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RESEARCH

Construction of Nonlocal Governing Operators with Local Boundary Conditions on a General Interval

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Abstract
We present nonlocal operators that enforce local boundary conditions. We extend the construction from $(-1, 1)$ to a general interval (a, b). This extension is nontrivial as it requires an in-depth understanding of the abstract convolution operator, a series representation. For implementation purposes, one has to find an integral representation. We accomplish this task by a careful extension of kernel functions from $(-1, 1)$ to (a, b) and subtle integral manipulations. We prove that the constructed operators and the original linearized peridynamic operator agree in the bulk of the domain for the six boundary conditions considered. Furthermore, they fully agree on the domain when the boundary condition (BC) is pure Neumann or periodic. We rigorously verify that BCs are satisfied by utilizing the Hilbert-Schmidt property of the abstract convolution operator.

Keywords Nonlocal operator · Local boundary condition · Functional calculus · Hilbert-Schmidt operator · Hilbert basis

1 Introduction

We present novel governing operators inspired by the theory of Peridynamics (PD), a nonlocal formulation of continuum mechanics developed by Silling [17]. The original (linearized) PD governing operator $\mathcal{M}_{\text{orig}}$ is a formal operator because it has no reference to a rigorous boundary condition (BC) [17, p. 201]. We addressed this issue over the years and have studied various aspects of local BCs in nonlocal problems [1–4, 6–8, 10–12]. We treat the kernel of $\mathcal{M}_{\text{orig}}$ more general than a convolution type and denote it by the bivariate function

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$K(x, x')$.[1] This treatment will to turn $\mathcal{M}_{\text{orig}}$ into a rigorous operator, thereby allowing us to incorporate local BC into the PD governing operator. Let us keep the domain as $\Omega = (-1, 1)$ for now. The PD governing operator is defined as

$$\mathcal{M}_{\text{orig}} u(x) = \int_\Omega K(x, x')\,\mathrm{d}x'\; u(x) - \int_\Omega K(x, x') u(x')\,\mathrm{d}x'.$$

Our governing operator is defined as

$$\mathcal{M}_{\text{BC}} u(x) = \int_\Omega K_{\text{BC}}(0, x')\,\mathrm{d}x'\; u(x) - \int_\Omega K_{\text{BC}}(x, x') u(x')\,\mathrm{d}x'.$$

Although the first term of $\mathcal{M}_{\text{orig}}$ is modified to obtain $\mathcal{M}_{\text{BC}}$, the operators are close to each other. We prove that the operators agree in the bulk of the domain for the six BCs considered. Furthermore, they fully agree on the domain when the BC is pure Neumann or periodic; see Theorem 4.1.

The main advantage that the operators $\mathcal{M}_{\text{BC}}$ provide is the ability to enforce local BC through the use of a forcing function only on the local boundary, *not in the interior* of the domain [1, 10]. There are several reasons why one wants to utilize local BCs. Probably the first and foremost reason is that the partial differential equation (PDE) formulations exclusively employ local BCs. In engineering and science, the BCs of practical use are predominantly local. The physical measurements are also local. The ability to incorporate such a widely accepted BC type into nonlocal formulations is quite valuable. Starting from the inception of the finite element method (FEM) in the 1960s, the discretization of PDEs employed local BCs. Hence, the legacy codes that utilize FEM, finite volume, or discontinuous Galerkin type discretization methods all are based on local BCs. The ability to inherit well-developed local BC based codes and re-use them for nonlocal formulations is prudent. For instance, the coupling of nonlocal and local problems along a local interface becomes completely natural. The ability to transfer well-established domain decomposition methods to nonlocal problems is a big gain for the nonlocal community.

In addition, surface applications such as contact, shear, and traction cannot be treated using nonlocal BC because their volume integration is zero. Such surface applications are formulated naturally with local BCs, and hence, can be treated in our nonlocal formulation. Finally, using local BCs, we aim to overcome the surface effects seen in PD simulations. Since our enforcement of BCs is rigorous, we hold that our construction has great potential in avoiding surface effects altogether by employing local BCs.

Our previous work considered the construction of operators on the master domain $(-1, 1)$. We prefer to use domain instead of interval throughout the paper. This study addresses the generalization of the operators from the master domain to the general domain (a, b). Changing the domain on which a differential operator is defined is straightforward. However, this may not be the case for an integral operator. At least in our construction, the BCs are encoded in the kernel of the integral operator. A change in the domain fundamentally affects the construction of the integral operator. Furthermore, our operators are derived from the series representation of the abstract convolution operator. The generalization to (a, b) becomes nontrivial because it requires an in-depth understanding of the abstract convolution operator and subtle integral manipulations as will be discussed in Sec. 6.

[1] In the literature, the linearized PD governing operator is stated as

$$\mathcal{M}_{\text{orig}} u(x) = \int_\Omega \widehat{C}(x' - x)\,\mathrm{d}x'\; u(x) - \int_\Omega \widehat{C}(x' - x) u(x')\,\mathrm{d}x'.$$

We simply see the convolution type kernel function $\widehat{C}(x' - x)$ as a general bivariate function $K(x, x')$.

In practical applications, one usually solves on a large domain. Cracks, however, form typically in a small part of this domain. Peridynamic models handle cracks much more effectively, but more expensively than local models. One potential application of the governing operators with local BCs on a general domain would be coupling nonlocal and local problems. This way, the coupled model would be taking advantage of best of both worlds. In coupling, one should have the freedom of solving a problem on an arbitrary subdomain. In addition, local BCs become a great advantage in this situation. In the presence of nonlocal BCs, the coupling process becomes very delicate and cumbersome. Hence, generalizing the governing operators facilitates coupling applications, which is a subject of our ongoing work. Aside from the coupling context, one must be able to handle a domain at arbitrary lengthscale for any reasonable practical application. In a separate paper [5], we studied the discretization of the operators developed for the general domain using kth order collocation method. In [5], we also present the convergence analysis and implementation details.

The rest of the paper is structured as follows. In Sec. 2, we provide a summary of functional calculus, which is the main concept used in constructing the governing operators. In Sec. 3, we provide the necessary ingredients required in the construction, especially the extensions of the kernel function. In Sec. 4, the proof of agreement of $\mathcal{M}_{\rm BC}$ with $\mathcal{M}_{\rm orig}$ is given. In Sec. 5, the essential ingredients in the construction such as eigenvalues, eigenfunctions, and even and odd projections are all generalized to (a, b). In Sec. 6, we provide the integral representation of the series based operator with antiperiodic and periodic BCs. In Sec. 7, the extensions of the kernel function are generalized to (a, b). In Sec. 8, we give the explicit expression of kernel functions. In Sec. 9, we rigorously verify the BC for each operator. Finally, we conclude in Sec. 10.

2 Functional Calculus

It is instructive to see how one utilizes functional calculus to construct novel matrices from an original one. Consider a symmetric matrix $A \in \mathbb{R}^{n\times n}$—in the operator theory language, this is a bounded self-adjoint operator. Let λ_k and v_k denote the eigenvalues and eigenvectors of A, respectively. Since $\{v_k\}_{k=1}^{n}$ form an orthonormal basis, the spectral decomposition of A is given by

$$A = \sum_{k=1}^{n} \lambda_k v_k v_k^t.$$

It is easy to see that any integer power m of A can be written

$$A^m = \sum_{k=1}^{n} \lambda_k^m v_k v_k^t.$$

Note that the eigenvectors of A^m are identical to those of A. A^m can be obtained from A only by changing the eigenvalues from λ_k to λ_k^m. One can generalize this construction of functions of A as

$$\varphi(A) = \sum_{k=1}^{n} \varphi(\lambda_k) v_k v_k^t,$$

where the *regulating function* φ is a holomorphic (basically infinitely differentiable in a neighborhood) function defined by

$$\varphi : \sigma(A) \to \mathbb{R}, \tag{2.1}$$

where $\sigma(A)$ is the spectrum of A. The eigenvalues of $\varphi(A)$ are $\varphi(\lambda_k)$. Hence, it is key to use the appropriate regulating function in the construction of operators.

Our earlier major result was that the peridynamic governing operator $\mathcal{M}_{\text{orig}}$ is a function of the classical (Laplace) operator $-\Delta$ on $\mathbb{R}^d$ [12]. We discussed the regulating function φ in Eq. 2.1 extensively in [3]. For a construction that is computationally amenable, a bounded domain Ω is a critical requirement. We reused the same regulating function to construct governing operators on Ω that enforce local BC on the boundary of Ω. The discovery of the explicit expression of the regulating function and the ensuing construction on Ω allowed us to incorporate local BCs into nonlocal problems, thereby, triggering this line of research.

An initial observation of the PD governing operator [17] is that it contains a convolution. Secondly, as mentioned above, practical applications call for a bounded domain. Hence, one starts the construction with convolution operators on a bounded domain that enforce the prescribed BC. The BC of main interest is either Dirichlet, Neumann, or mixed, denoted by DD, NN, DN, ND, respectively, all of which are constructed through antiperiodic and periodic BC, denoted by a, p, respectively. Hence, one starts with constructing the convolution operators $\mathcal{C}_{\mathtt{a}}$ and $\mathcal{C}_{\mathtt{p}}$ with antiperiodic and periodic BC, respectively, using the eigenfunctions on $\Omega = (-1, 1)$:

$$e_k^{\mathtt{a}}(x) := \frac{1}{\sqrt{2}} e^{i(2k+1)\frac{\pi}{2}x} \quad \text{and} \quad e_k^{\mathtt{p}}(x) := \frac{1}{\sqrt{2}} e^{i(2k)\frac{\pi}{2}x}, \quad k \in \mathbb{N},$$

of the classical operator $-\Delta_{\mathtt{a}}$ and $-\Delta_{\mathtt{p}}$ in which the BC information is already encoded. For a given kernel function $C \in L^2(\Omega)$, the convolution operator in series form, for $u \in L^2(\Omega)$, is defined as

$$\mathcal{C}_{\mathtt{BC}} u(x) := \sqrt{2} \sum_{k \in \mathbb{Z}} \langle e_k^{\mathtt{BC}} | C \rangle \langle e_k^{\mathtt{BC}} | u \rangle e_k^{\mathtt{BC}}(x), \quad \mathtt{BC} \in \{\mathtt{a}, \mathtt{p}\}, \tag{2.2}$$

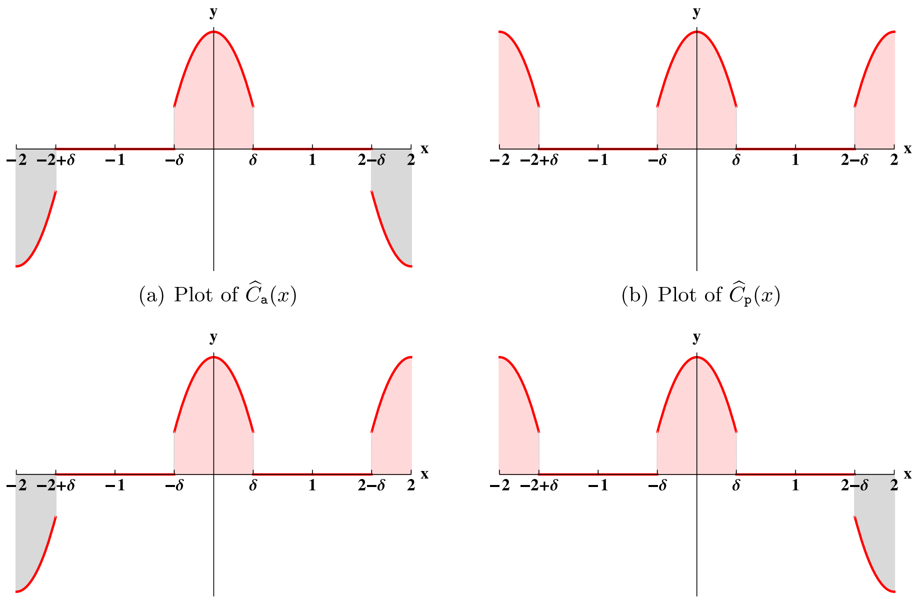
where $\langle \cdot | \cdot \rangle$ denotes the $L^2(\Omega)$ inner product. In [16], the convolution operator in series form appears in a different context (generated by Riesz bases instead of a Hilbert basis). Similar to our study, integral representations of the convolution operators in series form have also been studied in [16].

The question again comes to choosing the appropriate regulating function. Since we know which function of the classical operator the nonlocal governing operator is for the unbounded domain, we recycle that regulating function to be used for the bounded domain case. When the classical operator is considered on a bounded domain, a BC needs to be prescribed. This makes its spectrum $\sigma(-\Delta_{\mathtt{BC}})$ discrete. Hence, the sum in Eq. 2.2 is countable and the eigenfunctions of $-\Delta_{\mathtt{BC}}$ form a Hilbert (complete and orthonormal) basis for $L^2(\Omega)$. Practical applications call for an integral representation of Eq. 2.2, which was accomplished in our previous work on the master domain [2]. In this study, we aim to obtain integral representations of the generalized (2.2) on (a, b).

