



General classes of complementary distributions via random maxima and their discrete version

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Abstract

In this paper, we develop some new classes of complementary distributions generated from random maxima. This family contains many distributions of which complementary Weibull-geometric distribution is a special case. A three-parameter discrete complementary Weibull-geometric distribution is introduced and studied. This distribution is a generalization of the discrete Weibull distribution and contains many discrete submodels as particular cases. Its distributional properties including the hazard rate function, quantile function, random number generation, and probability generating function are investigated. The unknown parameters of the model are estimated using the method of maximum likelihood. The existence and uniqueness of the MLEs of the parameters are established. A simulation study is carried out to check the performance of the method. The new model is applied to a practical data set to prove empirically the flexibility in data modeling.

Keywords Complementary distributions · Estimation · Geometric distribution · Hazard rate function · Random maxima · Simulation

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1 Introduction

The complementary risk problems suggested by Basu and Klein (1982) arise in several areas, such as industrial reliability, public health, biomedical studies, demography, and actuarial science. In complementary risk analysis, the lifetime associated with a specific risk may not be observable; rather, we observe only the maximum lifetime value among all risks. In reliability studies, we observe only the maximum component lifetime of a parallel system. That is, the observable quantities for each component are the maximum lifetime value to failure among all risks, and the cause of failure. Whereas in competing risk scenario, in which the lifetime associated with a particular risk is not observable; rather, we observe only the minimum lifetime value among all risks. One can refer Lawless (2003), Crowder et al. (1991), and Cox and Oakes (1984) for the statistical procedures and extensive literature deal with these problems.

A difficulty arises if the risks are latent in the sense that there is no information about which factor was responsible for the component failure (or individual failure), that is the latent complementary risks, which can often be observed in field data. On many occasions, this information is not available, or it is impossible that the true cause of failure can be specified by an expert. In reliability, the components can be totally destroyed in the experiment. Furthermore, the true cause of failure can be masked from our view.

In modular systems, the need to keep a system running means that a module that contains many components can be replaced without the identification of the exact failing component. Goetghebeur and Ryan (1995) addressed the problem of assessing covariate effects based on a semi-parametric proportional hazards structure for each failure type when the failure type is unknown for some individuals. Reiser et al. (1995) considered statistical procedures for analyzing masked data, but their procedure cannot be applied when all observations have an unknown cause of failure. A comparison of two partial likelihood approaches for risk modeling with missing cause of failure is presented in Lu and Tsiatis (2005).

Louzada et al. (2011) introduced the complementary exponential geometric distribution, which is complementary to the exponential geometric model proposed by Adamidis and Loukas (1998), based on a complementary risk problem in the presence of latent risks, in the sense that there is no information about which factor was responsible for the component failure, but only the maximum lifetime value among all risks is observed. Louzada et al. (2013) introduced the complementary exponentiated exponential geometric distribution which considered a generalization to the complementary exponential geometric distribution. Jose et al. (2013) introduced the complementary exponential power series distribution with an increasing failure rate. Tojeiro et al. (2014) introduced the complementary Weibull geometric (CWG) as a complementary distribution to the Weibull-geometric (WG) model proposed by Barreto-Souza et al. (2011).

In this context, in this paper, we propose a new generalized family of complementary distributions conceived inside a latent complementary risk scenario, in which there is no information about which factor was responsible for the component failure (or individual failure), and only the maximum lifetime value among all risks is observed.

This paper is focused on the construction of discrete version of complementary distributions, which are not studied in the literature.

The rest of the paper is organized as follows: In Sect. 2, we develop a new generalization of complementary distributions with illustrations. In Sect. 3, we discuss the various methods of discretization of continuous distributions. In Sect. 4, we introduce the discrete complementary Weibull-geometric (DCWG) distribution and its properties are discussed in Sect. 5. The estimates of unknown parameters of the model using maximum-likelihood estimation method are derived and their existence and uniqueness are established. Also, a simulation study is carried out to check the performance of the MLEs of the model parameters. These results are presented in Sect. 6. In Sect. 7, the empirical study of the flexibility of DCWG distribution by means of a practical data set is carried out. Finally, conclusions are presented in Sect. 8.

2 A new generalization of complementary distributions

Let X_1, X_2, \dots , be independent and identically distributed (i.i.d) random variables having cumulative distribution function (cdf) $F(x)$ and let N be an integer-valued random variable independent of X_i 's.

Define $Z = \max(X_1, X_2, \dots, X_N)$. Then, the cdf of Z is

$$\begin{aligned}
 G(z) &= P(\max(X_1, X_2, \dots, X_N) \leq z) \\
 &= \sum_n P(\max(X_1, X_2, \dots, X_N) \leq z/N = n)P(N = n) \\
 &= \sum_n [P(X_i \leq z)]^n P(N = n) \\
 &= \sum_n F^n(z)P(N = n) = H(F(z)),
 \end{aligned}
 \tag{1}$$

where $F(z)$ is the common cdf of X_i 's and $H(s)$ is the probability generating function (pgf) of N . Thus, we have the random variable Z has complementary distribution. For example, if X_i 's are i.i.d Pareto random variables and N is negative binomial, then we say that Z has complementary Pareto negative binomial distribution.

Since N be any integer-valued random variable taking values on $\{1, 2, 3, \dots\}$, we may take N as power series family or power series distribution. We know that binomial, negative binomial or geometric, Poisson, etc. are members in power series family. Let N be an integer-valued random variable having power series distribution on $\{1, 2, 3, \dots\}$, and the probability mass function (pmf) of N is

$$P(N = n) = \frac{a_n \theta^n}{f(\theta) - a_0}, \quad n = 1, 2, \dots,
 \tag{2}$$

where $a_n > 0, \theta > 0$ and $f(\theta) = \sum_{n=0}^{\infty} a_n \theta^n$.

The pgf of N is

$$\begin{aligned} H(s) &= \sum_{n=1}^{\infty} s^n P(N = n) \\ &= \sum_{n=1}^{\infty} s^n \frac{a_n \theta^n}{f(\theta) - a_0} = \frac{f(\theta s) - a_0}{f(\theta) - a_0}. \end{aligned} \quad (3)$$

Hence, from Eq. (1), distribution function of $Z = \max(X_1, X_2, \dots, X_N)$, where X_i 's are i.i.d with cdf $F(x)$ and N has the power series distribution is

$$G(z) = H(F(z)) = \frac{f(\theta F(z)) - a_0}{f(\theta) - a_0}. \quad (4)$$

Thus, we can develop a number of families of distributions using complementary random variables. Some examples are:

2.1 Examples

1. Complementary family of distributions generated through Binomial(n, θ) distribution.

Based on Eq. (4), the family is defined through the cdf

$$G(z) = \frac{(1 + \theta F(z))^n - 1}{(1 + \theta)^n - 1}. \quad (5)$$

For example, complementary exponential binomial is defined by the cdf

$$G(z) = \frac{(1 + \theta(1 - e^{-\lambda z}))^n - 1}{(1 + \theta)^n - 1}, \quad \lambda > 0, \theta > 0, z > 0. \quad (6)$$

2. Complementary family of distributions generated through Poisson (θ) distribution. The family, based on Eq. (4), is defined through the cdf

$$G(z) = \frac{e^{\theta F(z)} - 1}{e^{\theta} - 1}, \quad \theta > 0, z > 0. \quad (7)$$

As an example, the complementary Pareto type III Poisson distribution is defined through the cdf

$$G(z) = \frac{e^{\theta \frac{z^\alpha}{1+z^\alpha}} - 1}{e^{\theta} - 1}, \quad \theta > 0, \alpha > 0, z > 0. \quad (8)$$

3. Complementary family of distributions generated through geometric (θ) distribution.

The family is defined through the cdf

$$G(z) = \frac{(1 - \theta F(z))^{-1} - 1}{(1 - \theta)^{-1} - 1}, \theta > 0, z > 0. \tag{9}$$

In this cdf $G(z)$, when $F(z)$ is Weibull, we get the complementary Weibull-geometric (CWG) distribution (Tojeiro et al., 2014).

From the illustrations, it can be seen that we may be able to develop a number of new families of distributions based on complementary random variables.

