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ORIGINAL PAPER

The Clique Number of the Intersection Graph of a Finite Group

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Abstract

For a nontrivial finite group G, the intersection graph $\Gamma(G)$ of G is the simple undirected graph whose vertices are the nontrivial proper subgroups of G and two vertices are joined by an edge if and only if they have a nontrivial intersection. In a finite simple graph Γ, the clique number of Γ is denoted by $\omega(\Gamma)$. In this paper we show that if G is a finite group with $\omega(\Gamma(G)) < 13$, then G is solvable. As an application, we characterize all non-solvable groups G with $\omega(\Gamma(G)) = 13$. Moreover, we determine all finite groups G with $\omega(\Gamma(G)) \in \{2, 3, 4\}$.

Keywords Finite group · Intersection graph · Clique number · Solvable group

Mathematics Subject Classification 20D60 · 05C25 · 20D99

1 Introduction and Main Results

Let G be a group. There are several ways to associate a graph to G (see [8] and the references therein). In this paper, we consider the intersection graph of G which is denoted by $\Gamma(G)$. The intersection graph $\Gamma(G)$ of a nontrivial group G is a simple and undirected graph defined as follows: the vertex set is the set of all proper non-trivial subgroups of G, and there is an edge between two distinct vertices H and K if and only if $H \cap K \neq 1$, where 1 denotes the trivial subgroup of G. The graph $\Gamma(G)$ has been extensively studied (see, for example, [1, 10, 11, 14, 19, 21, 22, 26]), when G is

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finite. Also in [12], the subgraph $\Gamma M(G)$ whose vertices are maximal subgroups of a finitely generated group G is investigated. Intersection graphs of subsemigroups of a semigroup, submodules of a module, and ideals of a ring, were investigated in [2, 7] and [9, 15], respectively.

Let Γ be a simple graph. The set of vertices of every complete subgraph of Γ is called a clique of Γ. The maximum size of a complete subgraph of Γ is called the clique number of Γ and it is denoted by $\omega(\Gamma)$. For convenience, we write $\omega(\Gamma) = 0$ if Γ has no vertices (i. e. Γ is the empty graph) and $\omega(\Gamma) = 1$ if Γ has a non-empty vertex set with no edges (i. e. Γ is null).

In group theory, it is well known that the quantitative properties of some special subgroups play an important role in characterizing the solvability of groups (see [16, 17, 25]). In this paper we give a criterion for solvability of G by $\omega(\Gamma(G))$.

Theorem 1.1 *Let G be a finite group such that* $\omega(\Gamma(G)) < 13$. *Then G is solvable.*

We point out that $\omega(\Gamma(A_5)) = 13$, where A_5 is the alternating group on 5 letters (see the proof of Lemma 3.3), therefore, the bound in Theorem 1.1 is the best possible.

As a consequence of Theorem 1.1, we give a characterization of A_5.

Corollary 1.2 *Let G be a non-solvable group. Then* $\omega(\Gamma(G)) = 13$ *if and only if* $G \cong A_5$.

Remark 1.3 For a finite group G, it is clear that $\omega(\Gamma(G)) = 0$ if and only if $G \cong C_p$, where C_p is a cyclic group of order p, for some prime number p. By Lemma 2.2 of [19], we see that $\omega(\Gamma(G) = 1$ if and only if G is isomorphic to one of the following groups:

$$C_{p^2}, \quad C_{pq}, \quad C_p \times C_p, \quad C_p \rtimes C_q,$$

where p and q are distinct prime numbers and the last group is the semidirect product of C_p by C_q.

In what follows, we determine groups G with $2 \leq \omega(\Gamma(G)) \leq 4$.

Theorem 1.4 *Let G be a finite group. Then* $\omega(\Gamma(G)) = 2$ *if and only if one of the following cases occurs::*

(a) *G is a cyclic group of order* p^3, *for a prime* p.

(b) $G = N \rtimes H$ *is Frobenius whose kernel N is the minimal normal subgroup of G such that* $N \cong C_p \times C_p$ *and* $H \cong C_q$, *where p and q are primes.*

Theorem 1.5 *Let G be a finite group. Then* $\omega(\Gamma(G)) = 3$ *if and only if one of the following statements holds:*

(a) *G is cyclic and* $G \cong C_{p^4}$, C_{p^2q} *or* C_{pqr}, *where p, q and r are distinct primes.*

(b) $G = N \rtimes H$ *is Frobenius whose kernel N is the (unique) minimal normal subgroup of G and we have either*

(1) $N \cong C_p$ *and* $H \cong C_{q^2}$ *or* C_{qr} *or*

(2) $N \cong C_p \times C_p$, $H \cong C_{q^2}$ *and G does not have any subgroup of order pq, where p, q and r are distinct primes.*

Theorem 1.6 *Let G be a finite group. Then $\omega(\Gamma(G)) = 4$ if and only if one of the following holds:*

(a) *G is abelian and $G \cong C_{p^5},\ C_2 \times C_2 \times C_q$ or $C_4 \times C_2$, where p, q are primes and q is odd..*

(b) *G is non-abelian and isomorphic to one of the followings:*

(1) *D_8 or Q_8 or*

(2) *$G = N \rtimes H$ is Frobenius whose kernel N is the (unique) minimal normal subgroup of G such that $N \cong C_p \times C_p$, $H \cong C_{qr}$ and G does not have subgroups of orders pr and pq, where p, q, r are distinct primes.*

Now we give some examples of groups, satisfying in Theorems 1.4, 1.5 and 1.6. Since $A_4 \cong (C_2 \times C_2) \rtimes C_3$ satisfies in Theorem 1.4(b),we have $\omega(\Gamma(A_4)) = 2$. In GAP-System [23], two groups

$$G_1 = AllSmallGroups(42, IsAbelian, false)[1] \cong C_7 \rtimes C_6$$

and

$$G_2 = AllSmallGroups(52, IsAbelian, false)[2] \cong C_{13} \rtimes C_4,$$

satisfy in Theorem 1.5(b)(1) and so $\omega(\Gamma(G_1)) = \omega(\Gamma(G_2) = 3$. Also, for $G_3 = AllSmallGroups(11^2.4, IsAbelian, false)[5] \cong (C_{11} \times C_{11}) \rtimes C_4$, we have $\omega(\Gamma(G_3)) = 3$, by Theorem 1.5(b)(2). Finally if $G_4 = AllSmallGroups(11^2.6, IsAbelian, false)[5] \cong (C_{11} \times C_{11}) \rtimes C_6$, then $\omega(\Gamma(G_4)) = 4$, by Theorem 1.6(b)(2).

In this paper all groups are finite and we use the usual notation, for example C_n, A_n, S_n, $PSL(2, q)$ and $Sz(q)$, respectively, denote the cyclic group of order n, the alternative group on n letters, the symmetric group on n letters, the projective special linear group of degree 2 over the finite field of size q and the Suzuki group over the field with q elements. For a group G, $x \in G$ and $H \le G$, we denote by $Z(G)$, $C_G(x)$, $N_G(H)$, H^x, the center of G, the centralizer of x in G, the normalizer of H in G and the conjugate of H in G by x, respectively. Also $\mathcal{S}(G)$ is the set of all subgroups of G. The rest of the notation is standard and can be found mainly in [18].

