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ORIGINAL PAPER

Approximations to the Euler–Mascheroni Constant

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Abstract
In this paper, we establish an asymptotic expansion for the Euler–Mascheroni constant. Based on this expansion, we establish a two-sided inequality and a continued fraction approximation for the Euler–Mascheroni constant.

Keywords Euler–Mascheroni constant · Psi (or Digamma) function · Polygamma functions · Asymptotic expansion · Inequality

Mathematics Subject Classification 41A60; 11Y60, 40A05

1 Introduction

Throughout this paper, $\mathbb{N}$ represents the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Euler–Mascheroni constant $\gamma = 0.577215664\ldots$ is defined as the limit of the sequence

$$D_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n \qquad (n \in \mathbb{N}). \tag{1.1}$$

The Euler–Mascheroni constant is a number that appears in analysis and number theory. It is not known yet whether the number is irrational or transcendental [20, 25]. The following two-sided inequality was presented in [24, 28]:

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$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad \text{for} \quad n \in \mathbb{N}, \tag{1.2}$$

which shows that the convergence of the sequence D_n to γ is very slow (like n^{-1}). By changing the logarithmic term in (1.1), faster approximation formulas to the Euler–Mascheroni constant were presented in [7, 10, 16, 17, 23]. For example, Chen and Mortici [7] established the following approximation formula:

$$\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right) = \gamma + O\left(n^{-5}\right) \quad \text{as} \quad n \to \infty \tag{1.3}$$

and posed an open question as follows: For a given $p \in \mathbb{N}_0$, find the constants a_i $(i = 0, 1, 2, \ldots, p)$, such that

$$\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \sum_{i=0}^{p} \frac{a_i}{n^i}\right)$$

is the fastest sequence which would converge to γ. This open problem has been considered by Yang [27], Gavrea and Ivan [18], and Lin [21]. Recently, Chen [5] determined the coefficients a_j and b_j, such that

$$\sum_{k=1}^{n} \frac{1}{k} - \ln \frac{n^p + \sum_{j=1}^{p} a_j n^{p-j}}{n^q + \sum_{j=1}^{q} b_j n^{q-j}} = \gamma + O\left(\frac{1}{n^{p+q+1}}\right) \qquad (n \to \infty),$$

where $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $p = q + 1$. This solves an open problem of Mortici [22]. There are a lot of formulas expressing γ as series, integrals, or products [4, 8, 9, 13, 14, 20]. For more information on the Euler–Mascheroni constant γ, please refer to survey papers [15, 26] and expository book [19].

In this paper, we consider the sequence $(A_n)_{n\in\mathbb{N}}$ defined by

$$A_n = \sum_{i=1}^{n} \sum_{j=0}^{n} \frac{1}{(i+j)^2} - 1 + \ln 2 - \ln n. \tag{1.4}$$

Using the computer program MAPLE 13, we find, as $n \to \infty$

$$A_n - \gamma \sim \frac{1}{2n} - \frac{11}{48n^2} + \frac{7}{48n^3} - \frac{31}{640n^4} - \frac{31}{960n^5} + \frac{635}{16128n^6} + \frac{127}{5376n^7} - \frac{3577}{61440n^8} - \frac{511}{15360n^9} + \cdots. \tag{1.5}$$

We here give a formula for determining these coefficients in the right-hand side of (1.5). Then, we establish a two-sided inequality and a continued-fraction approximation for the Euler–Mascheroni constant.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2 Lemmas and Preliminaries

The gamma function may be defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ $(x > 0)$. The logarithmic derivative of the gamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$ is known as the psi (or digamma) function. The derivatives of the psi function $\psi^{(n)}(x)$ $(n \in \mathbb{N})$ are called the polygamma functions. It is well known that the psi function has the duplication formula [1, p. 259, Eq. (6.3.8)]:

$$\psi(2x) = \frac{1}{2}\left[\psi(x) + \psi\left(x + \frac{1}{2}\right)\right] + \ln 2. \tag{2.1}$$

The following series expansion and asymptotic formula hold (see [1, p. 260]):

$$\psi'(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2} \tag{2.2}$$

and

$$\psi'(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{x^{2k+1}} = \frac{1}{x} + \frac{1}{2x^2} + \sum_{j=2}^{\infty} \frac{B_j}{x^{j+1}} \qquad (x \to \infty), \tag{2.3}$$

where B_n $(n \in \mathbb{N}_0)$ are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

