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ORIGINAL ARTICLES

Fractional Nonlocal Newton's Law of Motion and Emergence of Bagley-Torvik Equation

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Abstract
We construct a fractional nonlocal Newton's law by extending Suykens's nonlocal-in-time kinetic energy approach to its fractional counterpart by using a nonlocal Taylor series expansion. We derive the corresponding fractional order derivative Euler-Lagrange equations and we discuss some of their main consequences mainly for the case of a free particle and the case of an oscillator. Surprisingly, for the case of a time-dependent oscillator potential, the Bagley-Torvik equation used in viscoelasticity problems is obtained from nonlocal arguments. Some interesting features are obtained and discussed accordingly.

Keywords Fractional nonlocal-in-time kinetic energy · Fractional Taylor series · Fractional Newton's law of motion · Bagley-Torvik equation

Fractional calculus is an old branch of mathematics which has numerous applications in different branches of sciences and engineering [1]. It is devoted to the calculus and analysis of non-integer order derivatives. One important implication of fractional calculus concerns the study of nonconservative systems in mechanics. This is fruitfully studied by means of the fractional calculus of variations which is nowadays a subject of strong current research [2]. The validity of Noether's principle using the fractional calculus of variations is still obtained yet Noether's conservation law ceased to be valid in dissipative systems [3]. One important equation obtained in this new framework is the fractional Euler-Lagrange equation which usually takes a number of forms depending on the kind of fractional operators used and on the form of the action functional under consideration. It was proved in a large number of research studies that this equation has a large number of applications in physics and is helpful to model nonconservative mechanisms in a natural way (see [2] and references therein). It is noteworthy that fractional derivatives are nonlocal operators and describe asymptotic scaling systems and long-term memory effects in space and

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time [4]. The freedom in the definition of fractional derivatives allows us to introduce a large number of actions functional and therefore obtains a good number of fractional Euler-Lagrange equations. There are different definitions of fractional derivatives yet all of them are reduced to the standard derivative in the integer case, e.g., Riemann-Liouville, Caputo, and Erdelyi-Kober. They allow an ordinary interpolation among partial differential equations of very diverse properties [5]. On the other hand, nowadays, there is a large interest to deal with nonlocal dynamics since nonlocality may connect classical mechanics to quantum mechanics. In continuum classical mechanics, nonlocality appears in materials characterized by anisotropic nonlocality, elastic materials with couple-stress, strain-gradient in linear elasticity, micropolar elasticity, and elastic material surfaces, among others (see [6] and references therein). However, since nonlocality is one of the main characteristic aspects of quantum theory, its connection with classical mechanics is an interesting research topic. One motivating approach was introduced by Feynman in 1948 and based on the notion of nonlocal-in-time backward-forward coordinates (NTBFC) where the position differences are shifted with respect to each other. This idea was used in 1966 by Nelson in his seminal paper [7] which aims to derive the Schrödinger equation from classical mechanics and more specifically from a primary stochastic process in configuration space. The NTBFC motivated as well Nottale to construct his scale relativity which aims to combine quantum mechanics and relativity theory by introducing a nonlocal physical state of the coordinate system [8]. The NTBFC was also used in different frameworks, e.g., the extended Newtonian mechanics characterized by a nonlocal-in-time kinetic energy [9], nonconservative dynamical systems [10], and complexified Lagrangians dynamics [11], among others. In applied mathematics, nonlocal-in-time (NLT) is used in boundary value problems mainly in parabolic and hyperbolic equations where a grouping of initial and final values of the solution is taken into account [12–15]. An interesting class on NLT models is for us the one introduced by Suykens in [9] where the kinetic energy $T = m\dot{x}^2/2$ is replaced by its NLT counterpart $T = (m\dot{x}/2)(\dot{x}(t+\tau) + \dot{x}(t-\tau))/2$. Here, m is the mass of the particle and the value of τ is a tiny time parameter relative to the time scale of the model under study. This approach leads to a particular form of the Lagrangian holding higher-order derivatives and therefore to a hypothetical Newton's 2nd law of motion which contains in the limiting case advanced and retarded terms. These higher-order derivatives (HODS) are generated by the Taylor series expansions of terms $\dot{x}(t \pm \tau)$.