2 Preliminary Results

We frequently use the following results and we state it here for the reader's convenience.

Lemma 2.1 (see 1.3.11 of [18]) *Let G be a group and H and K be subgroups of G. Then $|G : H \cap K| \le |G : H||G : K|$, with equality if the indices $|G : H|$ and $|G : K|$ are coprime.*

Theorem 2.2 (see 8.5.5 of [18]) *If G is a finite group with a subgroup H such that $H \cap H^x = 1$, for all $x \in G \backslash H$, then $N = (G \backslash (\cup_{x \in G} H^x)) \cup \{1\}$ is a normal subgroup of G and $G = N \rtimes H$.*

A group G which has a proper nontrivial subgroup H, satisfying in the hypothesis of Theorem 2.2 is called a *Frobenius group*, H is called a Frobenius complement and N the Frobenius kernel. In this case, $Z(G) = 1$ and $|H|$ divides $|N| - 1$, by Exercises 8.5 (2) and (6) of [18] respectively. Note that if $1 \neq x \in N$, then $C_G(x) \leq N$, by Exercise 8.5 (5) of [18].

Recall that a subgroup H of a group G is *supplemented* in G if there is a subgroup K of G such that $G = HK$. Moreover if $H \cap K = 1$, then H is *complemented* in G by K.

Proposition 2.3 *Let G be a group and N be a proper minimal normal abelian subgroup of G. If $G = HN$, for some subgroups H of G and N is complemented in G by H, then H is a maximal subgroup of G.*

Proof Assume that there is a subgroup K of G such that $H \lneqq K$. Since $G = HN$, we have $K = H(K \cap N)$. Therefore, $K \cap N \neq 1$. Since N is the abelian normal subgroup of G and $G = KN$, we have $K \cap N \lhd G$. Thus $K \cap N = N$, by minimality of N and so $N < K$. Consequently $K = G$, as required. □

Recall that a finite group G is said to be *primitive* if it has a maximal subgroup M with $Core_G(M) := \cap_{g \in G} M^g = 1$. In this situation we call M a stabilizer of G. We need the following theorem of R. Baer on primitive groups.

Theorem 2.4 (see [3]) *Let G be a finite primitive group with a stabilizer M. Then one of the following three statements holds:*

(1) *G has a unique minimal normal subgroup N, this subgroup N is self-centralizing, and N is complemented by M in G.*
(2) *G has a unique minimal normal subgroup N, this N is non-abelian, and N is supplemented by M in G.*
(3) *G has exactly two minimal normal subgroups N and N^*, and each of them is complemented by M in G. Also $C_G(N) = N^*$, $C_G(N^*) = N$ and $N \cong N^* \cong NN^* \cap M$.*

The following lemma is straightforward but is useful in the sequel.

Lemma 2.5 *Let G be a finite group.*

(i) *If $1 < H < G$, then $\omega(\Gamma(H)) < \omega(\Gamma(G))$.*

(ii) *If $1 < N \lhd G$, then $|\mathcal{S}(\frac{G}{N})| - 1 + \omega(\Gamma(N)) \leq \omega(\Gamma(G))$.*

(iii) *Suppose that Ω is a clique in $\Gamma(G)$, and $H \in \Omega$. If $|H|$ is prime, then $H \leq K$, for every $K \in \Omega$.*

Proof The proofs of (i) and (iii) are clear. For (ii), if $\omega(\Gamma(N)) = k$, for some positive integer k, then there are proper subgroups $N_1, \ldots, N_k$ of N such that $N_i \cap N_j \neq 1$, for each $1 \leq i < j \leq k$. If T is a subgroup of $\frac{G}{N}$, then there exists a unique subgroup K of G such that $N \leq K$ and $T = \frac{K}{N}$. Now assume that $|\mathcal{S}(\frac{G}{N})| = l$, for some positive integer l. Then there exist subgroups $K_1 = N, K_2, \ldots, K_{l-1}, K_l = G$ containing N of G such that $\mathcal{S}(\frac{G}{N}) = \{\frac{K_1}{N}, \frac{K_2}{N}, \ldots, \frac{K_l}{N}\}$. So we have $\{N_1, \ldots, N_k, K_1, \ldots, K_{l-1}\}$ is a clique in $\Gamma(G)$. It follows that $\omega(\Gamma(G)) \geq k + l - 1$, as required. □

We denote by $m(G)$ the number of maximal subgroups of a group G. In the following, we give $m(G)$, for any p-group G, where p is a prime number.

Lemma 2.6 *Let G be a finite p-group. Then $m(G) = m(\frac{G}{\Phi(G)}) = \frac{|\frac{G}{\Phi(G)}|-1}{p-1}$.*

Proof Since $\Phi(G)$ is contained in any maximal subgroup of G, it is easy to see that $\frac{M}{\Phi(G)}$ is a maximal subgroup of $\frac{G}{\Phi(G)}$ whenever M is a maximal subgroup of G. On the other hand, similar to the proof of Lemma 2.5(ii), if T is a maximal subgroup of $\frac{G}{\Phi(G)}$, then there is a unique maximal subgroup M of G such that $T = \frac{M}{\Phi(G)}$. Thus we conclude that there exists a one-one correspondence from the set of all maximal subgroups of G to the set of all maximal subgroups of $\frac{G}{\Phi(G)}$ and then $m(G) = m(\frac{G}{\Phi(G)})$. By the proof of 5.3.2 of [18], we have $\frac{G}{\Phi(G)}$ is elementary abelian. Now if $|\frac{G}{\Phi(G)}| = p^m$, then every maximal subgroup of $\frac{G}{\Phi(G)}$ has order p^{m-1} and so by [5, Exercise 1(d), p. 81]] we have $m(\frac{G}{\Phi(G)}) = 1 + p + \cdots + p^{m-1}$. It follows that $m(\frac{G}{\Phi(G)}) = \frac{|\frac{G}{\Phi(G)}|-1}{p-1}$, as desired. □

3 A Criterion for Solvability by $\omega(\Gamma(G))$

In this section, we prove Theorem 1.1 and Corollary 1.2. The famous family of simple finite groups are minimal simple groups (i.e., finite non-abelian simple groups all of whose proper subgroups are solvable). The classification of minimal simple groups is given by Thompson (see Corollary 1 of [24]). As Thompson's classification of the minimal simple groups is a very useful tool to obtain solvability criteria in the class of finite groups. In Lemmas 3.3, 3.4, 3.5, 3.6 and 3.7, we show that if G is a minimal simple group non-isomorphic to A_5, then $\omega(\Gamma(G) > 13$. We need these results in the proof of Theorem 1.1 and Corollary 1.2.

In the following we give some facts about subgroups of $G = A_5$. The proofs are elementary (for example, one may check by software GAP [23]).

Fact 1. $|\mathcal{S}(A_5)| = 59$ and if n_k is the number of subgroups of G of order k, then $n_2 = 15, n_3 = n_6 = 10, n_4 = n_{12} = 5$ and $n_5 = n_{10} = 6$.