3 Background Material and Extensions

The univariate kernel function $C(x)$ is assumed to be nonnegative and even. Namely,

$$C(-x) = C(x).$$



(a) Plot of $\widehat{C}_{\mathtt{a}}(x)$ (b) Plot of $\widehat{C}_{\mathtt{p}}(x)$

(c) Plot of $\widehat{C}_{\mathtt{ap}}(x)$ (d) Plot of $\widehat{C}_{\mathtt{pa}}(x)$

Fig. 1 The kernel function $C(x)$ on the master domain $\Omega = (-1, 1)$ and its extensions on $(-2, 2)$. The periodic, antiperiodic, and mixed extensions of $C(x)$ are denoted by $\widehat{C}_{\mathtt{a}}(x)$, $\widehat{C}_{\mathtt{p}}(x)$, $\widehat{C}_{\mathtt{ap}}$, and $\widehat{C}_{\mathtt{pa}}$, respectively

The size of nonlocality is controlled by the characteristic function $\chi_\delta(x)$ whose role is the representation of the nonlocal neighborhood, called the *horizon*. More precisely, for $x \in \Omega$,

$$\chi_\delta(x) := \begin{cases} 1, & x \in (-\delta, \delta), \\ 0, & \text{otherwise}. \end{cases}$$

Hence, the size of nonlocality is determined by δ and we assume $\delta < 1$. Since the horizon is constructed by $\chi_\delta(x)$, a kernel function used in practice is in the form

$$C(x) = \chi_\delta(x)\nu(x),$$

where $\nu(x) \in L^2(\Omega)$ is even. We also assume that

$$u(x) \in L^2(\Omega).$$

The other ingredient to define the operators that enforce local BCs is the extension of $C(x)$ from $(-1, 1)$ to $(-2, 2)$:

$$\widehat{C}_{\mathtt{a}}(x) := \begin{cases} -C(x+2), & x \in (-2, -1), \\ C(x), & x \in (-1, 1), \\ -C(x-2), & x \in (1, 2), \end{cases} \qquad \widehat{C}_{\mathtt{p}}(x) := \begin{cases} C(x+2), & x \in (-2, -1), \\ C(x), & x \in (-1, 1), \\ C(x-2), & x \in (1, 2), \end{cases}$$

$$\widehat{C}_{\mathtt{ap}}(x) := \begin{cases} -C(x+2), & x \in (-2, -1), \\ C(x), & x \in (-1, 1), \\ C(x-2), & x \in (1, 2), \end{cases} \qquad \widehat{C}_{\mathtt{pa}}(x) := \begin{cases} C(x+2), & x \in (-2, -1), \\ C(x), & x \in (-1, 1), \\ -C(x-2), & x \in (1, 2). \end{cases} \tag{3.1}$$

Next, we will show that the extensions all agree in the Bulk $:= (-1+\delta, 1-\delta)$.

Lemma 3.1 *Let the kernel function $C(x)$ be in the form*

$$C(x) = \chi_\delta(x)\nu(x),$$

where $\nu(x) \in L^2(\Omega)$ is even. Let $\widehat{C}_{\mathrm{a}}(x)$, $\widehat{C}_{\mathrm{p}}(x)$, $\widehat{C}_{\mathrm{ap}}(x)$, and $\widehat{C}_{\mathrm{pa}}(x)$ denote the periodic, antiperiodic, and mixed extensions of $C(x)$ to $\widehat{\Omega} := (-2, 2)$, respectively, given in Eq. 3.1. *Then,*

$$\widehat{C}_{\mathrm{a}}(x) = \widehat{C}_{\mathrm{p}}(x) = \widehat{C}_{\mathrm{ap}}(x) = \widehat{C}_{\mathrm{pa}}(x), \quad x \in (-2+\delta, 2-\delta). \tag{3.2}$$

Furthermore, all kernel functions agree in the bulk. Namely, for $x \in$ Bulk,

$$\widehat{C}_{\mathrm{a}}(x'-x) = \widehat{C}_{\mathrm{p}}(x'-x) = \widehat{C}_{\mathrm{ap}}(x'-x) = \widehat{C}_{\mathrm{pa}}(x'-x), \quad x' \in (-1, 1). \tag{3.3}$$

Proof With a closer look at the definition of the functions given in Eq. 3.1, it is easy to see the following:

$$\widehat{C}_{\mathrm{a}}(x) = \begin{cases} -\nu(x-2), & x \in (-2, -2+\delta), \\ \nu(x), & x \in (-\delta, \delta), \\ -\nu(x+2), & x \in (2-\delta, 2), \\ 0, & \text{otherwise,} \end{cases} \qquad \widehat{C}_{\mathrm{p}}(x) = \begin{cases} \nu(x-2), & x \in (-2, -2+\delta), \\ \nu(x), & x \in (-\delta, \delta), \\ \nu(x+2), & x \in (2-\delta, 2), \\ 0, & \text{otherwise,} \end{cases}$$

$$\widehat{C}_{\mathrm{ap}}(x) = \begin{cases} -\nu(x-2), & x \in (-2, -2+\delta), \\ \nu(x), & x \in (-\delta, \delta), \\ \nu(x+2), & x \in (2-\delta, 2), \\ 0, & \text{otherwise.} \end{cases} \qquad \widehat{C}_{\mathrm{pa}}(x) = \begin{cases} \nu(x-2), & x \in (-2, -2+\delta), \\ \nu(x), & x \in (-\delta, \delta), \\ -\nu(x+2), & x \in (2-\delta, 2), \\ 0, & \text{otherwise,} \end{cases}$$

Hence, they differ only on $(-2, -2+\delta) \cup (2-\delta, 2)$, implying that the kernel functions coincide on $(-2+\delta, 2-\delta)$, i.e., (3.2) holds. It is easier to observe this from the plots; see Fig. 1. For $x \in$ Bulk and x' in the range of integration, i.e., $x' \in (-1, 1)$, one has $x - x' \in (-2+\delta, 2-\delta)$. Using this fact and (3.2), we conclude (3.3). □

4 Agreement with the Original Operator

Note that the first integral in the definition of $\mathcal{M}_{\mathrm{BC}}$ is a constant. Hence, we define it as

$$c := \int_\Omega K_{\mathrm{BC}}(0, x')\,\mathrm{d}x'.$$

To incorporate the local BC, we prepare $\mathcal{M}_{\mathrm{BC}}$ for BC by splitting $u(x)$ into its even and odd parts. For that, we define the orthogonal projections that give the even and odd parts, respectively, by $P_e, P_o : L^2(\Omega) \to L^2(\Omega)$

$$P_e u(x) := \frac{u(x) + u(-x)}{2}, \quad P_o u(x) := \frac{u(x) - u(-x)}{2}.$$

We rewrite $\mathcal{M}_{\mathrm{BC}}$ by choosing a generic convolution type kernel function $\widehat{C}(x'-x)$.

$$(\mathcal{M}_{\mathrm{BC}} - c)u(x) = -\int_\Omega \widehat{C}(x'-x)u(x')\,\mathrm{d}x'$$

$$
\begin{aligned}
&= -\int_{\Omega} \widehat{C}(x'-x)(P_e + P_o)u(x')\,\mathrm{d}x' \\
&= -\int_{\Omega} \big(\widehat{C}(x'-x)P_e + \widehat{C}(x'-x)P_o\big)u(x')\,\mathrm{d}x' \\
&= -\int_{\Omega} \big(\widehat{C}_{\mathtt{BC}_1}(x'-x)P_e + \widehat{C}_{\mathtt{BC}_2}(x'-x)P_o\big)u(x')\,\mathrm{d}x',
\end{aligned}
$$

where $\mathtt{BC}_1, \mathtt{BC}_2 \in \{\mathtt{a}, \mathtt{p}, \mathtt{ap}, \mathtt{pa}\}$. Choosing $\mathtt{BC}_1$ and $\mathtt{BC}_2$ appropriately will lead to the desired BC. By utilizing the agreement of kernel functions in Eq. 3.3, the governing operators all agree in the bulk and the following BCs can be enforced:

$$
\begin{aligned}
K_{\mathtt{a}}(x,x') &= \widehat{C}_{\mathtt{a}}(x'-x)P_e &&+ \widehat{C}_{\mathtt{a}}(x'-x)P_o, \\
K_{\mathtt{p}}(x,x') &= \widehat{C}_{\mathtt{p}}(x'-x)P_e &&+ \widehat{C}_{\mathtt{p}}(x'-x)P_o, \\
K_{\mathtt{DD}}(x,x') &= \widehat{C}_{\mathtt{a}}(x'-x)P_e &&+ \widehat{C}_{\mathtt{p}}(x'-x)P_o, \\
K_{\mathtt{NN}}(x,x') &= \widehat{C}_{\mathtt{p}}(x'-x)P_e &&+ \widehat{C}_{\mathtt{a}}(x'-x)P_o, \\
K_{\mathtt{DN}}(x,x') &= \widehat{C}_{\mathtt{ap}}(x'-x)P_e &&+ \widehat{C}_{\mathtt{pa}}(x'-x)P_o, \\
K_{\mathtt{ND}}(x,x') &= \widehat{C}_{\mathtt{pa}}(x'-x)P_e &&+ \widehat{C}_{\mathtt{ap}}(x'-x)P_o.
\end{aligned}
\tag{4.1}
$$

The normalized eigenfunctions on the master domain are as follows:

$$
\begin{aligned}
\underline{e}_k^{\mathtt{a}}(x) &:= \frac{1}{\sqrt{2}} e^{i(2k+1)\frac{\pi}{2}x}, && k \in \mathbb{Z}, \\
\underline{e}_k^{\mathtt{p}}(x) &:= \frac{1}{\sqrt{2}} e^{i(2k)\frac{\pi}{2}x}, && k \in \mathbb{Z}, \\
\underline{e}_k^{\mathtt{DD}}(x) &:= \sin\big((2k)\frac{\pi}{4}(x+1)\big), && k \in \mathbb{N}^*, \\
\underline{e}_k^{\mathtt{NN}}(x) &:= \begin{cases} \frac{1}{\sqrt{2}}, & k = 0, \\ \cos\big((2k)\frac{\pi}{4}(x+1)\big), & k \in \mathbb{N}^*, \end{cases} \\
\underline{e}_k^{\mathtt{DN}}(x) &:= \sin\big((2k+1)\frac{\pi}{4}(x+1)\big), && k \in \mathbb{N}^*, \\
\underline{e}_k^{\mathtt{ND}}(x) &:= \cos\big((2k+1)\frac{\pi}{4}(x+1)\big), && k \in \mathbb{N}^*,
\end{aligned}
\tag{4.2}
$$

where $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ These can be easily verified by evaluating them or their derivatives at the respective boundary points, and subsequently checking their second order derivatives. A comprehensive list can be found in [15, Chap. 13].

We give the corresponding eigenvalues as they are explicitly known [1, 9, 10]:

$$
\begin{aligned}
&\mathtt{BC} = \mathtt{a}, && \lambda_k^{\mathtt{a}} = \int_{-1}^{1} \big[1 - \cos((2k+1)\frac{\pi}{2}x)\big] C(x)\,\mathrm{d}x, \quad k \in \mathbb{Z}, \\
&\mathtt{BC} = \mathtt{p}, && \lambda_k^{\mathtt{p}} = \int_{-1}^{1} \big[1 - \cos((2k)\frac{\pi}{2}x)\big] C(x)\,\mathrm{d}x, \quad k \in \mathbb{Z}, \\
&\mathtt{BC} \in \{\mathtt{DD}, \mathtt{NN}\}, && \lambda_k^{\mathtt{BC}} = \int_{-1}^{1} \big[1 - \cos((2k)\frac{\pi}{4}x)\big] C(x)\,\mathrm{d}x, \quad k \in \mathbb{N}^* \quad \text{and} \quad \lambda_0^{\mathtt{NN}} = 0, \\
&\mathtt{BC} \in \{\mathtt{DN}, \mathtt{ND}\}, && \lambda_k^{\mathtt{BC}} = \int_{-1}^{1} \big[1 - \cos((2k+1)\frac{\pi}{4}x)\big] C(x)\,\mathrm{d}x, \quad k \in \mathbb{N}^*.
\end{aligned}
\tag{4.3}
$$

We establish the agreement of $\mathcal{M}_{\text{orig}}$ with $\mathcal{M}_{\text{BC}}$ as follows:

Theorem 4.1 *When $K(x, x') = K_{\text{BC}}(x, x')$, we have the following agreement:*

$$\mathcal{M}_{orig} u(x) = \mathcal{M}_{\text{BC}} u(x) \text{ when } \begin{cases} x \in \Omega & \text{if } \texttt{BC} \in \{\texttt{NN}, \texttt{p}\}, \\ x \in Bulk & \text{if } \texttt{BC} \in \{\texttt{a}, \texttt{DD}, \texttt{DN}, \texttt{ND}\}, \end{cases}$$

where $Bulk := (-1+\delta, 1-\delta)$ and $\mathcal{M}_{\text{BC}}$ enforces the local boundary condition BC.