Now, we consider the situation where N be a discrete random variable denoting the number of complementary risks related to the occurrence of an event of interest following a geometric distribution with pmf

$$P(N = n) = p(1 - p)^{n-1}, \tag{10}$$

where $0 < p < 1$ and $n = 1, 2, \dots$. Also, let $X_i, i = 1, 2, \dots$, denotes the time-to-event due to the i th complementary risk, which are independent of N . Then, for given $N = n$, the random variable X_i is assumed to be i.i.d and follows the Weibull distribution $W(\beta, \alpha)$ with scale parameter $\beta > 0$, shape parameter $\alpha > 0$, and probability density function (pdf):

$$f(x_i; \beta, \alpha) = \alpha\beta^\alpha x_i^{\alpha-1} e^{-(\beta x_i)^\alpha}, x_i > 0. \tag{11}$$

In the latent complementary risks scenario, the number of causes N and the lifetime X_i (associated with a particular cause) are not observable (latent variables), but only the maximum lifetime $Z = \max(X_1, X_2, \dots, X_N)$ among all the causes is usually observed. Then, using Eq. (9) with $\theta = 1 - p$, we get the CWG distribution. The cdf and survival function of CWG distribution, respectively, are

$$F(z) = \frac{p(1 - e^{-(\beta z)^\alpha})}{p + (1 - p)e^{-(\beta z)^\alpha}}, \tag{12}$$

and

$$S(z) = \frac{e^{-(\beta z)^\alpha}}{p + (1 - p)e^{-(\beta z)^\alpha}}. \tag{13}$$

This distribution is complementary to the Weibull-geometric (WG) distribution proposed in Barreto-Souza et al. (2011).

The aim of this paper is to introduce new discrete distributions arising from complementary type random variables. These types of discrete distributions are not studied in the literature.

3 Discretization of continuous distributions

Developing the discrete analogues of continuous distributions has drawn much attention among the researchers and a number of papers in this area are appeared in various journals. In reliability analysis and lifetime modeling, there are situations where the data show discrete behavior, such as, the number of rounds fired a weapon till the first failure; the number of cycles successfully completed prior to the first failure when a particular device works in cycles, etc.

It is well known that the geometric distribution and negative binomial distribution are the discrete analogue of the exponential distribution and gamma distribution, respectively. Discrete version of the normal distribution was introduced in Lisman and van Zuylem (1972) and studied in Kemp (1997). Another version of discrete normal distribution was studied in Roy (2003). Nakagawa and Osaki (1975), Stein and Dattero (1984), and Padgett and Spurrier (1985) have proposed three different analogues of the discrete Weibull distribution which are further studied in Khan et al. (1989) and Kulasekera (1994).

Recently, several forms of discrete lifetime distributions derived from continuous distributions are proposed by many authors. Some of them are: discrete half-normal distribution in Kemp (2008); discrete Burr and Pareto distributions in Krishna and Pundir (2009); discrete inverse Weibull distribution in Jazi et al. (2010); discrete generalized exponential distribution in Gomez-Deniz (2010) and Nekoukhou et al. (2012); discrete modified Weibull distribution in Nooghabi et al. (2011); discrete gamma distribution in Chakraborty and Chakravarty (2012); discrete additive Weibull distribution in Bebbington et al. (2012); discrete generalized exponential distribution of the second type in Nekoukhou et al. (2013); discrete reduced modified Weibull distribution in Almalki and Nadarajah (2014); discrete Lindley distribution in Bakouch et al. (2014); discrete generalized Rayleigh distribution in Alamatsaz et al. (2016); discrete Weibull-geometric distribution in Jayakumar and Babu (2018); and discrete additive Weibull-geometric distribution in Jayakumar and Babu (2019). For more details on discretization of continuous distributions, see Chakraborty (2015).

There are different methods available for obtaining the discrete analogue Y of a continuous random variable X , which can be classified as follows:

- (i) Difference equation analogues of Pearsonian differential equation.
- (ii) The pmf of Y retains the form of the pdf of X and support of Y is determined from full range of X .
- (iii) The pmf of Y retains the form of the pdf of X and support of Y is determined from a subset of the range of X .
- (iv) Survival function (sf) of Y retains the form of the survival function of X and support of Y is determined from full range of X .

In method (ii), the pmf of the discrete random variable Y is derived as an analogue of the continuous random variable X with pdf $f(x)$, $-\infty < x < \infty$ as

$$P(Y = y) = \frac{f(y)}{\sum_{i=-\infty}^{\infty} f(i)}, \quad y = 0, \pm 1, \pm 2, \dots \quad (14)$$

Because of the normalizing constant, the distribution generated through this technique may not always have a compact form. The discrete version of the complementary family of distributions generated through the Poisson (θ) distribution given in Eqn.(7) is obtained as

$$P(Y = y) = \frac{g(y)}{\sum_{i=0}^{\infty} g(i)} = \frac{e^{\theta F(y)}}{\sum_{i=0}^{\infty} e^{\theta F(i)}}, \quad y = 0, 1, 2, \dots \tag{15}$$

For different choices of the cdf $F(y)$, we will get different discrete families of complementary distributions generated through Poisson distribution.

The method (iv) is an easiest way to get the discrete version using the survival function. Suppose Y is the discrete version of a continuous random variable X , then its pmf is given by

$$P(Y = y) = S_X(y) - S_X(y + 1); \quad y = 0, 1, 2, \dots \tag{16}$$

Using this method, the discrete version of the complementary family of distributions generated through geometric (θ) is given by

$$\begin{aligned} P(Y = y) &= \frac{1 - F(y)}{1 - \theta F(y)} - \frac{1 - F(y + 1)}{1 - \theta F(y + 1)} \\ &= \frac{(1 - \theta)(F(y + 1) - F(y))}{(1 - \theta F(y))(1 - \theta F(y + 1))}. \end{aligned} \tag{17}$$

By selecting different distributions for $F(y)$ in Eq. (17), we get various discrete versions of complementary distributions generated through geometric compounding. Now, in the next section, we study extensively one of the members of such distributions called the discrete complementary Weibull geometric.

4 Discrete complementary Weibull-geometric distribution

Using the method (iv) explained in Sect. 3 and after a reparametrization, $\rho = e^{-\beta^\alpha}$, the pmf of the discrete analogue, say Y , of the CWG distribution is obtained as

$$P_Y(y; p, \rho, \alpha) = \frac{p(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})}{[p + (1 - p)\rho^{y^\alpha}][p + (1 - p)\rho^{(y+1)^\alpha}]}, \tag{18}$$

where $y = 0, 1, 2, \dots, p > 0, \alpha > 0$ and $0 < \rho < 1$. Here, the parameter p and ρ can be interpreted as the concentration parameters, while α is a shape parameter. We call this distribution as the discrete complementary Weibull-geometric (DCWG) distribution and is denoted as $DCWG(p, \rho, \alpha)$. Now, we present some submodels of the DCWG distributions.

In particular, when $\alpha = 1$, the pmf becomes

$$P_Y(y; p, \rho) = \frac{p(\rho^y - \rho^{(y+1)})}{[p + (1-p)\rho^y][p + (1-p)\rho^{(y+1)}]}, \quad (19)$$

which is called the discrete complementary exponential geometric distribution.

When $p = 1$, $P_Y(y; \rho, \alpha) = \rho^{y^\alpha} - \rho^{(y+1)^\alpha}$ which is the discrete Weibull distribution of Nakagawa and Osaki (1975).

When $p = 1$ and $\alpha = 2$, then $P_Y(y; \rho) = \rho^{y^2} - \rho^{(y+1)^2}$ which is the discrete Rayleigh distribution of Roy (2004).

When $p = 1$ and $\alpha = 1$, then $P_Y(y; \rho) = \rho^y - \rho^{(y+1)}$ which is the geometric distribution with parameter ρ .

When $p \rightarrow 0+$, the DCWG tends to a degenerate distribution at zero.

5 Structural properties of the DCWG(p, ρ, α) distribution

The shape of the pmf of $DCWG(p, \rho, \alpha)$ distribution for various choices of parameter values are shown in Fig. 1. It can be seen that the distribution is unimodal and highly positively skewed. For $0 < \alpha < 1$, $p > 0$, and $0 < \rho < 1$, we have $p_Y(0) > p_Y(1) > p_Y(2) > \dots$, and hence, $p_Y(y)$ is strictly decreasing function. Also note that when $\alpha = 1$, $p_Y(y)$ is geometric, and when $\alpha > 1$, $p_Y(y)$ is initially increasing to the maximum point and then decreasing.

5.1 Recurrence relation for probabilities

The recurrence relation for probabilities of $DCWG(p, \rho, \alpha)$ distribution is as follows:

$$P_Y(y+1) = \frac{(\rho^{(y+1)^\alpha} - \rho^{(y+2)^\alpha})(p + (1-p)\rho^{y^\alpha})}{(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})(p + (1-p)\rho^{(y+2)^\alpha})} P_Y(y). \quad (20)$$

From Gupta et al. (1997), the distribution is

- (a) log-concave if and only if $\left\{ \frac{P_Y(y+1)}{P_Y(y)} \right\}_{y \geq 0}$ is decreasing,
- (b) log-convex if and only if $\left\{ \frac{P_Y(y+1)}{P_Y(y)} \right\}_{y \geq 0}$ is increasing, and
- (c) geometric if $\left\{ \frac{P_Y(y+1)}{P_Y(y)} \right\}_{y \geq 0}$ is constant.

