Planar Quadratic Differential Systems with Invariants of the Form $ax^2 + bxy + cy^2 + dx + ey + c_1t$

Jaume Llibre[1] · Tayeb Salhi[2]

Abstract
A function $I(x, y, t)$ constant on the solutions of a differential system in $\mathbb{R}^2$ is called an invariant. We classify all planar quadratic differential systems having invariants of the form $I(x, y, t) = ax^2 + bxy + cy^2 + dx + ey + c_1t$ with $c_1 \neq 0$. There are 13 different families of quadratic systems having invariants of this form. As far as we know this is the first time that quadratic differential systems having an invariant different from a Darboux invariant have been classified

1 Introduction and statement of the main results

In this paper we shall work with polynomial differential systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

✉ Tayeb Salhi
  t.salhi@univ-bba.dz

  Jaume Llibre
  jllibre@mat.uab.cat

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona Bellaterra, Barcelona 08193, Catalonia, Spain

[2] Department of Mathematics, University Mohamed El Bachir El Ibrahimi, Bordj Bou Arreridj, El-anasser 34265, Algeria

where P and Q are polynomials of degree 2, called simply textit quadratic systems. As usual the dot denotes derivative with respect to the independent variable t, usually called the time.

Many natural phenomena in various branches of the sciences are modelized using quadratic systems. We can find in the literature more than one thousand published papers studying the quadratic systems, see for instance the books [3, 14, 15] and the hundreds of references which are cited therein.

Let U be an open and dense subset of $\mathbb{R}^2$, an textit invariant of system (1.1) in U is a non-constant C^1 function $I : U \times \mathbb{R} \to \mathbb{R}$ depending on t such that $I(x(t), y(t), t)$ is constant on all the solution curves $(x(t), y(t))$ of system (1.1) contained in U, i.e.

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}P + \frac{\partial I}{\partial x}Q + \frac{\partial I}{\partial t} = 0, \tag{1.2}$$

for all $(x, y) \in U$. For more details on the invariants, see [7, Chapter 8] or [13].

The objective of this paper is to classify all quadratic systems

$$\begin{aligned}\dot{x} &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2, \\ \dot{y} &= b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2,\end{aligned} \tag{1.3}$$

having invariants of the form

$$I(x, y, t) = ax^2 + bxy + cy^2 + dx + ey + c_1 t, \tag{1.4}$$

with $c_1 \neq 0$.

Quadratic planar differential systems play a pivotal role in mathematical modeling, offering valuable insights into the behavior of dynamic systems. Their significance extends across diverse fields, including physics, biology, and engineering, where they serve as powerful tools for understanding complex phenomena. For example the works [2, 5, 6] delve into the classification of quadratic planar differential systems and highlight their relevance in the analysis of dynamic behavior. These systems provide a mathematical framework for studying the intricate interplay of variables and offer a nuanced understanding of the underlying dynamics. The exploration and understanding of quadratic planar differential systems contribute not only to theoretical advancements but also to practical applications in various scientific disciplines.

We note that many different classes of quadratic systems have been classified as the structurally stable, their centers, their isochronous centers, their Hopf bifurcations, their Lotka-Volterra, their Bernoulli, their chordal, their Abel, their quadratic-linear, their with a unique finite singularity, their having a polynomial first integral, their having a Hamiltonian first integral, their homogeneous,..., see for details [3]. But there are few works on the quadratic systems having invariants, see [4, 10–13], but all the invariants of these works are Darboux invariants, i.e. invariants of the form $f(x, y)e^{st}$ with $s \in \mathbb{R} \setminus \{0\}$.

As far as we know this is the first time that quadratic systems having an invariant different from a Darboux invariant are studied. Thus our first main result is the following.

Theorem 1.1 *The quadratic systems* (1.3) *admitting an invariant of the form* (1.4) *are one of the following* 13 *families of quadratic systems:*

$$
\begin{aligned}
\dot{x} &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2, \\
\dot{y} &= b_0 - \frac{d}{e}(a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2), \\
I &= dx + ey - (da_0 + eb_0)t,
\end{aligned}
\tag{1.5}
$$

with $e \neq 0$ *and* $da_0 + eb_0 \neq 0$.

$$
\begin{aligned}
\dot{x} &= a_0 + a_1 x + \frac{1}{b^2}(b(2ca_1 + ea_4) - 4cea_3)y + a_3 x^2 + a_4 xy + \frac{2}{b^2}c(ba_4 - 2ca_3)y^2, \\
\dot{y} &= \frac{1}{2b^2 c}((ba_1 - ea_3)(be - 2cd) - b^3 a_0) - \frac{1}{2bc}(b^2 a_1 - bea_3 + 2cda_3)x \\
&\quad + \frac{1}{b^2}(bea_3 + 2cda_3 - b^2 a_1 - bda_4)y - \frac{1}{2c}ba_3 x^2 - \frac{1}{2c}ba_4 xy + \frac{1}{b}(2ca_3 - ba_4)y^2, \\
I &= dx + ey + \frac{b^2}{4c}x^2 + bxy + cy^2 + \frac{1}{2b^2 c}(b^2 a_0 + e(ea_3 - ba_1))(be - 2cd)t,
\end{aligned}
\tag{1.6}
$$

with $bc \neq 0$ *and* $(b^2 a_0 + e(ea_3 - ba_1))(be - 2cd) \neq 0$.

$$
\begin{aligned}
\dot{x} &= a_0 + \left(\frac{d}{bd-2ae}(2aa_2 + bb_2) + \frac{e}{b}a_3 - b_2\right)x + a_2 y + a_3 x^2 \\
&\quad + \left(\frac{b}{2ae-bd}(2aa_2 + bb_2) + \frac{b}{2a}a_3\right)xy \\
&\quad - \frac{b^2}{2abd-4a^2e}(2aa_2 + b_2 b)y^2, \\
\dot{y} &= -\frac{2}{b}a(a_0 b + a_2 d + b_2 e) + \frac{1}{b(bd-2ae)} \\
&\quad (-4a^2(a_2 d + b_2 e) - a_3 bd^2 + 2a_3 ade)x + b_2 y \\
&\quad - \frac{2}{b}aa_3 x^2 + \left(\frac{2a}{bd-2ae}(2aa_2 + bb_2) - a_3\right)xy + \frac{b}{bd-2ae}(2aa_2 + bb_2)y^2, \\
I &= dx + ey + ax^2 + bxy + \frac{b^2}{4a}y^2 - \frac{1}{b^2}((bd - 2ae)ba_0 - 2ae(da_2 + eb_2))t.
\end{aligned}
\tag{1.7}
$$

with $ab(bd - 2ae) \neq 0$ *and* $(bd - 2ae)ba_0 - 2ae(da_2 + eb_2) \neq 0$.