Proof Due to Eq. 3.3, we already know that the operators with the six listed BCs, i.e., $\texttt{BC} \in \{\texttt{a}, \texttt{p}, \texttt{DD}, \texttt{NN}, \texttt{DN}, \texttt{ND}\}$, agree with $\mathcal{M}_{\text{orig}}$ in the bulk. We will extend the agreement to the full domain Ω when $\texttt{BC} \in \{\texttt{p}, \texttt{NN}\}$. To prove this result, we utilize the fact that the eigenfunctions and eigenvalues of $\mathcal{M}_{\text{BC}}$ are explicitly known as given in Eqs. 4.2 and 4.3. When $\texttt{BC} \in \{\texttt{NN}, \texttt{p}\}$, $\lambda_0^{\text{BC}} = 0$ with the eigenfunctions $\underline{e}_0^{\text{p}} = \underline{e}_0^{\text{NN}} = \frac{1}{\sqrt{2}}$. Hence, the constant function $1(x) \equiv 1$ is an eigenfunction of the $\mathcal{M}_{\text{BC}}$ operator corresponding to the zero eigenvalue. In other words,

$$\mathcal{M}_{\text{BC}} 1 = 0, \quad \texttt{BC} \in \{\texttt{NN}, \texttt{p}\}. \tag{4.4}$$

More precisely, by substituting $u(x) \equiv 1$ in Eq. 4.4, one obtains

$$\mathcal{M}_{\text{BC}} 1(x) = \int_\Omega K_{\text{BC}}(0, x') 1(x) \, \mathrm{d}x' - \int_\Omega K_{\text{BC}}(x, x') 1(x') \, \mathrm{d}x' = 0.$$

This implies that

$$\int_\Omega K_{\text{BC}}(x, x') \, \mathrm{d}x' = \int_\Omega K_{\text{BC}}(0, x') \, \mathrm{d}x'.$$

Hence, the agreement of $\mathcal{M}_{\text{BC}}$ and $\mathcal{M}_{\text{orig}}$ holds for all $x \in \Omega$. When $\texttt{BC} \in \{\texttt{a}, \texttt{DD}, \texttt{DN}, \texttt{ND}\}$, the function $1(x)$ is not an eigenfunction, hence, we do not have agreement in all of Ω, but only in the bulk. For the proof that the operator $\mathcal{M}_{\text{BC}}$ enforces the corresponding local BC, see Sec. 9. For the proof in integral representation, see [8, Thm. 3.4]. □

Remark 4.2 The explicit expression of the kernel functions given in Eq. 4.1 are as follows:

$$\begin{aligned}
K_{\texttt{a}}(x, x') &= \widehat{C}_{\texttt{a}}(x' - x), \\
K_{\texttt{p}}(x, x') &= \widehat{C}_{\texttt{p}}(x' - x), \\
K_{\texttt{DD}}(x, x') &= \frac{1}{2}\left\{\left[\widehat{C}_{\texttt{a}}(x' - x) + \widehat{C}_{\texttt{a}}(x' + x)\right] + \left[\widehat{C}_{\texttt{p}}(x' - x) - \widehat{C}_{\texttt{p}}(x' + x)\right]\right\}, \\
K_{\texttt{NN}}(x, x') &= \frac{1}{2}\left\{\left[\widehat{C}_{\texttt{p}}(x' - x) + \widehat{C}_{\texttt{p}}(x' + x)\right] + \left[\widehat{C}_{\texttt{a}}(x' - x) - \widehat{C}_{\texttt{a}}(x' + x)\right]\right\}, \\
K_{\texttt{DN}}(x, x') &= \frac{1}{2}\left\{\left[\widehat{C}_{\texttt{ap}}(x' - x) + \widehat{C}_{\texttt{ap}}(x' + x)\right] + \left[\widehat{C}_{\texttt{pa}}(x' - x) - \widehat{C}_{\texttt{pa}}(x' + x)\right]\right\}, \\
K_{\texttt{ND}}(x, x') &= \frac{1}{2}\left\{\left[\widehat{C}_{\texttt{pa}}(x' - x) + \widehat{C}_{\texttt{pa}}(x' + x)\right] + \left[\widehat{C}_{\texttt{ap}}(x' - x) - \widehat{C}_{\texttt{ap}}(x' + x)\right]\right\}.
\end{aligned} \tag{4.5}$$

Remark 4.3 The agreement of operators on the master domain guaranteed in Theorem 4.1 easily extends to the general domain.

Remark 4.4 Identifying the eigenvalues for operators enforcing mixed BCs is involved and the details are discussed in an upcoming paper [9].

5 Construction on the General Domain

In section, we present the construction on the general domain. To distinguish the construction on different domains, we relabel the the master domain $(-1, 1)$ with an underbar as $\underline{\Omega} := (-1, 1)$. Define the classical operator $-\Delta_{\mathrm{p}}$ on $\underline{\Omega}$ with periodic BC as

$$\begin{aligned} -\Delta_{\mathrm{p}} u &= -\frac{2}{\pi^2} u'', \\ \lim_{x \to -1} u(x) &= \lim_{x \to 1} u(x), \\ \lim_{x \to -1} u'(x) &= \lim_{x \to 1} u'(x), \end{aligned}$$

where $'$ denotes the weak derivative. The function u is a restriction to $\underline{\Omega}$, of a periodic element of $W^2(\mathbb{R}, \mathbb{C})$, second order weakly differentiable functions. The operator $-\Delta_{\mathrm{p}}$ has a purely discrete spectrum consisting of the eigenvalues

$$\sigma(-\Delta_{\mathrm{p}}) = \{k^2 : k \in \mathbb{N}\}.$$

Note that k^2 is an eigenvalue of geometric multiplicity 2 with the corresponding eigenfunctions $\underline{e}_k^{\mathrm{p}}$ and $\underline{e}_{-k}^{\mathrm{p}}$. Recall from Eq. 4.2 that the normalized eigenfunctions of $-\Delta_{\mathrm{p}}$ are

$$\underline{e}_k^{\mathrm{p}}(x) = \frac{1}{\sqrt{2}} e^{i(2k)\frac{\pi}{2}x}, \quad k \in \mathbb{Z},$$

form a Hilbert (complete and orthonormal) basis for $L^2(\underline{\Omega})$.

We want to extend this construction from the master domain $\underline{\Omega}$ to the general domain $\Omega = (a, b)$. Define the length of the general domain as

$$L := b - a.$$

We first generalize the definition of the classical operator $-\Delta_{\mathrm{p}}$ with periodic BC

$$\begin{aligned} -\Delta_{\mathrm{p}} u &= -\frac{L^2}{4\pi^2} u'', \\ \lim_{x \to a} u(x) &= \lim_{x \to b} u(x), \\ \lim_{x \to a} u'(x) &= \lim_{x \to b} u'(x). \end{aligned}$$

The new basis e_k^{p} is defined in the following way through the use of a linear map τ:

$$e_k^{\mathrm{p}}(x) := \sqrt{\frac{2}{L}} \underline{e}_k^{\mathrm{p}}(\tau(x)). \tag{5.1}$$

Define the midpoint of the general domain by

$$m := \frac{b+a}{2}.$$

The midpoint m is and essential ingredient in our construction. By the definition (5.1), the domain and range of τ is determined. For x in the domain of e_k^{p}, which is Ω, its image $\tau(x)$ should be in the domain of $\underline{e}_k^{\mathrm{p}}$, which is $\underline{\Omega}$. Hence,

$$\tau : \Omega \to \underline{\Omega}.$$

The unique affine bijective map τ is

$$\tau(x) = \frac{2}{b-a}(x - \frac{a+b}{2}) = \frac{2}{L}(x-m).$$

Note that the scaling $\frac{2}{L}$ and the shift to m do not spoil orthonormal property of $\underline{e}_k^{\mathrm{p}}$, as shown in Eq. 5.4, due to its exponential nature. Consequently, the new basis takes the form

$$e_k^{\mathrm{p}}(x) = \frac{1}{\sqrt{L}} e^{i(2k)\frac{\pi}{2}\frac{2}{L}(x-m)}, \quad k \in \mathbb{Z}, \tag{5.2}$$

$$e_k^{\mathrm{a}}(x) = \frac{1}{\sqrt{L}} e^{i(2k+1)\frac{\pi}{2}\frac{2}{L}(x-m)}, \quad k \in \mathbb{Z}. \tag{5.3}$$

Let us explain in detail how the map τ helps to inherit the orthonormality of $\underline{e}_k^{\mathrm{p}}$.

$$\begin{aligned} \int_a^b e_k^{\mathrm{p}}(x)^* e_\ell^{\mathrm{p}}(x)\,\mathrm{d}x &= \frac{2}{L}\int_a^b \underline{e}_k^{\mathrm{p}}(\tau(x))^* \underline{e}_\ell^{\mathrm{p}}(\tau(x))\,\mathrm{d}x \\ &= \frac{2}{L}\int_{-1}^1 \underline{e}_k^{\mathrm{p}}(\underline{x})^* \underline{e}_\ell^{\mathrm{p}}(\underline{x}) \frac{L}{2}\,\mathrm{d}\underline{x} \\ &= \delta_{k\,\ell}. \end{aligned} \tag{5.4}$$

Note that the change of variable $\underline{x} = \tau(x)$ leads to $\mathrm{d}\underline{x} = \tau'(x)\,\mathrm{d}x = \frac{2}{L}\,\mathrm{d}x$, hence,

$$\mathrm{d}x\Big|_{x=a}^{x=b} = \frac{L}{2}\,\mathrm{d}\underline{x}\Big|_{\underline{x}=\tau(a)=-1}^{\underline{x}=\tau(b)=1}.$$

5.1 Eigenfunctions and Eigenvalues on the General Domain

The normalized eigenfunctions on the general domain are as follows:

$$\begin{aligned}
e_k^{\mathrm{a}}(x) &:= \frac{1}{\sqrt{L}} e^{i(2k+1)\frac{\pi}{2}\tau(x)}, && k \in \mathbb{Z}, \\
e_k^{\mathrm{p}}(x) &:= \frac{1}{\sqrt{L}} e^{i(2k)\frac{\pi}{2}\tau(x)}, && k \in \mathbb{Z}, \\
e_k^{\mathrm{DD}}(x) &:= \sqrt{\frac{2}{L}} \sin\big((2k)\frac{\pi}{4}(\tau(x)+1)\big), && k \in \mathbb{N}^*, \\
e_k^{\mathrm{NN}}(x) &:= \begin{cases} \frac{1}{\sqrt{L}}, & k = 0, \\ \sqrt{\frac{2}{L}} \cos\big((2k)\frac{\pi}{4}(\tau(x)+1)\big), & k \in \mathbb{N}^*, \end{cases} \\
e_k^{\mathrm{DN}}(x) &:= \sqrt{\frac{2}{L}} \sin\big((2k+1)\frac{\pi}{4}(\tau(x)+1)\big), && k \in \mathbb{N}^*, \\
e_k^{\mathrm{ND}}(x) &:= \sqrt{\frac{2}{L}} \cos\big((2k+1)\frac{\pi}{4}(\tau(x)+1)\big), && k \in \mathbb{N}^*.
\end{aligned} \tag{5.5}$$

The verification of Eq. 5.5 is similar to that of Eq. 4.2 as explained in Sec. 4. Here we provide the details of how one verifies the eigenfunction property on the general domain for the case e_k^{DN}. On the left boundary point $x = a$, one has

$$\begin{aligned}\lim_{x\to a} e_k^{\mathrm{DN}}(x) &= \lim_{x\to a}\sqrt{\frac{2}{L}}\sin\big((2k+1)\frac{\pi}{4}(\tau(x)+1)\big)\\ &= \sqrt{\frac{2}{L}}\sin\big((2k+1)\frac{\pi}{4}(\tau(a)+1)\big)\\ &= \sqrt{\frac{2}{L}}\sin\big((2k+1)\frac{\pi}{4}(0)\big), \quad \text{since } \tau(a)=-1\\ &= 0.\end{aligned}$$

On the right boundary point $x = b$, one has

$$\begin{aligned}\lim_{x\to b}\frac{\mathrm{d}e_k^{\mathrm{DN}}}{\mathrm{d}x}(x) &= \lim_{x\to b}\sqrt{\frac{2}{L}}(2k+1)\frac{\pi}{4}\frac{\mathrm{d}\tau}{\mathrm{d}x}(x)\cos\big((2k+1)\frac{\pi}{4}(\tau(x)+1)\big)\\ &= \sqrt{\frac{2}{L}}(2k+1)\frac{\pi}{4}\frac{\mathrm{d}\tau}{\mathrm{d}x}(b)\cos\big((2k+1)\frac{\pi}{4}(\tau(b)+1)\big)\\ &= \sqrt{\frac{2}{L}}(2k+1)\frac{\pi}{4}\frac{\mathrm{d}\tau}{\mathrm{d}x}(b)\cos\big((2k+1)\frac{\pi}{4}(1+1)\big)\\ &= \sqrt{\frac{2}{L}}(2k+1)\frac{\pi}{4}\frac{\mathrm{d}\tau}{\mathrm{d}x}(b)\cos\big((2k+1)\frac{\pi}{4}(2)\big), \quad \text{since } \tau(b)=1\\ &= 0, \quad \text{since } \cos\big((2k+1)\frac{\pi}{2}\big)=0.\end{aligned}$$