It follows from the known result (see [6, Eq. (3.26)]) that:

$$\psi'\left(x + \frac{1}{2}\right) \sim \frac{1}{x} - \sum_{k=2}^{\infty} \frac{(1 - 2^{1-k})B_k}{x^{k+1}} = \frac{1}{x} - \sum_{k=3}^{\infty} \frac{(1 - 2^{2-k})B_{k-1}}{x^k} \qquad (x \to \infty). \tag{2.4}$$

Lemma 2.1 (see [3, Theorem 9]) *Let $k \geq 1$ and $n \geq 0$ be integers. Then, for all real numbers $x > 0$*

$$S_k(2n; x) < (-1)^{k+1}\psi^{(k)}(x) < S_k(2n+1; x), \tag{2.5}$$

where

$$S_k(p;x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p}\left[B_{2i}\prod_{j=1}^{k-1}(2i+j)\right]\frac{1}{x^{2i+k}},$$

B_n *are the Bernoulli numbers.*

It follows from (2.5) that for $x > 0$:

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}. \quad (2.6)$$

Using the recurrence formula

$$\psi'(x+1) = \psi'(x) - \frac{1}{x^2}, \quad (2.7)$$

we deduce from (2.6) that for $x > 0$

$$\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x+1) < \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}. \quad (2.8)$$

Lemma 2.2 (see [2]) *For* $x > \frac{1}{2}$ *and* $N \in \mathbb{N}_0$

$$\frac{(n-1)!}{(x-\frac{1}{2})^n} + \sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}} < (-1)^{n+1}\psi^{(n)}(x) < \frac{(n-1)!}{(x-\frac{1}{2})^n} + \sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}} \qquad (n = 1, 2, \ldots), \quad (2.9)$$

where $B_n(t)$ *are the Bernoulli polynomials defined by the following generating function:*

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}.$$

Noting that

$$B_k(1/2) = -\left(1 - \frac{1}{2^{k-1}}\right)B_k \qquad (k \in \mathbb{N}_0),$$

we obtain from (2.9) that for $x > 0$

$$\frac{1}{x} - \frac{1}{12x^3} + \frac{7}{240x^5} - \frac{31}{1344x^7} < \psi'\left(x + \frac{1}{2}\right) < \frac{1}{x} - \frac{1}{12x^3} + \frac{7}{240x^5}. \quad (2.10)$$

Lemma 2.3 (see [11, 12]) *Let $a_1 \neq 0$ and $A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}$ $(x \to \infty)$ be a given asymptotic expansion. Then, the function $B(x) := a_1/A(x)$ has asymptotic expansion of the following form:*

$$B(x) \sim x + \sum_{j=0}^{\infty} \frac{b_j}{x^j} \qquad (x \to \infty),$$

where

$$b_0 = -\frac{a_2}{a_1}, \quad b_j = -\frac{1}{a_1}\left(a_{j+2} + \sum_{k=1}^{j} a_{k+1} b_{j-k}\right) \qquad (j \geq 1).$$

Remark 2.4 Lemma 2.3 provides a method to construct a continued-fraction approximation based on a given asymptotic expansion. The details of this method are given below.

Let $a_1 \neq 0$ and

$$A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j} \qquad (x \to \infty) \tag{2.11}$$

be a given asymptotic expansion. Then, the asymptotic expansion (2.11) can be transformed into the continued-fraction approximation of the form

$$A(x) \approx \cfrac{a_1}{x + b_0 + \cfrac{b_1}{x + c_0 + \cfrac{c_1}{x + d_0 + \ddots}}} \qquad (x \to \infty), \tag{2.12}$$

wherein the constants are given by the following recurrence relations:

$$\begin{cases} b_0 = -\dfrac{a_2}{a_1} \quad \text{and} \quad b_j = -\frac{1}{a_1}\left(a_{j+2} + \sum\limits_{k=1}^{j} a_{k+1} b_{j-k}\right) \\ c_0 = -\frac{b_2}{b_1} \quad \text{and} \quad c_j = -\frac{1}{b_1}\left(b_{j+2} + \sum\limits_{k=1}^{j} b_{k+1} c_{j-k}\right) \\ d_0 = -\frac{c_2}{c_1} \quad \text{and} \quad d_j = -\frac{1}{c_1}\left(c_{j+2} + \sum\limits_{k=1}^{j} c_{k+1} d_{j-k}\right) \\ \quad \cdots \quad \cdots \end{cases} \tag{2.13}$$

