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Liutex (vortex) core definition and automatic identification for turbulence vortex structures*

Hongyi Xu[1], Xiao-shu Cai[2], Chaoqun Liu[3]
1. *Aeronautics and Astronautics Department, Fudan University, Shanghai 200433, China*
2. *School of Energy and Power Engineering, University of Shanghai for Science and Technology, Shanghai 200093, China*
3. *Department of Mathematics, University of Texas at Arlington, Arlington 76019, Texas, USA*

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Abstract: As a milestone research in vortex identification (VI), the physical quantity of Liutex, including its forms of scalar, vector and tensor, was systematically explored and rigorously obtained as the third-generation (3G) of the vortex definition and identification methods distinguished from the first generation (1G) by vorticity and the second generation (2G) by the vortex identification (VI) criteria solely dependent on the velocity gradient tensor eigenvalues. Based on these findings, the vortex-core lines were abstracted from the well-defined Liutex, and for the first time, were automatically generated and massively visualized using computer. The distinctive characteristics of these vortex cores with the intriguing threshold-independency make them be the uniquely appropriate entity to represent and to depict the vortex structures in turbulence. The letter made use of the DNS data for the natural transition in a zero-pressure gradient flat-plate (Type-A turbulent boundary layer (TBL)) and the fully-developed turbulence in a square annular duct (Type-B TBL) to demonstrate the vortex structure represented by the vortex-core lines. The 3G VI approach based on the vortex-core lines is capable of profoundly uncovering the vortex natures. Moreover, the capability of automatically identifying the vortex cores and massively visualizing the large number of vortex-core behaviors in a transient way will enable the fluid-mechanics and other related-science communities to step into a new era to explore the intrinsic natures of the centennial puzzle of turbulence and other vortex-related phenomena in future.

Key words: Direct numerical simulation, turbulence structure, vortex core, vortex identification and automation

Introduction

Vortex is a common physical phenomenon observed every day and everywhere from smoke rings to galaxies. Intuitively, vortex is considered as the rotation/swirling motion of fluids. Surprisingly, it is rather difficult to give a rigorous mathematical definition and accurate identification method to quantitatively describe vortex. A satisfactory vortex definition has to answer the six core issues including the vortex absolute strength, relative strength, local vector direction, global rotation axis, vortex core, and vortex boundaries. Although vortices can be readily observed and intuitively regarded as the rotational/swirling motion of fluids, an unambiguous and generally accepted definition of a vortex has yet to be achieved and the definition of coherent structures in turbulence research is somewhat in a similar situation. According to Liu et al.[1], three generations of vortex identification (VI) methods were proposed. The first-generation (1G) VI method is represented by vorticity vector given as early as 1858[2]. Vorticity was later found incapable of properly representing the vortex behaviors[3] and therefore searching for more appropriate ways to identify vortex was started as the campaign of the second-generation (2G) VI methods which are nowadays well-known by the so-called vortex criteria including the Q criterion, the Δ criterion, the λ_2 criterion and the λ_{ci} criterion etc.[4-7]. Overall, most of them are the Eulerian local region-type VI methods which are scalar in nature and based on the local velocity gradient tensor. More specifically, these criteria are exclusively determined by the eigenvalues or invariants of the velocity gradient tensor and thereby can be classified as the eigenvalue-based criteria[8] which make use of the scalar iso-surface with an arbitrary threshold to

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* **Biography:** Hongyi Xu (1963-), Male, Ph. D., Professor, E-mail: hongyi_xu@fudan.edu.cn
Corresponding author: Chaoqun Liu, E-mail: cliu@uta.edu