As it was done in [2], in order to obtain integer eigenvalues, one has to use an appropriately scaled Laplace operator. Using $\frac{\mathrm{d}\tau}{\mathrm{d}x}(x) = \frac{2}{L}$, one obtains

$$\begin{aligned}-\Big(\frac{2L}{\pi}\Big)^2\frac{\mathrm{d}^2e_k^{\mathrm{DN}}}{\mathrm{d}x^2}(x) &= -\Big(\frac{2L}{\pi}\Big)^2\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{\frac{2}{L}}(2k+1)\frac{\pi}{4}\frac{2}{L}\cos\big((2k+1)\frac{\pi}{4}(\tau(x)+1)\big)\right]\\ &= \Big(\frac{2L}{\pi}\Big)^2\sqrt{\frac{2}{L}}\Big[(2k+1)\frac{\pi}{4}\frac{2}{L}\Big]^2\sin\big((2k+1)\frac{\pi}{4}(\tau(x)+1)\big)\\ &= \Big(\frac{2L}{\pi}\Big)^2\sqrt{\frac{2}{L}}\Big[(2k+1)\frac{\pi}{2L}\Big]^2\sin\big((2k+1)\frac{\pi}{4}(\tau(x)+1)\big)\\ &= \Big(\frac{2L}{\pi}\Big)^2\Big(\frac{\pi}{2L}\Big)^2(2k+1)^2e_k^{\mathrm{DN}}\\ &= (2k+1)^2e_k^{\mathrm{DN}}.\end{aligned}$$

We give the corresponding eigenvalues. On the general domain, the eigenvalues are as follows. We also give explicit expressions for the flat-top kernel.

$$\begin{aligned}\mathtt{BC}=\mathtt{a}, \quad \lambda_k^{\mathtt{a}} &= \int_a^b\big[1-\cos((2k+1)\frac{\pi}{2}\tau(x))\big]C(x)\,\mathrm{d}x\\ &= 2\delta\Big[1-\mathrm{sinc}\Big(\frac{(2k+1)\pi}{L}\delta\Big)\Big], \quad k\in\mathbb{Z},\end{aligned}$$

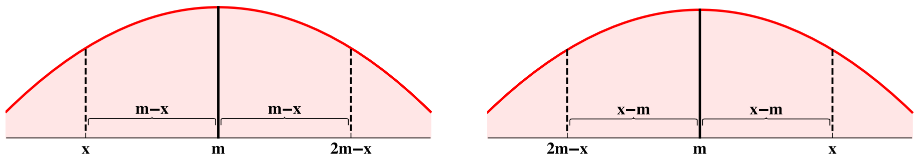


Fig. 2 An even function with respect to the midpoint m. Note that the symmetric partner of x is $2m - x$ regardless of $x < m$ or $x > m$

$$\begin{aligned}
\mathtt{BC} = \mathtt{p}, \qquad & \lambda_k^{\mathtt{p}} = \int_a^b \left[1 - \cos((2k)\frac{\pi}{2}\tau(x))\right] C(x)\,\mathrm{d}x \\
& = 2\delta\left[1 - \mathrm{sinc}\left(\frac{(2k)\pi}{L}\delta\right)\right], \quad k \in \mathbb{Z}, \\
\mathtt{BC} \in \{\mathtt{DD}, \mathtt{NN}\}, \qquad & \lambda_k^{\mathtt{BC}} = \int_a^b \left[1 - \cos\left((2k)\frac{\pi}{4}\tau(x)\right)\right] C(x)\,\mathrm{d}x \\
& = 2\delta\left[1 - \mathrm{sinc}\left(\frac{(2k)\pi}{2L}\delta\right)\right], \quad k \in \mathbb{N}^* \quad \text{and} \quad \lambda_0^{\mathtt{NN}} = 0, \\
\mathtt{BC} \in \{\mathtt{DN}, \mathtt{ND}\}, \qquad & \lambda_k^{\mathtt{BC}} = \int_a^b \left[1 - \cos((2k+1)\frac{\pi}{4}\tau(x))\right] C(x)\,\mathrm{d}x \\
& = 2\delta\left[1 - \mathrm{sinc}\left(\frac{(2k+1)\pi}{2L}\delta\right)\right], \quad k \in \mathbb{N}^*,
\end{aligned}$$

where

$$\mathrm{sinc}(x) := \frac{\sin x}{x}, \quad x > 0.$$

The crucial property guaranteed by the eigenfunctions is that they all form a Hilbert basis.

Theorem 5.1 *Any set of eigenfunctions among* $(e_k^{\mathtt{a}})_{k\in\mathbb{Z}}$, $(e_k^{\mathtt{p}})_{k\in\mathbb{Z}}$, $(e_k^{\mathtt{DD}})_{k\in\mathbb{N}^*}$, $(e_k^{\mathtt{NN}})_{k\in\mathbb{N}}$, $(e_k^{\mathtt{DN}})_{k\in\mathbb{N}^*}$, *and* $(e_k^{\mathtt{DN}})_{k\in\mathbb{N}^*}$ *given in* Eq. 5.5 *forms a Hilbert (complete and orthonormal) basis for* $L^2(\Omega)$.

Proof See [14, p.49 and Thm. 4.12.ii]. See [13, Example 7.5.5 on p. 204] for the role of a Hilbert-Schmidt operator when forming a Hilbert basis. For related properties of the eigenfunctions, see [2]. □

Since the eigenfunctions form Hilbert bases for $L^2(\Omega)$, the other crucial property that they all satisfy is Parseval's identity.

Theorem 5.2 *For* $u \in L^2(\Omega)$,

$$\sum_{k\in\mathfrak{I}_{\mathtt{BC}}} |\langle e_k^{\mathtt{BC}} | u \rangle|^2 = \|u\|_{L^2(\Omega)}^2, \tag{5.6}$$

where $\mathfrak{I}_{\mathtt{BC}}$ *is the index set corresponding to* $\mathtt{BC}$.

5.2 Even and Odd Functions and the Corresponding Projections on the General Domain

We extend the concept of even and odd functions to the general domain. Define the points that are equidistant to m as symmetric partners. If $x < m$, the symmetric partner of x

is $m + (m - x) = 2m - x$. Likewise, if $x > m$, the symmetric partner of m is again $m - (x - m) = 2m - x$. Hence, the symmetric partner of x with respect to the midpoint is $2m - x$ regardless of $x < m$ or $x > m$ as depicted in Fig. 2. Consequently, the notion of an even or odd function is defined using $2m - x$ as follows:

Definition 5.3 A function $u(x)$ is said to be even with respect to m when

$$u(x) = u(2m - x). \tag{5.7}$$

It is said to be odd with respect to m when

$$u(x) = -u(2m - x). \tag{5.8}$$

Note that instead of the midpoint m, $2m$ appears in the definitions (5.7) and (5.8). That's why manipulations such as change variables in our construction involve $2m$ instead of m. Even and odd parts of a function will used in the construction of the governing operators. One defines the self-adjoint orthogonal even and odd projection operators P_e and P_o in the following way:

Definition 5.4 The even and odd projections P_e and P_o with respect to m

$$P_e : L^2(\Omega) \to L^2(\Omega) \quad \text{and} \quad P_o : L^2(\Omega) \to L^2(\Omega)$$

are defined by

$$\begin{aligned} P_e u(x) &:= \frac{1}{2}(u(x) + u(2m - x)), \\ P_o u(x) &:= \frac{1}{2}(u(x) - u(2m - x)). \end{aligned}$$

6 Integral Representation of the Abstract Convolution

The functions on the master domain are labeled with the underbar. Using this convention, the abstract convolution defined in Eq. 2.2 on the master domain $\underline{\mathcal{C}}$ takes the form

$$\underline{\mathcal{C}}_{\mathtt{BC}}\underline{u}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} \langle \underline{e}_k^{\mathtt{BC}} | \underline{C} \rangle \langle \underline{e}_k^{\mathtt{BC}} | \underline{u} \rangle \, \underline{e}_k^{\mathtt{BC}}(x), \quad \mathtt{BC} \in \{\mathtt{a}, \mathtt{p}\},$$

where the inner product on $L^2(\underline{\Omega})$ is defined by

$$\langle \underline{f} | \underline{g} \rangle := \int_{-1}^{1} (\underline{f}(x))^* \underline{g}(x) \, \mathrm{d}x.$$

The abstract convolution is generalized to the general domain by

$$\mathcal{C}_{\mathtt{BC}} u(x) = \sqrt{L}\sum_{k\in\mathbb{Z}} \langle e_k^{\mathtt{BC}} | C \rangle \langle e_k^{\mathtt{BC}} | u \rangle \, e_k^{\mathtt{BC}}(x), \quad \mathtt{BC} \in \{\mathtt{a}, \mathtt{p}\},$$

where the inner product on $L^2(\Omega)$ is defined by

$$\langle f | g \rangle := \int_{a}^{b} (f(x))^* g(x) \, \mathrm{d}x. \tag{6.1}$$

Note that the inner product is antilinear in the first argument. Namely,

$$\langle sf|g\rangle = s^*\langle f|g\rangle, \quad s \in \mathbb{C}. \tag{6.2}$$

The integral representation of the abstract convolution with periodic BC is given as follows:

Theorem 6.1 *Let $C \in L^2(\Omega)$ be an even function with respect to the midpoint m. Namely,*

$$C(x) = C(2m - x). \tag{6.3}$$

Let $\mathcal{C}_{\mathtt{p}}$ be the abstract convolution with periodic BC defined by

$$\mathcal{C}_{\mathtt{p}}u(x) := \sqrt{L}\sum_{k\in\mathbb{Z}}\langle e_k^{\mathtt{p}}|C\rangle\langle e_k^{\mathtt{p}}|u\rangle\, e_k^{\mathtt{p}}(x).$$

Then, the integral representation of $\mathcal{C}_{\mathtt{p}}$ is

$$\mathcal{C}_{\mathtt{p}}u(x) = \int_a^b \widehat{C}_{\mathtt{p}}(x' - x + m)u(x')\,\mathrm{d}x'. \tag{6.4}$$

For notational convenience, denote

$$p_k := (2k)\frac{\pi}{2}\frac{2}{L},$$
$$\gamma := \frac{1}{\sqrt{L}}.$$

Then, the basis function $e_k^{\mathtt{p}}$ in Eq. 5.2 in the new notation simply becomes

$$e_k^{\mathtt{p}}(x) = \gamma e^{ip_k(x-m)}.$$

Now, we give the proof of Eq. 6.4.

Proof Using the antilinearity (6.2) of the inner product (6.1) in the first argument, rewrite the abstract convolution (2.2) in the following form:

$$\begin{aligned}\mathcal{C}_{\mathtt{p}}u(x) &= \frac{1}{\gamma}\sum_{k\in\mathbb{Z}}\langle e_k^{\mathtt{p}}|C\rangle\langle e_k^{\mathtt{p}}|u\rangle\, e_k^{\mathtt{p}}(x)\\ &= \frac{1}{\gamma}\Big\langle \sum_{k\in\mathbb{Z}}(e_k^{\mathtt{p}}(x))^*\langle C|e_k^{\mathtt{p}}\rangle\, e_k^{\mathtt{p}}\Big|u\Big\rangle.\end{aligned} \tag{6.5}$$

Concentrate on the term $(e_k^{\mathtt{p}}(x))^*\,\langle C|e_k^{\mathtt{p}}\rangle$. We have

$$\begin{aligned}(e_k^{\mathtt{p}}(x))^*\,\langle C|e_k^{\mathtt{p}}\rangle &= \gamma^2\, e^{-ip_k(x-m)}\int_a^b C^*(x')e^{ip_k(x'-m)}\,\mathrm{d}x'\\ &= \gamma^2\int_a^b C(x')e^{ip_k(x'-x)}\,\mathrm{d}x' \quad \text{because } C \text{ is real-valued}\\ &= \gamma^2\int_{a+x-m}^{b+x-m}\widehat{C}_{\mathtt{p}}(x')e^{ip_k(x'-x)}\,\mathrm{d}x'\end{aligned} \tag{6.6}$$

$$= \gamma^2 \int_a^b \widehat{C}_{\mathrm{p}}(x' + x - m) e^{i p_k (x' + x - m - x)} \,\mathrm{d}x'$$

$$= \gamma^2 \int_a^b \widehat{C}_{\mathrm{p}}(x - y' + m)\, e^{i p_k (-y' + m)} \,\mathrm{d}y' \tag{6.7}$$

$$= \gamma^2 \int_a^b e^{-i p_k (y' - m)}\, \widehat{C}_{\mathrm{p}}(-y' + m + x) \,\mathrm{d}y'$$

$$= \gamma \int_a^b \gamma e^{-i p_k (x' - m)}\, \widehat{C}_{\mathrm{p}}(x - x' + m) \,\mathrm{d}x'$$

$$= \gamma \int_a^b \gamma e^{-i p_k (x' - m)}\, \widehat{C}_{\mathrm{p}}(x' - x + m) \,\mathrm{d}x' \tag{6.8}$$

$$= \gamma \langle e_k^{\mathrm{p}} | \widehat{C}_{\mathrm{p}}(\cdot - x + m) \rangle. \tag{6.9}$$

The step (6.6) is obtained by using the fact that both e_k^{p} and the extension $\widehat{C}_{\mathrm{p}}$ are L-periodic and the value of the integral does not change on the $(x - m)$-shifted interval $(a + x - m, b + x - m)$. The step (6.7) is obtained by the change of variable

$$- y' + m = x' - m. \tag{6.10}$$

The change of variable (6.10) implies

$$y' = -x' + 2m,$$

which guarantees the correct limits of integration:

$$\mathrm{d}x' \Big|_{x'=a}^{x'=b} = -\,\mathrm{d}y' \Big|_{y'=-a+2m=b}^{y'=-b+2m=a}.$$