$$
\begin{aligned}
\dot{x} &= a_0 + a_1 x + a_2 y + \frac{2}{e}ca_1 xy - \frac{2}{d}cb_2 y^2, \\
\dot{y} &= -\frac{1}{2c}(a_2 d + b_2 e) - \frac{d}{e}a_1 x + b_2 y, \\
I &= dx + ey + cy^2 + \frac{1}{2c}(e(da_2 + eb_2) - 2cda_0)t.
\end{aligned}
\tag{1.8}
$$

with $cde \neq 0$ *and* $e(da_2 + eb_2) - 2cda_0 \neq 0$.

$$
\begin{aligned}
\dot{x} &= a_0 - \frac{e}{2a}b_3 x + a_2 y, \\
\dot{y} &= b_0 - \frac{1}{2aea_2}(e^2 b_2 b_3 + 4a^2 a_2 a_0)x + b_2 y - \frac{2}{e}aa_2 xy + b_3 x^2, \\
I &= -\frac{e}{a_2}b_2 x + ey + ax^2 + \left(\frac{e}{a_2}a_0 b_2 - eb_0\right)t.
\end{aligned}
\tag{1.9}
$$

with $aea_2 \neq 0$ and $\dfrac{e}{a_2}a_0b_2 - eb_0 \neq 0$.

$$\dot{x} = a_0 - \frac{e}{2q}b_3 x,$$
$$\dot{y} = b_0 + \frac{1}{2ae}(deb_3 - 4a^2 a_0)x + b_3 x^2, \tag{1.10}$$
$$I = dx + ey + ax^2 - (da_0 + eb_0)t.$$

with $ae \neq 0$ and $da_0 + eb_0 \neq 0$.

$$\dot{x} = a_0 + \frac{2}{b}aa_2 x + a_2 y + a_3 x^2 + \frac{1}{2ad}(bda_3 - 4a^2 a_2 - 2bab_2)xy$$
$$\quad - \frac{b}{2ad}(2aa_2 + bb_2)y^2,$$
$$\dot{y} = -\frac{2}{b^2}(ba_0 + da_2)a - \frac{1}{b^2}(4a^2 a_2 + bda_3)x + b_2 y - \frac{2}{b}aa_3 x^2 \tag{1.11}$$
$$\quad + \frac{1}{bd}(4a^2 a_2 - bda_3 + 2abb_2)xy + \frac{1}{d}(2aa_2 + bb_2)y^2,$$
$$I = dx + ax^2 + bxy + \frac{b^2}{4a}y^2 - da_0 t.$$

with $abd \neq 0$ and $a_0 \neq 0$.

$$\dot{x} = a_0 + \frac{b}{2c}a_2 x + a_2 y + a_3 x^2 + a_4 xy + \frac{2}{b^2}(ba_4 - 2ca_3)cy^2,$$
$$\dot{y} = -\frac{1}{2c}(ba_0 + da_2) - \frac{1}{4bc^2}(b^3 a_2 + 4c^2 da_3)x - \frac{1}{2cb^2}(2(ba_4 - 2ca_3)dc$$
$$\quad + b^3 a_2)y - \frac{b}{2c}a_3 x^2 - \frac{b}{2c}a_4 xy + \frac{1}{b}(2ca_3 - ba_4)y^2, \tag{1.12}$$
$$I = dx + \frac{b^2}{4c}x^2 + bxy + cy^2 - da_0 t.$$

with $bc \neq 0$ and $da_0 \neq 0$.

$$\dot{x} = a_0 + a_2 y + a_4 xy + a_5 y^2,$$
$$\dot{y} = -\frac{d}{2c}(a_2 + a_4 x + a_5 y), \tag{1.13}$$
$$I = dx + cy^2 - da_0 t.$$

with $c \neq 0$ and $da_0 \neq 0$.

$$\dot{x} = a_0 + a_2 y - \frac{2}{d}cb_2 y^2,$$
$$\dot{y} = b_0 + b_2 y, \tag{1.14}$$
$$I = dx - \frac{1}{b_2}(da_2 + 2cb_0)y + cy^2 + (\frac{b_0}{b_2}(da_2 + 2cb_0) - da_0)t.$$

with $db_2 \neq 0$ and $\dfrac{b_0}{b_2}(da_2 + 2cb_0) - da_0 \neq 0$.

$$
\begin{aligned}
\dot{x} &= a_0 + a_2 y + \frac{b}{e} a_2 xy + \frac{b^2}{2ea} a_2 y^2, \\
\dot{y} &= -\frac{2}{b} aa_0 - \frac{a_2}{e}(dy + 2axy + by^2), \\
I &= dx + ey + ax^2 + bxy + \frac{b^2}{4a} y^2 - \frac{a_0}{b}(db - 2ae)t.
\end{aligned}
\tag{1.15}
$$

with $abe \neq 0$ and $\dfrac{a_0}{b}(db - 2ae) \neq 0$.

$$
\begin{aligned}
\dot{x} &= \frac{bda_2^2}{b^2 a_2 + 2cda_4 + 2bcb_2} - \frac{b}{c} a_2 x + a_2 y + \frac{b}{4c^2 d} \\
&\quad (b^2 a_2 + 2cda_4 + 2bcb_2)x^2 + a_4 xy - \frac{1}{d}(ba_2 + 2cb_2)y^2, \\
\dot{y} &= -\frac{a_2 d(a_2 b^2 + a_4 cd + bcb_2)}{c(a_2 b^2 + 2c(a_4 d + bb_2))} - \frac{(a_2 b^2 + a_4 dc + cbb_2)}{2c^2} x + b_2 y - \frac{b^2}{8c^3 d} \\
&\quad (a_2 b^2 + 2c(a_4 d + bb_2))x^2 - \frac{b}{2c} a_4 xy + \frac{b}{2cd}(a_2 b + 2b_2 c)y^2, \\
I &= dx + \frac{b^2}{4c} x^2 + bxy + cy^2 - da_0 t.
\end{aligned}
\tag{1.16}
$$

with $cd(b^2 a_2 + 2cda_4 + 2bcb_2) \neq 0$ and $a_0 \neq 0$.

$$
\begin{aligned}
\dot{x} &= a_0 + a_4 xy + \frac{2}{b} ca_4 y^2, \\
\dot{y} &= -\frac{b}{2c} a_0 - a_4 (\frac{d}{b} y + \frac{b}{2c} xy + y^2), \\
I &= dx + \frac{b^2}{4c} x^2 + bxy + cy^2 - da_0 t.
\end{aligned}
\tag{1.17}
$$

with $bc \neq 0$ and $da_0 \neq 0$.

