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ORIGINAL PAPER

On the Gumbel–Barnett extended Celebioglu–Cuadras copula

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Abstract

A copula is a multivariate probability distribution function used to describe the dependence structure between random variables, independent of their marginal distributions. Among the numerous proposed copulas, renewed interest has recently been shown in the exponential-type copula, including the famous Gumbel–Barnett and Celebioglu–Cuadras copulas. The key reasons for this are their applicability potential, exploitable dependence qualities, and simplicity. In some works, rare attempts have been made to make a parametric compromise between these two copulas, but without searching for the optimal sets of admissible values for the parameters. In this article, we fill this gap by also considering an extended version of the Celebioglu–Cuadras copula, recently introduced in the literature. We therefore introduce a three-parameter copula that integrates both the Celebioglu–Cuadras and Gumbel–Barnett copulas. The main challenge is to analytically determine the widest range of acceptable values for the relevant parameters. Two new results are demonstrated in this regard, one of which significantly improves an existing theorem. The secondary functions related to this copula are exhibited. Some figures are produced to illustrate the validity and versatility of the proposal. The main copula properties are discussed, including symmetry, series expansions, analytical bounds, bivariate distribution generation, and concordance ordering. The theory is supported by a numerical analysis, which also demonstrates how the parameters under consideration have a favorable impact on the dependence structure.

Keywords Exponential-type copula · Symmetry · Correlation · Statistical modeling

Mathematics Subject Classification 60E15 · 62H99

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1 Introduction

Copulas are a powerful tool in probability and statistics that allow us to model the dependence structure among a set of random variables. Unlike traditional correlation-based methods, the copula method allows us to separate the joint distribution of random variables into two components: the marginal distributions of each random variable and the dependence structure between them. This separation enables us to capture the complex relationships between the random variables that cannot be adequately described by simple correlation measures. The function modeling the extracted dependence structure is called a copula. Some commonly used copulas include the Gaussian copula, which assumes that the variables are normally distributed and mainly have a linear relationship, and the Archimedean copula, which allows for more flexible dependence structures. Other copulas include the Student-T copula, the Clayton copula, and the Gumbel copula.

Historically, the concept of copulas originated in the early 1950s (see Sklar, 1959, 1973), but it was not until the 1980s that they gained wider recognition in the field of probability and statistics. Since then, copulas have become an essential tool in various areas, including finance, actuarial science, and multivariate analysis. They are particularly useful in modeling the dependence structure of financial instruments, risk management, portfolio optimization, and option pricing. The popularity of copulas has grown in recent years due to their flexibility and ability to capture complex dependence structures emerging from modern data analysis challenges. With a variety of copula functions available, each with its own unique properties and applications, researchers and practitioners have a broad range of tools at their disposal.

Understanding copulas and their applications is becoming increasingly important as we continue to face complex data and modeling challenges in various fields. This article provides theoretical contributions to the topic. Before describing these contributions, some precise facts must be recalled, beginning with the mathematical definition of a copula in the absolutely continuous bivariate case.

Definition 1 In the context of the absolutely continuous bivariate case, we define a copula as a differentiable function defined on $[0, 1]^2$ and denoted as $C(x, y)$, $(x, y) \in [0, 1]^2$, satisfying the following conditions:

Boundary (B) condition: For any $(x, y) \in [0, 1]^2$, we have

$$C(x, 0) = C(0, y) = 0, \quad C(x, 1) = x, \quad C(1, y) = y.$$

Positive derivative (PD) condition: For any $(x, y) \in [0, 1]^2$, we have

$$\frac{\partial^2}{\partial x \partial y} C(x, y) \geq 0.$$

Overall, in this article, the notion of copula will be understood in the absolutely continuous bivariate case. Most of the theoretical and practical information in this case is contained in Nelsen (2006), Durante and Sempi (2016), Joe (2015) and Cuadras (2006), modern applications are given in Safari-Katesari et al. (2020), Roberts and

Zewotir (2020), Tavakol et al. (2020) and Shiau and Lien (2021), and very recent theoretical and practical developments are described in Susam (2020a, b), Chesneau (2021a, b, 2022), Michimae and Emura (2022), Shih et al. (2022), El Ktaibi et al. (2022) and Yeh et al. (2023).

The purposes of this article are centered around two specific exponential-type copulas: the Gumbel–Barnett (GB) and Celebioglu–Cuadras (CC) copulas. A review of them is required to proceed. The GB copula is first shown as

$$C(x, y; a) = xy \exp\left[-a \log(x) \log(y)\right], \quad (x, y) \in [0, 1]^2, \tag{1}$$

or equivalently

$$C(x, y; a) = xy \left\{\exp\left[\log(y)\right]\right\}^{-a \log(x)} = xy^{1-a \log(x)}, \quad (x, y) \in [0, 1]^2,$$

with $a \in [0, 1]$. The GB copula is unique in that it is one of the most straightforward Archimedean copulas, covering the independence copula by taking $a = 0$, and is well suited for modeling various negative-type of dependence structures. However, it has no tail dependence. As an Archimedean copula, simplified formulas exist to determine various crucial quantities (tau of Kendal, rho of Spearman, tail dependence parameters, etc.). For more detail on these aspects, we refer to Nelsen (2006), Zhang et al. (2013) and Kularatne et al. (2021). On a similar but simple functional basis, the authors in Celebioglu (1997) and Cuadras (2009) have conjointly elaborated a modified version of the GB copula demonstrating a broader range of dependence. The CC copula is thus created. It is specified as

$$C(x, y; b) = xy \exp\left[b(1-x)(1-y)\right], \quad (x, y) \in [0, 1]^2, \tag{2}$$

with $b \in [-1, 1]$. Thus, the expression of the CC copula is quite simple. It also covers the independence copula, has controllable qualities, and is versatile in the sense of the dependence structure, including negative and positive-type dependences. It was used in recent studies, including Zhang et al. (2013), Bekrizadeh et al. (2017), Cuadras et al. (2020), Manstavičius and Bagdonas (2022), Diaz and Cuadras (2022) and Chesneau (2023a). In particular, several extended CC copulas were proposed in Chesneau (2023a) with the addition of diverse shape parameters.

On the other hand, employing the notion of weighted geometric mean, Zhang et al. (2013) proposed a compromise (or tradeoff) version of the GB and CC copulas of the following form:

$$C(x, y; a, b) = xy \exp\left[-a \log(x) \log(y) + b(1-x)(1-y)\right], \quad (x, y) \in [0, 1]^2. \tag{3}$$

In Zhang et al. (2013, Theorem 2), it is proved that, under the following precise parameter configuration: $a = \alpha\beta$ and $b = \alpha(1-\beta)$, with $\alpha \in [0, 1]$ and $\beta \in [0, 1]$, $C(x, y; a, b)$ is a valid copula. This implies that $a \in [0, 1]$ and $b \in [0, 1]$; some strict parameter conditions are here: b cannot be negative, and a and b are strongly related by the equation: $a + b = \alpha \in [0, 1]$. Despite these technical restrictions, this result is of great interest, because it shows that, with the use of only two parameters, a copula combining the diverse functionalities of the GB and CC copulas can be constructed.

