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Liutex theoretical system and six core elements of vortex identification *

Yi-qian Wang[1], Yi-sheng Gao[2], Hongyi Xu[3], Xiang-rui Dong[4], Jian-ming Liu[5], Wen-qian Xu[6], Meng-long Chen[7], Chaoqun Liu[8]
1. *School of Mathematical Science, Soochow University, Suzhou 215006, China*
2. *College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China*
3. *Aeronautics and Astronautics Department, Fudan University, Shanghai 200433, China*
4. *School of Energy and Power Engineering, University of Shanghai for Science and Technology, Shanghai 200093, China*
5. *School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, China*
6. *Faculty of Mechanical Engineering and Automation, Zhejiang Sci-Tech University, Hangzhou 310018, China*
7. *SimpleHPC LTD., Shanghai 200030, China*
8. *Department of Mathematics, University of Texas at Arlington, Arlington 76019, USA*

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Abstract: The third-generation vortex identification method of Liutex (previously called Rortex) was introduced by the team led by Prof. Chaoqun Liu from University of Texas at Arlington to mathematically extract the rigid rotation part from the fluid motion, and thus to define and visualize vortices. Unlike the vorticity-based first generation and the scalar-valued second generation, Q, λ_2, Δ and λ_{ci} methods for example, the Liutex vector provides a unique, mathematical and systematic way to define vortices and visualize vortical structures from multiple perspectives without ambiguity. In this article, we summarize the recent developments of the Liutex framework and discuss the Liutex theoretical system including its existence, uniqueness, stability, Galilean invariance, locality and globality, decomposition in tensor and vector forms, Liutex similarity in turbulence, and multiple Liutex-based vortex visualization methods including Liutex lines, Liutex magnitude iso-surfaces, Liutex-Ω method, and Liutex core line method, etc.. Thereafter, the six core elements of vortex identification, including (1) absolute strength, (2) relative strength, (3) local rotational axis, (4) vortex rotation axes, (5) vortex core size, (6) vortex boundary, are used as touchstones against which the Liutex vortex identification system is examined. It is demonstrated with illustrative examples that the Liutex system is able to give complete and precise information of all six core elements in contrast to the failure and inaccuracy of the first and second-generation methods. The important concept that vorticity cannot represent vortex and the superiority of the Liutex system over previous methods are reiterated and stated in appropriate places throughout the paper. Finally, the article concludes with future perspectives, especially the application of the Liutex system in studying turbulence mechanisms encouraged by the discovery of Liutex similarity law. As a newly defined physical quantity, Liutex may open a door for quantified vortex and turbulence research including Liutex (vortex) dynamics and lead the community out of the shadow of turbulence research which traditionally relies on observations, graphics, assumptions, hypotheses, and other qualitative analyses. An optimistic projection is that the Liutex system could be critical to investigation of the vortex dynamics in applications from hydrodynamics, aerodynamics, oceanography, meteorology, etc. and to research of the generation, sustenance, modelling and controlling of turbulence.

Key words: Vortex identification, Liutex vector, six core elements, Liutex core line, Liutex similarity, turbulence

Yi-qian Wang's Short Vita

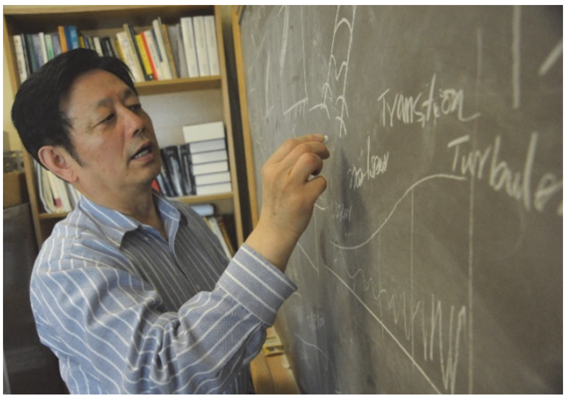
Dr. Yi-qian Wang is currently an Associate Professor at Soochow University, Suzhou, China. He obtained his B. E. and Ph. D. from Nanjing University of Aeronautics and Astronautics in 2010 and 2016. He had been a visiting student at the University of Texas at Arlington under the supervision of Prof. Chaoqun Liu from 2014 to 2015. He was a postdoctoral researcher at Tsinghua University for two years before joining School of Mathematical Science, Soochow university in 2019. He has published more than 20 journal and conference papers and has been one of the main contributors to the developments of the third

* **Biography:** Yi-qian Wang (1987-), Male, Ph. D., Associate Professor, E-mail: yiqianw@sina.com
Corresponding author: Chaoqun Liu,
E-mail: cliu@uta.edu

generation of vortex identification methods, especially the Liutex-based system and the Ω method. His research interests are focused on vortex identification, turbulence, and numerical methods.

Chaoqun Liu's Short Vita

Dr. Chaoqun Liu received both B. S. (1968) and M. S. (1981) from Tsinghua University, Beijing, China and Ph. D. (1989) from University of Colorado at Denver, USA. He is currently the Tenured and Distinguished Professor and the Director of Center for Numerical Simulation and Modeling at University of Texas at Arlington, Arlington, Texas, USA. He has worked on high order direct numerical simulation (DNS) and large eddy simulation (LES) for flow transition and turbulence for over 30 years since 1989. As PI, he has been awarded by NASA, US Air Force and US Navy with 50 federal research grants of over 5.7×10^6 US dollars in the United States. He has published 11 professional books, 117 journal papers and 145 conference papers. He is the founder and major contributor of the third generation of vortex identification methods including the Ω, Liutex/Rortex, Liutex-Ω, Modified Liutex-Ω, Liutex Core Lines Methods, RS vorticity decomposition and R-NR velocity gradient decomposition.



Introduction

Visualizing and defining vortices have always been a great challenge in the study of fluid mechanics. Though it is intuitively very simple that the vortex represents the rotational motion of fluids, a universally-accepted mathematical vortex definition has been elusive ever since the beginning of the subject. Ubiquitously seen in nature and viewed as the building blocks of turbulence, visualizing and defining vortices, however, are indispensable for researches in fluid mechanics.

The vorticity-based vortex identification methods, classified as the first generation[1], correlate vorticity tubes/filaments or vorticity concentrations to vortices, which, however, have been challenged by many literatures[2-3] in that the correlation between vorticity and vortex can be rather weak, especially in near wall turbulence. However, the confusion between vortex and vorticity has caused and is still causing misunderstanding towards fluid mechanics as vorticity tubes/filaments are regarded equivalent to vortices in many textbooks and research papers. Realizing that vorticity cannot represent vortices, multiple vortex identification methods, including Q, λ_2, Δ and λ_{ci} criteria[4-7], categorized as the second generation of vortex identification methods, have been proposed based on different theoretical reasoning. The parameters in these methods can be all somewhat viewed as rotational strength, and iso-surfaces of selected thresholds are then used to visualize vortical structures in flow fields.

Despite the widespread use of the second-generation methods, especially Q and λ_2 methods, many serious problems exist. First, a user-specified threshold is required when applying these methods from case to case, and the vortices visualized is vitally dependent on the selected threshold. A large threshold could lead to vortex structures discontinued and separated from each other, which is often called vortex breakdown, while a small one could lead to connected vortices without vortex breakdown. The selection of threshold is sensitive, case and time step related, empirical and hard to adjust. Second, the physical meaning of these parameters is unclear. Although they are all intended to represent rotational strength, they are really different from each other while the strength should be unique. Third, the rotational motion of fluid should have an axis to rotate about. However, these methods are all scalar-valued and thus provide no information regarding the rotation axis. In addition, they are all contaminated by shear and thus cannot be used to represent the pure fluid rotation.