represent the vortex structures. Unfortunately, both the 1G and 2G VI methods failed to give right answers to the above six core issues until the third-generation (3G) VI was proposed. The 3G VI methods can be classified into the Ω or Liutex-based and the Liutex-Ω methods etc. as given by Liu et al.[1, 8, 9-14]. Among these, the new physical quantity of Liutex is at the core and extremely important in turbulence research, but has been unfortunately ignored for over a century. Liutex is a systematical definition of the local fluid rigid rotation, including the scalar, vector and tensor interpretations. The scalar version or the magnitude of Liutex represents the local rotational strength, i.e., the local angular velocity. The direction of Liutex vector, which is determined by the real eigenvector of the velocity gradient tensor, represents the local rotation axis. The tensor form of Liutex, rather than the vorticity tensor, stands for the real rotational part of the velocity gradient tensor, which can be used for the decomposition of the velocity gradient tensor[15]. Meanwhile, Liutex can eliminate the contamination due to shearing and thus can accurately quantify the local rotational strength, as demonstrated for the incompressible flows[8]. Moreover, in contrast to the current eigenvalue-based criteria, the Liutex vector field and lines can be applied to study the vortical structures since Liutex has a vector form representing the local rotational axis. However, a user-specified threshold is still required for visualizing and identifying Liutex iso-surfaces. The idea combining the Ω method with Liutex was proposed as a potential solution to address the issue[13-14]. Since vortices are the building blocks of turbulence, the current turbulence theory, indeed, is lack of a serious research with regard to a quantitative VI so far. Within the context, the new findings and the potential research directions given in the current paper will lead the science communities into a new era of quantitatively studying the turbulence vortical natures. The aforementioned six core issues were satisfactorily addressed within the 3G VI framework: (1) The absolute strength which is Liutex magnitude (angular speed). (2) The relative strength which is Ω, (3) The local rotation axis which is the Liutex direction. (4) The vortex core which is a special Liutex where the gradient of Liutex is parallel to Liutex vector. (5) The vortex core size which is 95% of the local maximum value (see Ref. [1]). (6) The vortex boundary is concluded at Liutex magnitude $R > 0$.

As well known, the vortex structure is mostly generated and exhibited by the iso-surface strongly dependent on the associated threshold for almost all the current VI approaches. However, the determination of a threshold can be arbitrary, resulting in the very different vortex structures among all these VI methods. Actually, whether an appropriate threshold exists is unknown at all. The well-known fact is that a larger threshold might cause the loss of some important structure patterns qualitatively associated to the weak vortices and a smaller threshold could make the visualized vortex structure very vague and blurred. Therefore, a vortex structure exhibited by the vortex core line colored by the vortex strength is more scientifically appropriate in terms of the uniqueness, clarity and threshold independency. In this regard, although many efforts were made in Refs. [16-26], only very limited successes were obtained until the vortex-core line was successfully derived in Refs. [10, 27]. The vortex-core line in Ref. [27] was defined as a special Liutex line where $\nabla R \times \boldsymbol{r} = 0$ held. However, the vortex core line was generated manually and was tracked individually in Ref. [27]. Since the vortex structure can be very complicated for turbulence, it is not realistic and practical to track each vortex-core line by hand. An automatic Liutex core line identification becomes very critical in demand for visualizing turbulence coherent structures, which really means the vortex structure. This paper introduces an automatic vortex core line identification method and applies the method to the representative turbulences resolved by direct numerical simulation (DNS), including the DNS data for the natural transition in a zero-pressure gradient flat-plate (ZPGFP), i.e., the Type-A turbulent boundary layer (TBL) in Refs. [28-30] and the fully-developed turbulence in a square annular duct (SAD) or the pressure gradient-drive SAD (PGDSAD) TBL, i.e., the Type-B TBL in Refs. [31-33]. The vortex structures represented by the vortex-core lines symbolized a milestone in the VI research using Liutex. The VI approach based on the vortex cores are capable of profoundly uncovering the vortex natures and preserving the vortex-structure uniqueness with a distinctive threshold independency. Moreover, the capability of automatically identifying the vortex cores permits to massively visualize large number of vortex cores in the transient behaviors of turbulence, which will enable the fluid-mechanics and other related-science communities to enter into a new era to explore the intrinsic natures of the centennial puzzle of turbulence and other vortex-related phenomena.