In this article, we revisit this result and extend it by taking into account the more generic parametric form as follows:

$$C(x, y; a, b, c) = xy \exp\left[-a \log(x) \log(y) + b(1 - x^c)(1 - y^c)\right], \quad (x, y) \in [0, 1]^2, \quad (4)$$

where a and b are as arbitrary as possible, and c is a newly added shape parameter. The idea behind this shape parameter is to allow for greater flexibility in modeling the dependence structure, accommodating various degrees of tail dependence, and capturing more complex relationships between random variables. Moreover, its presence can enhance the possible values of a and b, beyond the ranges of values determined in Zhang et al. (2013); the interactions of the parameters can be quite profitable. Of course, by taking $c = 1$, the copula in Eq. (4) is reduced to the one in Eq. (3) with general a and b. Otherwise, it operates as a compromise between the GB copula and the extended CC copula as presented in Chesneau (2023a). Thus, the idea of the general copula in Eq. (4) and the three parameters involved is to offer a high degree of adaptability in dependence modeling, beyond those of the aforementioned reference.

Based on Eq. (4), a summary of the findings is as follows: In the first part, we determine wide admissible sets of values for a, b, and c making $C(x, y; a, b, c)$ a valid copula beyond the parameter restrictions considered in Zhang et al. (2013, Theorem 2). Two new results in this regard are presented: one assuming that $c \in [0, 1]$ only and with a direct proof using a general result of Liebscher (2008), and the other considering another approach, with possible negative values for c, but at the price of a complex and technical proof. This second result generalized Zhang et al. (2013, Theorem 2) on all the possible parameter values. In the second part, we determine the main functions related to the new copula and illustrate some of them with graphics. Finally, its properties are discussed, including symmetry, correlations, series expansions, analytical bounds, bivariate distribution generation, and concordance ordering. Also, a computational work reflects its interesting dependence properties; the rho of Spearman is considered a benchmark for this aim. Overall, we emphasize the role of the parameters a, b, and c, and the gain of the proposed copula from a dependence modeling perspective.

The next sections are as follows: Sect. 2 contains all the results of the article, along with the detailed proofs. Section 3 presents the main functions and displays some graphics of them. Section 4 focuses on the important properties of the proposed copula. A summary of the findings is given in Sect. 5. A minor result generalizing Zhang et al. (2013, Lemma 3) is also provided in Appendix, being of independent interest.

2 Main results

The main results of the article are in the form of two complementary propositions based on Eq. (4) and its mathematical validity with respect to the parameters, but with completely different approaches in terms of proofs.

The following proposition focuses on the case $c \in [0, 1]$ which can be derived from a general result established in Liebscher (2008); it is based on the product of two copulas composed of power functions with the addition of a shape parameter.

Proposition 2.1 *Let $(a, b, c) \in \mathbb{R}^3$. We consider the following bivariate function:*

$$C(x, y; a, b, c) = xy \exp\left[-a \log(x) \log(y) + b(1 - x^c)(1 - y^c)\right], \quad (x, y) \in [0, 1]^2.$$

Then, $C(x, y; a, b, c)$ is a copula for $c \in [0, 1]$, $a \in [0, (1 - c)^2]$ and $b \in [-1, 1]$.

Proof Let $C_{GB}(x, y; \alpha)$ be the GB copula as defined in Eq. (1) with $a = \alpha$, and $C_{CC}(x, y; \beta)$ be the CC copula as defined in Eq. (2) with $b = \beta$. For any $(x, y) \in [0, 1]^2$ and $c \in [0, 1]$, let us set

$$C_*(x, y; \alpha, \beta, c) = C_{GB}(x^{1-c}, y^{1-c}; \alpha) C_{CC}(x^c, y^c; \beta).$$

Then, according to Liebscher (2008, Theorem 2.1), since $C_{GB}(x, y; \alpha)$ is a copula for $\alpha \in [0, 1]$ and $C_{CC}(x, y; \beta)$ is a copula for $\beta \in [-1, 1]$, for any $c \in [0, 1]$, $C_*(x, y; \alpha, \beta, c)$ is a valid copula under these parameter conditions. Now, we remark that

$$\begin{aligned} C_*(x, y; \alpha, \beta, c) &= x^{1-c} y^{1-c} \exp\left\{-\alpha[\log(x^{1-c})][\log(y^{1-c})]\right\} \\ &\quad \times x^c y^c \exp\left[\beta(1 - x^c)(1 - y^c)\right] \\ &= xy \exp\left[-\alpha(1 - c)^2 \log(x) \log(y) + \beta(1 - x^c)(1 - y^c)\right] \\ &= C(x, y; a, b, c), \end{aligned}$$

with $a = \alpha(1 - c)^2 \in [0, (1 - c)^2]$ and $b = \beta \in [-1, 1]$, and no interdependence between a and b. The result is established. □

Proposition 2.1 must be viewed as the first step in our findings; it is of interest, since it demonstrates that an original compromise between the GB copula and the extended CC copula is possible. Furthermore, the parameter b can be chosen independently of a and c. However, the parameters a and c have a strong connection; if $c = 1$, then $a = 0$, which totally removes the product logarithmic term, and if $c = 0$, then $a \in [0, 1]$, but the product polynomial term is removed.

However, since Zhang et al. (2013, Theorem 2) focus on the case $c = 1$ with $a \in [0, 1]$, Proposition 2.1 does not contain this theorem; the used approach of Liebscher (2008) offers another viewpoint than the weighted geometric mean method.

With the same copula, the proposition below examines other parameter configurations. In particular, negative values for c are allowed, the same for $c > 1$, and we can have $c = 1$ with $a \neq 0$. The proof is totally self-contained; it uses the setting in Definition 1, differentiation techniques, well-chosen arrangements, and mathematical inequalities. We will show later that it contains Proposition 2.1, which is not apparent at first glance.

Proposition 2.2 *Let $(a, b, c) \in \mathbb{R}^3$. We consider the following bivariate function:*

$$C(x, y; a, b, c) = xy \exp\left[-a \log(x) \log(y) + b(1 - x^c)(1 - y^c)\right], \quad (x, y) \in [0, 1]^2.$$

Then, $C(x, y; a, b, c)$ is a copula for $a \geq 0$ and under one of the three distinct parameter configurations according to the signs of b and c:

PC1: $b \geq 0$, $c \geq 0$ and $1 - bc \geq a$,
PC2: $b \leq 0$, $c \in [-1, 0]$ and $1 + bc^2 \geq a$,
PC3: $b \in [-1, 0]$, $c \in [0, 1]$ and $(1 + bc)^2 \geq a$.

Proof Let us use Definition 1, and the B and PD conditions in particular to demonstrate that $C(x, y; a, b, c)$ is a valid copula.

Proof of the B condition: Under **PC1**, **PC2** or **PC3**, a limit study gives

$$\lim_{y \to 0} \left[-a \log(x) \log(y) + b(1 - x^c)(1 - y^c)\right] = -\infty,$$

implying that

$$C(x, 0; a, b, c) = \lim_{y \to 0} xy \exp\left[-a \log(x) \log(y) + b(1 - x^c)(1 - y^c)\right] = 0,$$

and similarly, we have $C(0, y; a, b, c) = 0$. On the other hand, it is clear that

$$\begin{aligned} C(x, 1; a, b, c) &= x \times 1 \times \exp\left[-a \log(x) \log(1) + b(1 - x^c)(1 - 1^c)\right] \\ &= x \times \exp(0) = x, \end{aligned}$$

and, in a similar way, we get $C(1, y; a, b, c) = y$. The B condition is thus demonstrated.
Proof of the PD condition: Using appropriate differentiation methods and factoring in a way that allows us to draw conclusions, we have