All the above three cases are justified to the DCWG distribution based on the values of the parameters as shown by plots of pmf.

The cdf of $DCWG(p, \rho, \alpha)$ distribution is obtained as

$$F(y) = P(Y \leq y) = 1 - S_Z(y+1) = \frac{p(1 - \rho^{(y+1)^\alpha})}{p + (1-p)\rho^{(y+1)^\alpha}}, \quad (21)$$

where $y = 0, 1, 2, \dots$; $p > 0$, $0 < \rho < 1$ and $\alpha > 0$.

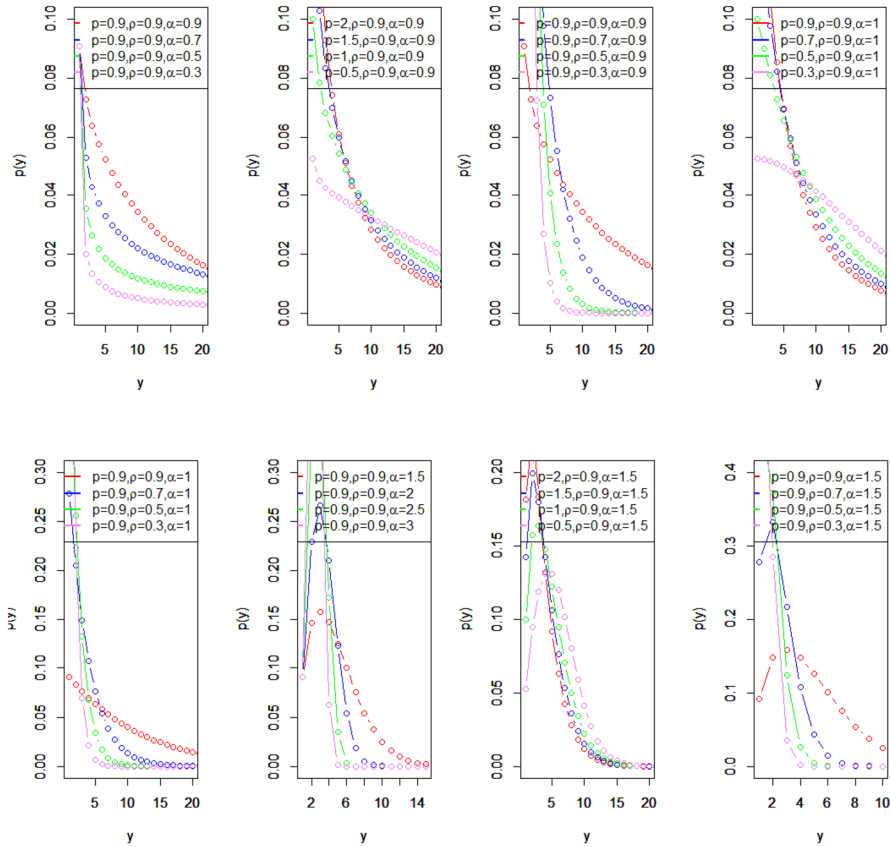


Fig. 1 Shape of pmf for various choices of parameter values

Remark 1 The cdf of $DCWG(p, \rho, \alpha)$ distribution can be expressed as

$$F(y) = \frac{1 - \rho^{(y+1)^\alpha}}{1 - \left(\frac{p-1}{p}\right)\rho^{(y+1)^\alpha}}. \tag{22}$$

This becomes the discrete Weibull-geometric distribution of Jayakumar and Babu (2018), if $0 < \frac{p-1}{p} < 1$ and is satisfied only if $p \in (1, \infty)$.

The survival function of $DCWG(p, \rho, \alpha)$ distribution is given by

$$S(y) = P(Y > y) = 1 - P(Y \leq y) = \frac{\rho^{(y+1)^\alpha}}{p + (1 - p)\rho^{(y+1)^\alpha}}. \tag{23}$$

Hazard rate function (hrf) is given by

$$h(y) = \frac{P(Y = y)}{P(Y \geq y)} = \frac{1 - \rho^{(y+1)^\alpha - y^\alpha}}{1 - \frac{p-1}{p}\rho^{(y+1)^\alpha}}, \quad (24)$$

provided that $P(Y \geq y) > 0$.

Here note that as $y \rightarrow 0$, $h(y) \rightarrow \frac{p(1-\rho)}{p+(1-p)\rho}$. When $p = 1$, the distribution becomes the discrete Weibull distribution and has the following properties:

- Decreasing failure rate for $0 < \alpha < 1$;
- Increasing failure rate for $\alpha > 1$;
- Constant failure rate for $\alpha = 1$.

For $\alpha = 1$, we have $\lim_{y \rightarrow \infty} h(y) \rightarrow 1 - \rho$, and for $0 < \alpha < 1$, $\lim_{y \rightarrow \infty} h(y) \rightarrow 0$, and for $\alpha > 1$, $\lim_{y \rightarrow \infty} h(y) \rightarrow 1$. The shape of hazard rate function for various choices of parameter values are shown in Fig. 2. The hazard rate function is also showing the bathtub and unimodal (upside down bathtub) shapes.

5.2 Quantile function

The u th quantile $\phi(u)$ of the $DCWG(p, \rho, \alpha)$ distribution is obtained as

$$\phi(u) = \lceil y_u \rceil = \left\lceil \left(\frac{\ln(p(1-u)) - \ln(p + (1-p)u)}{\ln(\rho)} \right)^{\frac{1}{\alpha}} - 1 \right\rceil, \quad (25)$$

where $\lceil y_u \rceil$ denotes the smallest integer greater than or equal to y_u .

The median is

$$\phi(0.5) = \lceil y_{0.5} \rceil = \left\lceil \left(\frac{\ln(p) - \ln(p+1)}{\ln(\rho)} \right)^{\frac{1}{\alpha}} - 1 \right\rceil. \quad (26)$$

Let u follow $U(0, 1)$ distribution, and then, using the expression given in Eq. (25), we can generate random samples from the $DCWG(p, \rho, \alpha)$ distribution.

5.3 Probability generating function

The pgf of $DCWG(p, \rho, \alpha)$ distribution is

$$P_Y(s) = 1 + (s-1) \sum_{y=1}^{\infty} \frac{s^{y-1} \rho^{(y+1)^\alpha}}{p + (1-p)\rho^{(y+1)^\alpha}}. \quad (27)$$

The expressions for mean and variance are

$$E(Y) = \sum_{y=1}^{\infty} \frac{\rho^{(y+1)^\alpha}}{p + (1-p)\rho^{(y+1)^\alpha}}, \quad (28)$$

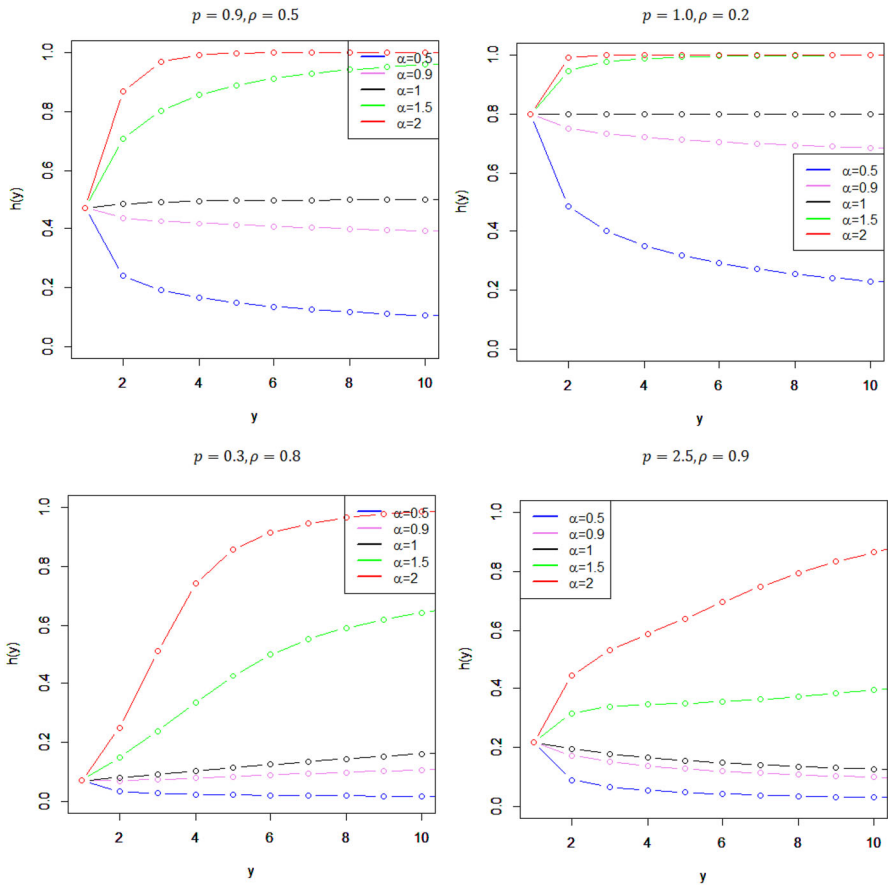


Fig. 2 Shape of hrf for various choices of parameter values

and

$$V(Y) = \sum_{y=1}^{\infty} \frac{(2y - 1)\rho^{(y+1)\alpha}}{p + (1 - p)\rho^{(y+1)\alpha}} - \left[\sum_{y=1}^{\infty} \frac{\rho^{(y+1)\alpha}}{p + (1 - p)\rho^{(y+1)\alpha}} \right]^2. \tag{29}$$