Clearly, since $a_j \Longrightarrow b_j \Longrightarrow c_j \Longrightarrow d_j \Longrightarrow \cdots$, the asymptotic expansion (2.11) is transformed into the continued-fraction approximation (2.12), and the constants in the right-hand side of (2.12) are determined by the system (2.13).

3 Asymptotic Expansion

We provide a recurrence relation for successively determining the coefficients of $\frac{1}{n^k}$ ($k \in \mathbb{N}$) in expansion (1.5) given by Theorem 3.1.

Theorem 3.1 *The sequence $(A_n)_{n\in\mathbb{N}}$, defined by (1.4), has the asymptotic expansion*

$$A_n - \gamma \sim \sum_{k=1}^{\infty} \frac{a_k}{n^k}, \qquad n \to \infty, \tag{3.1}$$

with the coefficients a_k given by the recurrence relation

$$a_1 = \frac{1}{2}, \quad a_k = \frac{1}{k}\Bigg\{ -\sum_{j=1}^{k-1} a_j (-1)^{k-j} \binom{k}{k-j+1} - \frac{(4-2^{1-k})B_k}{2} + (-1)^k \left(\frac{k}{2^k} - \frac{3k}{4} + \frac{1}{k+1} \right) \Bigg\} \quad (k \geq 2), \tag{3.2}$$

where B_n are the Bernoulli numbers.

Proof Denote

$$I_n = A_n - \gamma \quad \text{and} \quad J_n = \sum_{k=1}^{\infty} \frac{a_k}{n^k}.$$

In view of (1.5), we can let $I_n \sim J_n$ and

$$\Delta I_n := I_{n+1} - I_n \sim J_{n+1} - J_n =: \Delta J_n$$

as $n \to \infty$, where a_k are real numbers to be determined.

We obtain by (2.2) that

$$\sum_{j=0}^{n} \frac{1}{(i+j)^2} = \psi'(i) - \psi'(i+n+1). \tag{3.3}$$

Using (3.3) and (2.7), we have

$$\begin{aligned} \Delta I_n &= \sum_{i=1}^{n+1} \sum_{j=0}^{n+1} \frac{1}{(i+j)^2} - \sum_{i=1}^{n} \sum_{j=0}^{n} \frac{1}{(i+j)^2} - \ln\left(1 + \frac{1}{n}\right) \\ &= \sum_{i=0}^{n} \frac{1}{(i+n+2)^2} + \sum_{j=0}^{n} \frac{1}{(j+n+1)^2} - \ln\left(1 + \frac{1}{n}\right) \end{aligned}$$

$$= \psi'(n+2) - \psi'(2n+3) + \psi'(n+1) - \psi'(2n+2) - \ln\left(1+\frac{1}{n}\right)$$
$$= 2\psi'(n) - \frac{2}{n^2} - 2\psi'\big(2(n+1)\big) - \frac{3}{4(n+1)^2} - \ln\left(1+\frac{1}{n}\right). \tag{3.4}$$

We obtain from (2.1) that

$$2\psi'(2x) = \frac{1}{2}\psi'(x) + \frac{1}{2}\psi'\left(x+\frac{1}{2}\right). \tag{3.5}$$

We obtain by (3.5) and (2.7) that

$$2\psi'\big(2(n+1)\big) = \frac{1}{2}\psi'(n) - \frac{1}{2n^2} + \frac{1}{2}\psi'\left(n+\frac{1}{2}\right) - \frac{1}{2(n+\frac{1}{2})^2}. \tag{3.6}$$

Substituting (3.6) into (3.4), we obtain

$$\Delta I_n = \frac{3}{2}\psi'(n) - \frac{3}{2n^2} - \frac{1}{2}\psi'\left(n+\frac{1}{2}\right) + \frac{1}{2(n+\frac{1}{2})^2} - \frac{3}{4(n+1)^2} - \ln\left(1+\frac{1}{n}\right). \tag{3.7}$$