$$\begin{aligned} &\frac{\partial^2}{\partial x \partial y} C(x, y; a, b, c) = \exp\left[-a \log(x) \log(y) + b(1 - x^c)(1 - y^c)\right] \\ &\quad \times \Big\{ - a \log(y) \left[bc(x^c - 1)y^c + 1\right] + a \log(x) \left[a \log(y) - bcx^c y^c + bcx^c - 1\right] \\ &\quad - a + b^2c^2x^{2c}y^{2c} - b^2c^2x^{2c}y^c - b^2c^2x^c y^{2c} + b^2c^2x^c y^c + bc^2x^c y^c + 2bcx^c y^c \\ &\quad - bcx^c - bcy^c + 1 \Big\} \\ &= \exp\left[-a \log(x) \log(y) + b(1 - x^c)(1 - y^c)\right] \\ &\quad \times \left[T_1(x, y) + T_2(x, y) + T_3(x, y) + T_4(x, y)\right], \end{aligned}$$

where

$$T_1(x, y) = -a\log(y)\left[1 - bc(1 - x^c)y^c\right], \quad T_2(x, y) = a^2\log(x)\log(y),$$
$$T_3(x, y) = -a\log(x)\left[1 - bcx^c(1 - y^c)\right] \ (= T_1(y, x))$$

and, after some developments not detailed here, we have

$$T_4(x, y) = b^2c^2x^cy^c(1 - x^c)(1 - y^c) \\ +bc(c + 1)x^cy^c + bc(1 - x^c)(1 - y^c) + 1 - a - bc.$$

Therefore, to establish the PD condition, a possible way is to prove that, under **PC1**, **PC2**, and **PC3**, for any $(x, y) \in [0, 1]^2$, $T_1(x, y)$, $T_2(x, y)$, $T_3(x, y)$ and $T_4(x, y)$ are non-negative.

Proof of $T_1(x, y) \geq 0$: We recall that $a \geq 0$. Let us distinguish **PC1**, **PC2**, and **PC3**.

Under **PC1**: Since $b \geq 0$, $c \geq 0$ and $1 - bc \geq a \geq 0$, it is clear that $bc \in [0, 1]$. For any $(x, y) \in [0, 1]^2$, we have $-a\log(y) \geq 0$, $(1 - x^c)y^c \in [0, 1]$ and $1 - bc(1 - x^c)y^c \geq 1 - bc \geq 0$. Hence, we obtain $T_1(x, y) = -a\log(y)\left[1 - bc(1 - x^c)y^c\right] \geq 0$.

Under **PC2**: Since $b \leq 0$ and $c \in [-1, 0]$, we have $bc \geq 0$. For any $(x, y) \in [0, 1]^2$, we have $-a\log(y) \geq 0$, $x^c \geq 1$, $-(1 - x^c)y^c \geq 0$ and $1 - bc(1 - x^c)y^c \geq 0$. As a result, we get $T_1(x, y) = -a\log(y)\left[1 - bc(1 - x^c)y^c\right] \geq 0$.

Under **PC3**: Since $b \in [-1, 0]$, $c \in [0, 1]$, we have $-bc \geq 0$. For any $(x, y) \in [0, 1]^2$, we have $-a\log(y) \geq 0$, $(1 - x^c)y^c \geq 0$ and $1 - bc(1 - x^c)y^c \geq 0$. Hence, we obtain $T_1(x, y) = -a\log(y)\left[1 - bc(1 - x^c)y^c\right] \geq 0$.

Proof of $T_2(x, y) \geq 0$: Under **PC1**, **PC2** or **PC3**, for any $(x, y) \in [0, 1]^2$, we obviously have $a^2 \geq 0$ and $\log(x)\log(y) \geq 0$, implying that $T_2(x, y) \geq 0$.

Proof of $T_3(x, y) \geq 0$: Since, for any $(x, y) \in [0, 1]^2$, we have $T_1(x, y) \geq 0$, we also have $T_3(x, y) = T_1(y, x) \geq 0$.

Proof of $T_4(x, y) \geq 0$: We recall that $a \geq 0$. Let us distinguish **PC1**, **PC2**, and **PC3**.

Under **PC1**: The proof is direct. Since $b \geq 0$, $c \geq 0$ (so $bc \geq 0$), and $1 - bc \geq a$, for any $(x, y) \in [0, 1]^2$, we have $x^cy^c \geq 0$ and $(1 - x^c)(1 - y^c) \geq 0$, so

$$\begin{aligned} T_4(x, y) &= b^2c^2x^cy^c(1 - x^c)(1 - y^c) + bc(c + 1)x^cy^c \\ &\quad + bc(1 - x^c)(1 - y^c) + 1 - a - bc \\ &\geq 1 - a - bc \geq 0. \end{aligned}$$

Under **PC2**: Since $b \leq 0$, $c \in [-1, 0]$ (so $bc \geq 0$), and $1 + bc^2 \geq a$, for any $(x, y) \in [0, 1]^2$, we have $x^cy^c \geq 1$, so $bc(c+1)x^cy^c \geq bc(c+1) \geq 0$, and $(1-x^c)(1-y^c) \geq 0$, implying that

$$\begin{aligned} T_4(x, y) &= b^2c^2x^cy^c(1-x^c)(1-y^c) + bc(c+1)x^cy^c \\ &\quad + bc(1-x^c)(1-y^c) + 1 - a - bc \\ &\geq bc(c+1) + 1 - a - bc = bc^2 + 1 - a \geq 0. \end{aligned}$$

Under **PC3**: To begin, let us notice that we can rewrite $T_4(x, y)$ as

$$\begin{aligned} T_4(x, y) &= b^2c^2x^cy^c(1-x^c)(1-y^c) + bc(c+1)x^cy^c \\ &\quad + bc(1-x^c)(1-y^c) + 1 - a - bc \\ &= [1 + bc(1-x^c)(1-y^c)](1 + bcx^cy^c) \\ &\quad - bc(1 - cx^cy^c) - a. \end{aligned}$$

Since $b \in [-1, 0]$ and $c \in [0, 1]$ (so $bc \leq 0$), for any $(x, y) \in [0, 1]^2$, we have $(1-x^c)(1-y^c) \in [0, 1]$ and $x^cy^c \in [0, 1]$, implying that $1 + bc(1-x^c)(1-y^c) \geq 1 + bc \geq 0$, $1 + bcx^cy^c \geq 1 + bc \geq 0$, and $[1 + bc(1-x^c)(1-y^c)](1 + bcx^cy^c) \geq (1 + bc)^2$. Furthermore, since $cx^cy^c \in [0, 1]$, we have $-bc(1 - cx^cy^c) \geq 0$ and, since $(1 + bc)^2 \geq a$, we get

$$T_4(x, y) \geq (1 + bc)^2 - a \geq 0.$$

As a result, we obtain

$$\frac{\partial^2}{\partial x \partial y} C(x, y; a, b, c) \geq 0.$$

The PD condition is established.

The desired result is proved. □

The parameter configurations **PC1**, **PC2**, and **PC3** in Proposition 2.2 are determined to be as sharp as possible. However, we do not claim that they are the optimal ones in the mathematical sense.

Let us now discuss the importance of Proposition 2.2 in comparison to the existing results. To begin, let us notice that the CG copula is obtained by taking $b = 0$ or $c = 0$, the CC copula is obtained by taking $a = 0$ and $c = 1$, and the generalized version of the CC copula as described in Chesneau (2023a) is obtained by taking $a = 0$. In addition, the result in Zhang et al. (2013, Theorem 2) is obtained by taking $c = 1$, $a = \alpha\beta$ and $b = (1-\beta)\alpha$ with $\alpha \in [0, 1]$ and $\beta \in [0, 1]$, that is, $a \in [0, 1]$ and $b + a = \alpha \in [0, 1]$, so $1 - b \geq a$, which is covered by **PC1**. To the best of our knowledge, under **PC1**, the other values for a and b, as well as $c \in (0, +\infty)/\{1\}$, are unexplored cases. Furthermore, the parameter configurations **PC2** and **PC3** are exhibited for the first time in this setting, opening a new modeling horizon. This claim will be supported in Sects. 3 and 4 with graphical and numerical studies highlighting the importance of the parameters a, b, and c in the shape and correlation properties of the proposed copula.

