM and FAM of 27.021% and 57.286%, respectively (see Fig. 3b). As for the case of $N = 3$, the relative errors are higher for $x_2(t)$. This implies that a better approximation could required the incorporation of a $\cos 5wt$ term to the expansion in the solution $x_3(t)$. However, we note that despite observation, the solutions proved the effectiveness of the approach used here. For example, for the case of Fig. 2a, we find average values of the relative errors of 5.3801%, compared with the errors by the EBM and FAM of 59.2711% and 324.1187%, respectively (see Fig. 4a); while for the case of Fig. 2b, we found average values of the relative errors of 2.1643%, compared with the errors by the EBM and FAM of 26.1651% and 124.8371%, respectively (see Fig. 4b).

Concluding Remarks

In this paper we have achieved, for the first time, an efficient implementation of the modified technique of Energy Balance Method to find very accurate solutions, in this case for highly nonlinear oscillators based on the Liénard equation. We have shown that, by using this methodology, the solutions are better than others reported in the literature reviewed, including an improvement in several orders of magnitude for the estimation of relative errors compared with the numerical solution of the system. It has also been shown that the solutions using the MEBM can be understood as a generalization, in the strict sense of the term, respect to the Frequency-Amplitude method. This appreciation is justified because, under the same conditions for the values of the parameters in the Liénard equations, our solution matches the relative frequency for the case of Frequency-Amplitude method. In order to reproduce our solutions, we have written the explicit form of the result in the appendix.

The solutions we have achieved here have been obtained using completely analytic methods without the need to make reductions or impositions to the conditions of the system. However, because of the characteristics of the highly nonlinear nature of the terms involved in differential equations, some truncation have been made to preserve the smallest possible polynomial order to obtain exact solutions, without significantly affecting the accuracy of the solutions.

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Solutions for Case $N = 3$ and $p = 2$

For $k = 1, 2$ in (9) by using an ansatz solution (7) and keeping the terms of order $\mathcal{O}(\alpha^4)$ we have:

$$\begin{aligned} E_1 =&\, 64(22\alpha^2 + 4\alpha + 1)w^2 + 64(2\alpha^2 - 4\alpha - 1)a_0 \\ &+ 48(2\alpha^4 - 4\alpha^3 + 4\alpha^2 - 4\alpha - 1)A^2 a_1 \\ &+ 40(10\alpha^4 - 8\alpha^3 + 5\alpha^2 - 4\alpha - 1)A^4 a_2 \\ &+ 7(100\alpha^4 - 56\alpha^3 + 28\alpha^2 - 20\alpha - 5)A^6 a_3 = 0, \end{aligned} \tag{20}$$

$$\begin{aligned} E_2 =& -4608(2\alpha + 1)\alpha w^2 + 1536 a_0 \alpha \\ &+ 96(2\alpha^4 - 6\alpha^2 + 16\alpha + 1)A^2 a_1 \\ &+ 32(20\alpha^3 - 30\alpha^2 + 45\alpha + 4)A^4 a_2 + 3(-350\alpha^4 \\ &+ 448\alpha^3 - 392\alpha^2 + 448\alpha + 47)A^6 a_3 = 0, \end{aligned} \tag{21}$$

Eliminating w^2 from last equations, we have

$$C_3 A^6 a_3 + C_2 A^4 a_2 + C_1 A^2 a_1 + C_0 a_0 = 0, \tag{22}$$

where $C_3 = -3(7182\alpha^4 - 6720\alpha^3 + 2606\alpha^2 + 204\alpha - 47)$, $C_2 = -32(400\alpha^4 - 620\alpha^3 + 302\alpha^2 + 29\alpha - 4)$, $C_1 = 96(14\alpha^4 + 184\alpha^3 - 136\alpha^2 - 16\alpha + 1)$, and $C_0 = 3072\alpha(\alpha - 1)(6\alpha^2 + 8\alpha + 1)$. The previous equation is of high polynomial order in α, so it has no analytic solution, in this case it can be solved numerically. From the solution previously found for α, then the value of the frequency can be calculated by solving (20) or (21).

It is important to note here that Eq. (20) can be solved in terms of the angular frequency for the case of Frequency-Amplitude Method after calculating the value of α such that:

$$\begin{aligned} w^2 =& -\frac{2\alpha^2 - 4\alpha - 1}{22\alpha^2 + 4\alpha + 1} w_{\text{FAM}}^2 - \frac{3\alpha^2}{2}\frac{(\alpha - 1)^2 A^2 b}{22\alpha^2 + 4\alpha + 1} \\ &- \frac{5\alpha^2}{8}\frac{(10\alpha^2 - 8\alpha + 3)A^4 c}{22\alpha^2 + 4\alpha + 1}, \end{aligned} \tag{23}$$

when α is equal to zero then the value of the frequency matches the value of (19).