Theorem 1.1 is proved in section 2.

Roughly speaking the textit Poincaré disc $\mathbb{D}^2$ is the unit closed disc centered at the origin of coordinates whose interior is diffeomorphic with $\mathbb{R}^2$ and whose boundary, the circle $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$. Note that we can go to the infinity in the plane $\mathbb{R}^2$ in as many as directions as points has the circle $\mathbb{S}^1$. Any polynomial differential system can be extended analytically to the whole Poincaré disc, being the circle of the infinity invariant by this extended flow. This extension is called the textit Poincaré compactification, see for more details Chapter 5 of [7].

Now the singular points of a polynomial differential system which are in the circle of the infinity, i.e. which for its Poincaré compactification are in the circle of the infinity, are called the textit infinite singular points of the polynomial differential system, and the usual singular points of a polynomial differential system contained in $\mathbb{R}^2$, or equivalently in the open Poincaré disc are called the textit finite singular points of the polynomial differential system.

The textit chordal quadratic systems are the quadratic systems without finite singular points. The phase portraits in the Poincaré disc of all the chordal quadratic systems are classified in [8]. By the Poincaré-Bendixson Theorem (see for instance Theorem 1.25 and Corollary 1.30 of [7]) all the orbits of a chordal quadratic system starts at an infinite singular point and ends in an infinite singular point, eventually both singular points can coincide.

Corollary 1.2 *All quadratic systems having an invariant of the form $I(x, y, t) = ax^2 + bxy + cy^2 + dx + ey + c_1 t$ with $c_1 \neq 0$ are chordal quadratic systems.*

The proof of Corollary 1.2 is done at the end of section 2.

We say that two phase portraits in the Poincaré disc are textit topologically equivalent if there is a diffeomorphism from $\mathbb{D}^2$ into itself such that send orbits of one phase portrait onto the orbits of the other phase portrait preserving or reversing the sense of all the orbits.

As an exercise we have classified the phase portraits of the chordal quadratic systems given by system (1.14) with $a_0 \neq 0$ in section 3.

2 The proofs

In this section we shall prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1 To prove that the function $I(x, y, t) = dx + ey + ax^2 + bxy + cy^2 + c_1 t$ is an invariant of system (1.3) we must verify the partial derivative equation (1.2). Thus, the following polynomial must be a zero polynomial

$$
\begin{aligned}
&a_0 d + b_0 e + c_1 + (a_1 d + 2aa_0 + b_1 e + bb_0)x + (a_0 b + a_2 d + 2b_0 c + b_2 e)y \\
&+ (a_3 d + 2aa_1 + b_3 e + bb_1)x^2 + (a_1 b + a_4 d + 2aa_2 + 2b_1 c + b_4 e + bb_2)xy \\
&+ (a_2 b + a_5 d + 2b_2 c + b_5 e)y^2 + (2aa_3 + bb_3)x^3 \\
&+ (a_3 b + 2aa_4 + 2b_3 c + bb_4)x^2 y + (a_4 b + 2aa_5 + 2b_4 c + bb_5)xy^2 \\
&+ (a_5 b + 2b_5 c)y^3.
\end{aligned}
$$

Therefore the following algebraic system must be solved

$$
\begin{aligned}
&a_0 d + b_0 e + c_1 = 0, \\
&a_1 d + 2aa_0 + b_1 e + bb_0 = 0, \\
&a_0 b + a_2 d + 2b_0 c + b_2 e = 0, \\
&a_3 d + 2aa_1 + b_3 e + bb_1 = 0, \\
&a_1 b + a_4 d + 2aa_2 + 2b_1 c + b_4 e + bb_2 = 0, \\
&a_2 b + a_5 d + 2b_2 c + b_5 e = 0, \\
&2aa_3 + bb_3 = 0, \\
&a_3 b + 2aa_4 + 2b_3 c + bb_4 = 0, \\
&a_4 b + 2aa_5 + 2b_4 c + bb_5 = 0, \\
&a_5 b + 2b_5 c = 0.
\end{aligned}
\tag{2.1}
$$

By solving this algebraic system with the help of the Mathematica software we obtain 49 solutions, but we select only the minimal number of solutions because many solu-

tions provided by Mathematica are particular solutions of other solutions. In this work, we are interested in the case in which the independent variable t appears explicitly for that we neglect the solutions when $c_1 = 0$. Then, we get the following sets of independent solutions with $c_1 \neq 0$

$$s_1 = \left\{ b_1 = -\frac{d}{e}a_1, \ b_2 = -\frac{d}{e}a_2, \ b_3 = -\frac{d}{e}a_3, \ b_4 = -\frac{d}{e}a_4, \ b_5 = -\frac{d}{e}a_5, \ a = 0, \right.$$
$$\left. b = 0, \ c = 0, \ c_1 = -(da_0 + eb_0) \right\},$$

$$s_{11} = \left\{ a_2 = \frac{1}{b^2}(2bca_1 + bea_4 - 4cea_3), \ a_5 = \frac{2}{b^2}c(ba_4 - 2ca_3), \ b_3 = -\frac{b}{2c}a_3, \right.$$
$$b_0 = \frac{1}{2b^2c}((ba_1 - ea_3)(be - 2cd) - b^3a_0), \ b_5 = \frac{1}{b}(2ca_3 - ba_4),$$
$$b_1 = -\frac{1}{2bc}(b^2a_1 - bea_3 + 2cda_3), \ b_4 = -\frac{b}{2c}a_4,$$
$$a = \frac{b^2}{4c}, \ b_2 = \frac{1}{b^2}(bea_3 + 2cda_3 - b^2a_1 - bda_4),$$
$$\left. c_1 = \frac{1}{2b^2c}(be - 2cd)(b^2a_0 + e(ea_3 - ba_1)) \right\},$$

$$s_{12} = \left\{ a_1 = \frac{1}{b^2d - 2abe}(bdea_3 + 2abda_2 + 2abeb_2 - 2ae^2a_3), \ b_3 = -\frac{2}{b}aa_3, \right.$$
$$a_4 = \frac{b}{2a(2ae - bd)}(4a^2a_2 - bda_3 + 2a(ea_3 + bb_2)),$$
$$b_0 = -\frac{2}{b^2}(ba_0 + da_2 + eb_2)a, \ b_4 = \frac{1}{2ae - bd}(bda_3 - 4a^2a_2 - 2a(ea_3 + bb_2)),$$
$$b_1 = \frac{1}{b(bd - 2ae)}(-4a^2(da_2 + eb_2) + 2adea_3 - bd^2a_3),$$
$$a_5 = \frac{b^2}{2a(2ae - bd)}(2aa_2 + bb_2), \ c = \frac{b^2}{4a}, \ b_5 = \frac{b}{bd - 2ae}(2aa_2 + bb_2),$$
$$\left. c_1 = \frac{1}{b^2}((2ae - bd)ba_0 + 2(da_2 + eb_2)ae) \right\},$$