The mean, variance, skewness, and kurtosis of $DCWG(p, \rho, \alpha)$ distribution for various choices of parameter values are numerically computed and presented in Table 1. The results show that this distribution is suitable for modeling both over- and under-dispersed data.

Table 1 Mean, variance, skewness, and kurtosis of the $DCWG(p, \rho, \alpha)$ distribution for various parameter values

Parameters	$\alpha \rightarrow$	0.5	1.0	1.5	2.0
$p = 0.5$ $\rho = 0.5$	Mean	6.419	1.529	0.975	0.788
	Variance	149.154	2.814	0.785	0.418
	Skewness	5.144	1.589	0.760	0.323
	Kurtosis	54.088	6.696	3.435	2.695
$p = 1.0$ $\rho = 0.5$	Mean	3.788	1.00	0.672	0.564
	Variance	85.699	2.00	0.638	0.379
	Skewness	6.695	2.121	1.151	0.660
	Kurtosis	89.298	9.500	4.275	2.746
$p = 1.5$ $\rho = 0.8$	Mean	28.983	3.162	1.609	1.159
	Variance	5728.556	16.094	2.463	0.959
	Skewness	7.787	2.350	1.283	0.751
	Kurtosis	120.295	11.255	5.059	3.435
$p = 2.0$ $\rho = 0.2$	Mean	0.274	0.137	0.117	0.112
	Variance	1.441	0.181	0.114	0.101
	Skewness	9.973	3.846	2.832	2.537
	Kurtosis	188.106	22.100	10.624	7.741

6 Inference on parameters of the $DCWG(p, \rho, \alpha)$ distribution

Consider a random sample (y_1, y_2, \dots, y_n) of size n from $DCWG(p, \rho, \alpha)$. Then, the likelihood function is given by

$$L = \prod_{i=1}^n \frac{p \left[\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha} \right]}{\left[p + (1-p)\rho^{y_i^\alpha} \right] \left[p + (1-p)\rho^{(y_i+1)^\alpha} \right]}. \tag{30}$$

The log-likelihood function is

$$\begin{aligned} \ln L &= n \ln(p) + \sum_{i=1}^n \ln \left[\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha} \right] \\ &\quad - \sum_{i=1}^n \ln \left[p + (1-p)\rho^{y_i^\alpha} \right] - \sum_{i=1}^n \ln \left[p + (1-p)\rho^{(y_i+1)^\alpha} \right]. \end{aligned} \tag{31}$$

The likelihood equations are respectively

$$\frac{\partial \ln L}{\partial p} = \frac{n}{p} - \sum_{i=1}^n \frac{1 - \rho^{y_i^\alpha}}{p + (1-p)\rho^{y_i^\alpha}} - \sum_{i=1}^n \frac{1 - \rho^{(y_i+1)^\alpha}}{p + (1-p)\rho^{(y_i+1)^\alpha}} = 0, \tag{32}$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \rho} &= \sum_{i=1}^n \frac{y_i^\alpha \rho^{y_i^\alpha - 1} - (y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha - 1}}{\rho^{y_i^\alpha} - \rho^{(y_i + 1)^\alpha}} - (1 - p) \sum_{i=1}^n \frac{y_i^\alpha \rho^{y_i^\alpha - 1}}{p + (1 - p)\rho^{y_i^\alpha}} \\ &\quad - (1 - p) \sum_{i=1}^n \frac{(y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha - 1}}{p + (1 - p)\rho^{(y_i + 1)^\alpha}} = 0, \end{aligned} \tag{33}$$

and

$$\begin{aligned} \frac{\partial \ln L}{\partial \alpha} &= \sum_{i=1}^n \frac{\ln(\rho)[y_i^\alpha \rho^{y_i^\alpha} \ln(y_i) - (y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha} \ln(y_i + 1)]}{\rho^{y_i^\alpha} - \rho^{(y_i + 1)^\alpha}} \\ &\quad - (1 - p) \ln(\rho) \sum_{i=1}^n \frac{y_i^\alpha \rho^{y_i^\alpha} \ln(y_i)}{p + (1 - p)\rho^{y_i^\alpha}} \\ &\quad - (1 - p) \ln(\rho) \sum_{i=1}^n \frac{(y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha} \ln(y_i + 1)}{p + (1 - p)\rho^{(y_i + 1)^\alpha}} = 0. \end{aligned} \tag{34}$$

The likelihood equations do not have explicit solutions and have to be obtained numerically using statistical softwares like *nlm* package in **R** programming. Let the estimators be $\hat{\theta} = (\hat{p}, \hat{\rho}, \hat{\alpha})^T$. Here, the DCWG family satisfies the regularity conditions which are fulfilled for the parameters in the interior of the parameter space, but not on the boundary. Hence, the vector $\hat{\theta}$ is consistent and asymptotically normal. That is, $\sqrt{I_Y(\hat{\theta})}[\hat{\theta} - \theta]$ converges in distribution to multivariate normal with zero mean vector and identity covariance matrix. The Fisher’s information matrix can be computed using the approximation

$$I_Y(\hat{\theta}) \approx \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial p^2} \Big|_{\hat{\theta}} & -\frac{\partial^2 \ln L}{\partial p \partial \rho} \Big|_{\hat{\theta}} & -\frac{\partial^2 \ln L}{\partial p \partial \alpha} \Big|_{\hat{\theta}} \\ -\frac{\partial^2 \ln L}{\partial \rho \partial p} \Big|_{\hat{\theta}} & -\frac{\partial^2 \ln L}{\partial \rho^2} \Big|_{\hat{\theta}} & -\frac{\partial^2 \ln L}{\partial \rho \partial \alpha} \Big|_{\hat{\theta}} \\ -\frac{\partial^2 \ln L}{\partial \alpha \partial p} \Big|_{\hat{\theta}} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \rho} \Big|_{\hat{\theta}} & -\frac{\partial^2 \ln L}{\partial \alpha^2} \Big|_{\hat{\theta}} \end{bmatrix}, \tag{35}$$

where $\hat{\theta}$ is the MLE of θ . The existence and uniqueness of MLEs of the parameters of the DCWG(p, ρ, α) distribution are established when the other parameters are known as suggested in Popović et al. (2016) and are explained in the Theorems 1, 2, and 3 (see Appendix).

6.1 Simulation study

This section explains the performance of the MLEs of the model parameters of the DCWG distribution using Monte Carlo simulation for various sample sizes and for selected parameter values. The algorithm for the simulation study is given below:

Step 1. Input the value of replication (N);

- Step 2.** Specify the sample size n and the values of the parameters p , ρ and α ;
Step 3. Generate u_i from $U(0, 1)$, $i = 1, 2, \dots, n$;
Step 4. Obtain the random observations from the DCWG distribution using Eq. (25);
Step 5. Compute the MLEs of the three parameters;
Step 6. Repeat steps 3–5, N times;
Step 7. Compute the parameter estimate, standard error of estimate, average bias, mean square error (MSE), and coverage probability (CP) for each parameter.

Here, the expected value of the estimator is

$$E(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^N \hat{\theta}_i, E(SE(\hat{\theta})) = \sqrt{\frac{1}{N} \sum_{i=1}^N \left(-\frac{\partial^2 \log(L)}{\partial \theta_i^2} \right)},$$

Average Bias = $\frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)$, $MSE(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)^2$ and

CP = Probability of $\theta_i \in \left(\hat{\theta}_i \pm 1.96 \sqrt{-\frac{\partial^2 \log(L)}{\partial \theta_i^2}} \right)$.