Let us now discuss a configuration that intersects **PC1** and **PC3**. We can notice that, since $bc \in [0, 1]$, we have $1 - bc \geq (1 - bc)^2$. Therefore, with the following more restrictive condition on **PC1** : $(1 - bc)^2 \geq a$ instead of $1 - bc \geq a$, we arrive at the following result, which combines **PC1** and **PC3** in the case $c \in [0, 1]$:

$C(x, y; a, b, c)$ is a copula for $c \in [0, 1]$, $b \in [-1, 1]$, and $a \in [0, (1 - |b|c)^2]$.

Furthermore, for $c \in [0, 1]$ and $b \in [-1, 1]$, we have $(1 - |b|c)^2 \geq (1 - c)^2$. This shows that Proposition 2.2 offers a more suitable alternative to Proposition 2.1, under the same conditions on b and c.

To conclude this part, let us mention that an additional contribution to the work of Zhang et al. (2013) is given in the appendix, which can also be viewed as a complementary interest.

3 Functions and graphical work

Some important functions and graphics are given in this section.

3.1 Functions

Various useful functions, including original copulas, can be derived from a given copula. This section illustrates some of them using our theoretical findings.

To begin, we recall that the proposed copula is defined by

$$C(x, y; a, b, c) = xy \exp\left[-a \log(x) \log(y) + b(1 - x^c)(1 - y^c)\right], \quad (x, y) \in [0, 1]^2,$$

under one of the parameter configurations described in Proposition 2.2. For the rest of the article, let us call it the GB-ECC copula.

Upon differentiation of the GB-ECC copula, the GB-ECC copula density is obtained as

$$\begin{aligned} c(x, y; a, b, c) &= \frac{\partial^2}{\partial x \partial y} C(x, y; a, b, c) \\ &= \exp\left[-a \log(x) \log(y) + b(1 - x^c)(1 - y^c)\right] \\ &\quad \times \Big\{ - a \log(y) \left[bc(x^c - 1)y^c + 1\right] \\ &\quad + a \log(x) \left[a \log(y) - bcx^c y^c + bcx^c - 1\right] \\ &\quad - a + b^2c^2x^{2c}y^{2c} - b^2c^2x^{2c}y^c - b^2c^2x^c y^{2c} \\ &\quad + b^2c^2x^c y^c + bc^2x^c y^c + 2bcx^c y^c \\ &\quad - bcx^c - bcy^c + 1 \Big\}, \quad (x, y) \in [0, 1]^2. \end{aligned}$$

This copula density allows for the quantification of the strength and direction of the dependence structure and enables the calculation of various statistical measures (rho of Spearman, etc.). By using a graphical study, we will show in the next part that it accommodates various shapes thanks to the wide ranges of values for a, b and c, which reveal a great flexibility in dependence modeling.

On the other hand, based on the GB-ECC copula, the GB-ECC survival copula is given by

$$\begin{aligned}\hat{C}(x, y; a, b, c) &= x + y - 1 + C(1-x, 1-y; a, b, c) \\ &= x + y - 1 + (1-x)(1-y)\exp\{-a\log(1-x)\log(1-y) \\ &\quad + b[1-(1-x)^c][1-(1-y)^c]\}, \quad (x, y) \in [0, 1]^2. \end{aligned} \tag{5}$$

It is also a three-parameter exponential-type copula that does not appear in the existing literature.

Following the flipping technique described in De Baets et al. (2009), two other copulas may be described. The GB-ECC x-flipping copula is indicated as

$$\begin{aligned}\bar{C}(x, y; a, b, c) &= y - C(1-x, y; a, b, c) \\ &= y - (1-x)y\exp\{-a\log(1-x)\log(y) \\ &\quad + b[1-(1-x)^c](1-y^c)\}, \quad (x, y) \in [0, 1]^2.\end{aligned}$$

Similarly, the GB-ECC y-flipping GB-ECC copula is given as

$$\begin{aligned}\tilde{C}(x, y; a, b, c) &= x - C(x, 1-y; a, b, c) \\ &= x - x(1-y)\exp\{-a\log(x)\log(1-y) \\ &\quad + b(1-x^c)[1-(1-y)^c]\}, \quad (x, y) \in [0, 1]^2.\end{aligned}$$

These flipping copulas are also new three-parameter exponential-type copulas.

Furthermore, the GB-ECC copula is simple enough to think of using it in elaborate techniques to create new copulas. For instance, focusing on **PC1** of Proposition 2.2, one can take $a = 1 - bc$ with $b \in [0, 1]$ and $c \in [0, 1]$. Under this setting, the mixture copula technique using the uniform distribution over [0, 1] ensures that the following bivariate function is a valid copula:

$$\begin{aligned}C^{\dagger}(x, y; c) &= \int_0^1 C(x, y; 1-bc, b, c)\mathrm{d}b \\ &= yx^{1-\log(y)}\int_0^1 \exp\left\{b\left[c\log(x)\log(y) + (1-x^c)(1-y^c)\right]\right\}\mathrm{d}b \\ &= yx^{1-\log(y)}\frac{\exp[c\log(x)\log(y) + (1-x^c)(1-y^c)] - 1}{c\log(x)\log(y) + (1-x^c)(1-y^c)}, \quad (x, y) \in [0, 1]^2.\end{aligned}$$

More information on the mixture copula technique can be found in Nelsen (2006). One can also think of the copula product or something else. Further development in

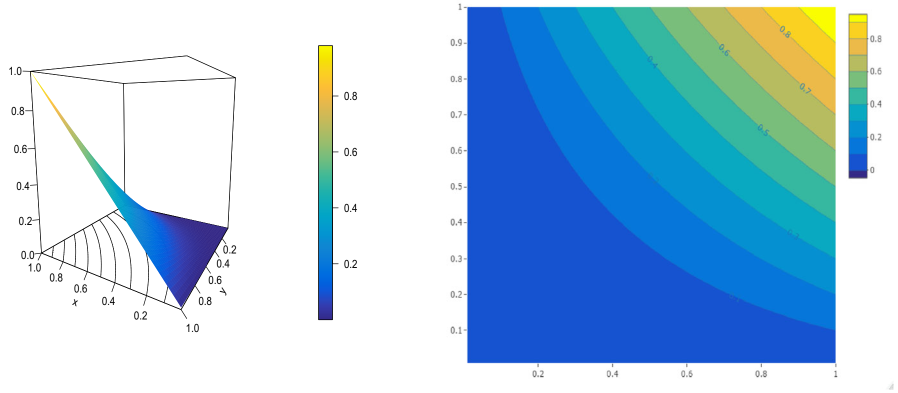

Fig. 1 Plots of the GB-ECC copula for $a = 3/4$ and $b = c = 1/2$, belonging to **PC1**: shape plot (left) and intensity plot (right)