Fact 2. If M is a maximal subgroup of G, then $|M| \in \{6, 10, 12\}$. Also all maximal subgroups of G of the same order are conjugate in G.

Fact 3. If M_1, M_2 and M_3 are maximal subgroups of G such that $|M_1| = 6$, $|M_2| = 10$ and $|M_3| = 12$, then $|M_i^x \cap M_j^y| \neq 1$, for every $1 \leq i < j \leq 3$ and $x, y \in G$. Moreover, $|M_2 \cap M_2^a| = 2$ and $|M_3 \cap M_3^b| = 3$, for each $a \in G \backslash M_2$ and $b \in G \backslash M_3$. Therefore, $\Lambda = \{M_1, M_2^g, M_3^g : g \in G\}$ is a clique in $\Gamma(G)$ and so $\omega(\Gamma(G)) \geq 1 + n_{10} + n_{12} = 12$.

Fact 4. Assume that M_1, M_2 and M_3 are the same as in Fact 3. If M_i, M_i^x and M_i^y are distinct conjugates of M_i in G, for some $x, y \in G$, then $|M_i \cap M_i^x \cap M_i^y| = 1$, for each i and $\{M_1, M_1^x, M_1^y\}$ does not form a clique in $\Gamma(G)$. Also we have $|M_1 \cap M_1^g| \leq 2$, for every $g \in G \backslash M_1$ and there is $z \in G$ such that $|M_1 \cap M_1^z| = 2$. Thus $\Lambda \cup \{M_1, M_1^z\}$ is a clique, by Fact 3, and so $\omega(\Gamma(G)) \geq 13$.

We use of above facts in the proof of the following lemma.

Lemma 3.1 *We have* $\omega(\Gamma(A_5)) = 13$.

Proof Let Ω be a clique in $\Gamma(A_5)$ of maximum size. Then $|\Omega| \geq 13$, by Fact 4. Now, we consider the following cases:

Case 1. In this case, we show that Ω does not have any member of size 2. Assume that there is $H_2 \in \Omega$ with $|H_2| = 2$. Then other subgroups of order 2, all subgroups of order 3 and all subgroups of order 5 of A_5 do not belong to Ω and so $|\Omega| \leq 57 - ((n_2 - 1) + n_3 + n_5) = 27$, by Fact 1. Since the intersection of any two distinct subgroups of order 4 is trivial, there is only one subgroup H_4 of order 4 containing H_2. Thus, $|\Omega| \leq 27 - (n_4 - 1) = 23$. Note that if $X \in \Omega$, then $H_2 \subseteq X$, by Lemma 2.5(iii). By Fact 4, Ω has at most two subgroups of order 6. Also, by Facts 3, 4 and Lemma 2.5 (iii), Ω has at most two subgroups of order 10. Since the intersection of any two distinct subgroups of order 12 has order 3 by Fact 3, Ω has at most one subgroup of order 12. It follows that $|\Omega| \leq 23 - ((10 - 2) + (6 - 2) + (5 - 1)) = 7$, a contradiction.

Case 2. In this case, we show that Ω does not have any member of size 3. Assume that there exists $H_3 \in \Omega$ with $|H_3| = 3$. Then Ω does not have any subgroup of order 2, 4, 5 and 10 and so $|\Omega| \leq 57 - (n_2 + n_4 + n_5 + n_{10}) = 25$, by Fact 1. Since Ω does not have two subgroups of order 3, we have $|\Omega| \leq 25 - 9 = 16$. Also if $X \in \Omega$, then $H_3 \subseteq X$, by Lemma 2.5(iii). By Fact 4, H_3 is contained in only one subgroup of order 6. Therefore, $|\Omega| \leq 16 - 9 = 7$, which is impossible.

Case 3. In this case, we show that Ω does not have any member of size 4. Assume that there exists $H_4 \in \Omega$ with $|H_4| = 4$. By Case 1, Ω does not have any subgroup of order 2. Since $n_2 = 15, n_3 = 10, n_4 = 5$ and $n_5 = 6$, we have $|\Omega| \leq 57-(15+10+4+6) = 22$. Since every clique can contain at most two subgroups of order 6 by fact 4, we have $|\Omega| \leq 22-(n_6-2) = 14$. Note that H_4 is contained in at least one maximal subgroup, which is certainly of order 12. By Fact 3, there is only one subgroup K of order 12 such that $H_4 \subset K$ and hence $|\Omega| \leq 14 - 4 = 10$, which is a contradiction.

Case 4. In this case, we show that Ω does not have any member of size 5. Assume that there exists $H_5 \in \Omega$ with $|H_5| = 5$. Since $n_2 = 15, n_3 = n_6 = 10, n_4 = n_{12} = 5$ and $n_5 = 6$, we have $|\Omega| \leq 57 - (15 + 20 + 10 + 5) = 7$, a contradiction.

Case 5. In this case, we first show that Ω has at least one member of size 6 and then prove that $|\Omega| = 13$. Suppose, for a contradiction, that Ω does not have any member of size 6. By Cases 1–4, we conclude that Ω does not have any subgroup of order 2, 3, 4 and 5. By Fact 1, every member of Ω has size either 10 or 12 and so $|\Omega| \leq n_{10} + n_{12} = 11$, a contradiction. It follows that there exists $H_6 \in \Omega$ with $|H_6| = 6$. Again, by Cases 1-4, we have $|\Omega| \leq n_6 + n_{10} + n_{12} = 16$. On the other hand Ω has at most two members of size 6 by Fact 4. Thus $|\Omega| = 13$ by Fact 4 and this completes the proof. □

In the proofs of Lemmas 3.3, 3.4 and 3.5, we need the following result about the Sylow p-subgroups of $PSL(2, p^n)$, where p is a prime number and n is a positive integer.

Proposition 3.2 *Let* $G = PSL(2, p^n)$. *Then a Sylow* p*-subgroup* P *of* G *is elementary abelian of order* p^n *and the number of Sylow* p*-subgroups of* G *is* $p^n + 1$ *(or equivalently, we have* $|G : N_G(P)| = p^n + 1$*).*

Proof See Satz 8.2, p. 191 of [13]. □

Lemma 3.3 *For any odd prime p, we have* $\omega(\Gamma(PSL(2, 2^p)) > 13$.

Proof If $p = 3$, then $G = PSL(2, 8)$ has a maximal subgroup M of index 28 in G. Since G is simple, M is not normal in G and so $N_G(M) = M$. It follows that M has 28 conjugates in G, say $M_1 = M, M_2, \ldots, M_{28}$. It is easily checked by GAP [23] that $|M_i \cap M_j| = 2$, for all distinct i, j and so $\omega(\Gamma(PSL(2, 2^m)) \geq 28$, as required.