We take random samples of size $n = 50, 100, 200,$ and 500 , respectively. The MLEs of the parameter vector $\theta = (p, \rho, \alpha)^T$ are determined by maximizing the log-likelihood function in Eq. (31) using the *optim* package in **R** programming based on each generated sample. This simulation is repeated 1000 times, and the average estimate and its standard error, average bias, MSE, and CP are computed and presented in Table 2. From Table 2, it can be seen that, as sample size increases, the estimates of bias and MSE decrease. Also note that the CP values are quite closer to the 95% nominal level.

7 Data application

In this section, we analyze a real-life data set to prove empirically the flexibility of the DCWG distribution. We compare the fit of the DCWG distribution with the following discrete life time distributions:

- (a) Discrete Weibull-geometric (DWG) distribution (Jayakumar & Babu, 2018) having pmf

$$P(X = x) = \frac{(1-p)(\rho^{x\alpha} - \rho^{(x+1)\alpha})}{(1-p\rho^{x\alpha})(1-p\rho^{(x+1)\alpha})};$$

$$\alpha > 0, 0 < p < 1, 0 < \rho < 1, x = 0, 1, 2, \dots$$

- (b) Exponentiated discrete Weibull (EDW) distribution (Nekoukhou & Bidram, 2015) having pmf

$$P(X = x) = (1 - p^{(x+1)\alpha})^\beta - (1 - p^{x\alpha})^\beta;$$

$$0 < p < 1, \alpha > 0, \beta > 0, x = 0, 1, 2, \dots$$

Table 2 The parameter estimate, standard error, average bias, MSE, and CP for given parameters

Parameter(θ)	Samples (n)	$E(\hat{\theta})(E(SE(\hat{\theta})))$	Average bias	MSE	CP
$p = 0.5$	50	0.3481(0.1202)	-0.151	0.049	86.4
	100	0.3992(0.0911)	-0.101	0.011	88.3
	200	0.4235(0.0531)	-0.077	0.006	90.6
	500	0.4807(0.0327)	-0.018	0.003	92.5
$\rho = 0.9$	50	0.5861(0.1925)	-0.324	0.099	83.8
	100	0.6032(0.1322)	-0.288	0.089	85.1
	200	0.7864(0.1103)	-0.104	0.013	90.6
	500	0.8429(0.1051)	-0.061	0.001	93.2
$\alpha = 0.5$	50	0.5886(0.1317)	0.091	0.008	90.3
	100	0.5432(0.0633)	0.034	0.003	92.6
	200	0.5277(0.0432)	0.028	0.002	93.8
	500	0.5013(0.021)	0.011	0.001	94.5
$p = 0.8$	50	0.4138(0.1316)	-0.391	0.147	88.2
	100	0.5883(0.1172)	-0.201	0.045	89.3
	200	0.7114(0.1082)	-0.089	0.008	91.6
	500	0.7938(0.0922)	-0.016	0.001	94.1
$\rho = 0.5$	50	0.4832(0.1132)	-0.017	0.019	90.7
	100	0.4891(0.1071)	-0.011	0.012	92.6
	200	0.4986(0.0192)	-0.009	0.002	93.1
	500	0.4993(0.0112)	-0.007	0.001	94.6
$\alpha = 1.5$	50	2.1336(0.2218)	0.534	0.402	87.9
	100	1.9817(0.1677)	0.451	0.238	89.8
	200	1.7926(0.1281)	0.283	0.088	91.3
	500	1.6013(0.0927)	0.101	0.010	93.9

(c) Discrete modified Weibull (DMW) distribution (Nooghabi et al., 2011) having pmf

$$P(X = x) = \rho^{x^\alpha c^x} - \rho^{(x+1)^\alpha c^{(x+1)}}; \\ 0 < \rho < 1, \alpha > 0, c \geq 1, x = 0, 1, 2, \dots$$

(d) Discrete Weibull (DW) distribution (Nakagawa & Osaki, 1975) having pmf

$$P(X = x) = \rho^{x^\alpha} - \rho^{(x+1)^\alpha}; \\ 0 < \rho < 1, \alpha > 0, x = 0, 1, 2, \dots$$

(e) Geometric (G) distribution having pmf

$$P(X = x) = (1 - \rho)\rho^x; 0 < \rho < 1, x = 0, 1, 2, \dots$$

The values of the log-likelihood function ($-\log L$), the K-S (Kolmogrov–Smirnov) statistic, AIC (Akaike Information Criterion), CAIC (Akaike Information Criterion with correction), BIC (Bayesian Information Criterion), and HQIC (Hannon–Quinn Information Criterion) are calculated for the six distributions to verify which distribu-

Table 3 Descriptive statistics for the data set

Samples(n)	Mean	SD	Min.	Max.	Skewness	Kurtosis
116	42.13	32.99	1.0	168	1.24	1.29

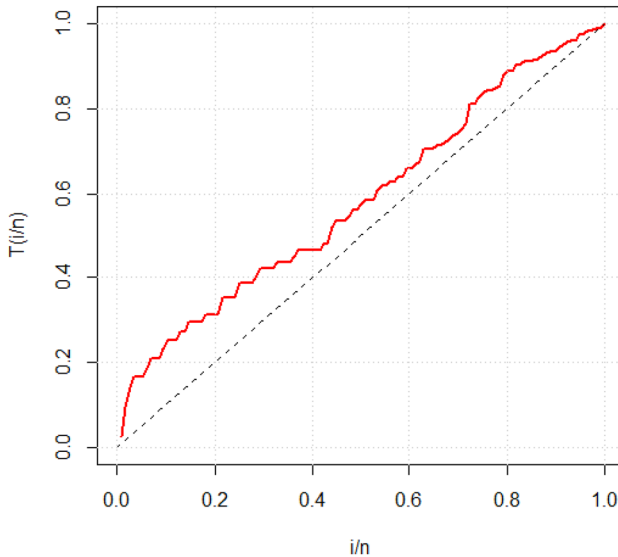


Fig. 3 The TTT plots of the data set

tion fits better to the data. The better distribution corresponds to smaller $-\log L$, $K-S$, AIC, CAIC, BIC, HQIC, and greater p value.

Here, $AIC = -2 \log L + 2k$, $CAIC = -2 \log L + \left(\frac{2kn}{n-k-1}\right)$, $BIC = -2 \log L + k \log n$, and $HQIC = -2 \log L + 2k \log(\log(n))$, where L is the likelihood function evaluated at the maximum-likelihood estimates, k is the number of parameters, and n is the sample size. We used the *AdequacyModel* package in **R** programming to obtain the MLE estimates and goodness-of-fit tests of the given data set.

The data set corresponds to daily ozone concentrations that were collected in New York during May–September, 1973 and were taken from Ferreira et al. (2012). This set of daily ozone-level measurements (in ppb = ppm × 1000) are as follows:

7 115 79 31 9 8 45 61 23 28 19 23 35 59 21 23 32 48 22 44 28 4 7 65 24 13 18 11 27
 44 21 73 12 1 10 110 23 28 36 30 85 89 20 80 41 6 97 122 32 135 34 21 82 73 16 14
 23 52 168 24 18 39 20 45 13 14 71 108 9 18 11 29 16 21 46 16 37 63 44 13 12 59 84
 7 20 64 118 36 37 50 76 23 13 39 85 14 49 9 96 30 32 16 78 14 64 78 91 18 40 35 47
 20 77 66 97 11.

The descriptive statistics of the data set are presented in Table 3. To identify the shape of the hrf of the data set, we consider the Total Time on Test (TTT) plot and it is shown in Fig. 3. From the TTT plot, we can see that the hrf is an increasing function, and therefore, the DCWG distribution is appropriate for this data set.

Table 4 The parameter estimates of data set

Model	ML estimates	$-\log L$
DCWG	$\hat{\rho} = 0.4480, \hat{\rho} = 0.9736, \hat{\alpha} = 1.0488$	547.008
DWG	$\hat{\rho} = 0.3140, \hat{\rho} = 0.9679, \hat{\alpha} = 0.9368$	564.514
EDW	$\hat{\rho} = 0.8688, \hat{\alpha} = 0.6621, \hat{\beta} = 1.7672$	558.830
DMW	$\hat{\rho} = 0.9772, \hat{\alpha} = 0.9113, \hat{c} = 1.0042$	549.488
DW	$\hat{\rho} = 0.9660, \hat{\alpha} = 0.9280$	555.843
G	$\hat{\rho} = 0.9768$	551.292

Table 5 The goodness-of-fit statistics for data set

Model	AIC	CAIC	BIC	HQIC	K-S	p value
DCWG	1097.629	1097.843	1105.890	1100.982	0.0833	0.3971
DWG	1135.029	1135.243	1143.289	1138.382	0.2721	6.95×10^{-8}
EDW	1123.660	1123.874	1131.921	1127.013	0.2174	3.46×10^{-5}
DMW	1104.977	1105.191	1113.237	1108.330	0.1210	0.0669
DW	1115.686	1115.792	1121.193	1117.921	0.1789	0.0012
G	1104.584	1104.620	1107.338	1105.702	0.1326	0.0338

The parameter estimates and goodness-of-fit statistics for the data set are presented in Tables 4 and 5 .