$$s_{13} = \left\{ a_3 = 0, \ a_4 = \frac{2}{e}ca_1, \ a_5 = -\frac{2}{d}cb_2, \ b_1 = -\frac{d}{e}a_1, \ b_0 = -\frac{1}{2c}(da_2 + eb_2), \right.$$
$$\left. b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ a = 0, \ b = 0, \ c_1 = \frac{e}{2c}(da_2 + eb_2) - da_0 \right\},$$

$$s_{20} = \left\{ a_1 = -\frac{e}{2a}b_3, \ a_3 = 0, \ a_4 = 0, \ a_5 = 0, \ b_1 = -\frac{1}{2aea_2}(e^2b_3b_2 + 4a^2a_2a_0), \right.$$
$$\left. b_4 = -\frac{2}{e}aa_2, \ b_5 = 0, \ b = 0, \ c = 0, \ d = -\frac{e}{a_2}b_2, \ c_1 = \frac{e}{a_2}a_0b_2 - eb_0 \right\},$$

$$s_{21} = \left\{ a_1 = -\frac{e}{2a}b_3, \ a_2 = 0, \ a_3 = 0, \ a_4 = 0, \ a_5 = 0, \ b_1 = \frac{1}{2ae}(deb_3 - 4a^2a_0), \right.$$
$$\left. b_2 = 0, \ b_4 = 0, \ b_5 = 0, \ c_1 = -(da_0 + eb_0) \right\},$$

$$s_{22} = \left\{ a_1 = \frac{2a}{bd - 2ae}(da_2 + eb_2), \ a_3 = 0, \ a_4 = -\frac{b}{bd - 2ae}(2aa_2 + bb_2), \right.$$

$$a_5 = -\frac{b^2}{2a(bd-2ae)}(2aa_2+bb_2), \; b_0 = -\frac{2}{b^2}(ba_0+da_2+eb_2)a, \; b_3 = 0,$$

$$b_1 = -\frac{4}{b(bd-2ae)}a^2(da_2+eb_2), \; b_4 = \frac{2}{bd-2ae}a(2aa_2+bb_2), \; c = \frac{b^2}{4a},$$

$$b_5 = \frac{b}{bd-2ae}(2aa_2+bb_2), \; c_1 = \frac{1}{b^2}(b(2ae-bd)a_0+2ae(da_2+eb_2)) \Bigg\},$$

$$s_{24} = \Bigg\{ a_1 = \frac{2}{b}aa_2, \; a_4 = \frac{1}{2ad}(ba_3-2aa_2-bb_2), \; a_5 = -\frac{b}{2ad}(2aa_2+bb_2),$$

$$b_0 = -\frac{2}{b^2}(ba_0+da_2)a, \; b_1 = -\frac{1}{b^2}(4a^2a_2+bda_3), \; b_3 = -\frac{2}{b}aa_3,$$

$$b_4 = \frac{1}{bd}(4a^2a_2+2abb_2-bda_3), \; b_5 = \frac{1}{d}(2aa_2+bb_2), \; c = \frac{b^2}{4a}, \; e = 0,$$

$$c_1 = -da_0 \Bigg\},$$

$$s_{27} = \Bigg\{ a_1 = \frac{b}{2c}a_2, \; a_5 = \frac{2}{b^2}(ba_4-2ca_3)c, \; b_0 = -\frac{1}{2c}(ba_0+da_2),$$

$$b_1 = -\frac{1}{4c^2b}(b^3a_2+4c^2da_3), \; b_3 = -\frac{b}{2c}a_3, \; b_4 = -\frac{b}{2c}a_4, \; b_5 = \frac{2}{b}ca_3-a_4,$$

$$b_2 = -\frac{d}{b^2}(ba_4-2ca_3)-\frac{b}{2c}a_2, \; a = \frac{b^2}{4c}, \; e = 0, \; c_1 = -da_0 \Bigg\},$$

$$s_{29} = \Bigg\{ a_1 = 0, \; a_3 = 0, \; b_0 = -\frac{d}{2c}a_2, \; b_1 = -\frac{d}{2c}a_4, \; b_2 = -\frac{d}{2c}a_5, \; b_3 = 0,$$

$$b_4 = 0, \; b_5 = 0, \; a = 0, \; b = 0, \; e = 0, \; c_1 = -da_0 \Bigg\},$$

$$s_{31} = \Bigg\{ a_1 = 0, \; a_3 = 0, \; a_4 = 0, \; a_5 = -\frac{2}{d}cb_2, \; b_1 = 0, \; b_3 = 0, \; b_4 = 0, \; b_5 = 0,$$

$$a = 0, \; b = 0, \; e = -\frac{1}{b_2}(da_2+2cb_0), \; c_1 = \frac{b_0}{b_2}(da_2+2cb_0)-da_0 \Bigg\},$$

$$s_{33} = \Bigg\{ a_1 = 0, \; a_3 = 0, \; a_4 = \frac{b}{e}a_2, \; a_5 = \frac{b^2}{2ae}a_2, \; b_0 = -\frac{2}{b}aa_0, \; b_1 = 0, \; b_3 = 0,$$

$$b_2 = -\frac{d}{e}a_2, \; b_4 = -\frac{2}{e}aa_2, \; b_5 = -\frac{b}{e}a_2, \; c = \frac{b^2}{4a}, \; c_1 = a_0(\frac{2}{b}ae-d) \Bigg\},$$

$$s_{36} = \Bigg\{ a_1 = \frac{2}{b}aa_2, \; a_3 = 0, \; a_4 = -\frac{1}{d}(2aa_2+bb_2), \; a_5 = -\frac{b}{2ad}(2aa_2+bb_2),$$

$$b_0 = -\frac{2}{b^2}a(ba_0+da_2), \; b_1 = -\frac{4}{b^2}a^2a_2, \; b_3 = 0, \; b_4 = \frac{2}{bd}a(2aa_2+bb_2),$$

$$b_5 = \frac{1}{d}(2aa_2+bb_2), \; e = 0, \; c = \frac{b^2}{4a}, \; c_1 = -da_0 \Bigg\},$$

$$s_{39} = \Bigg\{ a_1 = 0, \; a_2 = 0, \; a_4 = \frac{b}{2ad}(da_3-2ab_2), \; a_5 = -\frac{b^2}{2ad}b_2, \; b_0 = -\frac{2}{b}aa_0,$$