1. Liutex core line definition and automatic identification method

1.1 *Liutex vector definition*

Definition 1: The direction of Liutex vector is defined as the local rotation axis and the magnitude of Liutex vector is defined as the local angular speed, i.e.

$$\boldsymbol{R} = R\boldsymbol{r} \tag{1}$$

According to Liu et al.[8, 11], $\boldsymbol{r}$ is the eigenvector of the velocity gradient tensor $\nabla\boldsymbol{V}$, i.e., $\nabla\boldsymbol{V}\cdot\boldsymbol{r}=\lambda_r\boldsymbol{r}$.

Wang et al.[12] gave an explicit formula to R, i.e.

$$R=\boldsymbol{\omega}\cdot\boldsymbol{r}-\sqrt{(\boldsymbol{\omega}\cdot\boldsymbol{r})^2-4\lambda_{ci}^2} \qquad (2)$$

where $\boldsymbol{\omega}$ is the local vorticity vector, λ_{ci} is the imaginary part of the complex eigenvalue of $\nabla\boldsymbol{V}$.

1.2 *Liutex (vortex) rotation axis line definition*

Definition 2: A Liutex (vortex) rotation axis line is defined as the local maxima of Liutex which is a line without iso-surface.

Theorem 1: Any small element of a line on the Liutex iso-surface must be orthogonal to the gradient of Liutex if $\nabla R\neq 0$.

Proof: On the Liutex iso-surface, $\mathrm{d}R=\nabla R\cdot \mathrm{d}\boldsymbol{l}=0$ if $\nabla R\neq 0$, where $\mathrm{d}\boldsymbol{l}$ is an infinitesimal line element on the Liutex iso-surface. Therefore, ∇R, $\mathrm{d}\boldsymbol{l}$ are orthogonal.

Theorem 2: If $\nabla R\times \mathrm{d}\boldsymbol{l}=0$ at a point, the point must be located on the Liutex rotation axis.

Proof: Any point which is not located in the Liutex rotation core axis has to be in some Liutex iso-surface. If $\mathrm{d}\boldsymbol{l}$ is on a Liutex iso-surface, $\mathrm{d}R=\nabla R\cdot \mathrm{d}\boldsymbol{l}=0$ must hold and then $\nabla R\times \mathrm{d}\boldsymbol{l}\neq 0$. If $\nabla R\times \mathrm{d}\boldsymbol{l}=0$, $\mathrm{d}\boldsymbol{l}$ must not be on any Liutex iso-surface, which is the Liutex rotation axis because according to Definition 2, the Liutex rotation axis is a local maxima and has no Liutex iso-surface.

Definition 3: The vortex core line is defined as a special Liutex line which passes the points satisfying the condition of $\nabla R\times\boldsymbol{r}=0$, $\boldsymbol{r}>0$ where $\boldsymbol{r}$ represents the direction of the Liutex vector. The Definition 3 is used to find the Liutex (vortex) rotation core lines in the flow field which is uniquely defined without any threshold requirement. Therefore, the Liutex core rotation axis lines with the Liutex strength are derived uniquely and are believed the only entity that is capable of cleanly and unambiguously representing the vortex structures