The $-\log L$ value shown in Table 4 is minimum for DCWG distribution. From Table 5, the values of AIC, CAIC, BIC, HQIC, and K–S are minimum, and p value is maximum for DCWG distribution. This shows that the DCWG distribution is a better model for this data set. Figure 4 shows the fitted pdf and cdf with the empirical distribution of the given data set.

8 Conclusion

In this paper, we introduced a new generalized family of distributions generated from random maxima, which contains many distributions of which complementary Weibull-geometric distribution is a special case. Then, we proposed a new three-parameter discrete complementary Weibull-geometric (DCWG) distribution. This distribution is a generalization of discrete Weibull and geometric distributions. We studied the main mathematical and statistical properties including the quantile function and probability generating function of the DCWG distribution. The model parameters were estimated using maximum-likelihood estimation method and a simulation study is presented to illustrate the performance of the method. The existence and uniqueness of the MLE estimates were investigated. The DCWG distribution was applied to a practical data set to show its flexibility for data modeling.

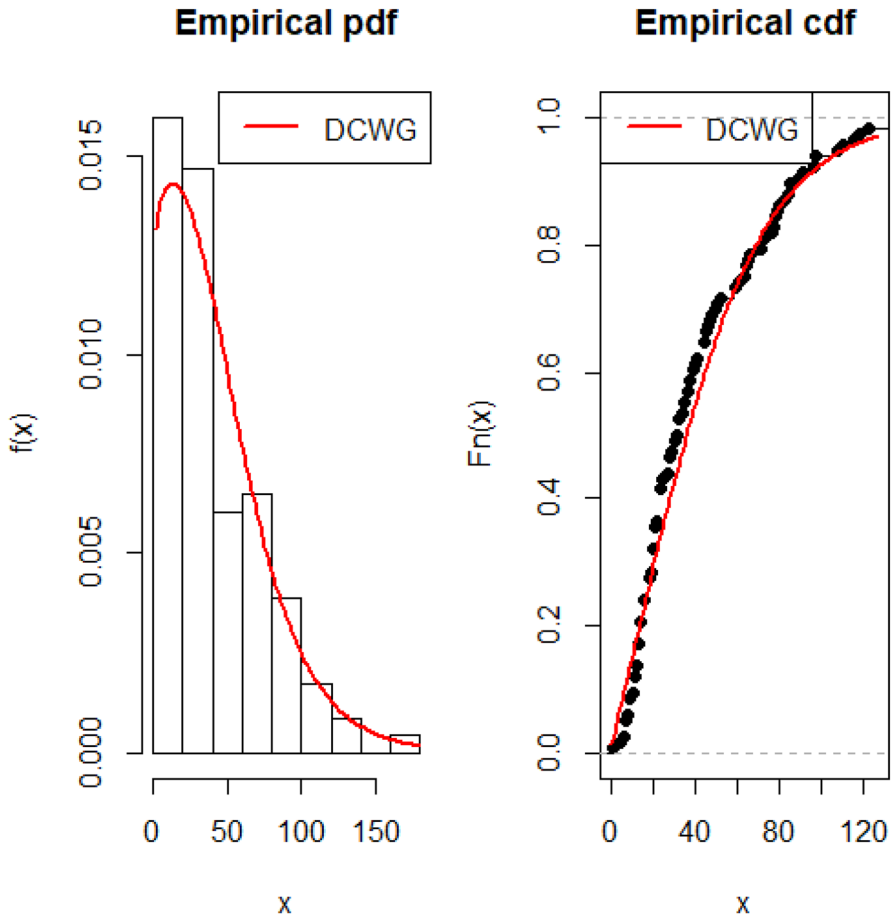


Fig. 4 Fitted pdf and cdf plots for the data set

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix

Theorem 1 From Eq. (32), let $f_1(p; \rho, \alpha, y) = \frac{\partial \ln(L)}{\partial p}$, where ρ and α are the true values of the parameters. Then, there exists a unique solution for $f_1(p; \rho, \alpha, y) = 0$,

for $\hat{p} \in (0, \infty)$ when

$$\sum_{i=1}^n \frac{(1 - \rho^{y_i^\alpha})^2}{(p + (1 - p)\rho^{y_i^\alpha})} + \sum_{i=1}^n \frac{(1 - \rho^{(y_i+1)^\alpha})^2}{(p + (1 - p)\rho^{(y_i+1)^\alpha})} < \frac{n}{p^2}.$$

Proof We have

$$f_1(p; \rho, \alpha, y) = \frac{n}{p} - \sum_{i=1}^n \frac{1 - \rho^{y_i^\alpha}}{p + (1 - p)\rho^{y_i^\alpha}} - \sum_{i=1}^n \frac{1 - \rho^{(y_i+1)^\alpha}}{p + (1 - p)\rho^{(y_i+1)^\alpha}}. \tag{36}$$

The limiting values of $f_1(p; \rho, \alpha, y)$ as $p \rightarrow 0$ and $p \rightarrow \infty$ are obtained as follows:

$$\lim_{p \rightarrow 0} f_1(p; \rho, \alpha, y) = \infty + 2n - \left[\sum_{i=1}^n \frac{1}{\rho^{y_i^\alpha}} + \sum_{i=1}^n \frac{1}{\rho^{(y_i+1)^\alpha}} \right] = \infty, \tag{37}$$

and

$$\lim_{p \rightarrow \infty} f_1(p; \rho, \alpha, y) = 0, \tag{38}$$

since $\lim_{p \rightarrow \infty} \sum_{i=1}^n \frac{1 - \rho^{y_i^\alpha}}{p + (1 - p)\rho^{y_i^\alpha}} = 0$ and $\lim_{p \rightarrow \infty} \sum_{i=1}^n \frac{1 - \rho^{(y_i+1)^\alpha}}{p + (1 - p)\rho^{(y_i+1)^\alpha}} = 0$.

Thus, there exist at least one root, say $\hat{p} \in (0, \infty)$, such that $f_1(p; \rho, \alpha, y) = 0$.

Now, to show the uniqueness, we have to show that $\frac{\partial f_1(p; \rho, \alpha, y)}{\partial p} < 0$, that is

$$-\frac{n}{p^2} + \sum_{i=1}^n \frac{(1 - \rho^{y_i^\alpha})^2}{(p + (1 - p)\rho^{y_i^\alpha})} + \sum_{i=1}^n \frac{(1 - \rho^{(y_i+1)^\alpha})^2}{(p + (1 - p)\rho^{(y_i+1)^\alpha})} < 0, \tag{39}$$

and this is possible when

$$\sum_{i=1}^n \frac{(1 - \rho^{y_i^\alpha})^2}{(p + (1 - p)\rho^{y_i^\alpha})} + \sum_{i=1}^n \frac{(1 - \rho^{(y_i+1)^\alpha})^2}{(p + (1 - p)\rho^{(y_i+1)^\alpha})} < \frac{n}{p^2}. \tag{40}$$

This completes the proof. □

Theorem 2 From Eq. (33), let $f_2(\rho; p, \alpha, y) = \frac{\partial \ln(L)}{\partial \rho}$, where p and α are the true values of the parameters. Then, there exist a unique solution for $f_2(\rho; p, \alpha, y) = 0$, for $\hat{\rho} \in (0, 1)$.