In this study, we propose a generalization of Suyken's approach for the derivation of a fractional nonlocal 2nd Newton law of motion. In fact, a generalization of Newton's law by replacing the classical derivative with Riemann-Liouville fractional derivative was addressed in [16]. Nevertheless, our approach differs completely since, in [16], a fractional Newton's law was obtained by means of the fractional virial theorem which is free from NLT aspect. We will show that the generalized fractional Newton's law obtained in this work can be applicable to a large number of nonlocal dynamical systems and will lead to many interesting fractional differential equations like the celebrated Bagley-Torvik equation used to model viscoelastic behavior of geological materials. Although Riemann-Liouville and Caputo fractional derivative operators are frequently used, in this work, we restrict ourselves to the fractional Taylor series expansion based on the Jumarie's modified Riemann-Liouville fractional derivative [17, 18]. In fact, Jumarie's fractional derivative is interesting for three main reasons: first, the derivative of a constant is 0 in contrast to the Riemann-Liouville fractional derivative which results in a time-dependent function; second, the function to be differentiated is not essentially differentiable in contrast to Caputo's fractional derivative; and third, a fractional Taylor series expansion was introduced by means of this novel fractional derivative.

In Suykens's approach, the kinetic energy $T = m\dot{x}^2/2$ for a particle of constant mass m and moving with velocity $\dot{x}(t)$ is replaced by its NLT counterpart $T = (m\dot{x}/2)(\dot{x}_n(t+\tau) + \dot{x}_n(t-\tau))/2, n \in N$. Performing first Taylor expansions of $x_n(t \pm \tau)$ as follows [9]:

$$x_n(t+\tau) \approx x(t) + \sum_{k=1}^{n} \frac{\tau^k}{k!} x^{(k)}(t) \equiv \sum_{k=0}^{n} \frac{\tau^k}{k!} x^{(k)}(t), \tag{1}$$

$$x_n(t-\tau) \approx x(t) + \sum_{k=1}^{n} \frac{(-\tau)^k}{k!} x^{(k)}(t) \equiv \sum_{k=0}^{n} \frac{(-\tau)^k}{k!} x^{(k)}(t), \tag{2}$$

and deriving both Eqs. (1) and (2) with respect to time, we find:

$$T_{\tau,n} = \underbrace{\frac{m\dot{x}^2}{2}}_{\text{local term}} + \underbrace{\frac{m\dot{x}}{4} \sum_{k=1}^{n} \frac{(1+(-1)^k)}{k!} \tau^k x^{(k+1)}(t)}_{\text{nonlocal terms}}.$$

Since the kinetic energy contains nonlocal terms, the Lagrangian of the system defined by $L_{\tau,n} = T_{\tau,n} - U$ is nonlocal and accordingly the Euler-Lagrange equations contain higher-order derivatives, which is the stationary solution to the action functional $S = \int_{t_0}^{t_f} L_{\tau,n} dt$ under the assumption that the action functional is subject to given boundary conditions $\delta x^{(j)}(t_0) = \delta x^{(j)}(t_f) = 0, j = 0, 1, 2, ..., N-1$:

$$\sum_{j=0}^{n+1} (-1)^j \frac{d^j}{dt^j} \frac{\partial L_{\tau,n}}{\partial q^{(j)}} = 0, \tag{4}$$

where $q_i(t)$ are independent variables such that $q_i = \dot{q}_{i-1}, i = 1, 2, ..., N-1$, $q_0 = x$, and $q = x_0$. This gives after some algebra:

$$m q + m \sum_{k=1}^{n} \frac{\left(1-(-1)^{k+1}\right)\left(1+(-1)^k\right)}{4k!} \tau^k q^{(k+2)} = -\frac{\partial U}{\partial q} = F, \tag{5}$$

U is the kinetic potential and F is the force. These are the main points introduced by Suykens in his nonlocal approach which was proved to be successful to explain a number of quantum discrete phenomena. In this section, a fractional generalization of this approach will be addressed. Let us recall first that if f is a continuous function defined on R, the Riemann-Liouville fractional derivative is defined by [5]:

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{6}$$

where $0 \le n-1 \le \alpha < n$, $n \in \mathbb{N}^*$. Jumarie's modified Riemann-Liouville fractional derivative is defined by:

$$f^{(\alpha)}(t) \equiv \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau) - f(0)}{(t-\tau)^{\alpha+1-n}} d\tau. \tag{7}$$

Since ${}_0D_t^{\alpha} t^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{n-\alpha}, \alpha \in \mathbb{R}^+$, then $\forall t \in \mathbb{R}^{*+}$:

$$f^{(\alpha)}(t) = {}_0D_t^{\alpha} f(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0). \tag{8}$$

Given an open interval $I \subset \mathbb{R}$ and $x \in C^n(I)$, the generalized Taylor series are defined as follows:

$$x_{n,\alpha}(t+\tau) = \sum_{k=0}^{n} \frac{\tau^k}{k!} x^{(k)}(t) + \sum_{k=1}^{\infty} \frac{\tau^{k(\alpha-n)+n}}{\Gamma(k(\alpha-n)+n+1)} x^{(k(\alpha-n)+n)}(t), \tag{9}$$

$$x_{n,\alpha}(t-\tau) = \sum_{k=0}^{n} \frac{(-1)^k \tau^k}{k!} x^{(k)}(t) + \sum_{k=1}^{\infty} \frac{(-1)^{k(\alpha-n)+n} \tau^{k(\alpha-n)+n}}{\Gamma(k(\alpha-n)+n+1)} x^{(k(\alpha-n)+n)}(t), \tag{10}$$

$\forall \tau \in I,\ t \in \mathbb{R}^+$. The following fractional Leibniz derivative rule $(fg)^{(\alpha)} = f^{(\alpha)} g + f g^{(\alpha)}$ holds as well and besides the following integration by parts rule holds: $\int_a^b f^{(\alpha)}(t) g(t) (dt)^{\alpha} = \alpha!$ $[f(t)g(t)]_a^b - \int_a^b f(t) g^{(\alpha)}(t) (dt)^{\alpha}$ where (f, g) are continuous functions on $[a, b]$ and the $(dt)^{\alpha}$ integral of functions $f(t)$ and $f^{(\alpha)}(t)$ are respectively defined such that $\int_0^t f(t)(dt)^{\alpha} = \alpha \int_0^t$ $(t-\tau)^{\alpha-1} f(t) dt$ and $\int_0^t f^{(\alpha)}(t)(dt)^{\alpha} = \Gamma(\alpha+1)(f(x) - f(0))$ [5].

In order to extend fractionally Suyken's approach based on Jumarie's modified Riemann-Liouville fractional derivative, we define the fractional NLT kinetic energy of particle with constant mass m by:

$$T_{n,\alpha} = \frac{m}{2} \dot{x} \frac{\dot{x}_{n,\alpha}(t+\tau) + \dot{x}_{n,\alpha}(t-\tau)}{2}, \tag{11}$$

where:

$$\dot{x}_{n,\alpha}(t+\tau) \approx \sum_{k=0}^{n} \frac{\tau^k}{k!} x^{(k+1)}(t) + \sum_{k=1}^{\infty} \frac{\tau^{k(\alpha-n)+n}}{\Gamma(k(\alpha-n)+n+1)} x^{(k(\alpha-n)+n+1)}(t), \tag{12}$$

and:

$$\dot{x}_{n,\alpha}(t-\tau) \approx \sum_{k=0}^{n} \frac{(-1)^k \tau^k}{k!} x^{(k+1)}(t) + \sum_{k=1}^{\infty} \frac{(-1)^{k(\alpha-n)+n} \tau^{k(\alpha-n)+n}}{\Gamma(k(\alpha-n)+n+1)} x^{(k(\alpha-n)+n+1)}(t). \tag{13}$$

The associated fractional NLT kinetic energy is therefore given by:

$$T_{n,\alpha} = \frac{m}{4} \dot{x} \left(\sum_{k=0}^{n} \frac{\left(1+(-1)^k\right) \tau^k}{k!} x^{(k+1)}(t) + \sum_{k=1}^{\infty} \frac{\left(1+(-1)^{k(\alpha-n)+n}\right) \tau^{k(\alpha-n)+n}}{\Gamma(k(\alpha-n)+n+1)} x^{(k(\alpha-n)+n+1)}(t) \right). \tag{14}$$

The proof is obtained directly after replacing Eqs. (12) and (13) into Eq. (11).