Solutions for Case $N = 3$ and $p = 2$

For $k = 1, 2, 3$ in (9) using an ansatz solution (10) and keeping the terms of order $\mathcal{O}((\alpha\beta)^4)$ we have the next algebraic system:

$$\begin{bmatrix} 14M_1 & 20M_2 & 16M_3 & 32M_4 & 32M_5 \\ 21N_1 & 16N_2 & 48N_3 & 768N_4 & 768N_5 \\ 3P_1 & 4P_2 & 32P_3 & 192P_4 & 192P_5 \end{bmatrix} \begin{bmatrix} A^6 a_3 \\ A^4 a_2 \\ A^2 a_1 \\ a_0 \\ w^2 \end{bmatrix} = 0, \tag{24}$$

where

$$\begin{aligned} M_1 =&\, 25\alpha^4 + (125\beta - 14)\alpha^3 + (300\beta^2 - 66\beta + 7)\alpha^2 \\ &+ (355\beta^3 - 123\beta^2 + 28\beta - 5)\alpha + 170\beta^4 - 83\beta^3 \\ &+ 31\beta^2 - 13\beta - \frac{35}{28}, \\ M_2 =&\, 10\alpha^4 + 8(5\beta - 1)\alpha^3 + (90\beta^2 - 32\beta + 5)\alpha^2 \\ &+ 2(50\beta^3 - 28\beta^2 + 9\beta - 2)\alpha + 46\beta^4 - 36\beta^3 \\ &+ 19\beta^2 - 10\beta - 1, \\ M_3 =&\, 3\alpha^4 + 2(4\beta - 3)\alpha^3 + 6(3\beta^2 - 3\beta + 1)\alpha^2 \\ &+ 6(3\beta^3 - 5\beta^2 + 3\beta - 1)\alpha + 8\beta^4 - 18\beta^3 \\ &+ 18\beta^2 - 14\beta - \frac{3}{2}, \\ M_4 =&\, 2\alpha^2 + 4(\beta - 1)\alpha + 4\beta^2 - 8\beta - 1, \\ M_5 =&\, 22\alpha^2 + 4(19\beta + 1)\alpha + 116\beta^2 + 8\beta + 1, \\ N_1 =& -25\alpha^4 + 4(-75\beta + 8)\alpha^3 \\ &+ 2(-465\beta^2 + 114\beta - 14)\alpha^2 \\ &+ 4(-325\beta^3 + 126\beta^2 - 34\beta + 8)\alpha \\ &- 695\beta^4 + 380\beta^3 - 166\beta^2 + 88\beta + \frac{47}{14}, \\ N_2 =&\, 10(-15\beta + 2)\alpha^3 + 30(-15\beta^2 + 6\beta - 1)\alpha^2 \\ &+ 15(-42\beta^3 + 26\beta^2 - 10\beta + 3)\alpha - 330\beta^4 + 290\beta^3 \\ &- 180\beta^2 + 123\beta + 4, \\ N_3 =&\, 2\alpha^4 - 8\alpha^3\beta + 6(-3\beta^2 + 4\beta - 1)\alpha^2 \\ &+ 4(-7\beta^3 + 12\beta^2 - 9\beta + 4)\alpha \\ &- 14\beta^4 + 36\beta^3 - 42\beta^2 + 44\beta + 1, \\ N_4 =& -(\beta - 1)\alpha - \beta^2 + 3\beta, \\ N_5 =& -6\alpha^2 - (37\beta + 3)\alpha - 45\beta^2 - 5\beta, \\ P_1 =&\, 525\alpha^4 + 238(10\beta - 1)\alpha^3 \\ &+ 2(2415\beta^2 - 441\beta + 28)\alpha^2 \\ &+ 2(2380\beta^3 - 609\beta^2 + 49\beta + 28)\alpha \\ &+ 1855\beta^4 - 574\beta^3 + 184\beta + \frac{5}{2}, \\ P_2 =&\, 225\alpha^4 + 20(45\beta - 8)\alpha^3 \\ &+ 30(57\beta^2 - 18\beta + 2)\alpha^2 \\ &+ 12(135\beta^3 - 60\beta^2 + 10\beta + 3)\alpha + 615\beta^4 \\ &- 340\beta^3 + 45\beta^2 + 132\beta + 1, \\ P_3 =&\, 6\alpha^4 + (21\beta - 12)\alpha^3 + 9(4\beta^2 - 4\beta + 1)\alpha^2 \\ &+ (33\beta^3 - 45\beta^2 + 18\beta + 3)\alpha \\ &+ 12\beta^4 - 21\beta^3 + 9\beta^2 + 15\beta, \\ P_4 =&\, \alpha^2 + 2\alpha\beta + \beta^2 + 2\beta, \\ P_5 =& -9\alpha^2 - 50\alpha\beta - 65\beta^2 - 10\beta. \end{aligned}$$