If $p \geq 5$, then $q = 2^p \geq 32$. By Proposition 3.2, we have $|G : N_G(P)| = q + 1$, where $P \in Syl_2(G)$. Since $N_G(N_G(P)) = N_G(P)$, the number of conjugates of $N_G(P)$ is $q+1$. It follows that $N_G(P)$ has $q+1$ conjugates in G. If $N_G(P) \neq N_G(P)^g$, for some $g \in G$, then $|G : N_G(P) \cap N_G(P)^g| \leq (q + 1)^2$, by lemma 2.1. Since $|G| = \frac{q(q^2-1)}{2}$, we have $|N_G(P) \cap N_G(P)^g| \neq 1$. Therefore, $\{N_G(P)^x : x \in G\}$ is a clique in the $\Gamma(G)$ and so $\omega(\Gamma(G)) \geq q + 1 > 13$, as claimed. □

Lemma 3.4 $\omega(\Gamma(PSL(2, 3^p)) > 13$, *for any odd prime p.*

Proof By Proposition 3.2, we have $|G : N_G(P)| = 3^p + 1$, where $P \in Syl_3(G)$. Similar to the proof of Lemma 3.3, $\{N_G(P)^x : x \in G\}$ is a clique in the $\Gamma(G)$ and so $\omega(\Gamma(G)) \geq 3^p + 1 > 13$ and this completes the proof. □

Lemma 3.5 $\omega(\Gamma(PSL(2, p)) > 13$, *where $p > 3$ is a prime with $p^2 + 1$ divisible by 5.*

Proof If $G = PSL(2, 7)$, then G has 14 (maximal) subgroups of index 7 (one can check it by GAP [23]), say $M_1, M_2, \ldots, M_{14}$. Since $|G| = 168$ and $|G : M_i \cap M_j| \leq 49$, for every $1 \leq i \neq j \leq 14$, we have $|M_i \cap M_j| \neq 1$, by Lemma 2.1 and so $\{M_1, \ldots, M_{14}\}$ is a clique in the $\Gamma(G)$ and so $\omega(\Gamma(G)) \geq 14$, as wanted.

Now if $p \geq 13$, then by the similar argument in the proof of Lemmas 3.3 and 3.4, we see that $\omega(\Gamma(G)) \geq p + 1 > 13$ and the proof is complete. □

Lemma 3.6 *We have* $\omega(\Gamma(PSL(3, 3)) > 13$.

Proof By GAP [23], G has 26 maximal subgroups of index 13, say $M_1, M_2, \ldots, M_{26}$. Since $|G| = 5616$ and $|G : M_i \cap M_j| \leq 169$, for every $1 \leq i \neq j \leq 26$, we have $|M_i \cap M_j| \neq 1$ and so $\omega(\Gamma(G)) \geq 26$, as required. □

Lemma 3.7 *We have* $\omega(\Gamma(Sz(2^m))) > 13$, *where m is an odd prime.*

Proof Suppose that $q = 2^m$, $G = Sz(q)$ and $F \in Syl_2(G)$. Then it is well-known that $|F| = q^2$ and $Z(F)$ is elementary abelian of order q. It follows from Lemma 5.9 in Chapter XI of [13] that $|C_F(g) : Z(F)| = 2$, for all $g \in F \backslash Z(F)$ and so $C_F(g)$ is abelian. Assume that $\{C_F(x_1), C_F(x_2), \ldots, C_F(x_n)\}$ is the set of all proper centralizers of elements in F. Then $F = \cup_{i=1}^{n} C_F(x_i)$ and $C_F(x_i) \cap C_F(x_j) = Z(F)$, for each $i \neq j$. Since $|F| = q^2$, $|Z(F)| = q$ and $|C_F(x_i)| = 2q$, we have $n = q - 1$. Also $Z(F)$ has $q - 1$ maximal subgroups, by Lemma 2.6 and the intersection of any pair of these maximal subgroups are non-trivial, by Lemma 2.1. If $Z_1, \ldots, Z_{q-1}$ are maximal subgroups of $Z(F)$, then $\{Z_1, \ldots, Z_{q-1}, C_F(x_1), \ldots, C_F(x_{q-1})\}$ is a clique in $\Gamma(G)$. Since $q \geq 8$, we have $\omega(\Gamma(Sz(2^m))) \geq 2(q - 1) \geq 14$. This completes the proof. □

Now we are ready to prove the first main result:

Proof of Theorem 1.1 Suppose, on the contrary, that there exists a non-solvable finite group G of the least possible order with $\omega(\Gamma(G)) < 13$. If there exists a non-trivial proper normal subgroup N of G, then $\omega(\Gamma(N)) < 13$ and $\omega(\Gamma(\frac{G}{N})) < 13$, by Lemma 2.5(i-ii). By minimality of $|G|$, we have $\frac{G}{N}$ and N are solvable, which implies that G is solvable, a contradiction. Thus G is simple and so by Theorem 1 of [4] and Lemma 2.5(i), G is a minimal simple group. By Thompson's classification of minimal simple groups [24], G is isomorphic to one of the following simple groups: A_5, $PSL(2, 2^m)$, where m is an odd prime; $PSL(2, 3^m)$, where m is an odd prime; $PSL(2, p)$, where $p > 3$ is a prime with $p^2 + 1$ divisible by 5; $PSL(3, 3)$; or $Sz(2^m)$, where m is an odd prime. By Lemmas 3.1, 3.3, 3.4, 3.5, 3.6 and 3.7, we have $\omega(\Gamma(G)) \geq 13$, which is impossible. □

Proof of Corollary 1.2 If $G \cong A_5$, then $\omega(\Gamma(G)) = 13$, by Lemma 3.1. Conversely, assume that G is non-solvable with $\omega(\Gamma(G)) = 13$. If M is a maximal subgroup of G, then $\omega(\Gamma(M)) < 13$, by Lemma 2.5(i). So M is solvable by Theorem 1.1. Thus every maximal subgroup of G is solvable and so G is a minimal non-solvable group. Therefore, G is a minimal simple group. By the proof of Theorem 1.1 and Lemmas 3.1, 3.3, 3.4, 3.5, 3.6 and 3.7, we conclude that $G \cong A_5$, as desired. □

4 Groups G with $2 \leq \omega(\Gamma(G)) \leq 4$

In this section, we prove Theorems 1.4, 1.5 and 1.6. First, we obtain $\omega(\Gamma(G))$, for some groups which are needed later.

Lemma 4.1 *For any distinct primes p, q and r and positive integer n, we have*

(1) $\omega(\Gamma(C_{p^n})) = n - 1$.
(2) $\omega(\Gamma(C_p \times C_p \times C_p)) = p^2 + p + 1$.
(3) $\omega(\Gamma(C_{pqr})) = \omega(\Gamma(C_{p^2q})) = 3$.
(4) $\omega(\Gamma(C_{p^2} \times C_p)) = \omega(\Gamma(C_{pq} \times C_p)) = p + 2$.
(5) *If G is a non-abelian group of order p^3, then* $\omega(\Gamma(G)) = p + 2$.

Proof (1) If $G = C_{p^n}$, then G has a unique subgroup H_i of order p^i, for each $0 \leq i \leq n$ and so $H_i < H_{i+1}$, for every i. Therefore, $\{H_1, H_2, \ldots, H_{n-1}\}$ is the unique clique in $\Gamma(G)$ with maximum size, as required.