$$b_1 = -\frac{d}{b}a_3, \; b_3 = -\frac{2}{b}aa_3, \; b_4 = \frac{2}{d}ab_2-a_3, \; b_5 = \frac{b}{d}b_2, \; c = \frac{b^2}{4a}, e = 0, \; c_1 = -da_0 \Bigg\},$$

$$s_{42} = \Bigg\{ a_1 = 0, \; a_2 = 0, \; a_5 = \frac{2}{b^2}(ba_4-2ca_3)c, \; b_0 = -\frac{b}{2c}a_0, \; b_1 = -\frac{d}{b}a_3,$$

$$b_2 = -\frac{d}{b^2}(ba_4 - 2ca_3), \ b_3 = -\frac{b}{2c}a_3, \ b_4 = -\frac{b}{2c}a_4, \ b_5 = \frac{2}{b}ca_3 - a_4,$$

$$a = \frac{b^2}{4c}, \ e = 0, \ c_1 = -da_0 \Big\},$$

$$s_{43} = \Big\{ a_0 = \frac{bda_2^2}{b^2a_2 + 2c(da_4 + bb_2)}, \ a_3 = \frac{b}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2)),$$

$$a_1 = -\frac{b}{2c}a_2, \ a_5 = -\frac{1}{d}(ba_2 + 2cb_2), \ b_0 = d\Big(\frac{a_2}{2c} - \frac{2aa_2^2}{b^2a_2 + 2c(da_4 + bb_2)}\Big),$$

$$b_1 = \frac{1}{2c}\Big(2aa_2 - \frac{1}{2c}(b^2a_2 + 2c(da_4 + bb_2))\Big), \ a = \frac{b^2}{4c}, \ e = 0,$$

$$b_4 = \frac{b}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2))\Big(\frac{4}{b^2}ac - 1\Big) - \frac{2}{b}aa_4, \ c_1 = -da_0,$$

$$b_3 = -\frac{2}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2))a, \ b_5 = \frac{1}{2cd}(b^2a_2 + 2c(da_4 + bb_2)) - a_4 \Big\},$$

$$s_{44} = \Big\{ a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_5 = \frac{2}{b}ca_4, \ b_0 = -\frac{b}{2c}a_0, \ b_1 = 0, \ b_2 = -\frac{d}{b}a_4,$$

$$b_3 = 0, \ b_4 = -\frac{b}{2c}a_4, \ b_5 = -a_4, \ e = 0, \ a = \frac{b^2}{4c}, \ c_1 = -da_0 \Big\},$$

$$s_{45} = \Big\{ a_0 = \frac{bda_2^2}{b^2a_2 + 2dca_4 + 2bcb_2},$$

$$a_1 = -\sqrt{\frac{a_2^2bd}{a_2b^2 + 2a_4cd + 2bb_2c}}\sqrt{\frac{b\left(a_2b^2 + 2a_4cd + 2bb_2c\right)}{c^2d}},$$

$$a_3 = -\frac{b}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2)), \ a_5 = -\frac{1}{d}(a_2b + 2b_2c), \ b_4 = -\frac{ba_4}{2c},$$

$$b_3 = -\frac{b^2}{8c^3d}(b^2a_2 + 2c(da_4 + bb_2)),$$

$$b_1 = -\frac{1}{4c^2}\Big(-bc\sqrt{\frac{a_2^2bd}{a_2b^2 + 2a_4cd + 2bb_2c}}$$

$$\sqrt{\frac{b\left(a_2b^2 + 2a_4cd + 2bb_2c\right)}{c^2d}} + a_2b^2 + 2a_4cd + 2bb_2c\Big),$$

$$b_0 = \frac{d}{2b}\Big(\sqrt{\frac{a_2^2bd}{a_2b^2 + 2a_4cd + 2bb_2c}}\sqrt{\frac{b\left(a_2b^2 + 2a_4cd + 2bb_2c\right)}{c^2d}}$$

$$- \frac{a_2^2b^3}{c\left(a_2b^2 + 2c(a_4d + bb_2)\right)}\Big),$$

$$b_5 = \frac{b}{2cd}(ba_2 + 2cb_2), \ a = \frac{b^2}{4c}, \ e = 0, \ c_1 = -da_0 \Big\},$$

In what follows we explain how we have eliminated the other solutions provided by Mathematica. The solutions $s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9$ and s_{10} are particular solutions

of s_1 when one or more parameters of s_1 are zero, then all these solutions provides system (1.5).

The solutions s_{11}, s_{12} and s_{13} produce systems (1.6), (1.7) and (1.8), respectively.

The solution s_{14} is a particular solution of s_1 when $d = 0$ and $a_3 = 0$, so it contained in system (1.5).

The solutions s_{15}, s_{16}, s_{17}, s_{18} and s_{19} are particular solutions of the solution s_{13} when one or more parameters vanish.

The solutions s_{20}, s_{21} and s_{22} yield systems (1.9), (1.10) and (1.11), respectively.

The solution s_{23} is a particular solution of s_{22} when we take $2aa_2 = bb_2$.

The solution s_{24} produces system (1.11).

The solution s_{25} is a particular solution of the solution s_{24} when $2aa_2 + bb_2 = 0$, so we do not consider it.

The solution s_{26} is a particular solution of s_{24} when $2aa_2 + bb_2 = 0$ and $a_3 = 0$, again we do not need to consider it.

The solution s_{27} gives system (1.12).

The solution s_{28} is a particular solution of the solution s_{27} when $a_2 = 0$.

The solution s_{29} yields system (1.13).

The solutions s_{30}, s_{32} and s_{34} produce linear differential systems instead of quadratic ones.

The solutions s_{31} and s_{33} correspond to system (1.14) and (1.15), respectively.

The solutions s_{36} and s_{39} represent particular cases of the solution s_{24} when $a_3 = 0$ and $a_2 = 0$, respectively. Therefore, we will not consider them.

The solutions s_{35}, s_{38} and s_{41} are particular solutions of the solution s_{36}, the first when $a_2 = 0$, the second when $2aa_2 + bb_2 = 0$ and third when $a_2 = (2a_1c)/b$, $b_2 = -(a_1b + a_4d)/b$ and $a = b^2/(4c)$.

The solutions s_{35} and s_{37} are same solution.

The solution s_{40} is a specific case of s_{39} when $b_2 = 0$, and s_{42} is a specific case of s_{27} when $a_2 = 0$. Hence, we shall omit them from consideration.

The solutions s_{43} and s_{44} and give rise to systems (1.16) and (1.17), respectively.

In the solution s_{45} we simplify the value $(a_2^2bd/(a_2b^2+2a_4cd+2bb_2c))^{1/2}(b(a_2b^2 + 2a_4cd + 2bb_2c)/c^2d)^{1/2}$ and it becomes $-a_2b/c$ that to get the simple form appears in system (1.16). The solution s_{46} is a particular solution of s_{45} when $a_3 = 0$.