1.3 *Automatic identification of Liutex core lines*

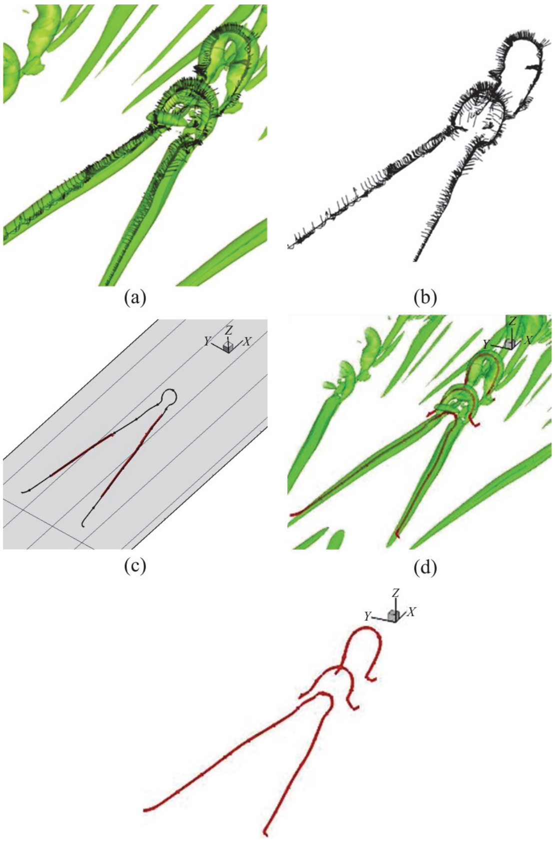
These definitions and theorems provide the theoretical base for generating the Liutex core line to represent the vortex structures, which essentially resolve the six major issues aforementioned in a threshold-independent way. However, to make use of the Liutex core lines to exhibit the complex vortex structures in a transient vortical flow, such as a turbulence, these definitions and theorems have to be fully implemented into a computing program to let computer identify and generate the core lines. The automation procedures are particularly important in turbulence research since the Liutex is a quantity constantly being generated and destroyed in any locations with a strongly transient nature. Therefore, the manual generation of Liutex core lines is only meaningful for the concept proof and demonstration. The automatic identification has to be conducted for a turbulence field at any time instant and the automation needs to be implemented in a temporally sequential way to further capture the transient nature of turbulence evolution. As guided by the theoretical framework set by these definitions and theorems, the seeding points on the Liutex core lines are first picked out at a given temporal instant and then are integrated along the spatial directions regulated by the Liutex vector field. The Liutex core lines are thereby generated and moreover, through progressively advancing the identification steps in time, the evolutions of the vortex structures, as uniquely represented and visualized by the Liutex core lines, will uncover the rotational characteristic natures and permit the dynamics of the vortex-bone structures of the turbulence being interrogated.

2. Vortex core line structures based on 3G VI approach

2.1 *Manual identification of Liutex core lines*

The manual generation of vortex-core lines was first given in Gao et. al.[34] using the DNS data at early transition stage of ZPGFP TBL. As demonstrated in Fig. 1, the vortex structures at the stage are distinctively characterized by the sharp hairpin pattern which remains at a quite large scales and has yet developed into the turbulent multi-scales. As a result, the Liutex core line was identified pretty easily and cleanly with the procedures including: (1) Calculating Liutex vector $\boldsymbol{r}$ and Liutex magnitude gradient ∇R where R is the magnitude of Liutex. (2) Plotting the iso-surface of R and finding the points on the iso-surface, see Fig. 1(a). (3) Plotting the integral curves of ∇R based on the points of the iso-surface R and deriving the limiting curve that all the integral curves are collapsed, see Fig. 1(b), and (4) finding a point on the limiting curve and integrating the Liutex vector $\boldsymbol{r}$ based on the point to obtain the Liutex core line, see Fig. 1(c). By repeating the same procedures for other iso-surfaces, the vortex core lines at other locations can thus be tracked individually, as seen in Fig. 1(d), 1(e).

Although quite successful in identifying the Liutex core line, the manual procedure was recognized only valuable for demonstrating the concept of vortex core lines and was not able to be applied in any meaningful 3G VI practice. The major drawback was caused by the fact that the theoretical foundation of the vortex core line identification, i.e., the abovementioned Definitions and Theorems, was not adequately implemented in the manual operation steps. As explained before, the automatic Liutex core line identification was integrally achieved by fully implementing the Definitions and Theorems into a computer program and therefore, satisfactorily overcame the drawback in the manual procedures.



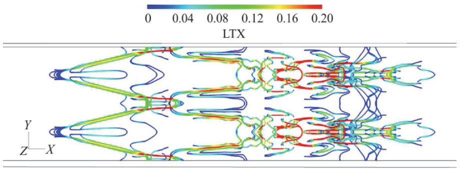
(a) (b) (c) (d) (e)

Fig. 1 (Color online) Manual generation of the vortex structures in the early transition stage of ZPGFP TBL

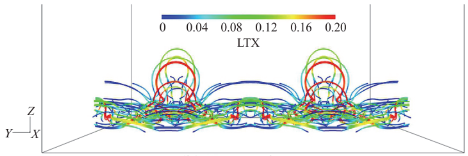
2.2 *Automatic identification of Liutex core lines for the natural transition of ZPGFP TBL*

Based on the theoretical foundation setup by Section 1.1, the automatic generation of vortex core lines was developed by fully implementing the definition and theorem into a computer program through the proper algorithms. The letter makes use of the DNS data for two typical TBLs, namely the Type-A (ZPGFP TBL) and -B TBL (PGDSAD TBL)[33], to demonstrate the automation capability of the current 3G VI approach.