(2) Suppose that $G = C_p \times C_p \times C_p$ and Ω is a clique in $\Gamma(G)$. If $H \in \Omega$, with $|H| = p$, then, by Lemma 2.5 (iii), $H < K$, for every $K \in \Omega \backslash \{H\}$ and so $|K| = p^2$. Since $\frac{G}{H} \cong C_p \times C_p$ and the number of subgroups of $C_p \times C_p$ of order p is $p+1$, by the proof of Lemma 2.5(ii),the number of proper subgroups of G containing H is $p+2$ and so $|\Omega| \leq p+2$. Now suppose that Ω does not contain any subgroup of G of order p. Then $|K| = p^2$, for every $K \in \Omega$ and so K is a maximal subgroup of G. Since $\Phi(G) = 1$, the number of maximal subgroups of G is $\frac{|\frac{G}{\Phi(G)}|-1}{p-1} = p^2+p+1$, by Lemma 2.6. Therefore $|\Omega| \leq p^2+p+1$. It follows from Lemma 2.1 that the intersection of any two distinct maximal subgroups of

G are non-trivial. This implies that all maximal subgroups of G forms a clique with maximum size in $\Gamma(G)$, as wanted.

(3) The proof is clear.

(4) Suppose that $G = C_{p^2} \times C_p$ and Ω is a clique in $\Gamma(G)$. If $H \in \Omega$ such that $|H| = p$, then $H < K$, for every $K \in \Omega \setminus \{H\}$, by Lemma 2.5 (iii) and so $|K| = p^2$. Since the number of proper subgroups of G containing H is at most $p+2$ and so $|\Omega| \leq p+2$. On the other hand $|\Phi(G)| = p$ and so the number of maximal subgroups of G is $p+1$. It follows that these maximal subgroups of G together with $\Phi(G)$ forms a clique with maximum size in $\Gamma(G)$ and so $\omega(\Gamma(G)) = p+2$, as desired.

Suppose that $G = C_{pq} \times C_p$ and Ω is a clique in $\Gamma(G)$. If Ω does not contain any subgroup of G of prime order, then $\Omega = \{T\}$ such that $|T| = p^2$. If there is a subgroup H of G such that $H \in \Omega$ and $|H| = p$, then $\frac{G}{H} \cong C_{pq}$ and so $|\Omega| \leq 3$. Finally assume that $Q \in \Omega$ such that $|Q| = q$. Then $Q \leq K$, for every $K \in \Omega$ by Lemma 2.5 (iii). On the other hand the number of proper subgroups of G containing Q is $p+2$ and so $|\Omega| \leq p+2$. Hence $\omega(\Gamma(G)) = p+2$, as required.

(5) The proof is similar to the first case of (4).

□

Lemma 4.2 *Let $G = N \rtimes H$ be a Frobenius group such that N is a minimal normal subgroup of G and H is a Frobenius complement. Also assume that p, q and r are distinct primes.*

(1) *If $N \cong C_p$ and $H \cong C_{q^2}$ or C_{qr}, then $\omega(\Gamma(G)) = 3$.*
(2) *If $N \cong C_p \times C_p$ and $H \cong C_q$, then $\omega(\Gamma(G)) = 2$.*
(3) *If $N \cong C_p \times C_p$ and $H \cong C_{q^2}$ such that $q \nmid p-1$, then $\omega(\Gamma(G)) = 3$.*
(4) *If $N \cong C_p \times C_p$ and $H \cong C_{qr}$ such that $q \nmid p-1$ and $r \nmid p-1$, then $\omega(\Gamma(G)) = 4$.*

Proof (1) Suppose that $N \cong C_p$ and $H \cong C_{q^2}$ and $1 < H_1 < H$. Since $G = N \rtimes H$ is a Frobenius group, $H \cap H^x = 1$ and so $H_1 \cap H_1^x = 1$, for every $1 \neq x \in N$. Therefore, G has p subgroups of order q and so all of them are contained in the non-abelian subgroups $K = NH_1$. It follows that $K = NH_1$ is the unique subgroup of G of order pq. It is clear that $K \cap T \neq 1$, for every non-trivial proper subgroup T of G. Thus K belongs to any clique with maximum size. Note that every clique contains at most one subgroup of order q^2 and also at most one subgroup of order q. If N belongs to a clique Ω, then H_1^x and H^x do not belong to Ω, for every $x \in N$. Thus $\{H_1^x, H^x, K\}$ is a clique with maximum size in $\Gamma(G)$, for every $x \in N$, as required.

Now suppose that $H \cong C_{qr}$. Then H contains (unique) subgroups Q and R of order q and r respectively. Since $G = N \rtimes H$ is a Frobenius group, we have $H \cap H^x = 1$, $Q \cap Q^x = 1$ and $R \cap R^x = 1$, for every $1 \neq x \in N$. Therefore $|Syl_q(G)| = |Syl_r(G)| = |G : N_G(H)| = |G : H| = p$. Since G is Frobenius, $Z(G) = 1$ and so NQ and NR are non-abelian. Therefore NQ and NR are the only subgroups of G of orders pq and pr, respectively. Note that the only proper subgroups of G containing properly Q^x, for some $x \in N$ are H^x and NQ, the only proper subgroups of G containing properly R^x, for some $x \in N$ are H^x and NR and also the only proper subgroups of G containing properly N are NQ and NR. Now suppose that Ω is a clique in $\Gamma(G)$ with maximum size.

If $Q^x \in \Omega$, for some $x \in N$, then $\{Q^x, H^x, NQ\} = \Omega$. If $R^x \in \Omega$, for some $x \in N$, then $\{R^x, H^x, NR\} = \Omega$. If $N \in \Omega$, then $\{N, NQ, NR\} = \Omega$. If none of the three cases does not occur, for Ω, then $\Omega = \{H^x, NQ, NR\}$, for some $x \in N$. So we conclude that Ω has 3 members, as wanted.

(2) Since N is a minimal normal abelian subgroup of G, we conclude that H^x is a maximal subgroup of G, for all $x \in N$ by Proposition 2.3 and so G does not have any subgroup of order pq. Therefore,

$$\mathcal{S}(G) = \{1, G, N, N_1, \ldots, N_{p+1}, H^x : \ x \in N\},$$

where $|N_i| = p$, for each i. Since $G = N \rtimes H$ is Frobenius, $\{N_i, N\}$ is a clique with maximum size in $\Gamma(G)$, for each i, as wanted.