Using (2.3) and (2.4), we find as $n \to \infty$

$$\Delta I_n \sim \frac{3}{2}\left(\frac{1}{n} + \frac{1}{2n^2} + \sum_{j=2}^{\infty}\frac{B_j}{n^{j+1}}\right) - \frac{3}{2n^2} - \frac{1}{2}\left(\frac{1}{n} - \sum_{k=3}^{\infty}\frac{(1-2^{2-k})B_{k-1}}{n^k}\right)$$
$$+ \sum_{k=2}^{\infty}\frac{(-1)^k(k-1)}{2^{k-1}n^k} - \sum_{k=2}^{\infty}\frac{(-1)^k 3(k-1)}{4n^k} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{kn^k},$$

which can be written as

$$\Delta I_n \sim \sum_{k=2}^{\infty}\left\{\frac{(4-2^{2-k})B_{k-1}}{2} + (-1)^k\left(\frac{k-1}{2^{k-1}} - \frac{3(k-1)}{4} + \frac{1}{k}\right)\right\}\frac{1}{n^k} \qquad (n \to \infty). \tag{3.8}$$

Direct computation yields

$$\sum_{k=1}^{\infty}\frac{a_k}{(n+1)^k} = \sum_{k=1}^{\infty}\frac{a_k}{n^k}\left(1+\frac{1}{n}\right)^{-k} = \sum_{k=1}^{\infty}\frac{a_k}{n^k}\sum_{j=0}^{\infty}\binom{-k}{j}\frac{1}{n^j}$$
$$= \sum_{k=1}^{\infty}\frac{a_k}{n^k}\sum_{j=0}^{\infty}(-1)^j\binom{k+j-1}{j}\frac{1}{n^j} = \sum_{k=1}^{\infty}\sum_{j=1}^{k}a_j(-1)^{k-j}\binom{k-1}{k-j}\frac{1}{n^k}.$$

We then obtain

$$\Delta J_n = \sum_{k=2}^{\infty} \left\{ \sum_{j=1}^{k} a_j (-1)^{k-j} \binom{k-1}{k-j} - a_k \right\} \frac{1}{n^k}. \tag{3.9}$$

Equating coefficients of the term n^{-k} on the right-hand sides of (3.8) and (3.9) yields

$$\frac{(4-2^{2-k})B_{k-1}}{2} + (-1)^k \left(\frac{k-1}{2^{k-1}} - \frac{3(k-1)}{4} + \frac{1}{k} \right) = \sum_{j=1}^{k} a_j (-1)^{k-j} \binom{k-1}{k-j} - a_k$$

for $k \geq 2$. For $k = 2$ we obtain $a_1 = \frac{1}{2}$, and for $k \geq 3$, we have

$$\begin{aligned} &\frac{(4-2^{2-k})B_{k-1}}{2} + (-1)^k \left(\frac{k-1}{2^{k-1}} - \frac{3(k-1)}{4} + \frac{1}{k} \right) \\ &= \sum_{j=1}^{k-2} a_j (-1)^{k-j} \binom{k-1}{k-j} - (k-1) a_{k-1}, \end{aligned}$$

which gives the desired result (3.2). The proof of Theorem 3.1 is complete. □

4 A Two-Sided Inequality

Motivated by (1.5), we establish a two-sided inequality for the Euler–Mascheroni constant given by Theorem 4.1.

Theorem 4.1 *Let the sequence $(A_n)_{n\in\mathbb{N}}$ be defined by (1.4). Then, for $n \geq 1$*

$$\frac{1}{2n} - \frac{11}{48n^2} < A_n - \gamma < \frac{1}{2n} - \frac{11}{48n^2} + \frac{7}{48n^3}. \tag{4.1}$$

Proof We consider the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ defined by

$$x_n := A_n - \gamma - \frac{1}{2n} + \frac{11}{48n^2} \quad \text{and} \quad y_n := A_n - \gamma - \frac{1}{2n} + \frac{11}{48n^2} - \frac{7}{48n^3}.$$