Proof We have

$$f_2(\rho; p, \alpha, y) = \sum_{i=1}^n \frac{y_i^\alpha \rho^{y_i^\alpha - 1} - (y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha - 1}}{\rho^{y_i^\alpha} - \rho^{(y_i + 1)^\alpha}} - (1 - p) \sum_{i=1}^n \frac{y_i^\alpha \rho^{y_i^\alpha - 1}}{p + (1 - p)\rho^{y_i^\alpha}} - (1 - p) \sum_{i=1}^n \frac{(y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha - 1}}{p + (1 - p)\rho^{(y_i + 1)^\alpha}}. \tag{41}$$

Now, we can see that

$$\lim_{\rho \rightarrow 0} f_2(\rho; p, \alpha, y) = \infty, \tag{42}$$

because

$$\lim_{\rho \rightarrow 0^+} \sum_{i=1}^n \frac{y_i^\alpha \rho^{y_i^\alpha - 1} - (y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha - 1}}{\rho^{y_i^\alpha} - \rho^{(y_i + 1)^\alpha}} = \infty, \quad \lim_{\rho \rightarrow 0} \sum_{i=1}^n \frac{y_i^\alpha \rho^{y_i^\alpha - 1}}{p + (1 - p)\rho^{y_i^\alpha}} = 0$$

and

$$\lim_{\rho \rightarrow 0} \sum_{i=1}^n \frac{(y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha - 1}}{p + (1 - p)\rho^{(y_i + 1)^\alpha}} = 0.$$

Also

$$\lim_{\rho \rightarrow 1} f_2(\rho; p, \alpha, y) = -\infty - (1 - p) \left[\sum_{i=1}^n y_i^\alpha + \sum_{i=1}^n (y_i + 1)^\alpha \right] = -\infty. \tag{43}$$

Hence, there exist a root for $\rho \in (0, 1)$.

The first derivative of $f_2(\rho; p, \alpha, y)$ is given by

$$\begin{aligned} \frac{\partial f_2(\rho; p, \alpha, y)}{\partial \rho} &= \sum_{i=1}^n \left[\frac{y_i^\alpha \rho^{y_i^\alpha} (y_i^\alpha - 1) - (y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha} [(y_i + 1)^\alpha - 1]}{\rho^{y_i^\alpha} - \rho^{(y_i + 1)^\alpha}} \right. \\ &\quad \left. - \frac{[y_i^\alpha \rho^{y_i^\alpha - 1} - (y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha - 1}]^2}{[\rho^{y_i^\alpha} - \rho^{(y_i + 1)^\alpha}]^2} \right] \\ &\quad - (1 - p) \sum_{i=1}^n \left[\frac{y_i^\alpha \rho^{y_i^\alpha - 1} (y_i^\alpha - 1)}{\rho [p + (1 - p)\rho^{y_i^\alpha}]} - \frac{(1 - p) y_i^{2\alpha} \rho^{2(y_i^\alpha - 1)}}{[p + (1 - p)\rho^{y_i^\alpha}]^2} \right] \\ &\quad - (1 - p) \sum_{i=1}^n \left[\frac{(y_i + 1)^\alpha \rho^{(y_i + 1)^\alpha - 1} [(y_i + 1)^\alpha - 1]}{\rho [p + (1 - p)\rho^{(y_i + 1)^\alpha}]} \right. \\ &\quad \left. - \frac{(1 - p)(y_i + 1)^{2\alpha} \rho^{2[(y_i + 1)^\alpha - 1]}}{[p + (1 - p)\rho^{(y_i + 1)^\alpha}]^2} \right]. \tag{44} \end{aligned}$$

The roots are unique when

$$\begin{aligned}
 & \sum_{i=1}^n \left[\frac{y_i^\alpha \rho^{y_i^\alpha} (y_i^\alpha - 1) - (y_i + 1)^\alpha \rho^{(y_i+1)^\alpha} [(y_i + 1)^\alpha - 1]}{\rho^{y_i^\alpha} - \rho^{(y_i + 1)^\alpha}} \right. \\
 & \quad \left. - \frac{[y_i^\alpha \rho^{y_i^\alpha - 1} - (y_i + 1)^\alpha \rho^{(y_i+1)^\alpha - 1}]^2}{[\rho^{y_i^\alpha} - \rho^{(y_i + 1)^\alpha}]^2} \right] \\
 & < (1 - p) \sum_{i=1}^n \left[\frac{y_i^\alpha \rho^{y_i^\alpha - 1} (y_i^\alpha - 1)}{\rho[p + (1 - p)\rho^{y_i^\alpha}]} - \frac{(1 - p)y_i^{2\alpha} \rho^{2(y_i^\alpha - 1)}}{[p + (1 - p)\rho^{y_i^\alpha}]^2} \right] \\
 & + (1 - p) \sum_{i=1}^n \left[\frac{(y_i + 1)^\alpha \rho^{(y_i+1)^\alpha - 1} [(y_i + 1)^\alpha - 1]}{\rho[p + (1 - p)\rho^{(y_i+1)^\alpha}]} \right. \\
 & \quad \left. - \frac{(1 - p)(y_i + 1)^{2\alpha} \rho^{2[(y_i+1)^\alpha - 1]}}{[p + (1 - p)\rho^{(y_i+1)^\alpha}]^2} \right]. \tag{45}
 \end{aligned}$$

This completes the proof. □

Theorem 3 From Eq. (34), let $f_3(\alpha; p, \rho, y) = \frac{\partial \ln(L)}{\partial \alpha}$, where p and ρ are the true values of the parameters. Then, there exist a unique solution for $f_3(\alpha; p, \rho, y) = 0$, for $\hat{\alpha} \in (0, \infty)$.

Proof We have

$$\begin{aligned}
 f_3(\alpha; p, \rho, y) &= \sum_{i=1}^n \frac{\ln(\rho)[y_i^\alpha \rho^{y_i^\alpha} \ln(y_i) - (y_i + 1)^\alpha \rho^{(y_i+1)^\alpha} \ln(y_i + 1)]}{\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha}} \\
 & - (1 - p) \ln(\rho) \sum_{i=1}^n \frac{y_i^\alpha \rho^{y_i^\alpha} \ln(y_i)}{p + (1 - p)\rho^{y_i^\alpha}} \\
 & - (1 - p) \ln(\rho) \sum_{i=1}^n \frac{(y_i + 1)^\alpha \rho^{(y_i+1)^\alpha} \ln(y_i + 1)}{p + (1 - p)\rho^{(y_i+1)^\alpha}}. \tag{46}
 \end{aligned}$$

Then, for $y_i > 0$, we have

$$\lim_{\alpha \rightarrow 0} f_3(\alpha; p, \rho, y) = \infty, \tag{47}$$

since

$$\begin{aligned}
 & \lim_{\alpha \rightarrow 0} \sum_{i=1}^n \frac{\ln(\rho)[y_i^\alpha \rho^{y_i^\alpha} \ln(y_i) - (y_i + 1)^\alpha \rho^{(y_i+1)^\alpha} \ln(y_i + 1)]}{\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha}} = \infty, \\
 & \lim_{\alpha \rightarrow 0} (1 - p) \ln(\rho) \sum_{i=1}^n \frac{y_i^\alpha \rho^{y_i^\alpha} \ln(y_i)}{p + (1 - p)\rho^{y_i^\alpha}} = \frac{(1 - p) \ln(\rho)}{p + (1 - p)\rho} \sum_{i=1}^n \ln(y_i)
 \end{aligned}$$

and

$$\lim_{\alpha \rightarrow 0} (1 - p) \ln(\rho) \sum_{i=1}^n \frac{(y_i + 1)^\alpha \rho^{(y_i+1)^\alpha} \ln(y_i + 1)}{p + (1 - p)\rho^{(y_i+1)^\alpha}} = \frac{(1 - p) \ln(\rho)}{p + (1 - p)\rho} \sum_{i=1}^n \ln(y_i + 1).$$

Also

$$\lim_{\alpha \rightarrow \infty} f_3(\rho; p, \alpha, y) = 0, \tag{48}$$

since

$$\lim_{\alpha \rightarrow \infty} \sum_{i=1}^n \frac{\ln(\rho)[y_i^\alpha \rho^{y_i^\alpha} \ln(y_i) - (y_i + 1)^\alpha \rho^{(y_i+1)^\alpha} \ln(y_i + 1)]}{\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha}} = 0,$$

$$\lim_{\alpha \rightarrow \infty} (1 - p) \ln(\rho) \sum_{i=1}^n \frac{y_i^\alpha \rho^{y_i^\alpha} \ln(y_i)}{p + (1 - p)\rho^{y_i^\alpha}} = 0$$

and

$$\lim_{\alpha \rightarrow \infty} (1 - p) \ln(\rho) \sum_{i=1}^n \frac{(y_i + 1)^\alpha \rho^{(y_i+1)^\alpha} \ln(y_i + 1)}{p + (1 - p)\rho^{(y_i+1)^\alpha}} = 0.$$