(3) Since $G = N \rtimes H \cong (C_p \times C_p) \rtimes C_{q^2}$ is Frobenius with complement H, the group G has $p + 1$ subgroups of order p (contained in N), say $N_1, \ldots, N_{p+1}$ and p^2 subgroups H_1^x of order q, for each $x \in N$, where $1 < H_1 < H$. Note that H is a maximal subgroup of G by Proposition 2.3. Therefore, G does have any subgroup of order pq^2. Clearly, $H_1^x \leq NH_1$, for every $x \in N$ and so NH_1 is the unique non-abelian subgroup of G of order p^2q. If G has a subgroup K of order pq, then K is not abelian since $C_G(x) = N$, for $1 \neq x \in N$ by Exercise 8.5 (5) of [18]. It follows that $q|p - 1$, which is contrary to our assumption in this part. Hence the order of a non-trivial proper subgroup of G belongs to $\{p, p^2, q, p^2q, q^2\}$ and so we have

$$\mathcal{S}(G) = \{1, G, N, N_1, \ldots, N_{p+1}, H^x, H_1^x, NH_1 : \ x \in N\}.$$

Since $N \cap H^y = H \cap H^x = H_1 \cap H_1^x = N_i \cap N_j = 1$, for every $y \in N$, $x \in N \backslash \{1\}$ and $i \neq j$, we have $\{N_i, N, NH_1\}$ is a clique with maximum size in $\Gamma(G)$, for each i. Also $\{H_1^x, H^x, NH_1\}$ is a clique with maximum, for every $x \in N$. This completes the proof.

(4) The proof is similar to the part (3).

□

Proof of Theorem 1.4: Since $\omega(G) = 2$, G contains two proper subgroups H_1 and H_2 such that $H_1 \cap H_2 \neq 1$ and so $\{H_1, H_2, H_1 \cap H_2\}$ is a clique in $\Gamma(G)$, which implies that either $H_1 < H_2$ or $H_2 < H_1$. Without loss of generality, assume that $H_1 < H_2$. Since $\omega(G) = 2$, we have $|H_1| = p$ is prime, H_1 is a maximal subgroup of H_2 and H_2 is a maximal subgroup of G. Now we claim that $|H_2 : H_1|$ and $|G : H_2|$ are primes.

If $H_1 \lhd H_2$ and $H_2 \lhd G$, then the claim is valid. If H_1 is not normal in H_2, then $N_{H_2}(H_1) = H_1$ and since $|H_1|$ is prime, we have $H_1 \cap H_1^x = 1$, for all $x \in H_2 \backslash H_1$. It follows from Theorem 2.2 that H_2 is Frobenius and so there is a normal subgroup N such that $H_2 = N \rtimes H_1$. If $|N|$ is not prime, then N has a proper non-trivial subgroup N_1 and so $\{N_1, N, H_2\}$ is a clique in $\Gamma(G)$ of size 3, which is a contradiction. Therefore, $|N| = |H_2 : H_1|$ is prime. Now if $H_2 \lhd G$, then $|G : H_2|$ is prime by maximlity of H_2 in G. So assume that H_2 is not normal in G. Then, by a similar argument to $H_1 \ntriangleleft H_2$, we conclude that G is Frobenius and $|G : H_2|$ is prime. Thus $|G| = |H_1||H_2 : H_1||G : H_2| = pqr$, for some primes p, q and r.

First suppose that G is abelian. If p, q and r are distinct, then $G \cong C_{pqr}$ and so by Lemma 4.1(3) we have $\omega(\Gamma(G)) = 3$, a contradiction. If $p = r \neq q$, then $G \cong C_{p^2q}$

or $C_p \times C_p \times C_q$, which implies that $\omega(\Gamma(G)) \neq 2$ by Lemma 4.1(3-4). If $p = q = r$, then $G \cong C_{p^3}$, $C_{p^2} \times C_p$ or $C_p \times C_p \times C_p$. Since $\omega(G) = 2$, we have $G \cong C_{p^3}$ by Lemma 4.1.

Now suppose that G is not abelian. If $p = q = r$, then $|G| = p^3$ and we have a contradiction by Lemma 4.1(5). If p, q and r are distinct, then at least one of Sylow subgroups of G is normal, which implies that $\omega(\Gamma(G)) \geq 3$, a contradiction. Finally suppose that $p = r \neq q$. If $Syl_q(G) = \{Q\}$ and $P \in Syl_p(G)$, then $\{P_1, P, P_1Q\}$ is a clique in $\Gamma(G)$, where P_1 is a proper non-trivial subgroup of P, again a contradiction. Hence we have $Syl_p(G) = \{P\}$. If $Q \in Syl_q(G)$, then $G = P \rtimes Q$. Since $\omega(\Gamma(G)) = 2$, we see that P must be a minimal normal subgroup of G and so $P \cong C_p \times C_p$, which implies that Q is a maximal subgroup of G Proposition 2.3. Since Q is not normal in G, we have $N_G(Q) = Q$. It follows $Q \cap Q^g = 1$, for every $g \in G \backslash Q$ and so G is a Frobenius group. This completes the proof.

The converse follows from Lemmas 4.1(1) and 4.2(2). □

Proposition 4.3 *Let G be a finite abelian group. Then $\omega(\Gamma(G)) = 3$ if and only if G is isomorphic to one of the following groups:*

$$C_{p^4}, \quad C_{p^2q}, \quad C_{pqr},$$

where p, q, r are distinct primes.

Proof Assume that $\omega(\Gamma(G)) = 3$. We consider two cases:

(i) Suppose that G is a p-group, for some prime p. Since $\omega(\Gamma(G)) = 3$, we have $|G| \geq p^3$. If $\Phi(G) = 1$, then G is elementary abelian and so G has a subgroup isomorphic to $E = C_p \times C_p \times C_p$. It follows from Lemma 4.1(2) that $\omega(\Gamma(E)) = p^2 + p + 1 > 3$, a contradiction. Hence $\Phi(G) \neq 1$. Moreover, the number of maximal subgroups of G is $t = \frac{|\frac{G}{\Phi(G)}|-1}{p-1}$, by Lemma 2.6. Therefore, either $t = 1$, which follows that $G \cong C_{p^4}$, by Lemma 4.1(1) or G has at least $p + 1$ maximal subgroups, say $M_1, M_2, \ldots, M_{p+1}$. By Lemma 2.1, $\{M_1, M_2, \ldots, M_{p+1}, \Phi(G)\}$ is a clique and so $\omega(\Gamma(G)) \geq p + 2$, which is impossible.

(ii) Now suppose that $|G|$ has at least two distinct prime divisors, p and q. Then G has two maximal subgroups M_1 and M_2 such that $|G : M_1| = p$ and $|G : M_2| = q$. Since $\omega(\Gamma(G)) = 3$, we have $M_1 \cap M_2 \neq 1$ and so $\{M_1, M_2, M_1 \cap M_2\}$ is a clique in $\Gamma(G)$. Note that G has a maximal subgroup different from M_1 and M_2, say M_3 since $G \neq M_1 \cup M_2$. We conclude that $\{M_1, M_2, M_3\}$ is a clique in $\Gamma(G)$. Since $\omega(\Gamma(G)) = 3$, we have $\cap_{i=1}^{3} M_i = 1$, which follows that $G \hookrightarrow \frac{G}{M_1} \times \frac{G}{M_2} \times \frac{G}{M_3}$. Consequently $|G| = pqr$, where r is prime. By Lemma 4.1(3-4), we have $G \cong C_{pqr}$ or C_{p^2q}, as wanted.