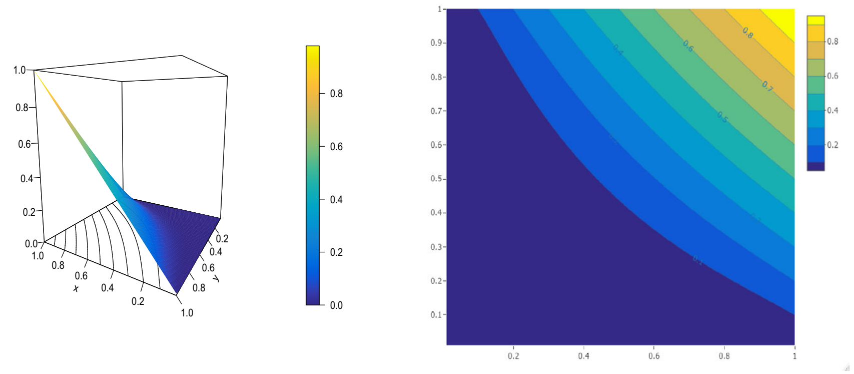

Fig. 2 Plots of the GB-ECC copula for $a = 1/2$, $b = -1/2$ and $c = -1$, belonging to **PC2**: shape plot (left) and intensity plot (right)

the copula creation based on the GB-ECC copula can be done; we leave this aspect for other works.

3.2 Graphical work

In this part, we complete our theoretical findings with visual material. Some figures of the GB-ECC copula and copula density are displayed. The graphics are made using the R software, in particular the packages `plot3D` and `plotly` (see CoreTeam (2016)).

To begin, with a focus on the result in Proposition 2.2, we plot the GB-ECC copula for selected values of the parameters satisfying either **PC1**, **PC2**, or **PC3**. Figure 1 shows the GB-ECC copula for $a = 3/4$ and $b = c = 1/2$, belonging to **PC1**.

Figure 2 displays the GB-ECC copula for $a = 1/2$, $b = -1/2$ and $c = -1$, belonging to **PC2**.

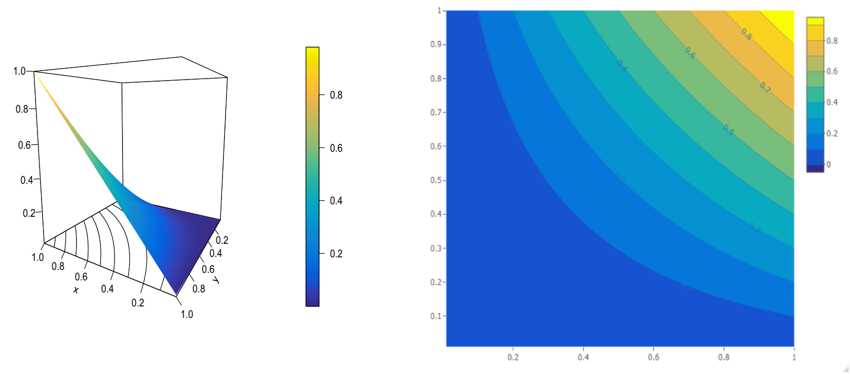

Fig. 3 Plots of the GB-ECC copula for $a = 1/4$, $b = -1/2$ and $c = 1$, belonging to **PC3**: shape plot (left) and intensity plot (right)

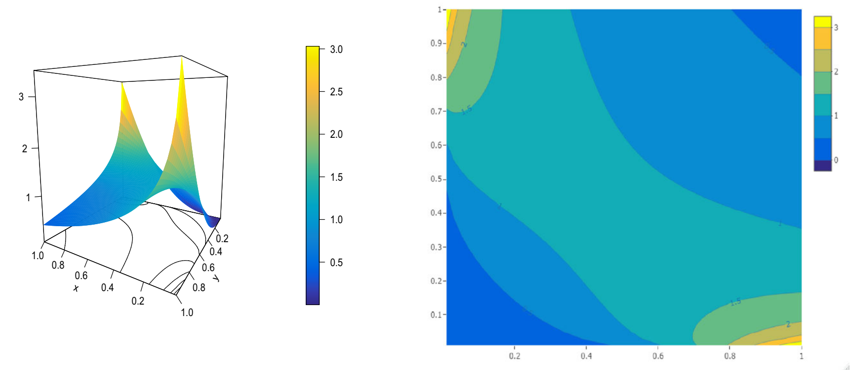

Fig. 4 Plots of the GB-ECC copula density for $a = 3/4$ and $b = c = 1/2$, belonging to **PC1**: shape plot (left) and intensity plot (right)

Figure 3 displays the GB-ECC copula for $a = 1/4$, $b = -1/2$ and $c = 1$, belonging to **PC3**.

Figures 1, 2, and 3 reveal the expected shapes of a valid copula, more or less skewed depending on the values of the parameters. A symmetry in x and y is observed. These figures visually underline the Proposition 2.2 in a punctual parametric way.

In general, copula densities can have a wide range of shapes and characteristics, including symmetry, asymmetry, tail dependence, and concavity or convexity. These shapes constitute the functional print of the involved dependence structure. In light of this, we plot the GB-ECC copula density for the same previously selected parameters. Thus, Fig. 4 displays the GB-ECC copula density for $a = 3/4$ and $b = c = 1/2$, belonging to **PC1**.

Figure 5 shows the GB-ECC copula density for $a = 1/2$, $b = -1/2$, and $c = -1$, belonging to **PC2**.

Figure 6 displays the GB-ECC copula density for $a = 1/4$, $b = -1/2$, and $c = 1$, belonging to **PC3**.

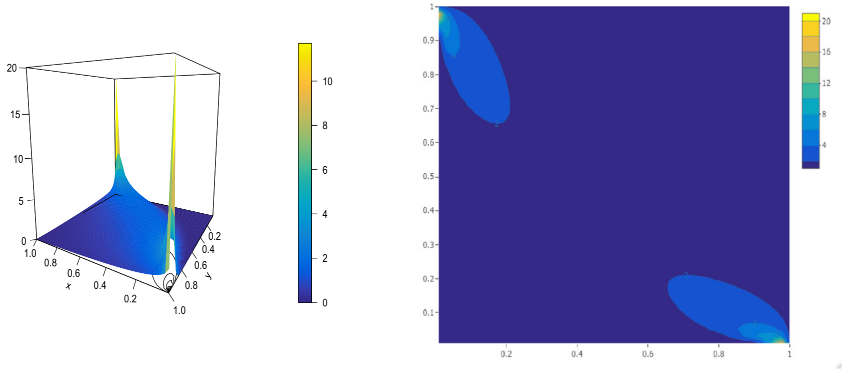

Fig. 5 Plots of the GB-ECC copula density for $a = 1/2$, $b = -1/2$, and $c = -1$, belonging to **PC2**: shape plot (left) and intensity plot (right)

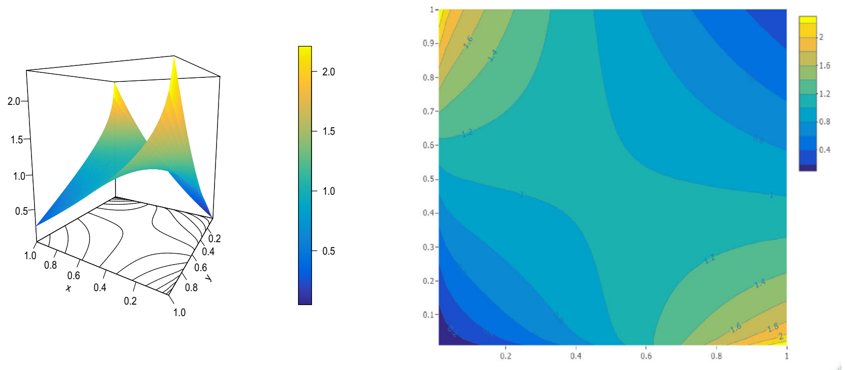

Fig. 6 Plots of the GB-ECC copula density for $a = 1/4$, $b = -1/2$, and $c = 1$, belonging to **PC3**: shape plot (left) and intensity plot (right)

In Figs. 4, 5 and 6, various shapes are observed, illustrating the functional versatility of the GB-ECC copula density. In particular, from Fig. 5 illustrating the case where c is negative, we see that the GB-ECC copula density is highest near the points of coordinates $(0, 1)$ and $(1, 0)$, indicating a strong dependence structure. Such a property is not observed when $c > 0$ (and $c = 1$ in particular, with reference to the CC and CG copulas).