Those difficulties prompted Prof. Chaoqun Liu of the University of Texas at Arlington to create a new generation of vortex identification method especially for turbulence[8]. Liu et al.[9] introduce the Ω method which, instead of capturing the absolute strength, aims at representing the relative rotation strength. In this respect, the Ω method overcomes the threshold issue and can generally capture both strong and weak vortices simultaneously with an almost universal threshold of $\Omega = 0.52$. It is also proposed that, based on the observation that vorticity concentrates at the near wall region of laminar boundary layer while the streamlines and pathlines are all straight, vorticity should be decomposed into a rotational part and a non-rotational part, i.e., $\boldsymbol{\omega} = \boldsymbol{R} + \boldsymbol{S}$. Although the accurate decomposition was not arrived, the Ω method proved to be efficient and easy-to-use without the threshold problem in various applications[10-14].

It was at the First Conference on Vortex and Turbulence at Shanghai on December 16-17, 2017 that the systematical extraction of the rigid rotation from fluid motion was reported by Prof. Chaoqun Liu. The vortex vector was named as Rortex with its direction as the local rotational axis and the magnitude as twice the angular velocity of the rigid rotation part, which was later renamed Liutex. The original formulation involves the use of real Schur decomposition to rotate the coordinate system so that the local rotational motion is confined in the *XY* plane after rotation[15]. Gao and Liu[16] substantially improved the method by pointing out that the local rotational axis is the real eigenvector of the velocity gradient tensor given that the other two eigenvalues are a pair of complex conjugates, which is the necessary and sufficient condition to justify the existence of local fluid rotation. Then, the swirling strength is determined in the plane perpendicular to the rotational axis, which could be interpreted as the rotational part of vorticity, or $\boldsymbol{R}$, with a residual shearing part $\boldsymbol{S}$. Subsequently, an explicit expression for the Liutex vector in terms of vorticity, eigenvalues and eigenvectors of the velocity gradient tensor in the original coordinate system without coordinate rotation is proposed by Wang et al.[17] which leads to further efficiency improvement and physical intuitive comprehension of the Liutex vector. Both approaches will be discussed below since they are essential in understanding the Liutex system.

Since the introduction of Liutex and the Ω method, considerable developments have been made in the research of vortex definition and identification. Multiple methods to visualize vortices from different perspective have been developed, including the Liutex iso-surfaces, Liutex-Ω method[18-19], Liutex core line method[20-21], objective Liutex method[22], objective Ω method[23]. In addition, the vorticity decomposition and velocity gradient tensor decomposition based on Liutex provide a powerful tool to analyze the interaction between shear and rotation in flows, especially in turbulence, which consists of numerous multiscale vortices. It is also demonstrated by Xu et al.[24] in a low Reynolds number turbulent boundary layer that the frequency and wavenumber spectrum of Liutex magnitude follow the -5/3 law almost exactly, while the spectrum of vorticity and parameters of other vortex identification methods deviate from the power law spectrum of any order. In addition, the energy spectrum of the same flow only marginally matches the −5/3 law in a much smaller frequency range. The discovery of Liutex similarity law reveals that the Liutex vector might be directly connected to turbulence generation and sustenance, and might lead to a deeper understanding towards turbulence. These series of work based on the Liutex vector is collectively classified as the third generation of the vortex identification methods, which is capable of representing vortices from different perspectives while the second generation of vortex identification methods can only provide iso-surface based visualization.

The six core elements of vortex identification methods, including (1) absolute strength, (2) relative strength, (3) local rotational axis, (4) vortex rotation axes, (5) vortex core size and (6) vortex boundary, are brought up to be touchstones to test vortex definition and visualization methods. In Liu et al.[1], the second element of relative strength is measured by Ω, and thus the Ω method was also viewed as part of the third generation of vortex identification method. However, the Ω method still relies on the Cauchy-Stokes decomposition of velocity gradient tensor, which is inaccurate to represent the physical rigid-rotation part of the flow. Although the idea of splitting vorticity into a rotational part and a non-rotational part had been raised in the Ω method, it is the Liutex vector that correctly extract the rigid-rotation out and decompose vorticity into $\boldsymbol{R}$ and $\boldsymbol{S}$ mathematically and systematically. In addition, the Liutex-Ω method combines the Liutex method which decomposes the flow motion correctly and the idea of Ω method to use a proportion to represent the rigidity of fluid motion, and thus can better describe the relative strength of a vortex. Therefore, the Liutex can form a complete system which answers all the six core elements of vortex identification methods. In this spirit, the Ω method is degraded to the second generation of vortex identification methods, although it still could be viewed as the best second-generation method without threshold problem and with the capability of capturing both the strong and weak vortices. The Liutex system alone, including its theoretical foundations and the multiple methods based on the Liutex vector, are then categorized as the third generation of vortex identification methods.

Further information regarding the vorticity-based and scalar-valued vortex identification methods will not be given in this paper in order to focus on the third generation of the Liutex-based vortex definition and its recent developments. For an introduction and brief summary of the first and second generation of vortex identification methods, readers are referred to Liu et al.[1] and Wang and Gui[25]. The rest of the paper is organized as follows: In Section 1, the mathematical definition of a vortex, i.e., the Liutex vector and its calculation is reviewed. Next, the Liutex theoretical system, including its existence, uniqueness, Galilean invariance, stability, locality and globality, applicability in compressible flows and decompositions in both tensor and vector forms, are presented in Section 2. Multiple vortex identification methods based on Liutex are then discussed in Section 3, with emphasis on the Liutex iso-surfaces, Liutex-Ω method and Liutex core line method. How the Liutex system gives full information regarding the six core elements of vortex identification methods is detailed in Section 4. Section 5 introduces a very important discovery, the Liutex similarity in turbulence, which gives a glance of the powerfulness of Liutex in turbulence research. Finally, conclusions and future perspectives are given in Section 6.

1. The Liutex vector definition

Based on the ideas that the local fluid rotation should have an axis and vorticity should be further decomposed into a rotational part and non-rotational part, Liu et al.[15] introduced the Rortex vector, which was later renamed as the Liutex vector, by coordinate rotations. Thereafter, Gao and Liu[16] substantially improved the computational procedure of Liutex by pointing out the local rotational axis is actually the real eigenvector of velocity gradient tensor, given that the other two corresponding eigenvalues are a pair of complex conjugates. An explicit formula to calculate the Liutex vector in terms of vorticity, eigenvalues and eigenvectors of velocity gradient tensor was given by Wang et al.[17] While the coordinate rotation approach is more intuitive in prompting the systematic decomposition of the fluid motion including tensor form and vector form, the explicit formula of Liutex is also physically revealing and brings computational efficiency improvement. Therefore, both approaches with the main results will be described here.

1.1 *Coordinate rotation approach*

The computational procedure of the coordinate rotation approach of Liutex consists of the following steps:

(1) Get the direction of Liutex vector

In the original xyz-coordinates, evaluate the eigenvalues of the velocity gradient tensor $\nabla \boldsymbol{v}$. If all three eigenvalues are real, only stretch and compression exist in the eigenvector directions with no fluid rotation, and thus set Liutex to zero. On the other hand, if there are one real eigenvalue λ_r and two complex conjugate eigenvalues $\lambda_{cr} \pm \mathrm{i}\lambda_{ci}$, calculate the corresponding unit real eigenvector $\boldsymbol{r}$, which represents the local rotational axis and thus the direction of Liutex vector.

An argument for $\boldsymbol{r}$ to be the local rotational axis is that velocity can only vary along the axis and no cross-velocity change persists. In mathematical expression, $\mathrm{d}\boldsymbol{v} = \nabla\boldsymbol{v}\mathrm{d}\boldsymbol{r} = \lambda\mathrm{d}\boldsymbol{r}$, which means λ is the real eigenvalue of $\nabla\boldsymbol{v}$, while $\mathrm{d}\boldsymbol{r}$ is the corresponding real eigenvector $\boldsymbol{r}$ given that the other two eigenvalues are a pair of complex conjugates. In addition, we require $(\nabla\times\boldsymbol{v})\cdot\boldsymbol{r} > 0$ to ensure that the fluid rotation is clockwise when viewed in the direction of $-\boldsymbol{r}$.