The converse of the lemma follows from Lemma 4.1. □

Proposition 4.4 *Let G be a finite non-abelian group. Then $\omega(\Gamma(G)) = 3$ if and only if $G = N \rtimes H$ is Frobenius whose kernel N is the unique minimal normal subgroup of G and we have either*

(1) *$N \cong C_p$ and $H \cong C_{q^2}$ or C_{qr} or*

(2) *$N \cong C_p \times C_p$, $H \cong C_{q^2}$ and G does not have any subgroup of order pq, where p, q, r are distinct primes.*

Proof Suppose that $\omega(\Gamma(G)) = 3$. Since G is not cyclic, G has at least three maximal subgroups M_1, M_2 and M_3. By hypothesis, $\cap_{i=1}^{3} M_i = 1$. If $M_i \lhd G$, for each i, then $|G : M_i|$ is prime, for each i and so G is abelian, a contradiction. Without loss of generality, assume that $M := M_1$ is not normal in G. Then $N_G(M) = M$ and $|G : M| \geq 3$. Thus M has at least three conjugates M, M^x, M^y, for some $x, y \in G$. If $Core_G(M) \neq 1$, then $\omega(\Gamma(G)) \geq 4$, which is a contradiction. So $Core_G(M) = 1$, which implies that G is a primitive group. It follows from Theorem 1.1 that G is solvable and by Theorem 2.4, G has a unique minimal normal subgroup N such that $C_G(N) = N$ and $G = N \rtimes M$. Since N is a principal factor of G, we have N is elementary abelian of order p^n, for some prime p, by 5.4.3 of [18]. Since $\omega(\Gamma(G)) = 3$, we have $n = 1$ or 2, by Lemmas 4.1(2) and 2.5. Also note that if $M \cap M^g \neq 1$, for some $g \in G \backslash M$, then $\{M, M^g, M \cap M^g, N(M \cap M^g)\}$ is a clique in $\Gamma(G)$, a contradiction. Thus G is a Frobenius group.

Subcase 1. If $n = 1$, then $G = N \rtimes M$, where $N \cong C_p$. Since $Aut(C_p)$ is cyclic of order $p - 1$, we have M is cyclic of order divisor of $p - 1$. It follows from Lemma 2.5(i) that $\omega(\Gamma(M)) < \omega(\Gamma(G)) = 3$. If $\omega(M) = 2$, then by Theorem 1.4, we have $M \cong C_{q^3}$, for some prime q. Therefore, M has two subgroups H and K of orders q and q^2 respectively and so $\{H, K, M, NH\}$ is a clique, which is impossible. Hence $\omega(\Gamma(M)) = 1$, which follows that $M \cong C_{qr}$ or C_{q^2}, by Remark 1.3, as required.

Subcase 2. If $n = 2$, then $G = N \rtimes M$, where $N \cong C_p \times C_p$. Therefore, $|M|$ is not prime, by hypothesis and Theorem 1.4. We conclude that $|\mathcal{S}(M)| \geq 3$. Since $\omega(\Gamma(G)) = 3$ and $\omega(\Gamma(N)) = 1$, we have $|\mathcal{S}(M)| \leq 3$, by Lemma 2.5(ii). Therefore, $M \cong C_{q^2}$, as wanted. It remains to show that G does not have any (non-abelian) subgroup of order pq.

Suppose, for a contradiction, that G has a subgroup K of order pq. Since $C_G(x) = N$, for every $1 \neq x \in N$, by Exercise 8.5 (5) of [18], we have K is non-abelian and so there are subgroups $1 < N_1 < N$ and $1 < M_1 < M$ such that $K = N_1 \rtimes M_1$. It follows that $\{N_1, N, K, NM_1\}$ is a clique in $\Gamma(G)$ with four elements, which is impossible.

The converse follows from Lemmas 4.2(1) and 4.2(3). □

Proof of Theorem 1.5 The proof follows from Propositions 4.3 and 4.4. □

Proposition 4.5 *Let G be a finite abelian group. Then $\omega(\Gamma(G)) = 4$ if and only if G is isomorphic to one of the follwing groups:*

$$C_{p^5}, \quad C_2 \times C_2 \times C_q, \quad C_2 \times C_4,$$

where p and q are primes and q is odd.

Proof Assume that $\omega(\Gamma(G)) = 4$. Then, by Remark 1.3 and Lemma 4.1, $|G|$ has at least three prime divisors (not necessarily distinct), say p, q and r. Note that the number of maximal subgroups of G is either one or at least three. So we consider three cases in terms of the number of maximal subgroups of G in the following:

Case 1. If G has at least four maximal subgroups, M_i, for $1 \leq i \leq 4$, then $M_i \cap M_j \neq 1$ and since $\omega(\Gamma(G)) = 4$, we have $\cap_{i=1}^{4} M_i = 1$ and so $|G|$ has at most four prime divisors (not necessarily distinct). Therefore, $|G|$ divides $pqrs$, for some prime s.

First suppose that $|G| = pqrs$. If all prime divisors of $|G|$ are distinct, then G is cyclic of order $pqrs$ and so $\omega(\Gamma(G)) > 4$, a contradiction. If $|G| = p^2qr$ such that p, q, r are distinct, then $\{H, P, HQ, HR, PQ, PR\}$ is a clique in $\Gamma(G)$, where $|H| = p$, $P \in Syl_p(G)$, $Q \in Syl_q(G)$ and $R \in Syl_r(G)$, again a contradiction. By a similar argument, we see that $|G| \neq p^2q^2, p^3q, p^4$, where p, q are distinct. So such group G does not exist.

Now suppose that $|G| = pqr$. By the number of maximal subgroups of G, we have G is not cyclic. Also $G \ncong C_p \times C_p \times C_p$, by Lemma 4.1(2). Hence $G \cong C_{p^2} \times C_p$ or $C_p \times C_p \times C_q$. It follows from Lemma 4.1(4) that $p = 2$ and q is odd. But the number of maximal subgroups $C_4 \times C_2$ is 3, a contradiction. Thus $G \cong C_2 \times C_2 \times C_q$, as wanted.

Case 2. Suppose that G has exactly three maximal subgroups. If G is cyclic, then $|G|$ has three distinct prime divisors and so $G \cong C_{pqr}$. So we have a contradiction, by Lemma 4.1(3) since $\omega(\Gamma(G)) = 4$. Thus G is not cyclic and then G has a Sylow t-subgroup, which is not cyclic, for some prime t. Without loss of generality, assume $t = p$ and $P \in Syl_p(G)$. Therefore, P (or equivalently G) has at least $p+1$ maximal subgroups. By hypothesis in this case, we have $p = 2$ and G must be a 2-group. On the other hand, the number of maximal subgroups of G is $|\frac{G}{\Phi((G)}| - 1$, which implies that $\frac{G}{\Phi((G)} \cong C_2 \times C_2$. Since $|\mathcal{S}(C_2 \times C_2)| = 5$ and $\omega(\Gamma(G)) = 4$, we have $\omega(\Gamma(\Phi(G))) = 0$, by Lemma 2.5(ii). Thus $|\Phi(G)| = 2$ and so $|G| = 8$. Hence $G \cong C_4 \times C_2$, as desired.

Case 3. Suppose that G has exactly one maximal subgroup. Then G is cyclic of order the power of a prime p and so $G \cong C_{p^5}$, by Lemma 4.1(1). This completes the proof.