The step (6.8) holds because C is even with respect to m, so is $\widehat{C}_{\mathrm{p}}$. Using property (6.3),

$$\widehat{C}_{\mathrm{p}}(x - x' + m) = \widehat{C}_{\mathrm{p}}(2m - (x - x' + m)) = \widehat{C}_{\mathrm{p}}(x' - x + m). \tag{6.11}$$

Substituting (6.9) in (6.5), one arrives at the integral representation:

$$\begin{aligned}
\mathcal{C}_{\mathrm{p}} u(x) &= \frac{1}{\gamma} \Big\langle \sum_{k \in \mathbb{Z}} \gamma \langle e_k^{\mathrm{p}} | \widehat{C}_{\mathrm{p}}(\cdot - x + m) \rangle\, e_k^{\mathrm{p}} \Big| u \Big\rangle \\
&= \frac{\gamma}{\gamma} \Big\langle \sum_{k \in \mathbb{Z}} \langle e_k^{\mathrm{p}} | \widehat{C}_{\mathrm{p}}(\cdot - x + m) \rangle\, e_k^{\mathrm{p}} \Big| u \Big\rangle \\
&= \langle \widehat{C}_{\mathrm{p}}(\cdot - x + m) | u \rangle \\
&= \int_a^b \widehat{C}_{\mathrm{p}}(x' - x + m) u(x') \,\mathrm{d}x'.
\end{aligned}$$

Consequently, the integral representation takes the form

$$\mathcal{C}_{\mathrm{p}} u(x) = \int_a^b \widehat{C}_{\mathrm{p}}(x' - x + m) u(x') \,\mathrm{d}x'.$$

□

One can construct the integral representation of the abstract convolution with antiperiodic BC by following the steps of the periodic case, which we do next.

Theorem 6.2 *Let $C \in L^2(\Omega)$ be an even function with respect to the midpoint m. Let $\mathcal{C}_{\mathrm{a}}$ be the abstract convolution with antiperiodic BC defined by*

$$\mathcal{C}_{\mathrm{a}} u(x) := \sqrt{L} \sum_{k\in\mathbb{Z}} \langle e_k^{\mathrm{a}} | C \rangle \langle e_k^{\mathrm{a}} | u \rangle \, e_k^{\mathrm{a}}(x).$$

Then, the integral representation of $\mathcal{C}_{\mathrm{a}}$ is

$$\mathcal{C}_{\mathrm{a}} u(x) = \int_a^b \widehat{C}_{\mathrm{a}}(x' - x + m) u(x') \, \mathrm{d}x'. \tag{6.12}$$

Again, for notational convenience, denote

$$a_k := (2k+1)\frac{\pi}{2}\frac{2}{L}.$$

Then, the basis function e_k^{a} in Eq. 5.3 in the new notation becomes

$$e_k^{\mathrm{a}}(x) = \gamma e^{i a_k (x-m)}.$$

Now, we give the proof of Eq. 6.12.

Proof Using the antilinearity (6.2) of the inner product (6.1) in the first argument, rewrite the abstract convolution (2.2) in the following form:

$$\begin{aligned}
\mathcal{C}_{\mathrm{a}} u(x) &= \frac{1}{\gamma} \sum_{k\in\mathbb{Z}} \langle e_k^{\mathrm{a}} | C \rangle \langle e_k^{\mathrm{a}} | u \rangle \, e_k^{\mathrm{a}}(x) \\
&= \frac{1}{\gamma} \Big\langle \sum_{k\in\mathbb{Z}} (e_k^{\mathrm{a}}(x))^* \langle C | e_k^{\mathrm{a}} \rangle \, e_k^{\mathrm{a}} \Big| u \Big\rangle. \qquad (6.13)
\end{aligned}$$

Concentrate on the term $(e_k^{\mathrm{a}}(x))^* \, \langle C | e_k^{\mathrm{a}} \rangle$. We have

$$\begin{aligned}
(e_k^{\mathrm{a}}(x))^* \, \langle C | e_k^{\mathrm{a}} \rangle &= \gamma^2 \, e^{-i a_k (x-m)} \int_a^b C^*(x') e^{i a_k (x'-m)} \, \mathrm{d}x' \\
&= \gamma^2 \int_a^b C(x') e^{i a_k (x'-x)} \, \mathrm{d}x' \quad \text{because } C \text{ is real-valued} \\
&= \gamma^2 \int_{a+x-m}^{b+x-m} \widehat{C}_{\mathrm{a}}(x') e^{i a_k (x'-x)} \, \mathrm{d}x' \qquad (6.14) \\
&= \gamma^2 \int_a^b \widehat{C}_{\mathrm{a}}(x' + x - m) e^{i a_k (x'+x-m-x)} \, \mathrm{d}x' \\
&= \gamma^2 \int_a^b \widehat{C}_{\mathrm{a}}(x - y' + m) \, e^{i a_k (-y'+m)} \, \mathrm{d}y' \\
&= \gamma^2 \int_a^b e^{-i a_k (y'-m)} \, \widehat{C}_{\mathrm{a}}(-y' + m + x) \, \mathrm{d}y' \\
&= \gamma \int_a^b \gamma e^{-i a_k (x'-m)} \, \widehat{C}_{\mathrm{a}}(x - x' + m) \, \mathrm{d}x' \\
&= \gamma \int_a^b \gamma e^{-i a_k (x'-m)} \, \widehat{C}_{\mathrm{a}}(x' - x + m) \, \mathrm{d}x' \qquad (6.15) \\
&= \gamma \langle e_k^{\mathrm{a}} | \widehat{C}_{\mathrm{a}}(\cdot - x + m) \rangle. \qquad (6.16)
\end{aligned}$$

The step (6.14) is obtained by using the fact that both e_k^{a} and the extension $\widehat{C}_{\mathrm{a}}$ are L-antiperiodic, hence, their product is L-periodic. As a result, the value of the integral does not change on the $(x-m)$-shifted interval $(a+x-m, b+x-m)$. The step (6.15) holds because C is even with respect to m, so is $\widehat{C}_{\mathrm{a}}$. Using property (6.3),

$$\widehat{C}_{\mathrm{a}}(x-x'+m) = \widehat{C}_{\mathrm{a}}(2m-(x-x'+m)) = \widehat{C}_{\mathrm{a}}(x'-x+m). \tag{6.17}$$

Substituting (6.16) in (6.13), one arrives at the integral representation:

$$\begin{aligned}
\mathcal{C}_{\mathrm{a}}u(x) &= \frac{1}{\gamma}\Big\langle \sum_{k\in\mathbb{Z}} \gamma \langle e_k^{\mathrm{a}} | \widehat{C}_{\mathrm{a}}(\cdot - x + m)\rangle\, e_k^{\mathrm{a}} \Big| u \Big\rangle \\
&= \frac{\gamma}{\gamma}\Big\langle \sum_{k\in\mathbb{Z}} \langle e_k^{\mathrm{a}} | \widehat{C}_{\mathrm{a}}(\cdot - x + m)\rangle\, e_k^{\mathrm{a}} \Big| u \Big\rangle \\
&= \langle \widehat{C}_{\mathrm{a}}(\cdot - x + m) | u\rangle \\
&= \int_a^b \widehat{C}_{\mathrm{a}}(x'-x+m)u(x')\,\mathrm{d}x'.
\end{aligned}$$

Consequently, the integral representation takes the form

$$\mathcal{C}_{\mathrm{a}}u(x) = \int_a^b \widehat{C}_{\mathrm{a}}(x'-x+m)u(x')\,\mathrm{d}x'.$$

□

Remark 6.3 Define the bivariate kernel functions

$$\begin{aligned}
K_{\mathrm{a}}(x,x') &:= \widehat{C}_{\mathrm{a}}(x'-x+m), \\
K_{\mathrm{p}}(x,x') &:= \widehat{C}_{\mathrm{p}}(x'-x+m).
\end{aligned}$$

Recall that evenness (6.3) implies (6.11) and (6.17). More precisely,

$$\begin{aligned}
K_{\mathrm{a}}(x,x') &= \widehat{C}_{\mathrm{a}}(x'-x+m) = \widehat{C}_{\mathrm{a}}(x-x'+m) = K_{\mathrm{a}}(x',x), \\
K_{\mathrm{p}}(x,x') &= \widehat{C}_{\mathrm{p}}(x'-x+m) = \widehat{C}_{\mathrm{p}}(x-x'+m) = K_{\mathrm{p}}(x',x).
\end{aligned} \tag{6.18}$$

The Eq. 6.18 guarantees that the convolution operators $\mathcal{C}_{\mathrm{a}}$ and $\mathcal{C}_{\mathrm{p}}$ are both self-adjoint.

7 The Kernel Function and Its Extension

Integral representation of the abstract convolutions requires the periodic and antiperiodic extensions in Eqs. 6.4 and 6.12 of the kernel function. We explain the extension for the periodic BC and the antiperiodic case easily follows. The argument of the kernel function $\widehat{C}_p$ in Eq. 6.4 is $x'-x+m$. Since $x, x' \in (a,b)$, $x'-x \in (a-b, b-a)$. Hence,

$$x'-x+m \in \Big(a-b+\frac{a+b}{2}, b-a+\frac{a+b}{2}\Big) = \Big(\frac{3a-b}{2}, \frac{3b-a}{2}\Big).$$

Consequently, the kernel function sweeps $\widehat{\Omega} := (\widehat{a}, \widehat{b})$ where

$$\widehat{a} = \frac{3a-b}{2} \quad \text{and} \quad \widehat{b} = \frac{3b-a}{2}.$$

Since $b-a>0$, observe that $\widehat{\Omega}$ contains Ω because

$$(\widehat{a},\widehat{b}) = \Big(\frac{3a-b}{2}, \frac{3b-a}{2}\Big) = \frac{1}{2}((2a-(b-a)), (2b+(b-a)) \supset \frac{1}{2}(2a,2b) = (a,b).$$

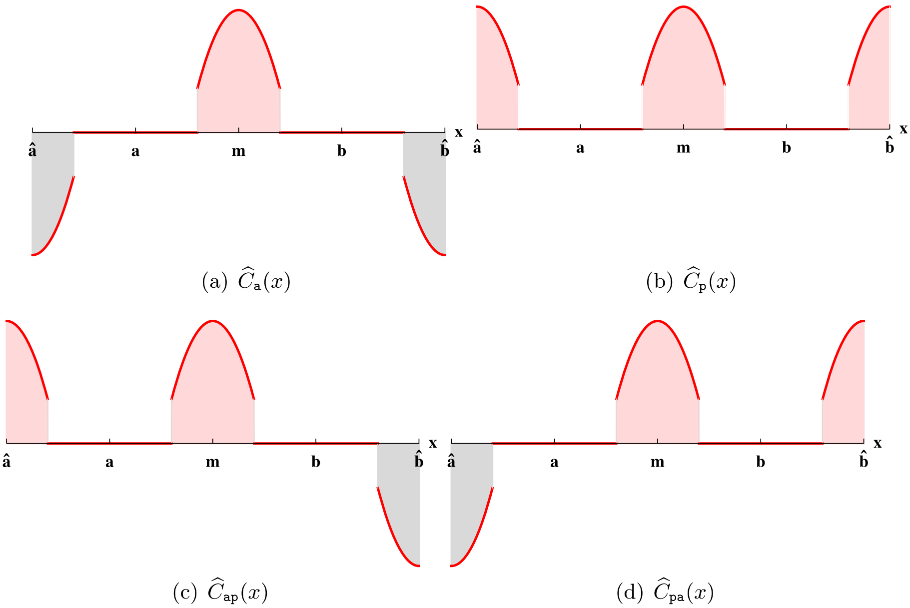


Fig. 3 The kernel function $C(x)$ on the general domain $\Omega = (a, b)$ and its extensions on $\widehat{\Omega} = (\widehat{a}, \widehat{b})$. The periodic, antiperiodic, and mixed extensions of $C(x)$ are denoted by $\widehat{C}_{\mathrm{a}}(x)$, $\widehat{C}_{\mathrm{p}}(x)$, $\widehat{C}_{\mathrm{ap}}$, and $\widehat{C}_{\mathrm{pa}}$, respectively

Observe that the length of $\widehat{\Omega}$ is

$$\frac{3b-a}{2} - \frac{3a-b}{2} = 2(b-a) = 2L$$

and the distance between the midpoint and the endpoints of $\widehat{\Omega}$ is

$$\frac{3b-a}{2} - m = m - \frac{3a-b}{2} = b - a = L.$$

Since $\widehat{C}_{\mathrm{a}}$ and $\widehat{C}_{\mathrm{p}}$ are the L-antiperiodic and L-periodic extensions of C, respectively, $\widehat{C}_{\mathrm{a}}$ and $\widehat{C}_{\mathrm{p}}$ are the same as C on Ω. For $x \in \widehat{\Omega} \setminus \Omega$, $\widehat{C}_{\mathrm{a}}$ and $\widehat{C}_{\mathrm{p}}$ are obtained by appropriate shifts of length L. More precisely, the extensions are expressed explicitly as

$$\widehat{C}_{\mathrm{a}}(x) := \begin{cases} -C(x+L), & x \in (\widehat{a}, a), \\ C(x), & x \in (a, b), \\ -C(x-L), & x \in (b, \widehat{b}), \end{cases} \qquad \widehat{C}_{\mathrm{p}}(x) := \begin{cases} C(x+L), & x \in (\widehat{a}, a), \\ C(x), & x \in (a, b), \\ C(x-L), & x \in (b, \widehat{b}), \end{cases}$$

$$\widehat{C}_{\mathrm{ap}}(x) := \begin{cases} -C(x+L), & x \in (\widehat{a}, a), \\ C(x), & x \in (a, b), \\ C(x-L), & x \in (b, \widehat{b}), \end{cases} \qquad \widehat{C}_{\mathrm{pa}}(x) := \begin{cases} C(x+L), & x \in (\widehat{a}, a), \\ C(x), & x \in (a, b), \\ -C(x-L), & x \in (b, \widehat{b}). \end{cases}$$

We depict the extended function $\widehat{C}_{\mathrm{a}}$, $\widehat{C}_{\mathrm{p}}$, $\widehat{C}_{\mathrm{ap}}$, and $\widehat{C}_{\mathrm{pa}}$ in Fig. 3.