Clearly

$$\lim_{n\to\infty} x_n = 0 \quad \text{and} \quad \lim_{n\to\infty} y_n = 0.$$

Noting that (3.7), we obtain by (2.8) and (2.10) that

$$x_{n+1} - x_n = \Delta I_n - \frac{1}{2(n+1)} + \frac{11}{48(n+1)^2} + \frac{1}{2n} - \frac{11}{48n^2}$$

$$\begin{aligned}
&= \frac{3}{2}\psi'(n+1) - \frac{1}{2}\psi'\left(n+\frac{1}{2}\right) + \frac{1}{2(n+\frac{1}{2})^2} - \frac{3}{4(n+1)^2} - \ln\left(1+\frac{1}{n}\right) \\
&\quad - \frac{1}{2(n+1)} + \frac{11}{48(n+1)^2} + \frac{1}{2n} - \frac{11}{48n^2} \\
&< \frac{3}{2}\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{6n^3} - \frac{1}{30n^5} + \frac{1}{42n^7}\right) \\
&\quad - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{12n^3} + \frac{7}{240n^5} - \frac{31}{1344n^7}\right) \\
&\quad + \frac{1}{2(n+\frac{1}{2})^2} - \frac{3}{4(n+1)^2} - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4}\right) \\
&\quad - \frac{1}{2(n+1)} + \frac{11}{48(n+1)^2} + \frac{1}{2n} - \frac{11}{48n^2} \\
&= -\frac{P_7(n-2)}{13440n^7(2n+1)^2(n+1)^2}
\end{aligned}$$

and

$$\begin{aligned}
y_{n+1} - y_n &= \Delta I_n - \frac{1}{2(n+1)} + \frac{11}{48(n+1)^2} - \frac{7}{48(n+1)^3} + \frac{1}{2n} - \frac{11}{48n^2} + \frac{7}{48n^3} \\
&= \frac{3}{2}\psi'(n+1) - \frac{1}{2}\psi'\left(n+\frac{1}{2}\right) + \frac{1}{2(n+\frac{1}{2})^2} - \frac{3}{4(n+1)^2} - \ln\left(1+\frac{1}{n}\right) \\
&\quad - \frac{1}{2(n+1)} + \frac{11}{48(n+1)^2} - \frac{7}{48(n+1)^3} + \frac{1}{2n} - \frac{11}{48n^2} + \frac{7}{48n^3} \\
&> \frac{3}{2}\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{6n^3} - \frac{1}{30n^5}\right) - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{12n^3} + \frac{7}{240n^5}\right) \\
&\quad + \frac{1}{2(n+\frac{1}{2})^2} - \frac{3}{4(n+1)^2} - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \frac{1}{5n^5}\right) \\
&\quad - \frac{1}{2(n+1)} + \frac{11}{48(n+1)^2} - \frac{7}{48(n+1)^3} + \frac{1}{2n} - \frac{11}{48n^2} + \frac{7}{48n^3} \\
&= \frac{P_5(n-2)}{480n^5(2n+1)^2(n+1)^3},
\end{aligned}$$

where

$$\begin{aligned}
P_7(n) &= 2152565 + 8661410n + 14190237n^2 + 12516340n^3 \\
&\quad + 6481224n^4 + 1980440n^5 + 331632n^6 + 23520n^7
\end{aligned}$$

and

$$P_5(n) = 6715 + 31135n + 36739n^2 + 18489n^3 + 4268n^4 + 372n^5.$$

We then obtain $x_{n+1} < x_n$ and $y_{n+1} > y_n$ for $n \geq 2$. Direct computations give

$$x_1 = -\frac{1}{48} + \ln 2 - \gamma = 0.095098\ldots, \quad x_2 = \frac{341}{576} - \gamma = 0.014798\ldots,$$

$$y_1 = -\frac{1}{6} + \ln 2 - \gamma = -0.050735\ldots, \quad y_2 = \frac{661}{1152} - \gamma = -0.003430\ldots.$$

We see that the sequence (x_n) is strictly decreasing and (y_n) is strictly increasing for $n \geq 1$, and we have

$$x_n = A_n - \gamma - \frac{1}{2n} + \frac{11}{48n^2} > \lim_{m\to\infty} x_m = 0 \qquad (n \in \mathbb{N})$$

and

$$y_n = A_n - \gamma - \frac{1}{2n} + \frac{11}{48n^2} - \frac{7}{48n^3} < \lim_{m\to\infty} y_m = 0 \qquad (n \in \mathbb{N}).$$

The proof of Theorem 4.1 is complete. □

5 Continued-Fraction Approximation

We convert the asymptotic expansion (3.1) into a continued-fraction approximation given by Theorem 5.1.