Hence, there exist a root for $\alpha \in (0, \infty)$. The first derivative of $f_3(\alpha; p, \rho, y)$ with respect to α is given by

$$\frac{\partial f_3(\alpha; p, \rho, y)}{\partial \alpha} = \ln(\rho)D_1 - (1 - p) \ln(\rho)(D_2 + D_3), \tag{49}$$

where

$$D_1 = \sum_{i=1}^n \left[\frac{y_i^\alpha \rho^{y_i^\alpha} [\ln(y_i)]^2 [1 + y_i^\alpha \ln(\rho)] - (y_i + 1)^\alpha \rho^{(y_i+1)^\alpha} [\ln(y_i + 1)]^2 [1 + (y_i + 1)^\alpha \ln(\rho)]}{\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha}} - \frac{[y_i^\alpha \rho^{y_i^\alpha} \ln(y_i) - (y_i + 1)^\alpha \rho^{(y_i+1)^\alpha} \ln(y_i + 1)]^2}{[\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha}]^2} \right],$$

$$D_2 = \sum_{i=1}^n \left[\frac{y_i^\alpha \ln(y_i) \rho^{y_i^\alpha} [1 + y_i^\alpha \ln(y_i) \ln(\rho)]}{p + (1 - p)\rho^{y_i^\alpha}} - \frac{(1 - p) \ln(\rho) y_i^{2\alpha} \rho^{2y_i^\alpha} [\ln(y_i)]^2}{[p + (1 - p)\rho^{y_i^\alpha}]^2} \right],$$

and

$$D_3 = \sum_{i=1}^n \left[\frac{(y_i + 1)^\alpha \ln(y_i + 1) \rho^{(y_i+1)^\alpha} [1 + (y_i + 1)^\alpha \ln(y_i + 1) \ln(\rho)]}{p + (1 - p)\rho^{(y_i+1)^\alpha}} - \frac{(1 - p) \ln(\rho) (y_i + 1)^{2\alpha} \rho^{2(y_i+1)^\alpha} [\ln(y_i + 1)]^2}{[p + (1 - p)\rho^{(y_i+1)^\alpha}]^2} \right].$$

The roots are unique when

$$D_1 < (1 - p)(D_2 + D_3). \quad (50)$$

This completes the proof. \square

References

- Adamidis, K., & Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics & Probability Letters*, 39, 35–42.
- Alamatsaz, M. H., Dey, S., Dey, T., & Harandi, S. S. (2016). Discrete generalized Rayleigh distribution. *Pakistan Journal of Statistics*, 32, 1–20.
- Almalki, S. J., & Nadarajah, S. (2014). A new discrete modified Weibull distribution. *IEEE Transactions on Reliability*, 63, 68–80.
- Bakouch, H. S., Jazi, M. A., & Nadarajah, S. (2014). A new discrete distribution. *Statistics*, 48, 200–240.
- Barreto-Souza, W., de Morais, A. L., & Cordeiro, G. M. (2011). The Weibull-geometric distribution. *Journal of Statistical Computation and Simulation*, 81, 645–657.
- Basu, A., & Klein, J. (1982). Some recent development in competing risks theory. In J. Crowley & R. A. Johnson (Eds.), *Survival analysis* (Vol. 1, pp. 216–229). IMS.
- Bebbington, M., Lai, C. D., Wellington, M., & Zitikis, R. (2012). The discrete additive Weibull distribution: A bathtub-shaped hazard for discontinuous failure data. *Reliability Engineering and System Safety*, 106, 37–44.
- Chakraborty, S. (2015). Generating discrete analogues of continuous probability distributions: A survey of methods and constructions. *Journal of Statistical Distributions and Applications*, 2, 1–30.
- Chakraborty, S., & Chakravarty, D. (2012). Discrete gamma distribution: properties and parameter estimation. *Communications in Statistics: Theory and Methods*, 41, 3301–3324.
- Cox, D. R., & Oakes, D. (1984). *Analysis of survival data*. London: Chapman and Hall.
- Crowder, M., Kimber, A., Smith, R., & Sweeting, T. (1991). *Statistical analysis of reliability data*. London: Chapman and Hall.
- Ferreira, M., Gomes, M. I., & Leiva, V. (2012). On an extreme value version of the Birnbaum-Saunders distribution. *REVSTAT-Statistical Journal*, 10, 181–210.
- Goetghebeur, E., & Ryan, L. (1995). A modified log rank test for competing risks with missing failure type. *Biometrics*, 77, 207–211.
- Gomez-Deniz, E. (2010). Another generalization of the geometric distribution. *Test*, 19, 399–415.
- Gupta, P. L., Gupta, R. C., & Tripathi, R. C. (1997). On the monotonic properties of the discrete failure rate. *Journal of Statistical Planning and Inference*, 65, 225–268.
- Jayakumar, K., & Babu, M. G. (2018). Discrete Weibull geometric distribution and its properties. *Communications in Statistics: Theory and Methods*, 47, 1767–1783.
- Jayakumar, K., & Babu, M. G. (2019). Discrete additive Weibull geometric distribution. *Journal of Statistical Theory and Applications*, 18, 33–45.
- Jazi, M. A., Lai, C. D., & Alamatsaz, M. H. (2010). A discrete inverse Weibull distribution and estimation of its parameters. *Statistical Methodology*, 7, 121–132.
- Jose, F. D., Borges, P., Cancho, V. G., & Louzada, F. (2013). The complementary exponential power series distribution. *Brazilian Journal of Probability and Statistics*, 27, 565–584.
- Kemp, A. W. (1997). Characterizations of a discrete normal distribution. *Journal of Statistical Planning and Inference*, 63, 223–229.
- Kemp, A. W. (2008). The discrete half-normal distribution, In: Birkha (Ed.), *Advances in mathematical and statistical modeling* (pp. 353–365).
- Khan, M. S. A., Khalique, A., & Abouammoh, A. M. (1989). On estimating parameters in a discrete Weibull distribution. *IEEE Transactions on Reliability*, 38, 348–350.
- Krishna, H., & Pundir, P. S. (2009). Discrete Burr and discrete Pareto distributions. *Statistical Methodology*, 6, 177–188.
- Kulasekera, K. B. (1994). Approximate MLEs of the parameters of a discrete Weibull distribution with type I censored data. *Microelectronics Reliability*, 34, 1185–1186.
- Lawless, J. F. (2003). *Statistical models and methods for lifetime data* (2nd ed.). New York: Wiley.

- Lisman, J. H. C., & van Zuylen, M. C. A. (1972). Note on the generation of most probable frequency distributions. *Statistica Neerlandica*, 26, 19–23.
- Louzada, F., Marchi, V., & Carpenter, J. (2013). The complementary exponentiated exponential geometric lifetime distribution. *Journal of Probability and Statistics*, 1–12.
- Louzada, F., Roman, M., & Cancho, V. G. (2011). The complementary exponential geometric distribution: Model, properties, and a comparison with its counterpart. *Computational Statistics & Data Analysis*, 55, 2516–2524.
- Lu, K., & Tsiatis, A. A. (2005). Comparison between two partial likelihood approaches for the competing risks model with missing cause of failure. *Lifetime Data Analysis*, 11, 29–40.
- Nakagawa, T., & Osaki, S. (1975). The discrete Weibull distribution. *IEEE Transactions on Reliability*, 24, 300–301.
- Nekoukhou, V., Alamatsaz, M. H., & Bidram, H. (2012). A discrete analogue of the generalized exponential distribution. *Communications in Statistics: Theory and Methods*, 41, 2000–2013.
- Nekoukhou, V., Alamatsaz, M. H., & Bidram, H. (2013). Discrete generalized exponential distribution of the second type. *Statistics*, 4, 476–887.
- Nekoukhou, V., & Bidram, H. (2015). The exponentiated discrete Weibull distribution. *SORT*, 39, 127–146.
- Nooghabi, M. S., Roknabadi, A. H. R., & Borzadaran, G. R. M. (2011). Discrete modified Weibull distribution. *Metron*, 69, 207–222.
- Padgett, W. J., & Spurrier, J. D. (1985). Discrete failure models. *IEEE Transactions on Reliability*, 34, 253–256.
- Popović, B. V., Ristić, M. M., & Cordeiro, G. M. (2016). A two-parameter distribution obtained by compounding the generalized exponential and exponential distributions. *Mediterranean Journal of Mathematics*, 13, 2935–2949.
- Reiser, B., Guttman, I., Lin, D., Guess, M., & Usher, J. (1995). Bayesian inference for masked system lifetime data. *Applied Statistics*, 44, 79–90.
- Roy, D. (2003). The discrete normal distribution. *Communications in Statistics: Theory and Methods*, 32, 1871–1883.
- Roy, D. (2004). Discrete Rayleigh distribution. *IEEE Transactions on Reliability*, 53, 255–260.
- Stein, W. E., & Dattero, R. (1984). On proportional odds models. *IEEE Transactions on Reliability*, 33, 196–197.
- Tojeiro, C., Louzada, F., Roman, M., & Borges, P. (2014). The complementary Weibull geometric distribution. *Journal of Statistical Computation and Simulation*, 84, 1345–1362.

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