4 Properties

In this section, the GB-ECC copula is discussed along with some of its major properties. It is supposed that the copula parameters belong to either **PC1**, **PC2**, or **PC3** of Proposition 2.2.

4.1 Basic properties

We now investigate some basic properties of the GB-ECC copula.

- The GB-ECC copula is diagonally symmetric since $C(x, y; a, b, c) = C(y, x; a, b, c)$ for any $(x, y) \in [0, 1]^2$.
- For $b \neq 0$ or $c \neq 0$, the GB-ECC copula does not belong to the Archimedean family; $C(x, y; a, b, c)$ does not satisfy the associative property.
- The GB-ECC copula is not radially symmetric, because, based on Eq. (5), there exists (x, y), such that $\hat{C}(x, y; a, b, c) \neq C(x, y; a, b, c)$.
- The GB-ECC copula satisfies the following product geometric property: For any $(x, y) \in [0, 1]^2$ and $\beta \in [0, 1]$,

$$\begin{aligned} &C(x, y; a_1, b_1, c)^{\beta} C(x, y; a_2, b_2, c)^{1-\beta} \\ &\quad = C\left(x, y; \beta a_1 + (1-\beta)a_2, \beta b_1 + (1-\beta)b_2, c\right), \end{aligned}$$

  which is also a common property to the GB and ECC copulas.
  However, $C\left(x, y; \beta a_1 + (1-\beta)a_2, \beta b_1 + (1-\beta)b_2, c\right)$ is a valid copula under some conditions on the parameters $\beta a_1 + (1-\beta)a_2$, $\beta b_1 + (1-\beta)b_2$ and c, implying an adjustment of the initial condition $\beta \in [0, 1]$ (see Proposition 2.2, for instance). This result can also be viewed as a generalization of Zhang et al. (2013, Theorem 2).
- The GB-ECC copula is negatively quadrant dependent for $b \in [-1, 0]$ (which is a common condition to **PC2** and **PC3**), since $C(x, y; a, b, c) \leq xy$ for any $(x, y) \in [0, 1]^2$.
- As a well-established copula fact, the Fréchet–Hoeffding bounds are satisfied. Hence, for any $(x, y) \in [0, 1]^2$, we have $\max(x + y - 1, 0) \leq C(x, y; a, b, c) \leq \min(x, y)$, that is

$$\begin{aligned} \max(x + y - 1, 0) &\leq xy \exp\left[-a \log(x) \log(y) + b(1 - x^c)(1 - y^c)\right] \\ &\leq \min(x, y). \end{aligned}$$

- Owing to the classical exponential and binomial series formulas, the following expansion is obtained:

$$C(x, y; a, b, c) = \sum_{k=0}^{+\infty} \sum_{\ell=0}^{k} \sum_{m=0}^{k-\ell} \sum_{n=0}^{k-\ell} \zeta_{k,\ell,m,n} \left\{x^{mc+1}[\log(x)]^{\ell}\right\} \left\{y^{nc+1}[\log(y)]^{\ell}\right\}, \tag{6}$$

  where

$$\zeta_{k,\ell,m,n} = \binom{k}{\ell}\binom{k-\ell}{m}\binom{k-\ell}{n}\frac{1}{k!}(-1)^{\ell+m+n} a^{\ell} b^{k-\ell}.$$

  This result can be of interest for computational manipulations of the GB-ECC copula, as seen later for the corresponding rho of Spearman.

- Using standard limit techniques, the lower and upper tail dependence parameters are calculated as follows:

$$\sigma_L = \lim_{x\to 0} \frac{C(x,x;a,b,c)}{x} = \lim_{x\to 0} x \exp\left\{-a[\log(x)]^2 + b(1-x^c)^2\right\} = 0$$

and

$$\sigma_U = \lim_{x\to 1} \frac{1-2x+C(x,x;a,b,c)}{1-x} = \lim_{x\to 1} \frac{1-2x+x^2\exp\left[-a[\log(x)]^2+b(1-x^c)^2\right]}{1-x} = 0,$$

respectively. As a result, the GB-ECC copula is free of tail dependence.
- The medial correlation of the GB-ECC copula is expressed as follows:

$$M = 4C\left(\frac{1}{2},\frac{1}{2};a,b,c\right) - 1 = \exp\left\{-a[\log(2)]^2 + b(1-2^{-c})^2\right\} - 1.$$

- The rho of Spearman associated with the GB-ECC copula is basically defined by

$$\begin{aligned}\rho &= 12\int_0^1\int_0^1 [C(x,y;a,b,c) - xy]\,\mathrm{d}x\mathrm{d}y \\ &= 12\int_0^1\int_0^1 xy\left\{\exp\left[-a\log(x)\log(y) + b(1-x^c)(1-y^c)\right] - 1\right\}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Due to the complexity of the integrated function, ρ lacks a closed-form expression. Based on Eq. (6), a series expansion is possible. Indeed, by interchanging the sums and integral signs, and using the following integral formula: $\int_0^1 x^\alpha[\log(x)]^\beta \mathrm{d}x = (-1)^\beta\beta!/(1+\alpha)^{\beta+1}$ where $\alpha > -1$ and β is an integer, we get

$$\rho = 12\sum_{k=1}^{+\infty}\sum_{\ell=0}^{k}\sum_{m=0}^{k-\ell}\sum_{n=0}^{k-\ell}\zeta_{k,\ell,m,n}\frac{(\ell!)^2}{(mc+2)^{\ell+1}(nc+2)^{\ell+1}}.$$

Hence, for a large integer K, the following finite approximation is admissible:

$$\rho \approx 12\sum_{k=1}^{K}\sum_{\ell=0}^{k}\sum_{m=0}^{k-\ell}\sum_{n=0}^{k-\ell}\zeta_{k,\ell,m,n}\frac{(\ell!)^2}{(mc+2)^{\ell+1}(nc+2)^{\ell+1}}.$$

Otherwise, numerical calculations of ρ can be investigated. These are performed below, with four decimals for the values. The software R is used, in particular the package `pracma`.
Table 1 presents its numerical values for $c = 1$, $a = (1-|b|c)^2$, and b varying on a grid of values into $[-1, 1]$. This configuration belongs to **PC1** for $b \in [0, 1]$ and **PC3** for $b \in [-1, 0]$.