(2) Perform Q coordinate rotation

Make a coordinate rotation (Q rotation) which rotates the original z-axis to the direction of the local rotational axis $\boldsymbol{r}$. The rotation matrix, which is orthonormal, is denoted by $\boldsymbol{Q}$ and the velocity gradient tensor $\nabla\boldsymbol{V}_Q$ in the resulting $X_QY_QZ_Q$ frame can be obtained by

$$\nabla\boldsymbol{V}_Q = \boldsymbol{Q}\nabla\boldsymbol{V}\boldsymbol{Q}^{\mathrm{T}} \tag{1}$$

The rotation matrix $\boldsymbol{Q}$ can be obtained by the Rodrigues' formula

$$\boldsymbol{Q} = \begin{bmatrix} \cos\psi + \gamma_x^2 d & \gamma_x\gamma_y d - \gamma_z\sin\psi & \gamma_x\gamma_z d + \gamma_y\sin\psi \\ \gamma_y\gamma_x d + \gamma_z\sin\psi & \cos\psi + \gamma_y^2 d & \gamma_y\gamma_z d - \gamma_x\sin\psi \\ \gamma_z\gamma_x d - \gamma_y\sin\psi & \gamma_z\gamma_y d + \gamma_x\sin\psi & \cos\psi + \gamma_z^2 d \end{bmatrix} \tag{2}$$

where $\psi = \arccos\boldsymbol{c}$ with $\boldsymbol{c} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^{\mathrm{T}}\cdot\boldsymbol{r}$, $d = 1-\cos\psi$ and γ_x, γ_y and γ_z are components of a unit vector $\boldsymbol{\gamma}$ defined by

$$\boldsymbol{\gamma} = \begin{bmatrix}\gamma_x \\ \gamma_y \\ \gamma_z\end{bmatrix} = \frac{\boldsymbol{Z}\times\boldsymbol{r}}{|\boldsymbol{Z}\times\boldsymbol{r}|} \tag{3}$$

with $\boldsymbol{Z} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^{\mathrm{T}}$.

After completing the Q-rotation, the obtained $\nabla\boldsymbol{V}_Q$ has the form

$$\nabla\boldsymbol{V}_Q = \begin{bmatrix} \dfrac{\partial U_Q}{\partial X_Q} & \dfrac{\partial U_Q}{\partial Y_Q} & 0 \\ \dfrac{\partial V_Q}{\partial x_Q} & \dfrac{\partial V_Q}{\partial Y_Q} & 0 \\ \dfrac{\partial W_Q}{\partial X_Q} & \dfrac{\partial W_Q}{\partial Y_Q} & \dfrac{\partial W_Q}{\partial Z_Q} \end{bmatrix} \tag{4}$$

It should be pointed out that for the new velocity gradient tensor $\nabla \boldsymbol{V}_Q$, the real eigenvalue is $\lambda_r = \partial W_Q / \partial Z_Q$, and the corresponding eigenvector is $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$, which is exactly the purpose of the Q-rotation–rotating the Z_Q-axis to coincide with the real eigenvector $\boldsymbol{r}$.

(3) Perform P coordinate rotation

A second P rotation is applied to rotate the coordinate around the Z_Q- axis, based on the idea that after Q-rotation, fluid rotation is confined in the $X_Q Y_Q$ plane. Thus, the in-plane rotation matrix $\boldsymbol{P}$ can be written as

$$\boldsymbol{P} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

where θ is the azimuth angle of coordinate rotation. The resulting velocity gradient tensor becomes

$$\nabla \boldsymbol{V}_P = \boldsymbol{P} \nabla \boldsymbol{V}_Q \boldsymbol{P}^{-1} \tag{6}$$

Since θ has not been specified, $\nabla \boldsymbol{V}_P$ is now a function of θ. In accordance with our assumption that along the Z_P (identical to Z_Q actually) direction, only stretch or compression exist, the components of the 2×2 upper left submatrix contains all the information regarding fluid rotation. Those entries are

$$\frac{\partial U_P}{\partial Y_P}(\theta) = \alpha \sin(2\theta + \varphi) - \beta \tag{7}$$

$$\frac{\partial V_P}{\partial X_P}(\theta) = \alpha \sin(2\theta + \varphi) + \beta \tag{8}$$

$$\frac{\partial U_P}{\partial X_P}(\theta) = -\alpha \cos(2\theta + \varphi) + \frac{1}{2}\left(\frac{\partial U_Q}{\partial X_Q} + \frac{\partial V_Q}{\partial Y_Q}\right) \tag{9}$$

$$\frac{\partial V_P}{\partial Y_P}(\theta) = \alpha \cos(2\theta + \varphi) + \frac{1}{2}\left(\frac{\partial U_Q}{\partial X_Q} + \frac{\partial V_Q}{\partial Y_Q}\right) \tag{10}$$

with

$$\alpha = \frac{1}{2}\sqrt{\left(\frac{\partial V_Q}{\partial Y_Q} - \frac{\partial U_Q}{\partial X_Q}\right)^2 + \left(\frac{\partial V_Q}{\partial X_Q} + \frac{\partial U_Q}{\partial Y_Q}\right)^2} \tag{11}$$

$$\beta = \frac{1}{2}\left(\frac{\partial V_Q}{\partial X_Q} - \frac{\partial U_Q}{\partial Y_Q}\right) \tag{12}$$

and φ is a constant angle determined by $\nabla \boldsymbol{V}_Q$. The two off-diagonal entries can be viewed as the angular velocity of fluid at particular azimuth angle θ with period of π. Then the rotational strength is defined as twice the minimal absolute value of the off-diagonal components of the 2×2 upper left submatrix, i.e., the minimal absolute value of $\partial U_P / \partial Y_P$ or $\partial V_P / \partial X_P$ and can be given by

$$R = 2(\beta - \alpha), \quad \alpha^2 - \beta^2 < 0 \tag{13}$$

$$R = 0, \quad \alpha^2 - \beta^2 \geq 0 \tag{14}$$

Actually, the condition $\alpha^2 - \beta^2 < 0$ is identical to require $\nabla \boldsymbol{v}$ has two complex conjugate eigenvalues, and note that eigenvalues are invariant to coordinate rotations specified above.

Finally, the Liutex vector is obtained as $\boldsymbol{R} = R\boldsymbol{r}$.

When the minimal absolute value of $\partial U_P / \partial Y_P$ is got, say the azimuth angle is θ^*, which leads to $\sin(2\theta^* + \varphi) = 1$, the velocity gradient tensor $\nabla \boldsymbol{V}_P$ becomes

$$\nabla \boldsymbol{V}_P = \begin{bmatrix} \frac{\partial U_P}{\partial X_P} & \frac{\partial U_P}{\partial Y_P} & 0 \\ \frac{\partial V_P}{\partial X_P} & \frac{\partial V_P}{\partial Y_P} & 0 \\ \frac{\partial W_P}{\partial X_P} & \frac{\partial W_P}{\partial Y_P} & \frac{\partial W_P}{\partial Z_P} \end{bmatrix} = \begin{bmatrix} \lambda_{cr} & -\phi & 0 \\ \phi + \varepsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix} \tag{15}$$

where $\phi = \beta - \alpha = R/2$, $\varepsilon = 2\alpha$, $\xi = (\partial W_P / \partial X_P).(\theta^*)$ and $\eta = (\partial W_P / \partial Y_P)(\theta^*)$. We will come back to this equation when discussing velocity gradient decompositions. The coordinate system after P rotation with angle θ^* is called the principal coordinate hereafter.