The solution s_{47} is a particular solution of s_{44} when $a_4 = -bb_2/d$ and $c = b^2/(4a)$.

The solutions s_{48} and s_{49} are particular solutions of s_{47}, when $b_2 = 0$ and $b_2 = -da_4/b$, respectively. So we do not consider the solutions s_{47}, s_{48} and s_{49}.

Hence all the solutions together produce the eighteen different quadratic systems with invariants of the form (1.4). This completes the proof of the theorem.

Proof of Corollary 1.2 All quadratic systems having an invariant of the form $I(x, y, t) = ax^2 + bxy + cy^2 + dx + ey + c_1t$ with $c_1 \neq 0$ have no finite singular points, because if (x_0, y_0) is a finite singular point then $I(x_0, y_0, t)$ would not be constant when t varies.
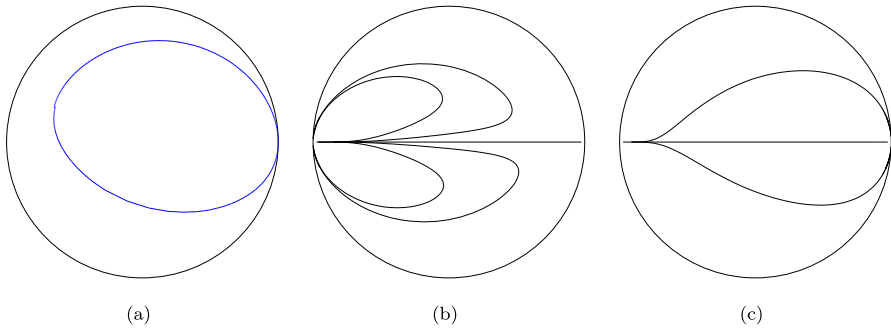
(a)   (b)   (c)

Fig. 1 The three different topological phase portrait of system (3.1) in the Poincaré disc

3 Phase portrait of system (1.14)

System (1.14) depends on five parameters a_0, a_2, c, d, b_0 and b_2. Clearly $cdb_2 \neq 0$, otherwise the differential system (1.14) would not be a quadratic system. Here we study the phase portraits of system (1.14) with $a_0 \neq 0$ in the Poincaré disc, from Corollary 1.2 we know that they are chordal quadratic systems.

Doing the rescaling $(x, y) = (a_0 X, b_2 Y)$ system (1.14) becomes

$$\dot{X} = 1 + aY + bY^2, \qquad \dot{Y} = c + Y, \tag{3.1}$$

where

$$a = \frac{a_2}{a_0}, \quad b = -\frac{2cb_2}{da_0}, \quad c = \frac{b_0}{b_2}.$$

The unique singular points of the differential system (3.1) are the infinite singular points localized at the endpoints of the x-axis, the local phase portraits at these points can be studied doing the changes of variables called blow-ups (see [1]) and using the coordinates of the Poincaré disc (see Chapter 5 of [7]). Then we obtain three topologically non-equivalent phase portraits in the Poincaré disc for the differential system (3.1), the one of Fig. 1(a) when $c = -1$ and $1 + a + b = 0$, the one of Fig. 1(b) when either $c > 1$ and $b < 0$, or $c = -1$, $1 + a + b = 0$ and $a > 0$, or $c = -1$, $1 + a + b < 0$ and $b > 0$, and finally the one of Fig. 1(c) otherwise.

4 Conclusion

In conclusion, this article successfully presents a classification of planar quadratic polynomial differential systems exhibiting invariants of the form $I(x, y, t) = ax^2 + bxy + cy^2 + dx + ey + c_1 t$ with $c_1 \neq 0$. The study identifies a total of 13 distinct families of quadratic systems exhibiting such kind of invariants. Notably, this research marks a pioneering effort, as it is the first instance of classifying quadratic differential systems with an invariant that deviates from a Darboux invariant. This achievement

opens avenues for further exploration and understanding in the field of planar quadratic systems with diverse invariants.

5 Appendix

Deleted solutions

Here we give the sets of solutions that we have eliminated in the proof of Theorem 1.1.

$$
s_2 = \left\{ a_2 = 0, \ b_1 = -\frac{d}{e}a_1, \ b_2 = 0, \ b_3 = -\frac{d}{e}a_3, \ b_4 = -\frac{d}{e}a_4, \ b_5 = -\frac{d}{e}a_5, \right.
$$
$$
\left. a = 0, \ b = 0, \ c = 0, \ c_1 = -(da_0 + eb_0) \right\},
$$

$$
s_3 = \{ b_1 = 0, \ b_2 = 0, \ b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ a = 0, \ b = 0, \ c = 0, \ d = 0,
$$
$$
c_1 = -eb_0 \},
$$

$$
s_4 = \left\{ a_2 = 0, \ a_4 = 0, \ b_1 = -\frac{d}{e}a_1, \ b_2 = 0, \ b_3 = -\frac{d}{e}a_3, \ b_4 = 0, \ b_5 = -\frac{d}{e}a_5, \right.
$$
$$
\left. a = 0, \ b = 0, \ c = 0, \ c_1 = -(da_0 + eb_0) \right\},
$$

$$
s_5 = \left\{ a_3 = 0, \ b_1 = -\frac{d}{e}a_1, \ b_2 = -\frac{d}{e}a_2, \ b_3 = 0, \ b_4 = -\frac{d}{e}a_4, \ b_5 = -\frac{d}{e}a_5, \right.
$$
$$
\left. a = 0, \ b = 0, \ c = 0, \ c_1 = -(da_0 + eb_0) \right\},
$$

$$
s_6 = \left\{ a_2 = 0, \ a_3 = 0, \ b_1 = -\frac{d}{e}a_1, \ b_2 = 0, \ b_3 = 0, \ b_4 = -\frac{d}{e}a_4, \ b_5 = -\frac{d}{e}a_5, \right.
$$
$$
\left. a = 0, \ b = 0, \ c = 0, \ c_1 = -(da_0 + eb_0), \right\}
$$

$$
s_7 = \{ a_3 = 0, \ b_1 = 0, \ b_2 = 0, \ b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ a = 0, \ b = 0, \ c = 0,
$$
$$
d = 0, \ c_1 = -eb_0 \},
$$

$$
s_8 = \left\{ a_1 = 0, \ a_3 = 0, \ a_4 = 0, \ b_1 = 0, \ b_2 = 0, \ b_3 = 0, \ b_4 = 0, \ b_5 = -\frac{d}{e}a_5, \right.
$$
$$
\left. a = 0, \ b = 0, \ c = 0, \ c_1 = -(da_0 + eb_0) \right\},
$$