Remark 1: *For* $n = 1$ *and* $\alpha = 1$, *Eq.* (14) *is reduced to*:

$$T_{1,1} = \frac{m}{4} \dot{x} \sum_{k=0}^{1} \frac{\left(1+(-1)^k\right) \tau^k}{k!} x^{(k+1)}(t) = \frac{m}{2} \dot{x}^2,,$$

which is the standard kinetic energy.

Remark 2:

The kinetic energy $T_{n,\,\alpha}$ *contains higher-order fractional derivative terms and therefore the fractional Lagrangian of the theory* $L_{n,\,\alpha} = T_{n,\,\alpha} - V$ *takes the general form*:

$$L_{n,\alpha} = \frac{m}{4}\dot{x}\left(\underbrace{2\dot{x} + \tau^2 x^{(3)} + \ldots}_{\text{integer derivatives}} + \underbrace{\frac{(1+(-1)^{\alpha})\tau^{\alpha}}{\Gamma(\alpha+1)}x^{(\alpha+1)} + \frac{(1+(-1)^{2\alpha-n})\tau^{2\alpha-n}}{\Gamma(2\alpha-n+1)}x^{(2\alpha-n+1)} + \ldots}_{\text{Jumarie's Riemann-Liouville fractional derivatives}}\right) - V\,.$$

It is then obvious that for $0 \le n-1 \le \alpha < n$, $n \in \mathbb{N}^*$, the fractional Lagrangian is the sum of two parts: the integer part which holds merely integer derivatives and the fractional part which just holds Jumarie's Riemann-Liouville fractional derivatives. Since τ is too small, the fractional Lagrangian of the theory may be approximated by:

$$L_{1,\alpha} = \frac{m}{2}\dot{x}\left(\dot{x} + \frac{(1+(-1)^{\alpha})\tau^{\alpha}}{2\Gamma(\alpha+1)}x^{(\alpha+1)}\right) - V. \tag{15}$$

The Lagrangian of the theory contains therefore classical and fractional derivatives consequently. Let $S_{1,\alpha} = \int_a^b L_{1,\alpha}\left(t, x(t), \dot{x}(t), x^{(\alpha+1)}(t)\right)(dt)^{\alpha}$ be the action functional with $0 \le \alpha < 1$, $x \in C^1[a,b]$, $L_{1,\,\alpha} \in C^2([a,b] \times \mathbb{R}^2; \mathbb{R})$ and subject to the boundary conditions $x(a) = x_a$ and $x(b) = x_b$. If $S_{1,\,\alpha}$ has an extremum and x (assumed to be admissible) is an extremizer (assumed to be admissible), then $x$satisfies the following fractional Euler-Lagrange equation:

$$\frac{\partial L_{1,\alpha}}{\partial x} - \frac{d}{dt}\left(\frac{\partial L_{1,\alpha}}{\partial \dot{x}}\right) - \frac{d^{\alpha+1}}{dt^{\alpha+1}}\left(\frac{\partial L_{1,\alpha}}{\partial x^{(\alpha+1)}}\right) = 0. \tag{16}$$

For the proof, the readers are referred to [19]. From $L_{1,\,\alpha}$, one has:

$$\frac{\partial L_{1,\alpha}}{\partial x} = -\frac{\partial V}{\partial x} = F_{\alpha}, \tag{17}$$

$$\frac{\partial L_{1,\alpha}}{\partial \dot{x}} = m\dot{x} + \frac{m}{2}\frac{(1+(-1)^{\alpha})\tau^{\alpha}}{2\Gamma(\alpha+1)}x^{(\alpha+1)}, \tag{18}$$

$$\frac{d}{dt}\left(\frac{\partial L_{1,\alpha}}{\partial \dot{x}}\right) = mx + \frac{m}{2}\frac{(1+(-1)^{\alpha})\tau^{\alpha}}{2\Gamma(\alpha+1)}x^{(\alpha+2)}, \tag{19}$$