1.2 *Explicit formula approach*

An investigation of the flow pattern induced by a given velocity gradient tensor reveals that vorticity along the direction of rotational axis $\boldsymbol{\omega} \cdot \boldsymbol{r} = (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{r}$ can be viewed as twice the spatial average angular velocity while the imaginary part of the complex eigenvalues λ_{ci} can be viewed as the pseudo-time average angular velocity. Together with the function form expressed in Eqs. (9) and (10), the minimum angular velocity can be associated with the spatial and pseudo-time average, and it can be derived that

$$\boldsymbol{R} = R\boldsymbol{r} = \left[\boldsymbol{\omega}\cdot\boldsymbol{r} - \sqrt{(\boldsymbol{\omega}\cdot\boldsymbol{r})^2 - 4\lambda_{ci}^2}\right]\boldsymbol{r} \tag{16}$$

which is an explicit formula to calculate the Liutex vector in terms of vorticity vector $\boldsymbol{\omega}$, imaginary part of the complex eigenvalue λ_{ci} and the real eigenvector of the velocity gradient tensor $\boldsymbol{r}$. First, if all the three eigenvalues are real, then $\lambda_{ci} = 0$. According to Eq. (16), we have $\boldsymbol{R} = 0$. Therefore, the explicit formula is valid for any point in the flow field. Also note that $\boldsymbol{\omega}\cdot\boldsymbol{r}$ has been required to be positive when specifying the direction of $\boldsymbol{r}$. Second, it's obvious that the Liutex vector $\boldsymbol{R}$ is actually part of the vorticity vector $\boldsymbol{\omega}$, more specifically, part of the vorticity in the direction of the real eigenvector, i.e., $\boldsymbol{\omega}\cdot\boldsymbol{r}$. It cannot be overestimated the importance of the concept that vorticity is not identical to the vortex, especially when the majority of textbooks nowadays still confuse vorticity with vortex. For a laminar boundary layer, vorticity is concentrated in the near wall region. However, the streamlines and pathlines are all straight with no rotational motion. It is much easier using the explicit formula to discuss the existence, uniqueness, Galilean invariance of the Liutex vector as shown in the following sections.

It is easy to find that although the Liutex definition based on the coordinate Q- and P-rotations may help understand the physics, the explicit Liutex formula is much easier to perform and, really, no coordinate rotation, either Q- or P-rotation, is needed for the calculation of Liutex. Both approaches are identical in extracting the rigid rotation part from the fluid motion, and will be used accordingly in this paper. However, in real calculation of Liutex, only the explicit formula is needed.

2. The Liutex theoretical system

The introduction of Liutex vector provides a whole new point of view towards the fluid motion, especially fluid rotation. Thus, an excursion of the Liutex theoretical foundation is mandatory before confidently utilizing it.

2.1 *Existence and uniqueness of Liutex vector*

In general, the eigenvalues of the velocity gradient tensor $\nabla\boldsymbol{v}$, as a 3×3 matrix, can be either three real numbers or one real and two complex conjugate numbers. For the first scenario with three real eigenvalues, the Liutex vector is set to zero according to Eq. (16). For the second scenario, the direction and magnitude of the Liutex vector is given by Eq. (16). It is iterated here we require $\boldsymbol{\omega}\cdot\boldsymbol{r} > 0$ to uniquely select the real unit eigenvector as the direction of Liutex. Thus, for any point in the flow field, the Liutex vector is uniquely defined, proving the existence and uniqueness of Liutex vector. Real Schur decomposition provides an alternative proof[15] that any 3×3 matrix can be similarly transformed to the form shown in Eq. (15). This clearly shows that as a physical quantity to represent fluid rotation, Liutex is defined and uniquely defined at every point in a flow field without any exception.

2.2 *Galilean invariance*

The importance of a vortex identification method to be Galilean invariant has been advocated by many researchers[5]. The streamlines and pathlines have been excluded to be used for vortex definition because when viewed from different reference frames, the pattern can be significantly different. The Galilean transformation between two reference frames can be represented by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \boldsymbol{Q}_c \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \boldsymbol{c}_1 t + \boldsymbol{c}_2 \tag{17}$$

where x, y and z represent the coordinates of the original reference frame while x', y' and z' represent the coordinates of the new reference frame. Here $\boldsymbol{Q}_c$ is assumed as a constant 3×3 orthonormal matrix, representing the spatial rotation of the reference frame with the property of $\boldsymbol{Q}_c^{-1} = \boldsymbol{Q}_c^{\mathrm{T}}$. $\boldsymbol{c}_1$ is the constant relative motion speed, and $\boldsymbol{c}_2$ represents the constant translation in space. Therefore, the velocity in the new reference frame can be written as

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \boldsymbol{Q}_c \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \boldsymbol{c}_1 \tag{18}$$

where u, v and w represent the velocity components in the original reference frame and u', v' and w' represent the velocity components in the new reference frame. Thus, the relationship between the velocity gradient tensors in two reference frames is

$$\nabla\boldsymbol{v}' = \boldsymbol{Q}_c \nabla\boldsymbol{v} \boldsymbol{Q}_c^{-1} \tag{19}$$

Suppose λ^* is an eigenvalue of $\nabla\boldsymbol{v}$ and $\boldsymbol{r}^*$ is the corresponding eigenvector, then

$$\nabla \boldsymbol{v}\boldsymbol{r}^* = \lambda^* \boldsymbol{r}^* \tag{20}$$

Based on Eqs. (19) and (20), we can obtain

$$\nabla \boldsymbol{v}'(\boldsymbol{Q}_c \boldsymbol{r}^*) = \boldsymbol{Q}_c \nabla \boldsymbol{v} \boldsymbol{Q}_c^{-1} \boldsymbol{Q}_c \boldsymbol{r}^* = \boldsymbol{Q}_c \nabla \boldsymbol{v}\boldsymbol{r}^* = \lambda^*(\boldsymbol{Q}_c \boldsymbol{r}^*) \tag{21}$$

which means that if λ^* and $\boldsymbol{r}^*$ are an eigenvalue and eigenvector pair of $\nabla \boldsymbol{v}$, and then λ^* and $\boldsymbol{Q}_c \boldsymbol{r}^*$ would be the corresponding eigenvalue and eigenvector of $\nabla \boldsymbol{v}'$. In addition, $\boldsymbol{Q}_c \boldsymbol{r}^*$ is the transformed vector in the new reference frame corresponding to $\boldsymbol{r}^*$ in the original reference frame. Therefore, the eigenvalues and eigenvectors of $\nabla \boldsymbol{v}$ are invariant under Galilean transformation.

According to Eq. (16), besides eigenvalues and eigenvectors of velocity gradient tensor $\nabla \boldsymbol{v}$, the Liutex vector also depends on vorticity, which can also be proved to be Galilean invariant. Thus, the Galilean invariance of the Liutex vector is proved. Wang et al.[26], on the other hand, discussed Liutex's Galilean invariance via the coordinate rotation approach.

2.3 *Stability*

By stability we mean how the Liutex vector responds to infinitesimal disturbances on the entries of velocity gradient tensor. When the Liutex vector is not zero, the local velocity gradient tensor $\nabla \boldsymbol{v}$ has one real eigenvalue λ_r and two complex conjugate eigenvalues $\lambda_{cr} \pm \mathrm{i}\lambda_{ci}$, i.e., three distinct values. Thus, the real eigenvalue is a simple eigenvalue, and both λ_r and the corresponding eigenvector continuously varies depends on the disturbance.