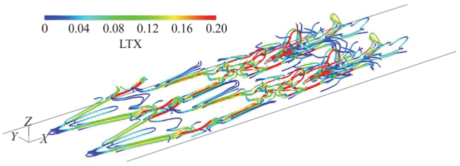
Figures 2, 3 plots the vortex core line generated by the automatic 3G VI method using the DNS data at the very early stage of the natural transition in the ZPGFP TBL. These strings exhibit a clearly symmetrical pattern representing the formation of horse-shoe or the Λ-shape vortex evolution. Comparing to the traditional VI methods, these structures are precisely identified by the strings of Liutex core lines instead of the iso-surfaces with certain threshold, and therefore lead to the uniquely-defined structure depicting the vortical characteristics of fluid motions, i.e., the vortex core lines with their tangential direction standing for the local rotating direction. These strings are colored by the Liutex magnitude R which distinctively represent the vortex rotational strengths, i.e., the local angular speeds. It can be seen clearly in Figs. 2(a)-2(c) that the vortex rotational strengths tend to be very non-uniformly distributed along these strings, which makes the iso-surface methods with a specific threshold being only capable of approximately, partially and evasively reflecting and representing the vortex structure. However, the 3G VI approach based on the vortex core lines provides a unique capability of representing the vortex structure definitively, systematically and completely. Comparing Figs. 2(a)-2(c) to Figs. 3(a)-3(c), it can be seen evidently that with the temporal evolution, the vortex structures develop progressively towards the more refined-scale structures from early to late transition. Consequently, the automatic identification has to be developed to investigate the more complex turbulences at the progressively refined-scales. Within the context, the current letter opens the door to make use of the vortex core lines to more quantitatively and rigorously study the strongly vortex-relevant coherent structures.



(a) Top-view



(b) Front-view

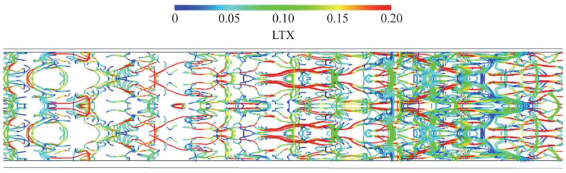


(c) 3-D-view

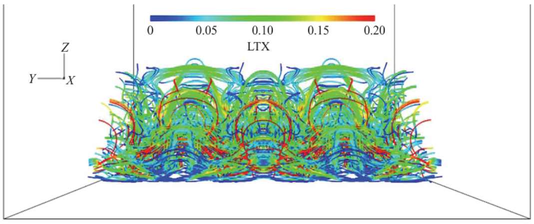
Fig. 2 (Color online) Automatic generation of the vortex structures in the early transition stage of ZPGFP TBL

2.3 *Automatic identification of Liutex core lines for the fully-developed PGDSAD TBL*

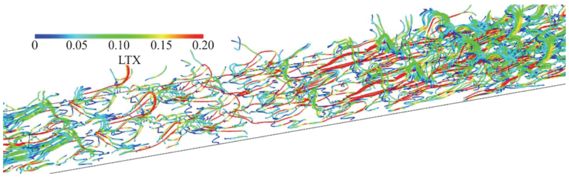
After successfully being applied to the transition