Thus the system can be solved numerically to find the values of α, β and w. It is interesting to note that when $\beta = \alpha = a_3 = 0$, then the system admits a possible solution for the frequency w that just coincides with w_{FAM} in (19).

References

1. Feng Z (2004) Exact solutions to the liénard equation and its applications. Chaos Solitons Fractals 21(2):343
2. Van Der Pol B, Van Der Mark J (1928) Lxxii. The heartbeat considered as a relaxation oscillation, and an electrical model of the heart. Lond Edinb Dublin Philos Mag J Sci 6(38):763
3. Nathan A (1977) The rayleigh-van der pol harmonic oscillator. Int J Electron Theor Exp 43(6):609
4. Sunday J (2017) The duffing oscillator: applications and computational simulations. Asian Res J Math 2(3):1
5. Hu N, Wen X (2003) The application of duffing oscillator in characteristic signal detection of early fault. J Sound Vib 268(5):917
6. Li C, Qu L (2007) Applications of chaotic oscillator in machinery fault diagnosis. Mech Syst Signal Process 21(1):257
7. Nayfeh AH, Sanchez NE (1989) Bifurcations in a forced softening duffing oscillator. Int J Non-Linear Mech 24(6):483
8. Nayfeh AH (2011) Introduction to perturbation techniques. Wiley, Hoboken
9. Nayfeh AH, Mook DT (2008) Nonlinear oscillations. Wiley, Hoboken
10. Sanchez NE, Nayfeh AH (1990) Prediction of bifurcations in a parametrically excited duffing oscillator. Int J Non-Linear Mech 25(2–3):163
11. Awrejcewicz J, Mrozowski J (1989) Bifurcations and chaos of a particular van der pol-duffing oscillator. J Sound Vib 132(1):89
12. Awrejcewicz J, Olejnik P (2003) Stick-slip dynamics of a two-degree-of-freedom system. Int J Bifurc Chaos 13(04):843
13. Andrianov I, Awrejcewicz J (2000) A role of initial conditions choice on the results obtained using different perturbation methods. J Sound Vib 236(1):161
14. Andrianov IV, Awrejcewicz J (2003) Asymptotical behaviour of a system with damping and high power-form non-linearity. J Sound Vib 267:1169
15. He JH (1999) Homotopy perturbation technique. Comput Methods Appl Mech Eng 178(3–4):257
16. Wang SQ, He JH (2008) Nonlinear oscillator with discontinuity by parameter-expansion method. Chaos Solitons Fractals 35(4):688
17. Hesameddini E, Latifizadeh H (2009) A new vision of the he's homotopy perturbation method. Int J Nonlinear Sci Numer Simul 10(11–12):1415
18. Hesameddini E, Latifizadeh H (2009) An optimal choice of initial solutions in the homotopy perturbation method. Int J Nonlinear Sci Numer Simul 10(11–12):1389
19. Beléndez A, Pascual C, Fernández E, Neipp C, Beléndez T (2008) Higher-order approximate solutions to the relativistic and duffing-harmonic oscillators by modified he's homotopy methods. Phys Scr 77(2):025004
20. Yıldırım A (2010) Application of the homotopy perturbation method for the fokker-planck equation. Int J Numer Methods Biomed Eng 26(9):1144
21. He JH (1999) Variational iteration method-a kind of non-linear analytical technique: some examples. Int J Non-linear Mech 34(4):699
22. He JH (2000) Variational iteration method for autonomous ordinary differential systems. Appl Math Comput 114(2–3):115
23. Momani S, Abuasad S (2006) Application of he's variational iteration method to helmholtz equation. Chaos Solitons Fractals 27(5):1119
24. Marinca V, Herişanu N (2008) Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer. Int Commun Heat Mass Transf 35(6):710
25. Marinca V, Herişanu N, Nemeş I (2008) Optimal homotopy asymptotic method with application to thin film flow. Open Phys 6(3):648
26. Marinca V, Herişanu N (2010) Determination of periodic solutions for the motion of a particle on a rotating parabola by means of the optimal homotopy asymptotic method. J Sound Vib 329(9):1450