8 Explicit Expression of the Kernel Functions

All of the necessary ingredients to construct the operators that enforce Dirichlet, Neumann, and mixed BCs are in place. More precisely, the integral representations, even and odd projections, and kernel extensions all have been extended the general domain. By mimicking the operators given in Eq. 4.1, we want to write the kernel functions explicitly as given in Eq. 4.5.

We present the Dirichlet case. The other BCs follow easily. Extending the operator definition to the general domain, the operator that enforces Dirichlet BC becomes

$$(\mathcal{M}_{\mathrm{DD}} - c)u(x) = -\int_a^b \big(\widehat{C}_{\mathrm{a}}(x' - x + m)P_e + \widehat{C}_{\mathrm{p}}(x' - x + m)P_o\big)u(x')\,\mathrm{d}x' \tag{8.1}$$
$$= -\frac{1}{2}\int_a^b \big(\widehat{C}_{\mathrm{a}}(x' - x + m)(u(x') + u(2m - x')) + \widehat{C}_{\mathrm{p}}(x' - x + m)((u(x') - u(2m - x'))\big)\,\mathrm{d}x'.$$

The form of $\mathcal{M}_{\mathrm{DD}}$ in Eq. 8.1 is an implication of d'Alembert's formula as explained in [1].

The goal is to obtain $u(x')$ as a common multiplier in the integrand. Concentrate on the $\widehat{C}_{\mathrm{a}}(x' - x + m)u(2m - x')$ term. Using the evenness of $\widehat{C}_{\mathrm{a}}$ with respect to m, we rewrite the term $\widehat{C}_{\mathrm{a}}(x' - x + m)$ as

$$\widehat{C}_{\mathrm{a}}(x' - x + m) = \widehat{C}_{\mathrm{a}}(2m - (x' - x + m)) = \widehat{C}_{\mathrm{a}}(-x' + x + m).$$

Using the change of variable $y' = 2m - x'$, the integral becomes

$$\int_a^b \widehat{C}_{\mathrm{a}}(-x' + x + m)u(2m - x')\,\mathrm{d}x' = \int_{2m-a}^{2m-b} \widehat{C}_{\mathrm{a}}((-2m + y') + x + m)u(y')\,\mathrm{d}(-y')$$
$$= \int_b^a \widehat{C}_{\mathrm{a}}(y' + x - m)u(y')\,\mathrm{d}(-y') \tag{8.2}$$
$$= \int_a^b \widehat{C}_{\mathrm{a}}(x' + x - m)u(x')\,\mathrm{d}x'. \tag{8.3}$$

The step (8.2) is due to the fact that $2m - a = (a+b) - a = b$ and $2m - b = (a+b) - b = a$.

Using evenness of $\widehat{C}_{\mathrm{a}}$ with respect to m and the same steps above, one gets

$$\int_a^b \widehat{C}_{\mathrm{p}}(x' - x + m)u(2m - x')\,\mathrm{d}x' = \int_a^b \widehat{C}_{\mathrm{p}}(x' + x - m)u(x')\,\mathrm{d}x'. \tag{8.4}$$

Combining (8.3) and (8.4), one arrives at the desired expression of the integrand with $u(x')$ as a common multiplier:

$$(\mathcal{M}_{\mathrm{DD}} - c)u(x) = -\frac{1}{2}\int_a^b \big(\widehat{C}_{\mathrm{a}}(x' - x + m) + \widehat{C}_{\mathrm{a}}(x' + x - m) + \widehat{C}_{\mathrm{p}}(x' - x + m) - \widehat{C}_{\mathrm{p}}(x' + x - m)\big)u(x')\,\mathrm{d}x'.$$

Similar to Eq. 4.1, one can now give the explicit expression of kernel functions on the general domain as follows:

$$K_{\mathrm{a}}(x, x') = \widehat{C}_{\mathrm{a}}(x' - x + m),$$
$$K_{\mathrm{p}}(x, x') = \widehat{C}_{\mathrm{p}}(x' - x + m),$$

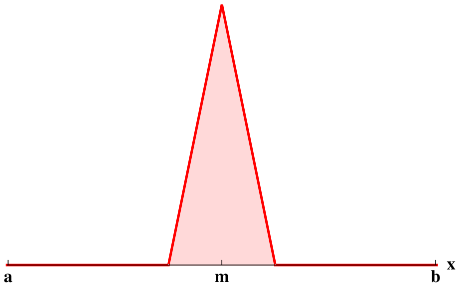


Fig. 4 The univariate kernel function $C(x)$ on the general domain $\Omega = (a, b)$, where $a = 2$, $b = 6$, and $\delta = 0.5$

$$\begin{aligned}
K_{\mathrm{DD}}(x, x') &= \frac{1}{2}\{[\widehat{C}_{\mathrm{a}}(x' - x + m) + \widehat{C}_{\mathrm{a}}(x' + x - m)] + [\widehat{C}_{\mathrm{p}}(x' - x + m) - \widehat{C}_{\mathrm{p}}(x' + x - m)]\},\\
K_{\mathrm{NN}}(x, x') &= \frac{1}{2}\{[\widehat{C}_{\mathrm{p}}(x' - x + m) + \widehat{C}_{\mathrm{p}}(x' + x - m)] + [\widehat{C}_{\mathrm{a}}(x' - x + m) - \widehat{C}_{\mathrm{a}}(x' + x - m)]\},\\
K_{\mathrm{DN}}(x, x') &= \frac{1}{2}\{[\widehat{C}_{\mathrm{ap}}(x' - x + m) + \widehat{C}_{\mathrm{ap}}(x' + x - m)] + [\widehat{C}_{\mathrm{pa}}(x' - x + m) - \widehat{C}_{\mathrm{pa}}(x' + x - m)]\},\\
K_{\mathrm{ND}}(x, x') &= \frac{1}{2}\{[\widehat{C}_{\mathrm{pa}}(x' - x + m) + \widehat{C}_{\mathrm{pa}}(x' + x - m)] + [\widehat{C}_{\mathrm{ap}}(x' - x + m) - \widehat{C}_{\mathrm{ap}}(x' + x - m)]\}.
\end{aligned}$$

The univariate kernel function $C(x)$ is depicted in Fig. 4 and the corresponding bivariate kernel functions $K_{\mathrm{BC}}(x, x')$ are depicted in Fig. 5.

9 Enforcement of Boundary Conditions

In this section, we prove that the operators $\mathcal{M}_{\mathrm{BC}}$ enforce the corresponding BC. Since the solution u belongs to $L^2(\Omega)$, it is not necessarily continuous. Hence, one verifies the BCs by using limits instead of function values. All of the six BC, either contain the limit of a function

$$\lim_{x\to a} u(x) \quad \text{and} \quad \lim_{x\to b} u(x) \tag{9.1}$$

or the limit of a derivative of a function

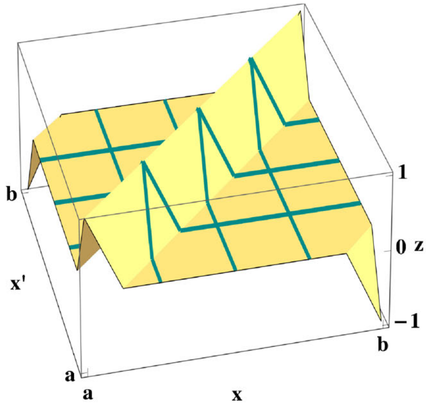
$$\lim_{x\to a} u'(x) \quad \text{and} \quad \lim_{x\to b} u'(x). \tag{9.2}$$

The forcing function f plays a critical role in enforcing the BC. Consider the governing equation

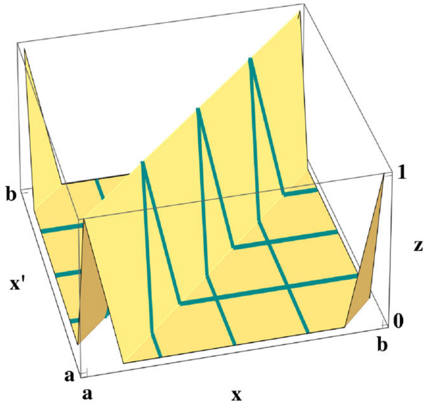
$$\mathcal{M}_{\mathrm{BC}} u(x) = f(x). \tag{9.3}$$

For simplicity of notation, we do not use a BC identifier for u and f since their BCs are determined by the operator preceding them. The following limits all exist, hence provide the foundation of the proof that the prescribed BCs are satisfied:

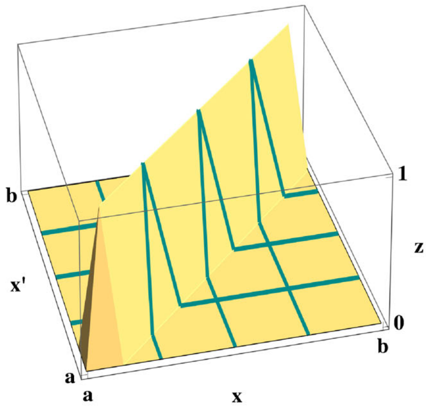
$$\begin{aligned}
&\lim_{x\to a} \mathcal{M}_{\mathrm{a}} u(x) = -\lim_{x\to b} \mathcal{M}_{\mathrm{a}} u(x) \quad \text{and} \quad \lim_{x\to a} \Big[\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathrm{a}} u\Big](x) = -\lim_{x\to b} \Big[\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathrm{a}} u\Big](x),\\
&\lim_{x\to a} \mathcal{M}_{\mathrm{a}} u(x) = -\lim_{x\to b} \mathcal{M}_{\mathrm{a}} u(x) \quad \text{and} \quad \lim_{x\to a} \Big[\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathrm{a}} u\Big](x) = -\lim_{x\to b} \Big[\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathrm{a}} u\Big](x),\\
&\lim_{x\to a} \mathcal{M}_{\mathrm{p}} u(x) = \lim_{x\to b} \mathcal{M}_{\mathrm{p}} u(x) \quad \text{and} \quad \lim_{x\to a} \Big[\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathrm{p}} u\Big](x) = \lim_{x\to b} \Big[\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathrm{p}} u\Big](x),
\end{aligned}$$



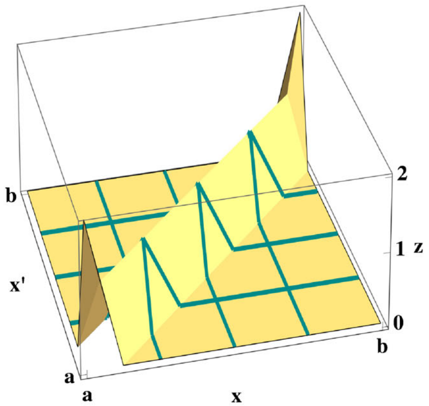
(a) $K_{\mathtt{a}}(x, x')$



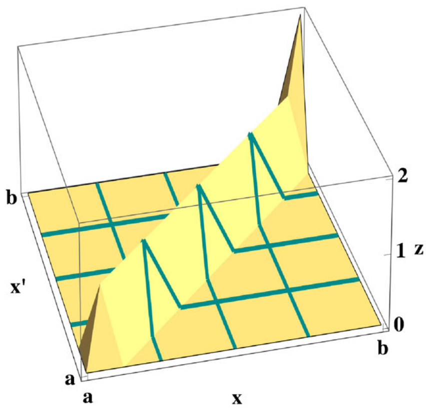
(b) $K_{\mathtt{p}}(x, x')$



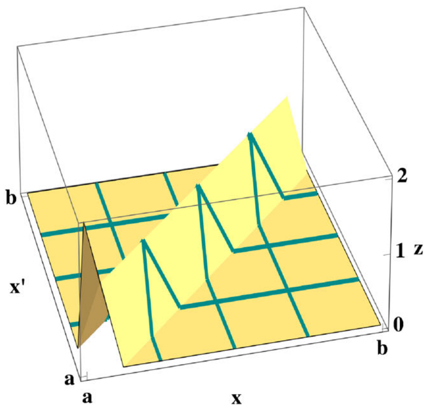
(c) $K_{\mathtt{DD}}(x, x')$



(d) $K_{\mathtt{NN}}(x, x')$



(e) $K_{\mathtt{DN}}(x, x')$



(f) $K_{\mathtt{ND}}(x, x')$

Fig. 5 The bivariate kernel functions $K_{\mathrm{BC}}(x, x')$ on the general domain $\Omega = (a, b)$, where $a = 2$, $b = 6$, and $\delta = 0.5$

$$\begin{aligned}
&\lim_{x\to a} \mathcal{M}_{\mathrm{DD}}u(x) = c \lim_{x\to a} u(x) && \text{and} && \lim_{x\to b} \mathcal{M}_{\mathrm{DD}}u(x) = c \lim_{x\to b} u(x),\\
&\lim_{x\to a} \Big[\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathrm{NN}}u\Big](x) = c \lim_{x\to a} u'(x) && \text{and} && \lim_{x\to b} \Big[\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathrm{NN}}u\Big](x) = c \lim_{x\to b} u'(x),\\
&\lim_{x\to a} \mathcal{M}_{\mathrm{DN}}u(x) = c \lim_{x\to a} u(x) && \text{and} && \lim_{x\to b} \Big[\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathrm{DN}}u\Big](x) = c \lim_{x\to b} u'(x),\\
&\lim_{x\to a} \Big[\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathrm{ND}}u\Big](x) = c \lim_{x\to a} u'(x) && \text{and} && \lim_{x\to b} \mathcal{M}_{\mathrm{ND}}u(x) = c \lim_{x\to b} u'(x).
\end{aligned} \tag{9.4}$$