Theorem 5.1 *It is asserted that*

$$A_n - \gamma \approx \cfrac{\frac{1}{2}}{n + \frac{11}{24} + \cfrac{-\frac{47}{576}}{n - \frac{5129}{5640} + \cfrac{\frac{112521}{55225}}{n + \frac{2058785551}{1480776360} + \ddots}}} \qquad (n \to \infty). \tag{5.1}$$

Proof By Remark 2.4, we can convert (3.1) into a continued-fraction approximation of the form

$$A_n - \gamma \approx \cfrac{a_1}{n + b_0 + \cfrac{b_1}{n + c_0 + \cfrac{c_1}{n + d_0 + \ddots}}} \qquad (n \to \infty),$$

where the constants in the right-hand side can be determined by using the system (2.13). We see from (1.5) that

$$a_1 = \frac{1}{2}, \quad a_2 = -\frac{11}{48}, \quad a_3 = \frac{7}{48}, \quad a_4 = -\frac{31}{640}, \quad a_5 = -\frac{31}{960}, \quad a_6 = \frac{635}{16128}, \ldots.$$

We obtain from the first recurrence relation in (2.13) that

$$b_0 = -\frac{a_2}{a_1} = \frac{11}{24},$$

$$b_1 = -\frac{a_3 + a_2 b_0}{a_1} = -\frac{47}{576},$$
$$b_2 = -\frac{a_4 + a_2 b_1 + a_3 b_0}{a_1} = -\frac{5129}{69120},$$
$$b_3 = -\frac{a_5 + a_2 b_2 + a_3 b_1 + a_4 b_0}{a_1} = \frac{163853}{1658880},$$
$$b_4 = -\frac{a_6 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0}{a_1} = \frac{2749273}{278691840}, \ldots.$$

We obtain from the second recurrence relation in (2.13) that

$$c_0 = -\frac{b_2}{b_1} = -\frac{5129}{5640},$$
$$c_1 = -\frac{b_3 + b_2 c_0}{b_1} = \frac{112521}{55225},$$
$$c_2 = -\frac{b_4 + b_2 c_1 + b_3 c_0}{b_1} = -\frac{2058785551}{726761000}, \ldots.$$

Continuing the above process, we get

$$d_0 = -\frac{c_2}{c_1} = \frac{2058785551}{1480776360}, \ldots.$$

The proof of Theorem 5.1 is thus completed. □

Remark 5.2 Based on (5.1), we find the following two-sided inequality:

$$\frac{\frac{1}{2}}{n + \frac{11}{24} + \cfrac{-\frac{47}{576}}{n - \frac{5129}{5640} + \cfrac{\frac{112521}{55225}}{n + \frac{2058785551}{1480776360}}}} < A_n - \gamma < \frac{\frac{1}{2}}{n + \frac{11}{24} + \cfrac{-\frac{47}{576}}{n - \frac{5129}{5640}}} \quad (n \in \mathbb{N}). \tag{5.2}$$

Following the same method as was used in the proof of Theorem 4.1, we can prove (5.2). We here omit the proof. Elementary calculations show that

$$\frac{\frac{1}{2}}{n + \frac{11}{24} + \cfrac{-\frac{47}{576}}{n - \frac{5129}{5640} + \cfrac{\frac{112521}{55225}}{n + \frac{2058785551}{1480776360}}}} - \left(\frac{1}{2n} - \frac{11}{48n^2}\right)$$
$$= \frac{1764329280n^2 + 1071186592n + 667905887}{48n^2(252047040n^3 + 236742280n^2 + 229857800n + 60718717)} > 0 (n \geq 1)$$

and

$$\frac{\frac{1}{2}}{n+\frac{11}{24}+\frac{-\frac{47}{576}}{n-\frac{5129}{5640}}}-\left(\frac{1}{2n}-\frac{11}{48n^2}+\frac{7}{48n^3}\right)$$
$$=-\frac{4371n-6559}{48n^3(1880n^2-848n-937)}<0 \quad (n\geq 2).$$

Hence, for $n \geq 2$, the two-sided inequality (5.2) is more accurate than the two-sided inequality (4.1).

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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