27. Sierra-Porta D (2018) Some algebraic approach for the second painlevé equation using the optimal homotopy asymptotic method (oham). Numer Anal Appl 11(2):170
28. Sierra-Porta D, Núnez L (2017) On the polynomial solution of the first painlevé equation. Int J Appl Math Res 6(1):34
29. Iqbal S, Javed A (2011) Application of optimal homotopy asymptotic method for the analytic solution of singular lane-emden type equation. Appl Math Comput 217(19):7753
30. Luo AC, Huang J (2012) Approximate solutions of periodic motions in nonlinear systems via a generalized harmonic balance. J Vib Control 18(11):1661
31. Sun W, Wu B (2008) Accurate analytical approximate solutions to general strong nonlinear oscillators. Nonlinear Dyn 51(1–2):277
32. Wu B, Sun W (2011) Construction of approximate analytical solutions to strongly nonlinear damped oscillators. Arch Appl Mech 81(8):1017
33. Guo Z, Leung A (2010) The iterative homotopy harmonic balance method for conservative helmholtz-duffing oscillators. Appl Math Comput 215(9):3163
34. He JH (2002) Preliminary report on the energy balance for nonlinear oscillations. Mech Res Commun 29(2–3):107
35. Khan Y, Mirzabeigy A (2014) Improved accuracy of he's energy balance method for analysis of conservative nonlinear oscillator. Neural Comput Appl 25(3–4):889
36. Alam M, Razzak MA, Hosen MA, Parvez MR (2016) The rapidly convergent solutions of strongly nonlinear oscillators. SpringerPlus 5(1):1258
37. Zhang HL, Xu YG, Chang JR (2009) Application of he's energy balance method to a nonlinear oscillator with discontinuity. Int J Nonlinear Sci Numer Simul 10(2):207
38. He JH (2004) Solution of nonlinear equations by an ancient chinese algorithm. Appl Math Comput 151(1):293
39. Molla MHU, Alam M (2017) Higher accuracy analytical approximations to nonlinear oscillators with discontinuity by energy balance method. Results Phys 7:2104
40. Molla M, Alam M, Alam M (2019) Rapidly convergent solution of nonlinear oscillators with general non-rational restoring force. J Vib Eng Technol. https://doi.org/10.1007/s42417-019-00142-z
41. Zaitsev VF, Polyanin AD (2002) Handbook of exact solutions for ordinary differential equations. CRC Press, Boca Raton
42. Dey B (1986) Dynamics of non-linear monatomic and diatomic chains at a t= 0 first-order phase transition point. J Phys C Solid State Phys 19(18):3365
43. Yang XL, Tang JS (2008) Exact solutions to the generalized lienard equation and its applications. Pramana 71(6):1231
44. Xu GQ (2010) New explicit exact solutions for the lieénard equation and its applications. arXiv preprint arXiv:1003.2921
45. Feng Z (2002) On explicit exact solutions for the lienard equation and its applications. Phys Lett A 293(1–2):50
46. Akbarzade M, Ganji D (2010) Coupled method of homotopy perturbation method and variational approach for solution to nonlinear cubic-quintic duffing oscillator. Adv Theor Appl Mech 3(7):329

47. Sedighi HM, Shirazi KH, Attarzadeh MA (2013) A study on the quintic nonlinear beam vibrations using asymptotic approximate approaches. Acta Astronaut 91:245
48. Lai SK, Lim CW, Wu B, Wang C, Zeng Q, He X (2009) Newton-harmonic balancing approach for accurate solutions to nonlinear cubic-quintic duffing oscillators. Appl Math Model 33(2):852
49. Ganji D, Gorji M, Soleimani S, Esmaeilpour M (2009) Solution of nonlinear cubic-quintic duffing oscillators using he's energy balance method. J Zhejiang Univ Sci A 10(9):1263
50. Younesian D, Askari H, Saadatnia Z, KalamiYazdi M (2010) Frequency analysis of strongly nonlinear generalized duffing oscillators using he's frequency-amplitude formulation and he's energy balance method. Comput Math Appl 59(9):3222

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