$$
s_9 = \{ b_1 = 0, \ b_2 = 0, \ b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ a = 0, \ b = 0, \ c = 0, \ d = 0,
$$
$$
c_1 = -eb_0 \},
$$

$$
s_{10} = \left\{ a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_4 = 0, \ a_5 = 0, \ a = 0, \ b = 0, \right.
$$
$$
\left. c = 0, \ e = 0, \ c_1 = -da_0 \right\},
$$

$$
s_{14} = \{ a_3 = 0, \ b_1 = 0, \ b_2 = 0, \ b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ a = 0, \ b = 0,
$$
$$
c = 0, \ d = 0, \ c_1 = -eb_0 \},
$$

$$
s_{15} = \left\{ a_1 = 0, \ a_3 = 0, \ a_4 = 0, \ a_5 = -\frac{2}{d}b_2 c, \ b_0 = -\frac{1}{2c}(da_2 + eb_2), \right.
$$

$$b_1 = -\frac{d}{e}a_1, \ b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ a = 0, \ b = 0, \ c_1 = -(da_0 + eb_0)\Big\},$$

$$s_{16} = \Big\{a_3 = 0, \ a_4 = \frac{2}{e}ca_1, \ a_5 = 0, \ b_0 = -\frac{d}{2c}a_2, \ b_2 = 0, \ b_1 = -\frac{d}{e}a_1, \ b_3 = 0,$$
$$b_4 = 0, \ b_5 = 0, \ a = 0, \ b = 0, \ c_1 = \frac{d}{2c}ea_2 - da_0\Big\},$$

$$s_{17} = \Big\{a_2 = -\frac{e}{d}b_2, \ a_3 = 0, \ a_4 = 0, \ a_5 = 0, \ b_1 = -\frac{d}{e}a_1, \ b_3 = 0, \ b_4 = 0,$$
$$b_5 = 0, \ a = 0, \ b = 0, \ c = 0, \ c_1 = -(da_0 + eb_0)\},$$

$$s_{18} = \{a_3 = 0, \ a_4 = 0, \ b_1 = 0, \ b_2 = 0, \ b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ a = 0, \ b = 0,$$
$$c = 0, \ d = 0, \ c_1 = -eb_0\},$$

$$s_{19} = \{b_1 = 0, \ b_2 = 0, \ b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ a = 0, \ b = 0, \ c = 0, \ d = 0,$$
$$c_1 = -eb_0\},$$

$$s_{23} = \Big\{a_1 = \frac{2}{b}aa_2, \ a_3 = 0, \ a_4 = 0, \ a_5 = 0, \ b_0 = \frac{1}{b^3}(4a^2ea_2 - 2ab(ba_0 + da_2)),$$
$$b_1 = -\frac{4}{b^2}a^2a_2, \ b_2 = -\frac{2}{b}aa_2, \ b_3 = 0, \ b_4 = 0, \ c = \frac{b^2}{4a}, \ b_5 = 0,$$
$$c_1 = \frac{1}{b^3}(2ae - bd)(b^2a_0 - 2aea_2)\Big\},$$

$$s_{25} = \Big\{a_1 = \frac{2}{b}aa_2, \ a_4 = \frac{b}{2a}a_3, \ a_5 = 0, \ b_0 = -\frac{2}{b^2}(ba_0 + da_2)a,$$
$$b_1 = -\frac{1}{b^2}(4a^2a_2 + bda_3), \ b_2 = -\frac{2}{b}aa_2, \ b_3 = -\frac{2}{b}aa_3, \ b_4 = -a_3, \ b_5 = 0,$$
$$c = \frac{b^2}{4a}, \ e = 0, \ c_1 = -da_0,\Big\},$$

$$s_{26} = \Big\{a_1 = \frac{2}{b}aa_2, \ a_3 = 0, \ a_4 = 0, \ a_5 = 0,$$
$$b_0 = -\frac{2}{b^2}(ba_0 + da_2)a, \ b_1 = -\frac{4}{b^2}a^2a_2, \ b_2 = -\frac{2}{b}aa_2,$$
$$b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ c = \frac{b^2}{4a}, \ e = 0, \ c_1 = -da_0\Big\},$$

$$s_{28} = \Big\{a_1 = 0, \ a_2 = 0, \ a_5 = \frac{2}{b^2}(ba_4 - 2ca_3)c, \ b_0 = -\frac{b}{2c}a_0, \ b_1 = -\frac{d}{b}a_3,$$
$$b_2 = -\frac{d}{b^2}(ba_4 - 2ca_3), \ b_3 = -\frac{b}{2c}a_3, \ b_4 = -\frac{b}{2c}a_4, \ b_5 = \frac{2}{b}ca_3 - a_4,$$
$$a = \frac{b^2}{4c}, \ e = 0, \ c_1 = -da_0\Big\},$$

$$s_{30} = \{a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_4 = 0, \ a_5 = 0, \ b_3 = 0, \ b_4 = 0, \ b_5 = 0,$$
$$a = 0, \ b = 0, \ c = 0, \ e = 0, \ c_1 = -da_0\},$$

$$s_{32} = \left\{ a_1 = 0, \ a_3 = 0, \ a_4 = 0, \ a_5 = 0, \ b_0 = -\frac{d}{2c}a_2, \ b_1 = 0, \ b_2 = 0, \ b_3 = 0, \right.$$
$$\left. b_4 = 0, \ b_5 = 0, \ a = 0, \ b = 0, \ c_1 = da_0 - \frac{d}{2c}ea_2, \right\},$$

$$s_{34} = \left\{ a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_4 = 0, \ a_5 = 0, \ b_0 = -\frac{2}{b}aa_0, \ b_1 = 0, \ b_2 = 0, \right.$$
$$\left. b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ c = \frac{b^2}{4a}, \ c_1 = a_0\left(\frac{2}{b}ae - d\right) \right\},$$

$$s_{35} = \left\{ a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_4 = -\frac{b}{d}b_2, \ a_5 = -\frac{b^2}{2ad}b_2, \ b_0 = -\frac{2}{b}aa_0, \right.$$
$$\left. b_1 = 0, \ b_3 = 0, \ b_4 = \frac{2}{d}ab_2, \ b_5 = \frac{b}{d}b_2, \ c = \frac{b^2}{4a}, \ e = 0, \ c_1 = -da_0 \right\},$$