$$\frac{\partial L_{1,\alpha}}{\partial x^{(\alpha+1)}} = \frac{m}{2}\frac{(1+(-1)^{\alpha})\tau^{\alpha}}{2\Gamma(\alpha+1)}\dot{x}, \tag{20}$$

$$\frac{d^{\alpha+1}}{dt^{\alpha+1}}\left(\frac{\partial L_{1,\alpha}}{\partial x^{(\alpha+1)}}\right) = \frac{m}{2}\frac{(1+(-1)^{\alpha})\tau^{\alpha}}{2\Gamma(\alpha+1)}\frac{d^{\alpha+1}\dot{x}}{dt^{\alpha+1}} \equiv \frac{m}{2}\frac{(1+(-1)^{\alpha})\tau^{\alpha}}{2\Gamma(\alpha+1)}x^{(\alpha+2)}. \tag{21}$$

Replacing Eqs. (16)–(21) into Eq. (16), we obtain the fractional Newton's 2nd law of motion:

$$F_{\alpha} = mq + \frac{m}{2}\frac{(1+(-1)^{\alpha})\tau^{\alpha}}{\Gamma(\alpha+1)}q^{(\alpha+2)}. \tag{22}$$

Remark 3:

For $\alpha = 1$, Eq. (22) is reduced to the standard Newton's 2nd law of motion $F = mq$. Besides, for n even and $\alpha = 1/2$, all fractional derivatives in the fractional Newton 2nd law of motion cancel and only the integer derivative term remains. In fact for n, even we can rewrite Eq. (14) as:

$$T_{n,\alpha} = \frac{m}{4}\dot{x}\left(\sum_{k=0}^{n/2}\frac{\left(1+(-1)^{2k}\right)\tau^{2k}}{(2k)!}x^{(2k+1)}(t) + \sum_{k=1}^{\infty}\frac{\left(1+(-1)^{2k\left(\alpha-\frac{n}{2}\right)+\frac{n}{2}}\right)\tau^{2k(\alpha-n)+n}}{\Gamma\left(2k\left(\alpha-\frac{n}{2}\right)+\frac{n}{2}+1\right)}x^{\left(2k\left(\alpha-\frac{n}{2}\right)+\frac{n}{2}+1\right)}(t)\right),$$

and for $\tau< <1$, the fractional Lagrangian is approximated:

$$L_{1,\alpha} = \frac{m}{2}\dot{x}\left(\dot{x} + \frac{\left(1+(-1)^{2\alpha}\right)\tau^{2\alpha}}{2\Gamma(2\alpha+1)}x^{(2\alpha+1)}\right) - V.$$

The action of the theory is $S_{1,\alpha} = \int_a^b L_{1,\alpha}\left(t, q(t), \dot{q}(t), q^{(2\alpha+1)}(t)\right)(dt)^{\alpha}$ and the resulting fractional Euler-Lagrange equation is now:

$$\frac{\partial L_{1,\alpha}}{\partial q} - \frac{d}{dt}\left(\frac{\partial L_{1,\alpha}}{\partial \dot{q}}\right) - \frac{d^{2\alpha+1}}{dt^{2\alpha+1}}\left(\frac{\partial L_{1,\alpha}}{\partial q^{(\alpha+1)}}\right) = 0.$$

Therefore, after evaluation of the corresponding partial derivatives, we find effortlessly:

$$F = mq + \frac{m}{2}\frac{\left(1+(-1)^{2\alpha}\right)\tau^{2\alpha}}{\Gamma(2\alpha+1)}q^{(2\alpha+2)}.$$

For $\alpha = 1/2$, we observe that all fractional derivatives cancel and only the integer derivative term remains.