In the principal coordinates, the velocity gradient has the form in Eq. (15). Disturbances on entries of, $\partial W_P / \partial X_P$, $\partial W_P / \partial Y_P$ and $\partial W_P / \partial Z_P$ will not change the Liutex vector as the velocity gradient tensor becomes

$$\nabla \boldsymbol{V}_P = \begin{bmatrix} \dfrac{\partial U_P}{\partial X_P} & \dfrac{\partial U_P}{\partial Y_P} & 0 \\ \dfrac{\partial V_P}{\partial X_P} & \dfrac{\partial V_P}{\partial Y_P} & 0 \\ \dfrac{\partial W_P}{\partial X_P} + \delta_1 & \dfrac{\partial W_P}{\partial Y_P} + \delta_2 & \dfrac{\partial W_P}{\partial Z_P} + \delta_3 \end{bmatrix} =$$

$$\begin{bmatrix} \lambda_{cr} & -\phi & 0 \\ \phi + \varepsilon & \lambda_{cr} & 0 \\ \xi + \delta_1 & \eta + \delta_2 & \lambda_r + \delta_3 \end{bmatrix} \tag{22}$$

the Liutex vector will not change since $R = 2\phi$. These three entries are stretch or compression in the Liutex direction ($\partial W_P / \partial Z_P$) and shears ($\partial W_P / \partial X_P$ and $\partial W_P / \partial Y_P$). Disturbances on $\partial U_P / \partial Y_P$ and $\partial V_P / \partial X_P$ will also not change the principal coordinates. Since we are considering infinitesimal disturbance, it can be concluded that disturbance on $\partial V_P / \partial X_P$ doesn't change R, while disturbance on $\partial U_P / \partial Y_P$, say δ_4 changes the Liutex vector from R to $R + 2\delta_4$. Disturbances on the entries of $\partial U_P / \partial X_P$, $\partial V_P / \partial Y_P$, $\partial U_P / \partial Z_P$ and $\partial V_P / \partial Z_P$ generally change the principal coordinates, and thus leading to changes of the Liutex vector. However, disturbances are generally assumed infinitesimally small or $\delta << R$, a vortex point where $R > 0$ cannot become a non-vortex point or $R = 0$ unless δ is very large, which implies the Liutex vector is stable. Stability means that change of Liutex by a small disturbance to any entry of $\nabla \boldsymbol{v}$ is small and small disturbance cannot change the rotational state to a non-rotational state unless $|\delta| \geq R$ which is not the case for stability analysis with small disturbance.

2.4 *Locality and globality*

As described above, the Liutex vector is locally defined while the vortex is actually a global phenomenon that fluid particles rotate around a line axis. The general strategy is to use iso-surfaces to visualize vortices, the appropriateness of which, however, has not been justified. Anyway, following this approach we can certainly visualize vortices with iso-surfaces of Liutex magnitude, which represents the global motion of the fluids.

Different from the scalar-valued second generation of vortex identification methods, the Liutex vector has information of both magnitude and direction. Thus, a Liutex line in the fluid field can be defined as a line which is everywhere tangent to the local Liutex vector. Note that, different from vorticity lines, Liutex lines could end inside the flow field (vorticity lines cannot terminate inside the flow field). In addition, it is found that certain Liutex lines can well capture the global rotational axis of vortices. Colored by the Liutex magnitude, these special Liutex lines can be viewed as better representations of vortices than iso-surfaces. We will come back to this Liutex core line method in Section 3.

2.5 *Applicability in compressible flows*

The Liutex vector is derived from a pure kinetic point of view, and thus is valid for both incompressible and compressible flows. In contrary, many second-generation vortex identification methods,

Q method for example, assume incompressibility, which need adjustments when applying in compressible flows.

2.6 *Decompositions based on Liutex*

Based on the explicit formula given by Eq. (16), the Liutex vector is part of vorticity. Thus, vorticity $\boldsymbol{\omega}$ can be decomposed into a rotational part, i.e., the Liutex vector $\boldsymbol{R}$ and a non-rotational part $\boldsymbol{S}$, which can be expressed as

$$\boldsymbol{\omega} = \boldsymbol{R} + \boldsymbol{S} \tag{23}$$

In the principal coordinates, it can be inferred from Eq. (15) that

$$\boldsymbol{\omega} = \begin{bmatrix} \eta \\ -\xi \\ 2\phi + \varepsilon \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} 0 \\ 0 \\ 2\phi \end{bmatrix} \tag{24}$$

Thus, we have

$$\boldsymbol{S} = \begin{bmatrix} \eta \\ -\xi \\ \varepsilon \end{bmatrix} \tag{25}$$

The entries of $\boldsymbol{S}$ represent the shear along three coordinate directions. Thus, Eq. (23) is actually Liutex-Shear, abbreviated as RS, decomposition of vorticity.

Also in the principal coordinates, the velocity gradient tensor can be decomposed as

$$\nabla \boldsymbol{V}_P = \begin{bmatrix} \lambda_{cr} & -\phi & 0 \\ \phi + \varepsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix} = \begin{bmatrix} 0 & -\phi & 0 \\ \phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ \varepsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix} \equiv \boldsymbol{R} + \boldsymbol{NR} \tag{26}$$

Here $\boldsymbol{R}$ denotes the tensor corresponding to the Liutex vector, i.e., the rigid rotational part of the local fluid motion, while $\boldsymbol{NR}$ is the non-rotational part. It is obvious that $\boldsymbol{NR}$ has three real eigenvalues, so that $\boldsymbol{NR}$ itself implies no local rotation. The non-rotational part $\boldsymbol{NR}$ can be further decomposed as

$$\boldsymbol{NR} = \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_r \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \varepsilon & 0 & 0 \\ \xi & \eta & 0 \end{bmatrix} \equiv \boldsymbol{C} + \boldsymbol{S} \tag{27}$$

$\boldsymbol{C}$ can be viewed as the compression-stretching part and $\boldsymbol{S}$ represents the pure shear part. The above decomposition is performed in the principal coordinates and could be recast to the original coordinates. The physical meaning of each part is clear without any ambiguity. In contrast, according to the traditional Cauchy-Stokes decomposition, the velocity gradient tensor is decomposed as

$$\nabla \boldsymbol{v} = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}) + \frac{1}{2}(\nabla \boldsymbol{v} - \nabla \boldsymbol{v}^{\mathrm{T}}) \equiv \boldsymbol{A} + \boldsymbol{B} \tag{28}$$

This decomposition leads to the false conclusion that the symmetric part $\boldsymbol{A}$ represents deformation while the anti-symmetric part $\boldsymbol{B}$, which corresponds vorticity, represents the fluid rotation. Thus, the Cauchy-Stokes decomposition can be viewed as the theoretical foundation of the first generation of vortex identification methods. In fact, the Q method can be written as

$$Q = \frac{1}{2}(\|\boldsymbol{B}\|_F - \|\boldsymbol{A}\|_F) \tag{29}$$

which can be viewed as a remedy to the fact that $\boldsymbol{B}$ by itself cannot represent the rigid rotation of fluid rotation. Note that Q is contaminated by shearing as discussed in Gao and Liu[16]. It should be pointed out, Eqs. (26) and (27) are a unique decomposition of the velocity gradient tensor while the Cauchy-Stokes decomposition terms are closely related to coordinate orientation and thus are not Galilean invariant although vorticity, $\|\boldsymbol{A}\|_F$ and $\|\boldsymbol{B}\|_F$ are Galilean invariant.

3. Liutex based vortex identification methods

Multiple vortex visualization methods have been developed based on the Liutex vector, including Liutex magnitude iso-surface, objective Liutex[22], Liutex-Ω method[18-19], and Liutex core line method[20-21], etc..

3.1 *Iso-surface of Liutex magnitude*

It is a tradition of the second generation of vortex identification methods to use iso-surfaces to represent vortices. This is also true for the Liutex magnitude, as shown in Fig. 1, in which the typical vortical structures of boundary layer transition, Λ and hairpin vortices, are well captured. A major advantage of the Liutex magnitude iso-surface over second-generation methods is that no contamination by shearing and/or stretching is contained in Liutex, while those previous methods are prone to be contaminated by shearing to different extent as detailed in Gao and Liu[16].

It should be acknowledged that, however, the threshold issue of the second-generation methods persists in the Liutex magnitude iso-surface method as the nature of iso-surface based methods. With a smaller threshold of $R = 0.005$, as shown in Fig. 1(a), the 2-D and 3-D T-S waves are well captured while the Λ and hairpin vortices are however smeared and blurred in the transition region. With $R = 0.020$, the Λ vortex around $x = 385$ are clearly visualized but the T-S waves are lost in Fig. 1(b). An even larger threshold of $R = 0.100$, as shown in Fig. 1(c), leads to a clearer representation of hairpin rings. However, both T-S waves and Λ vortices are lost. In addition, the legs of the hairpin chain around $x = 340$ are disconnected with hairpin rings. This inconsistency of the threshold problem roots from the fact that vortices are of different rotational strengths and almost every iso-surface based method has the threshold problem. It is thus concluded that iso-surfaces are not adequate to describe the vortical structures in a flow field.