For given f, using Eqs. 9.3 and 9.4, the following BCs are rigorously satisfied as will be shown in Sec. 9.2:

$$\begin{aligned}
\mathtt{a}:&\quad \lim_{x\to a} u(x) = -\lim_{x\to b} u(x), &&\quad \lim_{x\to a} u'(x) = -\lim_{x\to b} u'(x),\\
\mathtt{p}:&\quad \lim_{x\to a} u(x) = \lim_{x\to b} u(x), &&\quad \lim_{x\to a} u'(x) = \lim_{x\to b} u'(x),\\
\mathtt{DD}:&\quad \lim_{x\to a} u(x) = \frac{1}{c}\lim_{x\to a} f(x), &&\quad \lim_{x\to b} u(x) = \frac{1}{c}\lim_{x\to b} f(x),\\
\mathtt{NN}:&\quad \lim_{x\to a} u'(x) = \frac{1}{c}\lim_{x\to a} f'(x), &&\quad \lim_{x\to b} u'(x) = \frac{1}{c}\lim_{x\to b} f'(x),\\
\mathtt{DN}:&\quad \lim_{x\to a} u(x) = \frac{1}{c}\lim_{x\to a} f(x), &&\quad \lim_{x\to b} u'(x) = \frac{1}{c}\lim_{x\to b} f'(x),\\
\mathtt{ND}:&\quad \lim_{x\to a} u'(x) = \frac{1}{c}\lim_{x\to a} f'(x), &&\quad \lim_{x\to b} u(x) = \frac{1}{c}\lim_{x\to b} f(x).
\end{aligned}$$

9.1 The Hilbert-Schmidt Property of $\mathcal{C}_{\mathbf{BC}}$

The operator that is relevant to the BC is $\mathcal{C}_{\mathrm{BC}} = c - \mathcal{M}_{\mathrm{BC}}$. Note that the operator $\mathcal{C}_{\mathrm{BC}}$ is self-adjoint because both $\mathcal{M}_{\mathrm{BC}}$ and c are self-adjoint operators. We will utilize a crucial property: the operator $\mathcal{C}_{\mathrm{BC}}$ is Hilbert-Schmidt. The main tool to prove that the BCs are satisfied is this property. An operator that possesses the Hilbert-Schmidt property has a continuous extension to the boundary of the domain Ω. Hence, in our construction, each governing operator "feels the boundary." The type of BC determines the structure of the kernel function K_{BC}. This *a priori* determination of the structure of the kernel function is unique feature of our method. After encoding the type of BC through the kernel function, the boundary data is provided via the forcing function only on the local boundary.

Furthermore, the series defining the operator is uniformly convergent, thereby allowing the interchange of a limit with the summation. To prove that the BCs are satisfied in a convenient and unified way, we resort to the series representation of the convolution operator given by

$$\mathcal{C}_{\mathrm{BC}}u = \sum_{k\in\mathcal{I}_{\mathrm{BC}}} \mu_k^{\mathrm{BC}} \langle e_k^{\mathrm{BC}} | u\rangle e_k^{\mathrm{BC}},$$

where $\mathcal{I}_{\mathrm{BC}}$ is the index set corresponding to BC. The eigenvalues μ_k^{BC} are defined by

$$\mu_k^{\mathrm{BC}} := c - \lambda_k^{\mathrm{BC}}.$$

Instead, if one uses the integral representation of $\mathcal{M}_{\mathrm{BC}}$ to verify the BCs, then piecewise differentiability of the kernel function C becomes a requirement for BCs of type (9.2), which is an unnecessary restriction on C imposed by the stringent structure of integration. This is another reason why one should resort to the series representation and require C to be only in $L^2(\Omega)$.

The following theorem characterizes the Hilbert-Schmidt property.

Theorem 9.1 *$\mathcal{C}_{\mathrm{BC}}$ is Hilbert-Schmidt if and only if $\left(|\mu_k^{\mathrm{BC}}|^2\right)_{k\in\mathcal{I}_{\mathrm{BC}}}$ is summable.*

Proof See [3, Thm. 6 part (i)]. □

Next, we set out to prove that each $\mathcal{C}_{\mathrm{BC}}$ possesses the Hilbert-Schmidt property using the characterization provided in Theorem 9.1. The main idea is to prove the summability of $\left(|\mu_k^{\mathrm{BC}}|^2\right)_{k\in\mathcal{I}_{\mathrm{BC}}}$ by connecting it to $\|C\|_{L^2(\Omega)}$, which we know is finite due to $C \in L^2(\Omega)$.

Lemma 9.2 *For $\mathrm{BC} \in \{\mathrm{a}, \mathrm{p}\}$, the operator $\mathcal{C}_{\mathrm{BC}}u = \sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{BC}} \langle e_k^{\mathrm{BC}}|u\rangle e_k^{\mathrm{BC}}$ is Hilbert-Schmidt.*

Proof We need to prove that $\left(|\mu_k^{\mathrm{BC}}|^2\right)_{k\in\mathbb{Z}}$ is summable. Since $\mu_k^{\mathrm{BC}} = \langle e_k^{\mathrm{BC}}|C\rangle$, one readily obtains the summability using Parseval's identity (5.6):

$$\sum_{k\in\mathbb{Z}} |\mu_k^{\mathrm{BC}}|^2 = \sum_{k\in\mathbb{Z}} |\langle e_k^{\mathrm{BC}}|C\rangle|^2 = \|C\|^2_{L^2(\Omega)}.$$

□

Lemma 9.3 *The operators $\mathcal{C}_{\mathrm{DD}}u = \sum_{k\in\mathbb{N}^*} \mu_k^{\mathrm{DD}}\langle e_k^{\mathrm{DD}}|u\rangle e_k^{\mathrm{DD}}$ and $\mathcal{C}_{\mathrm{NN}}u = \sum_{k\in\mathbb{N}} \mu_k^{\mathrm{NN}}\langle e_k^{\mathrm{NN}}|u\rangle$ e_k^{NN} are Hilbert-Schmidt.*

Proof We proved the following property of μ_k^{DD} in [2, 10]:

$$\mu_k^{\mathrm{DD}} = \begin{cases} \langle e^{\mathrm{p}}_{k/2}|C\rangle & \text{if } k \in \mathbb{N}^* \text{ is even,} \\ \langle e^{\mathrm{a}}_{(k-1)/2}|C\rangle & \text{if } k \in \mathbb{N}^* \text{ is odd.} \end{cases}$$

Rewrite it as

$$\mu_k^{\mathrm{DD}} = \mu_{k,e}^{\mathrm{DD}} + \mu_{k,o}^{\mathrm{DD}},$$

where

$$\mu_{k,e}^{\mathrm{DD}} := \begin{cases} \langle e^{\mathrm{p}}_{k/2}|C\rangle & \text{if } k \in \mathbb{N}^* \text{ is even,} \\ 0 & \text{if } k \in \mathbb{N}^* \text{ is odd.} \end{cases} \quad \text{and} \quad \mu_{k,o}^{\mathrm{DD}} := \begin{cases} 0 & \text{if } k \in \mathbb{N}^* \text{ is even,} \\ \langle e^{\mathrm{a}}_{(k-1)/2}|C\rangle & \text{if } k \in \mathbb{N}^* \text{ is odd.} \end{cases}$$

Both $\left(|\mu_{k,e}^{\mathrm{DD}}|^2\right)_{k\in\mathbb{N}^*}$ and $\left(|\mu_{k,o}^{\mathrm{DD}}|^2\right)_{k\in\mathbb{N}^*}$ are summable due to Parseval's identity (5.6), because

$$\sum_{k\in\mathbb{N}^*} |\mu_{k,e}^{\mathrm{DD}}|^2 = \sum_{k\in\mathbb{N}^*} |\,\langle e_k^{\mathrm{p}}|C\rangle\,|^2 \le \sum_{k\in\mathbb{Z}} |\,\langle e_k^{\mathrm{p}}|C\rangle\,|^2 = \|C\|^2_{L^2(\Omega)} \tag{9.5}$$

and

$$\sum_{k\in\mathbb{N}^*} |\mu_{k,o}^{\mathrm{DD}}|^2 = \sum_{k\in\mathbb{N}} |\,\langle e_k^{\mathrm{a}}|C\rangle\,|^2 \le \sum_{k\in\mathbb{Z}} |\,\langle e_k^{\mathrm{a}}|C\rangle\,|^2 = \|C\|^2_{L^2(\Omega)}. \tag{9.6}$$

We can now establish the summability of $\left(|\mu_k^{\mathrm{DD}}|^2\right)_{k\in\mathbb{N}^*}$ using those of $\left(|\mu_{k,e}^{\mathrm{DD}}|^2\right)_{k\in\mathbb{N}^*}$ and $\left(|\mu_{k,o}^{\mathrm{DD}}|^2\right)_{k\in\mathbb{N}^*}$ as shown in Eqs. 9.5 and 9.6, respectively, as follows:

$$\begin{aligned} \sum_{k\in\mathbb{N}^*} |\mu_k^{\mathrm{DD}}|^2 &= \sum_{k\in\mathbb{N}^*} |\mu_{k,e}^{\mathrm{DD}}|^2 + \sum_{k\in\mathbb{N}^*} |\mu_{k,o}^{\mathrm{DD}}|^2 \\ &\le 2\|C\|^2_{L^2(\Omega)}. \end{aligned}$$

Note that $\mu_k^{\text{NN}} = \mu_k^{\text{DD}}$ for $k \in \mathbb{N}^*$ and $\mu_0^{\text{NN}} = 0$. Then,

$$\sum_{k\in\mathbb{N}} |\mu_k^{\text{NN}}|^2 = 0^2 + \sum_{k\in\mathbb{N}^*} |\mu_k^{\text{DD}}|^2 \leq 2\|C\|^2_{L^2(\Omega)}.$$

□

Lemma 9.4 *The operators* $\mathcal{C}_{\text{DN}}u = \sum_{k\in\mathbb{N}^*} \mu_k^{\text{DN}} \langle e_k^{\text{DN}}|u\rangle e_k^{\text{DN}}$ *and* $\mathcal{C}_{\text{ND}}u = \sum_{k\in\mathbb{N}^*} \mu_k^{\text{ND}} \langle e_k^{\text{ND}}|u\rangle e_k^{\text{ND}}$ *are Hilbert-Schmidt.*

Proof Note that since $\mu_k^{\text{DN}} = \mu_k^{\text{ND}}$, it is sufficient to prove the summability of $\left(|\mu_k^{\text{DN}}|^2\right)_{k\in\mathbb{N}^*}$. Note that

$$\begin{aligned} e_k^{\text{ND}}(x) &= \cos\left((2k+1)\frac{\pi}{4}(\tau(x)+1)\right) \\ &= \cos\left((2k+1)\frac{\pi}{4}\tau(x)\right)\cos\left((2k+1)\frac{\pi}{4}\right) - \sin\left((2k+1)\frac{\pi}{4}\tau(x)\right)\sin\left((2k+1)\frac{\pi}{4}\right). \end{aligned}$$

Hence, using the evenness of C and the fact that $1/\cos\left((2k+1)\frac{\pi}{4}\right) = \pm\sqrt{2}$, one gets

$$\begin{aligned} |\mu_k^{\text{DN}}|^2 &= |\left\langle \cos\left((2k+1)\frac{\pi}{4}\tau(x)\right)|C(x)\right\rangle|^2 \\ &= |\frac{1}{\cos\left((2k+1)\frac{\pi}{4}\right)}\langle e_k^{\text{ND}}|C\rangle|^2 \\ &= 2|\langle e_k^{\text{ND}}|C\rangle|^2. \end{aligned}$$

Consequently, using Parseval's identity (5.6), one arrives at the summability:

$$\sum_{k\in\mathbb{N}^*} |\mu_k^{\text{DN}}|^2 = 2\sum_{k\in\mathbb{N}^*} |\langle e_k^{\text{ND}}|C\rangle|^2 = 2\|C\|^2_{L^2(\Omega)}.$$

□

9.2 Rigorous Verification of Boundary Conditions

After establishing the Hilbert-Schmidt property of the operator $\mathcal{C}_{\text{BC}}$, we can now prove that the BCs are satisfied for each operator considered. For the BC related to the limit of a function given Eq. 9.1, write the expression of u involving $\mathcal{C}_{\text{BC}}u$:

$$u(x) = \frac{1}{c}f(x) + \frac{1}{c}\mathcal{C}_{\text{BC}}u(x). \tag{9.7}$$

Then, take the limit as x approaches to the related endpoint and use the associated BC of e_k^{BC}.