$$s_{37} = \left\{ a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_4 = -\frac{b}{d}b_2, \ a_5 = -\frac{b^2}{2ad}b_2, \ b_0 = -\frac{2}{b}aa_0, \right.$$
$$\left. b_1 = 0, \ b_3 = 0, \ b_4 = \frac{2}{d}ab_2, \ b_5 = \frac{b}{d}b_2, \ c = \frac{b^2}{4a}, \ e = 0, \ c_1 = -da_0 \right\},$$

$$s_{38} = \left\{ a_1 = \frac{2}{b}aa_2, \ a_4 = 0, \ a_3 = 0, \ a_5 = 0, \ b_0 \right.$$
$$= -\frac{2}{b^2}a(ba_0 + da_2), \ b_1 = -\frac{4}{b^2}a^2a_2, \ b_2 = -\frac{2}{b}aa_2, \ b_3 = 0,$$
$$\left. b_4 = 0, \ b_5 = 0, \ c = \frac{b^2}{4a}, e = 0, \ c_1 = -da_0 \right\},$$

$$s_{40} = \left\{ a_1 = 0, \ a_2 = 0, \ a_4 = \frac{b}{2a}a_3, \ a_5 = 0, \ b_0 = -\frac{2}{b}aa_0, \ b_2 = 0, \right.$$
$$\left. b_1 = -\frac{d}{b}a_3, \ b_3 = -\frac{2}{b}aa_3, \ b_4 = -a_3, \ b_5 = 0, \ c = \frac{b^2}{4a}, \ e = 0, c_1 = -da_0 \right\},$$

$$s_{41} = \left\{ a_2 = \frac{2}{b}ca_1, \ a_3 = 0, \ a_5 = \frac{2}{b}ca_4, \ b_0 = -\frac{1}{2cb}(b^2a_0 + 2cda_1), \ b_1 = -\frac{b}{2c}a_1, \right.$$
$$b_2 = -\frac{1}{b}(ba_1 + da_4), \ b_3 = 0, \ b_4 = -\frac{b}{2c}a_4, \ b_5 = -a_4, \ a = \frac{b^2}{4c}, \ e = 0,$$
$$\left. c_1 = -da_0 \right\},$$

$$s_{46} = \left\{ a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_5 = \frac{2}{b^2}bca_4, \ b_0 = -\frac{2}{b}aa_0, \ b_1 = 0, \ b_2 = -\frac{d}{b}a_4, \right.$$
$$\left. b_3 = 0, \ b_4 = -\frac{2}{b^2}aba_4, \ b_5 = -a_4, \ a = \frac{b^2}{4c}, \ e = 0, \ c_1 = -da_0 \right\},$$

$$s_{47} = \left\{ a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_4 = -\frac{b}{d}b_2, \ a_5 = -\frac{b^2}{2ad}b_2, \ b_0 = -\frac{2}{b}aa_0, \right.$$
$$\left. b_1 = 0, \ b_3 = 0, \ b_4 = \frac{2}{d}ab_2, \ b_5 = \frac{b}{d}b_2, \ c = \frac{b^2}{4a}, \ e = 0, \ c_1 = -da_0 \right\},$$

$$s_{48} = \left\{ a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_4 = 0, \ a_5 = 0, \ b_0 = -\frac{2}{b}aa_0, \ b_1 = 0, \right.$$

$$\left. b_2 = 0, \ b_3 = 0, \ b_4 = 0, \ b_5 = 0, \ c = \frac{b^2}{4a}, \ e = 0, \ c_1 = -da_0 \right\},$$

$$s_{49} = \left\{ a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_5 = \frac{2}{b}ca_4, \ b_0 = -\frac{b}{2c}a_0, \ b_1 = 0, \ b_2 = -\frac{d}{b}a_4, \right.$$

$$\left. b_3 = 0, \ b_4 = -\frac{b}{2c}a_4, \ b_5 = -a_4, \ a = \frac{b^2}{4c}, \ e = 0, \ c_1 = -da_0 \right\}.$$

References

1. Álvarez, M.J., Ferragut, A., Jarque, J.: A survey on the blow up technique. Int. J. Bifur. Chaos Appl. Sci. Eng. **21**, 3103–3118 (2011)
2. Alberich-Carramiñana, M., Ferragut, A., Llibre, J.: Quadratic planar differential systems with algebraic limit cycles via quadratic plane cremona maps. Adv. Math. **389**, 107924 (2021)
3. Artés, J.C., Llibre, J., Schlomiuk, D., Vulpe, N.: Geometric Configurations of Singularities of Planar Polynomial Differential Systems. A global classification on the quadratic case, Birkhäuser (2021)
4. Bolaños, Y., Llibre, J., Valls, C.: Phase portraits of quadratic Lotka–Volterra systems with a Darboux invariant in the Poincaré disc, Comm. in Contemporary Math. **16** , 1350041, 23 pp (2014)
5. Boularas, D.: A new classification of planar homogeneous quadratic systems. Qual. Theory Dyn. Syst. **2**, 93–110 (2001)
6. Chavarriga, J., García, B., Llibre, J., del Rio, J.S.P., Rodríguez, J.A.: Polynomial first integrals of quadratic vector fields. J. Differ. Equ. **230**(2), 393–421 (2006)
7. Dumortier, F., Llibre, J., Artés, J.C.: Qualitative Theory of Planar Differential Systems. UniversiText, Springer, New York (2006)
8. Gasull, A., Sheng, L.R., Llibre, J.: Chordal quadratic systems. Rock. Mt. J. Math. **16**(4), 751–782 (1986)
9. Kuznetzov, Y.A.: Elements of applied bifurcation theory, 2nd edn. Appl. Math. Sci. **112**, Springer, New York (1998)
10. Llibre, J., Messias, M., Reinol, A.C.: Darboux invariants for planar polynomial differential systems having an invariant conic. Zeitschrift fuer Angewandte Mathematik und Physik **65**, 1127–1136 (2014)
11. Llibre, J., Messias, M., Reinol, A.C.: Normal forms for polynomial differential systems in $\mathbb{R}^3$ having an invariant quadric and a Darboux invariant. Int. J. Bifurcat. Chaos **25** , 1550015, 16 pp (2015)
12. Llibre, J., Oliveira, R.D.S.: Quadratic systems with invariant straight lines of total multiplicity two having Darboux invariants. Commun. Contemp. Math. **17** , 1450018, 17 pp (2015)
13. Llibre, J., Oliveira, R.: Quadratic systems with an invariant conic having Darboux invariants. Commun. Contemp. Math. Vol. 20, No. 4 1750033 (15 pages) (2018)
14. Reyn, J.: Phase Portraits of Planar Quadratic Systems, Mathematics and Its Applications, vol. 583. Springer, New York (2007)
15. Yanqian, Y., et al.: Theory of Limit Cycles, Translations of Mathematics Monographs, vol. 66. American Mathematics Society, Providence, RI (1986)