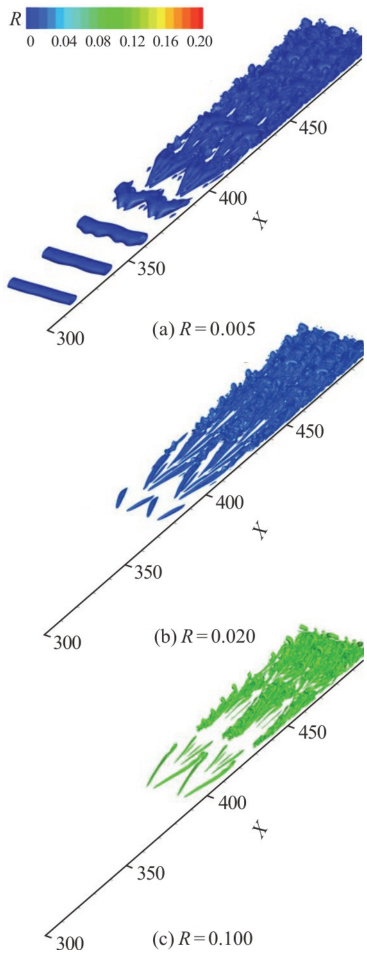


Fig. 1 Iso-surfaces in a boundary layer transition on a flat plate

3.2 *Liutex-Ω method*

The fact that vorticity is not vortex and the threshold issue of the second generation of vortex identification methods prompt Liu et al.[9] to introduce the Ω method. Rather than capturing the swirling strength, the Ω parameter is a measure of local vorticity density, or fluid rigidity, which can be viewed as relative vortex strength.

$$\Omega = \frac{\|\boldsymbol{B}\|_F}{\|\boldsymbol{A}\|_F + \|\boldsymbol{B}\|_F + \varepsilon} \tag{30}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are the symmetric and antisymmetric part from the Cauchy-Stokes decomposition given by Eq. (28). ε is a small positive number introduced to avoid division by zero. Dong et al.[27] suggests that ε could be determined at each time step by

$$\varepsilon = 0.001(\|\boldsymbol{B}\|_F - \|\boldsymbol{A}\|_F)_{\max} \tag{31}$$

The maximum in Eq. (31) is achieved over the whole fluid domain. According to the definition, the parameter Ω ranges from 0-1, and thus has been normalized. It has been demonstrated by many users[10-14] that the Ω method with a fixed threshold of 0.52 could well capture both strong and weak vortices in transient flows without adjustment of the threshold and thus could be viewed as a reliable, robust and easy-to-use vortex identification method in practice. However, the Liutex vector is more precise without any shear contamination from a theoretical perspective. The Liutex-Ω method is then developed to combine ideas from the Liutex vector and the Ω method by Dong et al.[18], and modified by Liu et al.[19] The parameter Ω_R is defined as

$$\Omega_R = \frac{(\boldsymbol{\omega} \cdot \boldsymbol{r})^2}{2\left[(\boldsymbol{\omega} \cdot \boldsymbol{r})^2 - 2\lambda_{ci}^2 + 2\lambda_{cr}^2 + \lambda_r^2\right] + \varepsilon} \tag{32}$$

The small positive number ε is also introduced to avoid non-physical noises. The Liutex-Ω method applied to direct numerical simulation (DNS) data in comparison with the Q method is shown in Fig. 2. Obviously, as the threshold of the Q method increases many vortex structures disappear in the downstream of the flow, especially for the vortex ring highlighted by red circles in Fig. 2. The Liutex-Ω method, on the other hand, maintains the vortical structures as the threshold increases from 0.52 to 0.70. Thus, the Liutex-Ω is insensitive to moderate threshold changes, and generally a threshold of 0.52 can well capture both strong and weak vortices simultaneously. So far, the Liutex- Ω method can be viewed as the best iso-surface based vortex identification method without the threshold problem and could be interpreted as the relative strength of fluid rotation.

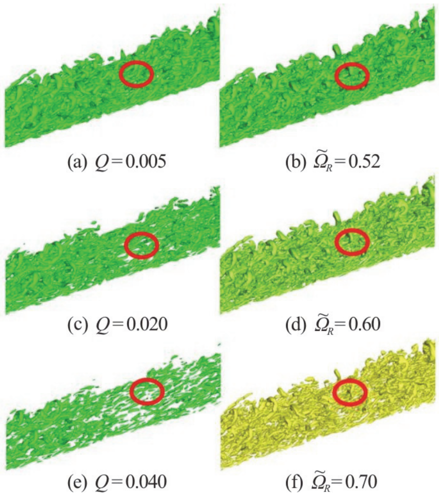


Fig. 2 Comparison of Liutex- Ω method and the Q method with different thresholds

The original Ω method was categorized into the third generation in Liu et al.[1] since it overcomes the threshold problem and in addition the idea of further decomposition of vorticity into a rotational part and a non-rotational part was introduced by using Ω as a parameter measuring the percentage of the rotational part to the vorticity. However, this inappropriately implies that the direction of the rotational part is identical to vorticity, which is generally not true according to Eq. (16). Furthermore, the calculation of parameter Ω relies on the Cauchy-Stokes decomposition, which we have criticized of not being able to represent the physics. Therefore, the Ω method should be degraded to a second-generation vortex identification method. However, the Ω method may be the best one in the second generation as it is normalized, threshold insensitive, and capable to capture both strong and weak vortices.

3.3 *Liutex core line method*

Since Liutex is a vector, the Liutex line can thus be defined as a line that is everywhere tangent to the local Liutex vector, similar to the definition of the vorticity line. In Fig. 3, the vorticity lines and Liutex lines through a Λ -vortex are shown with and without iso-surfaces. It can be seen that the vorticity lines are not aligned along but penetrate vortex structures and it is not always true that a vortex locates at the vorticity concentration. On the other hand, the Liutex lines follow the shape of the Λ - vortex almost exactly. A similar situation happens to the hairpin vortex as shown in Fig. 4. Thus, those Liutex lines colored by Liutex magnitude could be an alternative to visualize vortices.

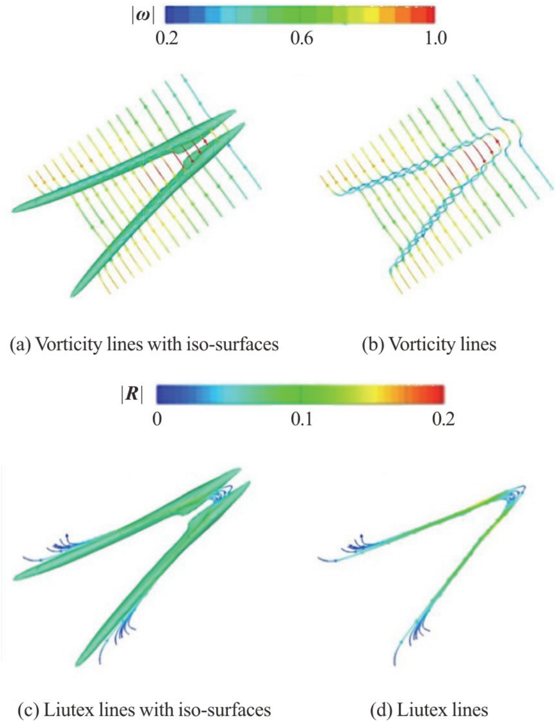


Fig. 3 Liutex lines colored by Liutex magnitude and vorticity lines colored by vorticity magnitude for a Λ - vortex