Table 1 Values of ρ for $c = 1$, $a = (1 - |b|c)^2$, and $b \in [-1, 1]$

b	−1.0	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8	1.0
ρ	−0.2962	−0.2656	−0.2799	−0.334	−0.4187	−0.5239	−0.3253	−0.1192	0.0824	0.2582	0.3806

Table 2 Values of ρ for $c = 1/2$, $a = (1 - |b|c)^2$, and $b \in [-1, 1]$

b	−1.0	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8	1.0
ρ	−0.2632	−0.3033	−0.3509	−0.4044	−0.4625	−0.5239	−0.4331	−0.3405	−0.2469	−0.1537	−0.0625

Table 3 Values of ρ for $c = -1$, $a = 1 + bc^2$, and $b \in [-1, 0]$

b	−1.0	−0.9	−0.8	−0.7	−0.6	−0.5	−0.4	−0.3	−0.2	−0.1	0.0
ρ	−0.8333	−0.8124	−0.7902	−0.7666	−0.7414	−0.7142	−0.6846	−0.6522	−0.6159	−0.5743	−0.5239

Table 2 is analogous to Table 1 but with $c = 1/2$.

Table 3 presents some values of ρ for $c = -1$, $a = 1 + bc^2$, and $b \in [-1, 0]$, corresponding to **PC2**.

These tables demonstrate the vast range of amplitudes (here, from −0.83 to 0.38) that the rho of Spearman of the GB-ECC copula can have. This flexibility is only possible thanks to the combined actions of the parameters a, b, and c, demonstrating their importance from a correlation viewpoint. Therefore, the GB-ECC copula is ideal for modeling various kinds of dependence. It offers much greater flexibility in dependence structure than the GB and extended CC copulas.

- Naturally, the GB-ECC copula can serve as a generator of bivariate distributions. Indeed, for any cumulative distribution functions of absolutely continuous distributions, say $F(x)$ and $G(x)$, the following bivariate function defines a new bivariate cumulative distribution function:

$$\begin{aligned} H(x, y; \tau) &= C(F(x), G(y); a, b, c) \\ &= F(x)G(y)\exp\Big\{-a\log[F(x)]\log[G(y)] \\ &\quad + b[1 - F(x)^c][1 - G(y)^c]\Big\}, \ (x, y) \in \mathbb{R}^2, \end{aligned} \tag{7}$$

where τ represents the vector of all the involved parameters, including a, b, and c. Let us call it the GB-ECC family of distributions. The corresponding probability density function is obtained as

$$h(x, y; \tau) = f(x)g(y)c(F(x), G(y); a, b, c), \quad (x, y) \in \mathbb{R}^2, \tag{8}$$

where $f(x)$ and $g(y)$ are the probability density functions associated with $F(x)$ and $G(y)$, respectively. There are therefore an infinite number of new bivariate distributions that could be produced. As an example, we can investigate the special

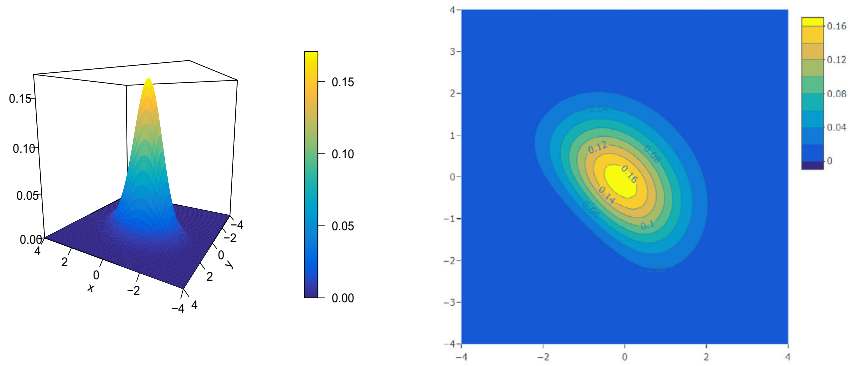

Fig. 7 Plots of the probability density function of the GB-ECC normal distribution for $a = 3/4$ and $b = c = 1/2$, belonging to **PC1**: shape plot (left) and intensity plot (right)

member of the GB-ECC family defined with the standard normal distribution as the marginal distributions. Thus, if we consider

$$F(x) = G(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t, \quad f(x) = g(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad x \in \mathbb{R},$$

and we insert them into the cumulative distribution and probability density functions in Eqs. (7) and (8), we introduce the GB-ECC normal distribution with $\tau = (a, b, c)$. To observe the effect of the parameters a, b, and c on the subjacent dependence model, we present the shapes of the probability density function of the GB-ECC normal distribution for parameters belonging to the configurations **PC1**, **PC2**, and **PC3** of Proposition 2.2.
Figure 7 displays the probability density function of the GB-ECC normal distribution for $a = 3/4$ and $b = c = 1/2$, belonging to **PC1**.

Figure 8 shows the probability density function of the GB-ECC normal distribution for $a = 1/2$, $b = -1/2$, and $c = -1$, belonging to **PC2**.

Figure 9 displays the probability density function of the GB-ECC normal distribution for $a = 1/4$, $b = -1/2$, and $c = 1$, belonging to **PC3**.

From Figs. 7, 8, and 9, we observe different shapes for the probability density distribution; the parameters a, b, and c clearly affect the skewness and kurtosis of the GB-ECC normal distribution.

On the other hand, for selected lifetime cumulative distribution functions depending on the context, we refer to Taketomi et al. (2022).

- We can consider the so-called omnibus estimation approach for the estimation of the parameters a, b, and c in a data analysis setting (see Genest et al. (1995) and Silvapulle et al. (2004)). To present this approach, let us consider n observations, say $(x_1, y_1), \ldots, (x_n, y_n)$, drawn from a continuous random vector, say (X, Y).

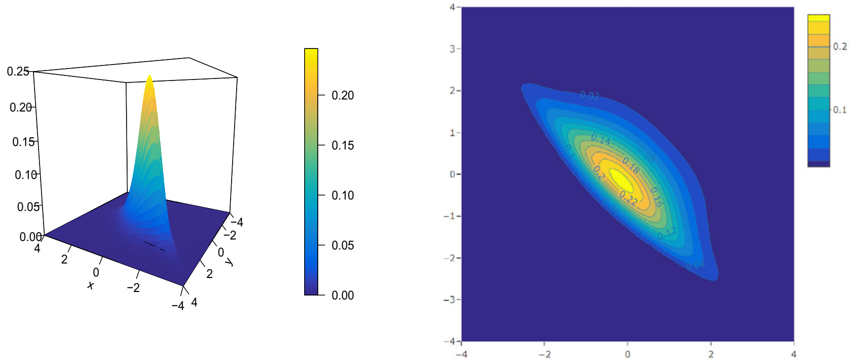

Fig. 8 Plots of the probability density function of the GB-ECC normal distribution for $a = 1/2$, $b = -1/2$, and $c = -1$, belonging to **PC2**: shape plot (left) and intensity plot (right)

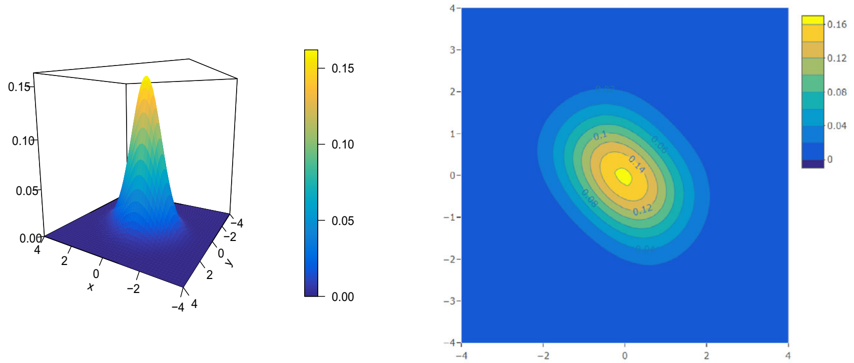

Fig. 9 Plots of the probability density function of the GB-ECC normal distribution for $a = 1/4$, $b = -1/2$, and $c = 1$, belonging to **PC3**: shape plot (left) and intensity plot (right)