1 Illustrations

1-(Complexified Fractional Differential Equations) As a first illustration, we consider the case of a free particle, i.e., $F_{\alpha} = 0$ and we set $\alpha = 1/2$. Equation (22) is accordingly reduced to the fractional differential equation:

$$x + (1+i)\sqrt{\frac{\tau}{\pi}}\dot{x}^{(5/2)} = 0. \tag{23}$$

Assuming the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$, and $x(0) = 0$, the solution is given by [20]:

$$x(t) = E_{7/2}\left(\sqrt{\frac{\pi}{\tau}}\frac{i-1}{2}t^{7/2}\right), \tag{24}$$

where $E_{\alpha}(at^{\alpha}) = 1 + \frac{a}{\Gamma(1+\alpha)}t^{\alpha} + \frac{a^2}{\Gamma(1+2\alpha)}t^{2\alpha} + \ldots$ is the 1-parameter Mittag-Leffler function [21].

This solution is complexified and this is expected since the Lagrangian for $0 \leq \alpha < 1$ is complexified. Equation (18) differs from the standard solution which gives $x(t) = t$ and from Suykens's approach which gives $x(t) = c_1 t + c_2 + \sum_l a_{1,l}\cos(\omega_l t) + \sum_l a_{2,l}\sin(\omega_l t)$ where c_1, c_2, $a_{1,l}$, $a_{2,l}$ are real coefficients.

It is notable that complexified dynamical systems and complexified classical mechanics which are characterized by a phase space spanned by complex canonical variables are discussed intensely in literature and the generation of complexified differential equations in general is somewhat motivating due to their important contributions in non-Hermitian quantum mechanical systems, and complexified Hamiltonian systems, among others [22]. It is therefore interesting to obtain fractional complexified classical mechanics since fractional classical mechanics was introduced in literature as a classical counterpart of fractional quantum mechanics. This is an open problem that deserves a future study.

2-(Bagley-Torvik Equation) As a second illustration, we consider the case $F_\alpha = -kx + f(t)$, $k \in R$ where $f(t)$ is a time-dependent function and we set again $\alpha = 1/2$. In the standard case, this equation describes an oscillator with a time-dependent potential. In fact, the study of oscillators with a time-dependent potential has attracted much interest in literature mainly in nonlinear optics and quantum mechanics [23]. Equation (22) results into the following fractional differential equation (after setting $k/m = 1$ for convenience):

$$x + (1+i)\sqrt{\frac{\tau}{\pi}}\dot{x}^{(3/2)} + x = f(t). \tag{25}$$

Surprisingly, Eq. (19) is the celebrated Bagley-Torvik equation used in the analysis of viscoelastically damped structures [24]. In most of literatures devoted to study and analyze this equation, the Riemann-Liouville or Caputo fractional derivatives were used and not the Jumarie's modified Riemann-Liouville fractional derivatives. The main differences are related to the values of the initial conditions used, i.e., the value $q(0) = 0$ leads to a Bagley-Torvik equation which traditionally is formulated with Riemann-Liouville fractional derivative rather than Caputo's fractional derivative, and besides in our argument, the fractional damping term is complexified. However, the complexified terms may be omitted if we replace the real nonlocal time term τ by an imaginary time, e.g., $\tau \to -2i\tau$ which reduces Eq. (19) to the fractional Bagley-Torvik equation: $x + 2\sqrt{\tau/\pi}\dot{x}^{(3/2)} + x = f(t)$ which is characterized usually by initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, and $0 \leq t \leq T$. The use of imaginary time is well-known in literature and is known as the imaginary time propagation method which is viewed as the power method used in numerical linear algebra [25]. It is notable that for $x(0) = 0$ and $\dot{x}(0) = 0$, the solution of Bagley-Torvik equation coincides for both the Riemann-Liouville fractional derivative and Caputo's fractional derivative. The solution of this equation was given in a large number of literatures using different computational and analytical methods [26–28]. Nevertheless, the substitution of fractional Riemann-Liouville or Caputo fractional derivatives by Jumarie's fractional derivative will not change the solution considerably. The analysis of $x + 2\sqrt{\tau/\pi}\dot{x}^{(3/2)} + x = f(t)$ with Jumarie's fractional derivative is motivating and it deserves to be explored in a future work. For $f(t) = 0$, the solution of Eq. (19) for $x(0) = 1$ is approximated by [29]:

$$\begin{aligned} x(t) \approx 1 - \frac{4}{3\sqrt{\pi}} + \frac{2\sqrt{\tau}}{\pi}(1+i) + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}{}_1\Psi_1\left[\left| \begin{matrix}(n+1,1)\\ \left(2n+1,\frac{1}{2}\right)\end{matrix} \right| -(1+i)\sqrt{\frac{\tau}{\pi}}t^{\frac{1}{2}}\right] \\ -(1+i)\sqrt{\frac{\tau}{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n+\frac{1}{2}}{}_1\Psi_1\left[\left| \begin{matrix}(n+1,1)\\ \left(2n+\frac{3}{2},\frac{1}{2}\right)\end{matrix} \right| -(1+i)\sqrt{\frac{\tau}{\pi}}t^{\frac{1}{2}}\right], \end{aligned} \tag{26}$$

where:

$$\Psi_1\left[\left|\begin{matrix}(n+1,1)\\ \left(2n+1,\dfrac{1}{2}\right)\end{matrix}\right|-(1+i)\sqrt{\frac{\tau}{\pi}}t^{\frac{1}{2}}\right]=\sum_{j=0}^{\infty}\frac{\Gamma(n+j+1)}{\Gamma\left(2n+1+\dfrac{3+j}{2}\right)j!}\left(-(1+i)\sqrt{\frac{\tau}{\pi}}t^{\frac{1}{2}}\right)^j, \tag{27}$$

and:

$$\Psi_1\left[\left|\begin{matrix}(n+1,1)\\ \left(2n+\dfrac{3}{2},\dfrac{1}{2}\right)\end{matrix}\right|-(1+i)\sqrt{\frac{\tau}{\pi}}t^{\frac{3}{2}}\right]=\sum_{j=0}^{\infty}\frac{\Gamma(n+j+1)}{\Gamma\left(2n+\dfrac{3+j}{2}\right)j!}\left(-(1+i)\sqrt{\frac{\tau}{\pi}}t^{\frac{1}{2}}\right)^j, \tag{28}$$

$\Psi_1[\cdot]$ being the Wright function. Yet, a numerical package is required to give an accurate solution; nevertheless, the lesson we learn to this stage is that fractional formulation of Suyken's NLT kinetic energy approach is motivating since the celebrated Bagley-Torvik equation may be obtained from a totally different argument used in the seminal paper [7]. Besides, the use of imaginary time is attractive since it can be helpful at the classical level. It is noteworthy that Nelson used as well the imaginary time in his construction of quantum fields from Markov fields [30]. Even in the complexified case, Eq. (19) is motivating since the extension of Noether's theorem in the complex domain was developed for the case of second-order differential equation [31–34]. It is therefore interesting to investigate about solutions of Eq. (19) in the future.

To conclude, we have developed a fractional version of Suyken's nonlocal kinetic energy approach based on Jumarie's fractional derivative which is a modified version of the Riemann-Liouville fractional derivative since the fractional Taylor series may be used safely. We restricted our analysis for $\alpha = 1/2$ and to τ^{α}-order. We have derived the fractional Newton's 2nd law of motion F_{α}, and we have discussed two independent cases: $F_{\alpha} = 0$ and $F_{\alpha} = -kx + f(t)$. For the first case, we have obtained a complexified fractional differential equation where the solution depends on the 1-parameter Mittag-Leffler function. Such a complexified solution is motivating since differential equations of integer orders in the complex domain are attractive and may have interesting consequences in classical mechanics, e.g., the dynamics of the complexified simple pendulum. One therefore expects that complexified fractional differential equations may also have interesting features that deserve to be deciphered. For the second case, surprisingly, a complexified version of the celebrated Bagley-Torvik equation is obtained. The complex damping term may be removed if the real-time parameter τ is replaced by an imaginary time term, a technique largely used in quantum theory and numerical linear algebra. Different values of α may be chosen and a number of fractional differential equations may be obtained and analyzed accordingly. We hypothesize that the fractional model constructed here which is based on Suyken's nonlocal kinetic energy approach is motivating and deserves future considerations. A number of points are required in the future, e.g., the fractional Hamiltonian of the theory, the study of Bagley-Torvik equation based on Jumarie's fractional derivatives, and the analysis of complexified fractional differential equations, among others.

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