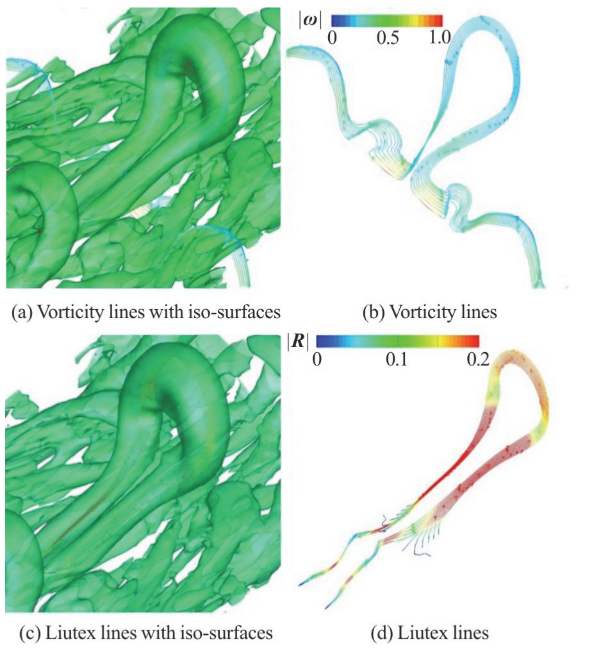


Fig. 4 Liutex lines colored by Liutex magnitude and vorticity lines colored by vorticity magnitude for a hairpin vortex

An even better strategy would be representing the vortices by vortex core lines colored by the vortex strength. The efforts in this regard, however, had not been very successful until the introduction of the axis line definition of vortex rotation based on the Liutex vector. For the rotational axes of vortices, their rotational strength, i.e., the Liutex magnitude, should be a maximum in the plane normal to the direction of the Liutex vector. This requirement can be expressed as

$$\nabla R \times \boldsymbol{r} = 0 \,, \quad R > 0 \tag{33}$$

which means that the gradient of the Liutex magnitude ∇R is in the same direction as the local Liutex vector direction $\boldsymbol{r}$. The second condition of $R > 0$ ensures the considered point is inside a vortex. The idea is that on the iso-surfaces of R, ∇R is perpendicular to any small line element $\boldsymbol{dl}$ that lays down on the iso-surface. As R becomes larger and larger until the iso-surface reaches the vortex core, ∇R would be in the same direction as $\boldsymbol{R}$ or $\boldsymbol{r}$. The condition of Eq. (33) is actually used to detect seeding points through which the Liutex lines are drawn to represent the vortex core lines. These Liutex lines are called Liutex core lines. The general procedure to visualize Liutex core lines consists of first cutting through the vortices with a slice, then detecting seeding points using the condition of Eq. (33), and finally drawing Liutex lines through these seeding points. Figure 5 shows the Liutex core lines during boundary layer transition. It can be shown in Figs. 5 (a) and 5(b) that the Liutex core lines can well capture the vortices and additional swirling strength information can be added to these lines as shown in Fig. 5(c).

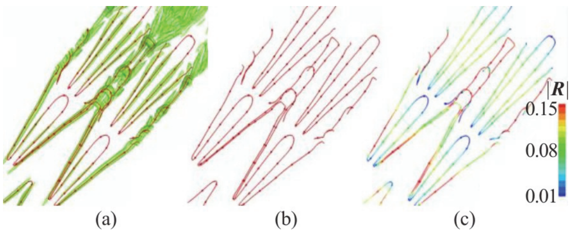


Fig. 5 Liutex core lines (a) with iso-surfaces, (b) without iso-surfaces, and (c) colored by Liutex magnitude

The manual procedure suggested by Gao et al.[20], however, is difficult to implement when the vortical structures are complicated with twists and tangles. Accordingly, an automatic version is proposed by Xu et al. [21], which is utilized to show the vortex cores in Fig. 6. The Liutex core lines can be viewed as the skeleton of the vortices in the flow field and can thus be used to understand the physics and mechanisms of fluid flows. It should also be pointed out that the Liutex core lines are unique and the method is free from the threshold problem, both strong and weak vortices can be visualized and the color in Fig. 5(c) and Fig. 6 represents the swirling strength, i.e., the Liutex magnitude. It should be pointed out that the vortex structure must be unique. Liutex core line method is the only method to show vortex structure uniquely and free from the threshold problem.

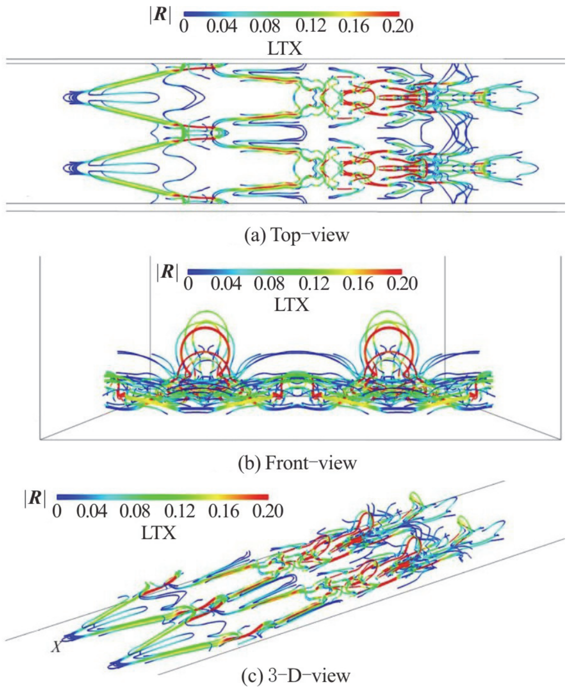


Fig. 6 Liutex core lines in boundary layer transition

With the automatic method, highly twisted and tangled Liutex core lines are shown in Fig. 7 in turbulence. Despite the successful representation of vortical structures, the automatic approach has difficulty to distinguish physically very close vortex cores and faked ones caused by numerical errors. The algorithm to address this issue remains for future development. However, it should be recognized that the Liutex core line method is by far the best method to educe smooth vortex core lines from the flow field, and the approach uniquely defines the flow skeletons without the threshold problem and provides a different perspective from the traditional iso-surface based methods towards the vortical structures as they are all inevitably threshold-dependent.

4. The six core elements of vortex identification

It is the communications between Prof. Lian-di Zhou and the present authors that enlightened the introduction of the six core elements of vortex identi-

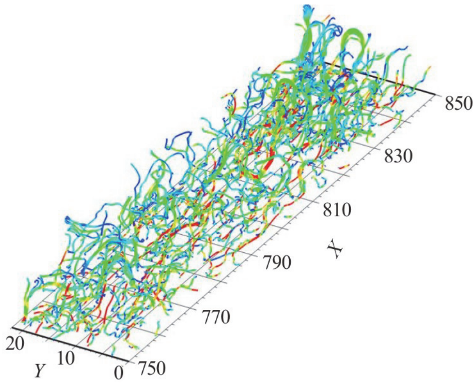


Fig. 7 Liutex core lines in turbulent boundary layer

fication, which serve as touchstones against which a successful vortex identification method needs to be benchmarked. The six core elements include (1) absolute strength, (2) relative strength, (3) local rotational axis, (4) vortex rotation axes, (5) vortex core size and (6) vortex boundary. The vorticity-based first-generation methods fail to represent vortices, especially in the near wall region, thus cannot answer any of the six core elements. Vorticity and vortex are not directly correlated. The second-generation methods can generally give a threshold-dependent vortex boundary, despite parameters of second-generation methods are all contaminated by shears to different extent. However, information of other elements cannot be provided since they are all scalar-value based methods. Only the third-generation vortex identification of the Liutex system can give full information about the six core elements.

(1) The absolute strength is the Liutex magnitude, which is twice the angular speed of the rigid rotation part of the fluid motion. According to the definition, the Liutex magnitude is the only correct measurement of the rigid rotation part, while all other methods are contaminated by shearing as detailed in Gao et al.[16]

(2) The relative strength is defined by the parameter of the Liutex-Ω method, i.e., Ω_R, which can be understood as the density of Liutex or the rigidity of fluid motion. Actually, the relative strength is more important than the absolute strength, analogous to the importance of relative error over absolute error. For example, there could exist both strong vortices rotating at 1 000 circles/second and weak vortices rotating at 10 circles/second. With the absolute strength, a threshold of 100 circles/second might capture the strong vortices but ignore the weak ones. Thus, the use of relative strength, i.e., the Liutex-Ω method, can capture both the strong and weak vortices at the same time and could be viewed as the best strategy to visualize vortices using iso-surfaces.