Then, the following argmax-values give the omnibus estimates of a, b, and c:

$$(\hat{a}, \hat{b}, \hat{c}) = \text{argmax}_{(a,b,c)\in\Pi} \sum_{i=1}^{n} \log\left\{c\left[\hat{F}(x_i), \hat{G}(y_i); a, b, c\right]\right\},$$

where Π represents the ranges of values on the parameters making $C(x, y; a, b, c)$ a valid copula,

$$\hat{F}(x) = \frac{1}{n}\sum_{j=1}^{n} 1\{x_j \le x\}, \quad \hat{G}(y) = \frac{1}{n}\sum_{j=1}^{n} 1\{y_j \le y\},$$

and $1\mathcal{S}$ denotes the indicator function over a set $\mathcal{S}$. In other words, the omnibus estimates $\hat{a}$, $\hat{b}$, and $\hat{c}$ are equivalent to the maximum-likelihood estimates determined with the transformed data $(\hat{F}(x_1), \hat{G}(y_1)), \ldots, (\hat{F}(x_n), \hat{G}(y_n))$. This approach has assured global efficiency (see Genest et al. (1995) and Silvapulle et al. (2004)).

4.2 Copula comparisons

This part is devoted to some copula comparisons involving the GB-ECC copula and other well-referenced copulas. These comparisons show how the GB-ECC copula can be considered an alternative in terms of dependence modeling.

- For $b \in [-1, 0]$, for any $(x, y) \in [0, 1]^2$, we have

$$C(x, y; a, b, c) \leq C_{GB}(x, y; a),$$

where $G_{GB}(x, y; a)$ is the GB copula as given in Eq. (1). For $b \in [0, 1]$ (which belongs to **PC1**), the reverse inequality holds.
- For any $(x, y) \in [0, 1]^2$, we have

$$C(x, y; a, b, 1) \leq C_{CC}(x, y; b),$$

where $G_{CC}(x, y; b)$ is the CC copula as given in Eq. (2). Of course, this result can be extended by taking $c \in [-1, 1]$ instead of $c = 1$, and the extended CC copula.
- Owing to the following exponential inequality: $e^z \geq 1 + z$ for any $z \in \mathbb{R}$, for any $(x, y) \in [0, 1]^2$, we get

$$\begin{aligned}
&\exp\left[a \log(x) \log(y) - b(1 - x^c)(1 - y^c)\right] \\
&\quad \geq 1 + a \log(x) \log(y) - b(1 - x^c)(1 - y^c) \\
&\quad \geq 1 - b(1 - x^c)(1 - y^c).
\end{aligned}$$

Therefore, since $b(1 - x^c)(1 - y^c) \leq 1$, we have

$$C(x, y; a, b, c) \leq C_{AMH}(x, y; b),$$

where

$$C_{AMH}(x, y; b) = \frac{xy}{1 - b(1 - x^c)(1 - y^c)},$$

which is a modified version of the Ali–Mikhail–Haq (AMH) copula with the parameter b (see Nelsen (2006) for the former AMH copula). As a result, under some parameter conditions, $C(x, y; a, b, c)$ is smaller than $C_{AMH}(x, y; b)$ in the concordance ordering sense.
- On the other hand, for any $(x, y) \in [0, 1]^2$ and $b \in [-1, 0]$, we have

$$\begin{aligned}
&\exp\left[a \log(x) \log(y) - b(1 - x^c)(1 - y^c)\right] \\
&\quad \geq 1 + a \log(x) \log(y) - b(1 - x^c)(1 - y^c) \\
&\quad \geq 1 + a \log(x) \log(y).
\end{aligned}$$

This implies that

$$C(x, y; a, b, c) \le C_{Ch}(x, y; a),$$

where

$$C_{Ch}(x, y; a) = \frac{xy}{1 + a\log(x)\log(y)},$$

which is a copula introduced in Chesneau (2023b), defined with the parameter a. As a result, under some parameter conditions, $C(x, y; a, b, c)$ is smaller than $C_{Ch}(x, y; a)$ in the concordance ordering sense.

These copula orderings also illustrate how the parameters a, b, and c offer a higher degree of flexibility compared to the aforementioned copulas.

5 Conclusion

This article contributes to the understanding of copula theory by significantly improving a theorem established in Zhang et al. (2013). More precisely, we considered a copula defined as a compromise between the GB copula and an extended version of the CC copula with the use of three parameters. It is named the GB-ECC copula. We determine the ranges of values allowed for these parameters, making mathematical efforts to have them as wide as possible. Then, we discussed the obtained results in relation to the existing findings. The related functions were given, namely the copula density, survival copula, (x and y) flipping copulas, and a special mixture copula. We perform a graphical study of the GB-ECC copula and copula density. The discussion covered the fundamental copula features, such as symmetry, series expansions, analytical bounds, bivariate distribution generation, and concordance ordering. The theory was supported by a numerical analysis, which also demonstrated how the parameters under consideration have an effect on the dependent structure.

The logical perspectives of this work are:

- The application of the GB-ECC copula to the analysis of real-life bivariate data.
- The development of new bivariate distributions with various supports.
- The creation of original "compromise or tradeoff copulas" based on the mathematical techniques developed in this article, such as the following mixed GB-AMH copula

$$C(x, y; a, b) = \frac{xy\exp[-a\log(x)\log(y)]}{1 - b(1-x)(1-y)}, \quad (x, y) \in [0, 1]^2,$$

among other ideas.

- The investigation of the multivariate GB-ECC copula, which naturally takes the following form:

$$C(x_1, \ldots, x_n; a, b, c) = \left(\prod_{i=1}^{n} x_i\right) \exp\left[-a\left(\prod_{i=1}^{n} \log(x_i)\right) + b\left(\prod_{i=1}^{n}(1 - x_i^c)\right)\right], \quad (x, \ldots, x_n) \in [0, 1]^n,$$

where n represents the dimension. For this expression, one can prove that the multivariate B condition is satisfied, but the multivariate PD condition remains challenging, since the precise conditions on the parameters a, b, and c need to be determined.

These points need more work, which we leave for the future.

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Declarations

Conflict of interest The author declares no conflict of interest.

Appendix

In the lemma below, we provide an extension of Zhang et al. (2013, Lemma 3) with an alternative proof; the proof is much more direct, short, and without the use of differentiation. This lemma is an ingredient of Zhang et al. (2013, Theorem 2). It can be viewed as a minor result of independent interest.

Lemma 5.1 *Let $\beta \leq 1$. Then, for any $(x, y) \in [0, 1]^2$, we have*

$$3(1 - \beta)xy - (1 - \beta)(x + y) - \beta \geq -1.$$

Proof The proof is based on a proper arrangement of the terms in the main equation. Since $1 - \beta \geq 0$, and, for any $(x, y) \in [0, 1]^2$, $xy \geq 0$ and $(1 - x)(1 - y) \geq 0$, we have

$$\begin{aligned} 3(1 - \beta)xy - (1 - \beta)(x + y) - \beta &= 2(1 - \beta)xy + (1 - \beta)(1 - x)(1 - y) - 1 \\ &\geq -1. \end{aligned}$$

The desired result is proved. □

Hence, in comparison with Zhang et al. (2013, Lemma 3), we have the condition $\beta \leq 1$ (possibly negative) instead of $\beta \in [0, 1]$, and the proof is considerably simplified.

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