The converse is clear, by Lemma 4.1.

□

Proposition 4.6 *Let G be a finite non-abelian group. Then $\omega(\Gamma(G)) = 4$ if and only if G is one of the follwing groups:*

(i) $G \cong D_8,\ Q_8$. *or*
(ii) $G = N \rtimes H$ *is Frobenius whose kernel N is the (unique) minimal normal subgroup of G such that $N \cong C_p \times C_p$, $H \cong C_{qr}$ and G does not have subgroups of orders pr and pq*

Proof Assume that $\omega(\Gamma(G)) = 4$. Suppose first that every maximal subgroup is normal in G. Then G is nilpotent. Since G is non-abelian, there is a Sylow p-subgroup P of G such that P is non-abelian, for some prime p. Therefore, $\Phi(P) \neq 1$ and so $\omega(\Gamma(P)) \geq m(P)+1 \geq p+1+1$, where $m(P)$ is the number of maximal subgroups of P. Thus, by Lemma 2.5(i), we see that $4 = \omega(\Gamma(G)) \geq \omega(\Gamma(P)) \geq 4$, which implies that $G = P$ and then we have $p = 2$ and $m(P) = p + 1 = 3$. Now if M_1, M_2 and M_3 are maximal subgroups of G, then $\{\Phi(G), M_1, M_2, M_3\}$ is a clique with maximum size and so $|\Phi(G)| = 2$ and $M_i \cap M_j = \Phi(G)$, for every $i \neq j \leq 3$. Since $|G : M_i| = 2$, for each i, we have $|G| = 8$. Thus $G \cong D_8$ or Q_8, as wanted.

Now assume that G has a maximal subgroup M, which is not normal in G. Then $N_G(M) = M$ and $|G : M| \geq 3$. We claim that $Core_G(M) = 1$.

Suppose, for a contradiction, that $M_G := Core_G(M) \neq 1$. If M has at least four conjugates, say M, M^x, M^y, M^z, for some $x, y, z \in G$, then $\omega(\Gamma(G)) \geq 5$, a contradiction. So we have $|G : M| = 3$. It follows that $\frac{G}{M_G} \hookrightarrow S_3$. If $\frac{G}{M_G} \cong S_3$, then by Lemma 2.5(ii), $\omega(\Gamma(G)) \geq |\mathcal{S}(S_3)| - 1 = 5$, which is a contradiction. Since 3 divides $|G : M_G|$, we have $M_G = M$ and so $M \lhd G$, a contradiction. This implies that $M_G = 1$, as claimed. Thus G is primitive. By Theorems 1.1, G is solvable and by Theorem 2.4, G has a unique minimal normal subgroup N such that $C_G(N) = N$ and $G = N \rtimes M$. It follows that N is elementary abelian p-group, for some prime p. If $|N| \geq p^3$, then N contains a subgroup $E \cong C_p \times C_p \times C_p$ and so $\omega(\Gamma(E)) > 4$, by Lemma 4.1(2), which is impossible, by Lemma 2.5(i). Thus $|N| = p$ or p^2.

If $|N| = p$, then, by Normalizer-Centralizer Theorem we have $\frac{G}{C_G(N)} = \frac{G}{N} \hookrightarrow Aut(C_p) \cong C_{p-1}$ and so M is cyclic. It follows from Lemma 2.5(i) that $\omega(\Gamma(M)) < 4$. If $\omega(\Gamma(M)) = 2$ or 3, then by Theorems 1.4 and 4.3, we have $M \cong C_{q^3},\ C_{q^2r},\ C_{q^4}$ or C_{qrs}. Thus M has at least two proper subgroups of different orders, which are divisible by q, say M_1, M_2 and so $\{M_1, M_2, NM_1, NM_2, M\}$ is a clique in $\Gamma(G)$, a contradiction. Also if $\omega(\Gamma(M)) = 1$, then $M \cong C_{q^2}$ or C_{qr}, where q, r are distinct primes. Since M is cyclic and maximal subgroup of G, we have $M \cap M^x \lhd \langle M, M^x \rangle = G$, for any $x \in G \backslash M$. Now since G has a unique minimal subgroup N, we see that $M \cap M^x = 1$, for every $x \in G \backslash M$ and then G is Frobenius. Therefore, $G = N \rtimes M$, where $N \cong C_p$ and $M \cong C_{q^2}$ or C_{qr}, which follows $\omega(\Gamma(G)) = 3$ by Theorem 4.4, another contradiction. Finally if $\omega(\Gamma(M)) = 0$, then $|M| = q$ is prime and so $G \cong C_p \rtimes C_q$. It follows that $\omega(\Gamma(G)) = 1$, by Remark 1.3, our final contradiction.

Now suppose that $N \cong C_p \times C_p$. Then $\omega(\Gamma(M)) < \omega(\Gamma(G)) = 4$, by Lemma 2.5(i). Also by Remark 1.3 and Theorem 1.4, $\omega(\Gamma(M)) \neq 0$. If $\omega(\Gamma(M)) = 1$, then $M \cong C_{q^2}, C_{qr}, C_q \times C_q$ or $C_q \rtimes C_r$, where q and r are distinct primes. In two last cases, M (or equivalently $\frac{G}{N}$) has $q + 1$ non-trivial proper subgroups and so $\{H, N, K_1, K_2, \ldots, K_{q+1}\}$ is a clique in $\Gamma(G)$, where $1 < H < N$ and K_i is a proper subgroup of G containing N, for each i. It follows that $\omega(\Gamma(G)) \geq q + 3$, a contradiction. If $M \cong C_{q^2}$, then $M \cap M^x \lhd \langle M, M^x \rangle = G$, for any $1 \neq x \in N$, we have $M \cap M^x = 1$, for every $1 \neq x \in N$ and then G is Frobenius. By Theorem 4.4, we have $\omega(\Gamma(G)) = 3$. Hence $M \cong C_{qr}$ and similarly G is Frobenius, as wanted.

If $\omega(\Gamma(M)) = 2$ and $\{H_1, H_2\}$ is a clique in $\Gamma(M)$ such that $H_1 < H_2$, then $\{H_1, H_2, M, NH_1, NH_2\}$ is a clique in $\Gamma(G)$), which is impossible.

If $\omega(\Gamma(M)) = 3$ and $\{M_1, M_2, M_3\}$ is a clique in $\Gamma(M)$), then

$\{M_1, M_2, M_3, M, NM_1\}$ is a clique in $\Gamma(G)$), a contradiction. It remains to prove that G does not have subgroups of orders pq and pr.

Suppose, for a contradiction, that G has a subgroup K of order pq. Then there are subgroups $N_1 < N$ and $Q \in Syl_q(G)$ such that $K = N_1 \rtimes Q$. It follows that $\{N_1, N, K, NQ, NR\}$ is a clique in $\Gamma(G)$, where R is a Sylow r-subgroup of G. This is a contradiction with our assumption. Similarly G does not have any subgroup of order pr.

The converse of the lemma follows from Lemmas 4.1(5) and 4.2(4). □

Proof of Theorem 1.6 The proof follows from Propositions 4.5 and 4.6. □

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