(3) The local rotational axis is in the direction of the Liutex vector, i.e., the local real eigenvector of velocity gradient tensor given that the other two corresponding eigenvalues are complex conjugate. First, it is very important to note that the second generation of vortex identification methods is all scalar-valued, i.e., it ignored the information of local rotational axis. Second, the information of local axis direction makes it possible to draw smooth Liutex lines which gives the opportunity to extract global information regarding the vortex cores.

(4) The vortex rotation axes are defined as the connected points that satisfies $\nabla R \times \boldsymbol{r} = 0$, $R > 0$, which means the gradient of Liutex magnitude has the same direction with the Liutex vector direction. More intuitively, it means that vortex cores locate where the Liutex magnitude is locally maximum in the plane normal to $\boldsymbol{r}$, which is the direction of Liutex vector. In practice, we only use this condition to detect seeding points and then draw Liutex lines through these seeding points which are called Liutex core lines. The equivalence of these two, however, has not been elaborated, which should be studied more.

(5) The vortex core size is defined as a place where the relative rotational strength Ω_R (parameter of Liutex-Ω method) has been declined to 95% of the corresponding vortex core center point. This region is the central region where fluid rotation happens. Since it is based on the identified vortex cores, it doesn't have the threshold issue. Actually, the vortex core size can be considered as complementary information of length scales to the Liutex core line method. Note that the definition of the vortex core is still empirical here.

(6) The boundary of a vortex is referred as a separation line/surface between rotation and non-rotation area. It is the nature of fluid motion that combines rotation and shearing. Generally, all second-generation vortex identification methods use a non-zero threshold to define the boundary of vortices. However, the selection of threshold is arbitrary and subjective. Therefore, we propose use the iso-surface of $\Omega_R = 0.52$ as the boundary of vortices, which can be applied to various flows without adjustment of thresholds.

In retrospect, the vorticity-based first generation of vortex identification methods fails to correctly represent vortices, while the scalar-valued second-generation methods can only approximately give the vortex boundary by iso-surfaces of a user-specified threshold, which implicitly refers to the rotational strength, despite the contamination by shearing. Only the third-generation vortex identification method of the Liutex system is capable of representing vortices from different perspectives and thus give full information regarding all the six-core elements of a qualified vortex identification method.

5. The Liutex similarity law in turbulence

Kolmogorov's 1941 theory (K41) of similarity hypotheses and the −5/3 law for energy spectrum are considered as the most applauded theoretical achievement in turbulence research and the origin of the modern turbulence theory. However, these deductions are based on the assumption of sufficient high Reynolds number and isotropy of turbulence. Generally, these assumptions cannot be exactly fulfilled in boundary layers where the viscosity is important, and the energy spectrum usually deviates from the −5/3 law.

As stated, the Liutex vector is designed to extract the rigid rotation part from the fluid motion, which is not influenced by viscous dissipation and independent of Reynolds number, thus relaxing the very high Reynolds number assumption. It is demonstrated by Xu et al.[24] that the spectrum of Liutex magnitude follows the −5/3 law almost exactly, while the traditional energy spectrum only marginally matches the −5/3 law in a much smaller frequency range as shown in Fig. 8.

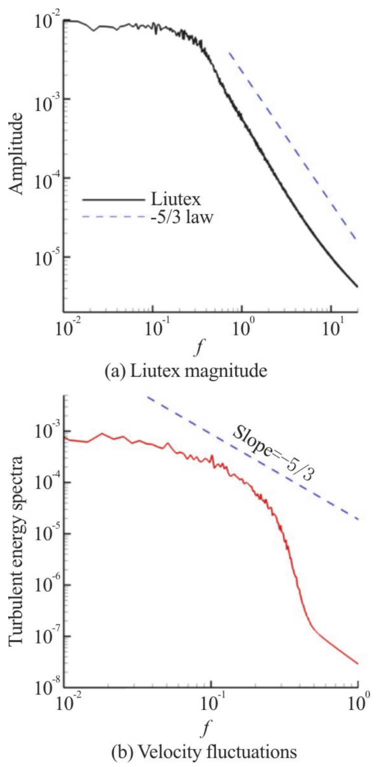


Fig. 8 Liutex spectrum and turbulent energy spectrum

The much stronger universal similarity of Liutex's −5/3 law over that of K41 comes from the fact that Liutex represents the rigid rotation part of fluid motion, which is shear-free and thus not influenced by viscous effect and independent of Reynolds number. On the other hand, vorticity and other popular second generation of vortex identification methods, Q criterion for example, do not possess this feature due to the shearing and stretching contamination. Not only does this work promote the current understanding towards the physics of turbulence, but also promisingly lead to a more solid and universal subgrid model in large eddy simulation.

6. Conclusion and future perspectives

The Liutex system with its recent developments is summarized in this article, and main conclusions can be described as follows:

(1) Vorticity is not vortex. Vorticity is mathematically defined as the curl of velocity, while a vortex is a physical phenomenon of fluid rotation. The association between vorticity and vortex has been proved to be very weak, especially near the wall surface. Therefore, the vorticity-based first generation of vortex identification methods generally fails to represent vortices correctly. The second-generation methods ignore the directional information of vortices and represent vortices by iso-surfaces of an empirically selected threshold, which however is contaminated by shearing. The outstanding one is the Ω method which can be considered as the best second-generation method without threshold problem and with the capability of capturing both strong and weak vortices simultaneously.

(2) Liutex is a mathematical and systematical approach to extract rigid rotation from the fluid motion without any shear contamination. Its existence, uniqueness, stability, Galilean invariance, locality and globality, decomposition in tensor and vector forms are discussed as parts of the Liutex theoretical system, providing a solid foundation upon which we can confidently apply the Liutex system to study fluid mechanics. One highlight of Liutex is that it is a vector, in contrast to the scalar-valued second-generation methods, with the direction as the local rotational axis and the magnitude as twice the angular velocity of the rigid rotation part of fluid motion. This fact makes Liutex a promising tool to identify vortices and a new physical quantity to describe fluid rotation.

(3) Multiple vortex identification methods have been proposed based on the Liutex vector, including Liutex lines, Liutex iso-surfaces, Liutex-Ω method, and Liutex core line method. With those methods, the six core elements of vortex identification including absolute strength, relative strength, local rotational axis, vortex core axes, vortex core size and vortex boundary are all provided. Thus, the Liutex system offers a complete description of the vortices from

different perspectives.

(4) The discovery of Liutex similarity of −5/3 law in turbulence is astonishing since it proves that Liutex is a fundamental physical quantity in turbulence. Turbulence is made up of numerous vortices with different sizes and rotational strength. Often in the study of near-wall turbulence, vorticity is used to derive statistical information about vortices. However, the argument that vorticity cannot represent vortex has been pointed out over and over again. All the second-generation methods are seldom used in the statistical study of turbulence due to their contamination by shear and stretching (compression). Generally, only a rough representation of vortices with iso-surfaces are provided by these second-generation methods. Therefore, the Liutex system can be viewed as a new key to study the physics of turbulence structure, generation and sustenance.

Considerable developments, including both theoretical and practical aspects, have been made in the last two years since the introduction of the Liutex vector. However, since Liutex is a newly introduced concept, there are still many questions remained for research, the equivalence between the vortex core line and the Liutex line, for example. As a newly defined physical quantity, Liutex may open a door for quantified vortex and turbulence research including Liutex (vortex) dynamics and lead the community out of the shadow of turbulence research which traditionally relies on observations, graphics, assumptions, hypotheses, and other qualitative analyses. As a projection, the Liutex system could be critical to investigations of the vortex dynamics in applications of hydrodynamics, aerodynamics, oceanography, meteorology, etc. and to researches of the generation, sustenance, modelling and controlling of turbulence.

The software of the third generation of vortex generation method developed by the UTA team has been published online at **https://www.uta.edu/math/cnsm/public_html/cnsm/cnsm.html** for free download with a short agreement for users to sign.

Acknowledgments

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