(a) Top-view



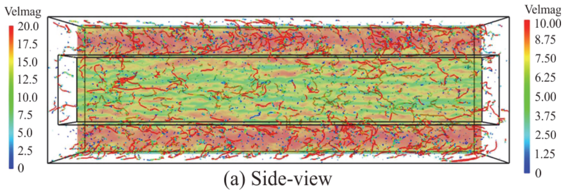
(b) Front-view



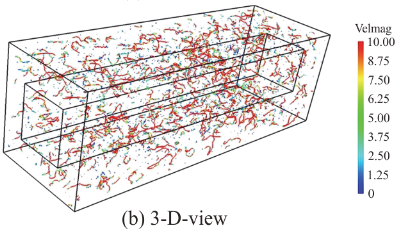
(c) 3-D-view

Fig. 3 (Color online) Automatic generation of the vortex structures in the late transition stage of ZPGFP TBL

of ZPGFP TBL, the automatic 3G VI approach was further pushed to the Type-B TBL as represented by the fully-developed PGDSAD TBL. The DNS data were presented in Xu[32] and was validated by the various experiments and other related DNS data. The 3G VI approach based on the vortex core lines was conducted, for the first time, in the PGDSAD TBL. Qualitatively, the visualizations of these core strings were very reminiscent for the existing knowledges of the wall-bounded turbulence, such as the near-wall streak structures and the ejection or sweeping characteristic motions of a wall-turbulence, but were refreshingly represented and exhibited by the vortex core strings as seen in Figs. 4(a), 4(b). For example, the vortex strings near-wall, as seen in Fig. 4(a), tend to be generally lifted-up from the wall at the downstream end and to be suppressed down towards the wall at the upstream end, indicating an ejection and sweeping must exists around the two ends of the strings, respectively. The strings are all elongated in the streamwise direction, giving rise to the nature of the streak structures mentioned in many existing coherent-structure studies. However, in order to find the detailed mechanisms behind these observations, the temporal evolutions of these vortex core strings have to be time-sequentially generated and transiently visualized so that the dynamics of these strings can be investigated. In this regard, the definition of Liutex provides a quantity permitting a quantitative interrogation of the vortex dynamics as represented by the evolution of the vortex core lines, and apparently, the automatic 3G VI capability definitely plays a vitally important role in the future researches towards achieving the goal.



(a) Side-view



(b) 3-D-view

Fig. 4 (Color online) Automatic generation of the vortex structures in the fully-developed Type-B TBL in a square annular duct

3. Conclusions

Based on the above discussions and visualization demonstrated by the DNS data for the typical Type-A and -B TBLs, the following conclusions can be made:

(1) As a vector, Liutex is an innovative and uniquely quantitative definition to depict the fluid rotational nature with its direction as the local rotation axis and its magnitude as the angular speed of rotation[10-12], which can never be matched by any other existing VI methods.

(2) The vortex structure of turbulence is strongly dependent on the arbitrary selection of the threshold for all the current scalar-based VI methods including the one based on Liutex-related scalars. In fact, the investigation of Liutex core lines suggests that the campaign searching for such an ad hoc threshold turn out of being illusive since these core strings are featured by the very non-uniformly distributed rotational strengths with no specific characteristic value being appropriately selected as the threshold.

(3) The vortex core line structure is uniquely defined by Definition 3, which is a special Liutex line in nature with $\nabla R \times \boldsymbol{r} = 0$, $\boldsymbol{r} > 0$ and these Liutex core lines are not associated with any scalar iso-surfaces and therefore, are threshold independent. The significance of the 3G VI based on vortex core line is twofold:

1) The Liutex core lines along with the Liutex strength is uniquely capable of depicting and reflecting the vortex structure of turbulence in its vortical nature, which is deemed to be the only appropriate entity representing the vortex structure, or more

specifically the vortex core string structure.

2) An automatic vortex core line identification developed by the current research can capture the vortex core structures massively distributed in complex turbulences and thereby be strongly applicable for future fluid mechanics research, particularly for the study of turbulence vortex structures, or the so-called coherent structures. The software, which is capable of automatically generating the unique vortex structure in a time-sequential way to depict and capture the transient vortex evolutions and dynamics, is therefore of extremely high value for the fluid mechanics community and the related science research as well as the engineering applications.

The software of the 3G VI methods has been published on line at
https://www.uta.edu/math/cnsm/public_html/cnsm/cnsm.html
for free download with a short agreement for users to sign. An automatic Liutex core line identification software is ready to be released and for purchase with more information at www.rortex.org.

Acknowledgments

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