For the BC related to the limit of a derivative given in Eq. 9.2, differentiate both sides of Eq. 9.7. The series in the definition of $\mathcal{C}_{\text{BC}}$ is uniformly convergent due to its Hilbert-Schmidt property. Hence, termwise differentiation of the series is possible, meaning that the differentiation and summation can be interchanged in the following way:

$$\begin{aligned} u'(x) &= \frac{1}{c} f'(x) + \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{C}_{\mathrm{BC}} u(x) \\ &= \frac{1}{c} f'(x) + \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\in\mathfrak{I}_{\mathrm{BC}}} \mu_k^{\mathrm{BC}} \langle e_k^{\mathrm{BC}}|u\rangle e_k^{\mathrm{BC}}(x) \\ &= \frac{1}{c} f'(x) + \frac{1}{c}\sum_{k\in\mathfrak{I}_{\mathrm{BC}}} \mu_k^{\mathrm{BC}} \langle e_k^{\mathrm{BC}}|u\rangle \frac{\mathrm{d}e_k^{\mathrm{BC}}}{\mathrm{d}x}(x). \end{aligned} \tag{9.8}$$

Then, take the limit as x approaches to the related endpoint and use the associated BC of $\frac{\mathrm{d}e_k^{\mathrm{BC}}}{\mathrm{d}x}$.

• **The operators** $\mathcal{M}_{\mathrm{a}}$ **and** $\mathcal{M}_{\mathrm{p}}$**:** The antiperiodic and periodic BCs assume the following compatibility condition on f:

$$\lim_{x\to a} f(x) = -\lim_{x\to b} f(x) \quad \text{and} \quad \lim_{x\to a} f'(x) = -\lim_{x\to b} f'(x), \tag{9.9}$$

and

$$\lim_{x\to a} f(x) = \lim_{x\to b} f(x) \quad \text{and} \quad \lim_{x\to a} f'(x) = \lim_{x\to b} f'(x), \tag{9.10}$$

respectively. Use Eqs. 9.7 and 9.9, the BC related to the function value is satisfied:

$$\begin{aligned} \lim_{x\to a} u(x) &= \frac{1}{c}\lim_{x\to a} f(x) + \frac{1}{c}\lim_{x\to a}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{a}}\langle e_k^{\mathrm{a}}|u\rangle e_k^{\mathrm{a}}(x) \\ &= \frac{1}{c}\lim_{x\to a} f(x) + \frac{1}{c}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{a}}\langle e_k^{\mathrm{a}}|u\rangle \lim_{x\to a} e_k^{\mathrm{a}}(x) \\ &= -\frac{1}{c}\lim_{x\to b} f(x) - \frac{1}{c}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{a}}\langle e_k^{\mathrm{a}}|u\rangle \lim_{x\to b} e_k^{\mathrm{a}}(x) \\ &= -\frac{1}{c}\lim_{x\to b} f(x) - \frac{1}{c}\lim_{x\to b}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{a}}\langle e_k^{\mathrm{a}}|u\rangle e_k^{\mathrm{a}}(x) \\ &= -\lim_{x\to b} u(x). \end{aligned}$$

Use Eqs. 9.8 and 9.9, the BC related to the derivative value is satisfied:

$$\begin{aligned} \lim_{x\to a} u'(x) &= \frac{1}{c}\lim_{x\to a} f'(x) + \frac{1}{c}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{a}}\langle e_k^{\mathrm{a}}|u\rangle \lim_{x\to a}\frac{\mathrm{d}e_k^{\mathrm{a}}}{\mathrm{d}x}(x) \\ &= -\frac{1}{c}\lim_{x\to b} f'(x) - \frac{1}{c}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{a}}\langle e_k^{\mathrm{a}}|u\rangle \lim_{x\to b}\frac{\mathrm{d}e_k^{\mathrm{a}}}{\mathrm{d}x}(x) \\ &= -\frac{1}{c}\lim_{x\to b} f'(x) - \frac{1}{c}\lim_{x\to b}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{a}}\langle e_k^{\mathrm{a}}|u\rangle e_k^{\mathrm{a}}(x) \\ &= -\lim_{x\to b} u'(x). \end{aligned}$$

Use Eqs. 9.7 and 9.10, the BC related to the function value is satisfied:

$$\begin{aligned}\lim_{x\to a} u(x) &= \frac{1}{c}\lim_{x\to a} f(x) + \frac{1}{c}\lim_{x\to a}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{p}}\langle e_k^{\mathrm{p}}|u\rangle e_k^{\mathrm{p}}(x)\\ &= \frac{1}{c}\lim_{x\to a} f(x) + \frac{1}{c}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{p}}\langle e_k^{\mathrm{p}}|u\rangle \lim_{x\to a} e_k^{\mathrm{p}}(x)\\ &= \frac{1}{c}\lim_{x\to b} f(x) + \frac{1}{c}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{p}}\langle e_k^{\mathrm{p}}|u\rangle \lim_{x\to b} e_k^{\mathrm{p}}(x)\\ &= \frac{1}{c}\lim_{x\to b} f(x) + \frac{1}{c}\lim_{x\to b}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{p}}\langle e_k^{\mathrm{p}}|u\rangle e_k^{\mathrm{p}}(x)\\ &= \lim_{x\to b} u(x).\end{aligned}$$

Use Eqs. 9.8 and 9.10, the BC related to the derivative value is satisfied:

$$\begin{aligned}\lim_{x\to a} u'(x) &= \frac{1}{c}\lim_{x\to a} f'(x) + \frac{1}{c}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{p}}\langle e_k^{\mathrm{p}}|u\rangle \lim_{x\to a} \frac{\mathrm{d}e_k^{\mathrm{p}}}{\mathrm{d}x}(x)\\ &= \frac{1}{c}\lim_{x\to b} f'(x) + \frac{1}{c}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{p}}\langle e_k^{\mathrm{p}}|u\rangle \lim_{x\to b} \frac{\mathrm{d}e_k^{\mathrm{p}}}{\mathrm{d}x}(x)\\ &= \frac{1}{c}\lim_{x\to b} f'(x) + \frac{1}{c}\lim_{x\to b}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\in\mathbb{Z}} \mu_k^{\mathrm{p}}\langle e_k^{\mathrm{p}}|u\rangle e_k^{\mathrm{p}}(x)\\ &= \lim_{x\to b} u'(x).\end{aligned}$$

• **The operator** $\mathcal{M}_{\mathrm{DD}}$**:** Use Eq. 9.7. Denote the generic endpoint by x_0 where either $x_0 = a$ or $x_0 = b$. The BCs are satisfied because one can move $\lim_{x\to x_0}$ inside the series due to the Hilbert-Schmidt property of $\mathcal{C}_{\mathrm{DD}}$. Then use the fact that

$$\lim_{x\to a} e_k^{\mathrm{DD}}(x) = \lim_{x\to b} e_k^{\mathrm{DD}}(x) = 0,$$

and arrive at the desired BCs:

$$\begin{aligned}\lim_{x\to x_0} u(x) &= \frac{1}{c}\lim_{x\to x_0} f(x) + \frac{1}{c}\lim_{x\to x_0}\sum_{k\in\mathbb{N}^*} \mu_k^{\mathrm{DD}}\langle e_k^{\mathrm{DD}}|u\rangle e_k^{\mathrm{DD}}(x)\\ &= \frac{1}{c}\lim_{x\to x_0} f(x) + \frac{1}{c}\sum_{k\in\mathbb{N}^*} \mu_k^{\mathrm{DD}}\langle e_k^{\mathrm{DD}}|u\rangle \lim_{x\to x_0} e_k^{\mathrm{DD}}(x)\\ &= \frac{1}{c}\lim_{x\to x_0} f(x).\end{aligned}$$

• **The operator** $\mathcal{M}_{\mathrm{NN}}$**:** Use Eq. 9.8. The BCs are satisfied because one can move $\lim_{x\to x_0}$ inside the series again due to the Hilbert-Schmidt property of $\mathcal{C}_{\mathrm{NN}}$. Then use the fact that

$$\lim_{x\to a} \frac{\mathrm{d}e_k^{\mathrm{NN}}}{\mathrm{d}x}(x) = \lim_{x\to b} \frac{\mathrm{d}e_k^{\mathrm{NN}}}{\mathrm{d}x}(x) = 0,$$

and arrive at the desired BCs:

$$\begin{aligned}
\lim_{x\to x_0} u'(x) &= \frac{1}{c}\lim_{x\to x_0} f'(x) + \frac{1}{c}\lim_{x\to x_0} \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\in\mathbb{N}} \mu_k^{\mathrm{NN}}\langle e_k^{\mathrm{NN}}|u\rangle e_k^{\mathrm{NN}}(x) \\
&= \frac{1}{c}\lim_{x\to x_0} f'(x) + \frac{1}{c}\sum_{k\in\mathbb{N}} \mu_k^{\mathrm{NN}}\langle e_k^{\mathrm{NN}}|u\rangle \lim_{x\to x_0}\frac{\mathrm{d}e_k^{\mathrm{NN}}}{\mathrm{d}x}(x) \\
&= \frac{1}{c}\lim_{x\to x_0} f'(x).
\end{aligned}$$

• **The operator** $\mathcal{M}_{\mathrm{DN}}$**:** For the BC of type (9.1), use Eq. 9.7. The BC related to $\lim_{x\to a}$ is satisfied because one can move the limit inside the series, due to the Hilbert-Schmidt property of $\mathcal{C}_{\mathrm{DN}}$. Then use the fact that

$$\lim_{x\to a} e_k^{\mathrm{DN}}(x) = 0,$$

and arrive at the desired BC:

$$\begin{aligned}
\lim_{x\to a} u(x) &= \frac{1}{c}\lim_{x\to a} f(x) + \frac{1}{c}\lim_{x\to a} \sum_{k\in\mathbb{N}^*} \mu_k^{\mathrm{DN}}\langle e_k^{\mathrm{DN}}|u\rangle e_k^{\mathrm{DN}}(x) \\
&= \frac{1}{c}\lim_{x\to a} f(x) + \frac{1}{c}\sum_{k\in\mathbb{N}^*} \mu_k^{\mathrm{DN}}\langle e_k^{\mathrm{DN}}|u\rangle \lim_{x\to a} e_k^{\mathrm{DN}}(x) \\
&= \frac{1}{c}\lim_{x\to a} f(x).
\end{aligned}$$

For the BC of type (9.2), use Eq. 9.8. The BC related to $\lim_{x\to b}$ is satisfied because one can move the limit inside the series, due to the Hilbert-Schmidt property of $\mathcal{C}_{\mathrm{DN}}$. Then use the fact that

$$\lim_{x\to b} \frac{\mathrm{d}e_k^{\mathrm{DN}}}{\mathrm{d}x}(x) = 0,$$

and arrive at the desired BC:

$$\begin{aligned}
\lim_{x\to b} u'(x) &= \frac{1}{c}\lim_{x\to b} f'(x) + \frac{1}{c}\sum_{k\in\mathbb{N}^*} \mu_k^{\mathrm{DN}}\langle e_k^{\mathrm{DN}}|u\rangle \lim_{x\to b}\frac{\mathrm{d}e_k^{\mathrm{DN}}}{\mathrm{d}x}(x) \\
&= \frac{1}{c}\lim_{x\to b} f'(x).
\end{aligned}$$

• **The operator** $\mathcal{M}_{\mathrm{ND}}$**:** The proof is similar to the case of $\mathcal{M}_{\mathrm{DN}}$ where the roles of a and b are swapped.

10 Conclusion

Unlike the case for a differential operator, the extension to a general domain for an integral operator proved to be nontrivial. Since the BCs are encoded in the kernel of the integral operator in our construction, a change in the domain fundamentally changes the process by which our operators are constructed. They are derived from the series representation of the abstract convolution operator. The generalization to (a, b) becomes challenging because it requires an in-depth understanding of the abstract convolution operator and ensuing delicate integral manipulations. Our construction guarantees the rigorous verification of local BCs, which is its distinguishing feature. A potential application of the present work is the coupling of nonlocal and local problems where the freedom of solving a problem on an arbitrary domain is essential and employment of local BCs is advantageous.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Ethical Approval Not applicable.

Competing interests The authors declare no competing interests.

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