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ORIGINAL PAPER

Strong Stability Preserving Runge–Kutta and Linear Multistep Methods

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Abstract

This paper reviews strong stability preserving discrete variable methods for differential systems. The strong stability preserving Runge–Kutta methods have been usually investigated in the literature on the subject, using the so-called Shu–Osher representation of these methods, as a convex combination of first-order steps by forward Euler method. In this paper, we revisit the analysis of strong stability preserving Runge–Kutta methods by reformulating these methods as a subclass of general linear methods for ordinary differential equations, and then using a characterization of monotone general linear methods, which was derived by Spijker in his seminal paper (SIAM J Numer Anal 45:1226–1245, 2007). Using this new approach, explicit and implicit strong stability preserving Runge–Kutta methods up to the order four are derived. These methods are equivalent to explicit and implicit RK methods obtained using Shu–Osher or generalized Shu–Osher representation. We also investigate strong stability preserving linear multistep methods using again monotonicity theory of Spijker.

Keywords Runge–Kutta methods · Linear multistep methods · General linear methods · Monotonicity · Strong stability preserving · SSP coefficient · Shu–Osher representation

Mathematics Subject Classification 65Lxx · 65L05 · 65L06

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1 Introduction

It is a purpose of this paper to systematically investigate strong stability preserving (SSP) Runge-Kutta (RK) methods and linear multistep methods (LMMs), for numerical solution of initial-value problems for ordinary differential equations (ODEs)

$$\begin{cases} y'(t) = f\big(t, y(t)\big), \ t \in [t_0, t_{\text{end}}], \\ y(t_0) = y_0 \in \mathbb{R}^m, \end{cases} \tag{1.1}$$

where the function $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is assumed to be sufficiently smooth. To define SSP property of numerical schemes, we assume that the discretization of (1.1) by the forward Euler method

$$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1}), \tag{1.2}$$

$n = 1, 2, \ldots, N$, $Nh = t_{\text{end}} - t_0$, $t_n = t_0 + nh$, satisfies the condition

$$\|y_n\| \le \|y_{n-1}\|, \tag{1.3}$$

$n = 1, 2, \ldots, N$, in some norm or semi-norm $\|\cdot\|$, for all stepsizes h such that

$$h \le h_{FE}. \tag{1.4}$$

We then search for methods, which preserve the monotonicity property (1.3) of the forward Euler method (1.2) under the restriction on the stepsize h, of the form

$$h \le \mathcal{C} \cdot h_{FE}. \tag{1.5}$$

We discuss first explicit RK methods. Such methods, in Butcher representation, take the form

$$\begin{cases} Y_i^{[n]} = y_{n-1} + h \displaystyle\sum_{j=1}^{i-1} a_{ij} f\big(t_{n-1} + c_j h, Y_j^{[n]}\big), \quad i = 1, 2, \ldots, s, \\ y_n = y_{n-1} + h \displaystyle\sum_{j=1}^{s} b_j f\big(t_{n-1} + c_j h, Y_j^{[n]}\big), \end{cases} \tag{1.6}$$

$n = 1, 2, \ldots, N$, and can be represented by the Butcher table

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} = \begin{array}{c|ccccc} c_1 = 0 & & & & & \\ c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ c_s & a_{s,1} & a_{s,2} & \cdots & a_{s,s-1} & \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array} .$$

RK formulas (1.6) which satisfy (1.3) under the restriction (1.5) are called SSP RK methods, and the constant $\mathcal{C}$ is called SSP coefficient. To compare schemes with different number of stages s, we also define the effective SSP coefficient $\mathcal{C}_{\text{eff}}$ by

$$\mathcal{C}_{\text{eff}} = \mathcal{C}/s.$$

As discussed in [12, 14], the search for explicit SSP RK methods (1.6) is facilitated by SSP revealing representation of these methods as convex combinations of forward Euler steps. This so-called Shu–Osher representation proposed in [35], has the form

$$\begin{cases} Y_1^{[n]} = y_{n-1}, \\ Y_i^{[n]} = \sum_{j=1}^{i-1} \Big(\alpha_{ij} Y_j^{[n]} + h\beta_{ij} f\big(t_{n-1} + c_j h, Y_j^{[n]}\big)\Big), \qquad i = 2, 3, \ldots, s, \\ y_n = \sum_{j=1}^{s} \Big(\alpha_{s+1,j} Y_j^{[n]} + h\beta_{s+1,j} f\big(t_{n-1} + c_j h, Y_j^{[n]}\big)\Big) \end{cases} \tag{1.7}$$

$n = 1, 2, \ldots, N$, where α_{ij} are scalars such that

$$\sum_{j=1}^{i-1} \alpha_{ij} = 1, \quad i = 2, 3, \ldots, s+1, \tag{1.8}$$

and β_{ij} are given by

$$\begin{cases} \beta_{ij} = a_{ij} - \sum_{k=j+1}^{i-1} \alpha_{ik} a_{kj}, \qquad i = 2, 3, \ldots, s, \ j = 1, 2, \ldots, i-1, \\ \beta_{s+1,j} = b_j - \sum_{k=j+1}^{s} \alpha_{s+1,k} a_{kj}, \quad j = 1, 2, \ldots, s. \end{cases} \tag{1.9}$$

The Shu–Osher representation of RK methods (1.6) is not unique, and the SSP coefficient of (1.7), which depends on the choice of α_{ij}, subject to (1.8), and the resulting β_{ij}, can be characterized by the following result, essentially due to Shu and Osher [35] (see also [12, 14]).

Theorem 1.1 *Assume that the forward Euler method* (1.2) *applied to* (1.1) *is strongly stable, i.e., the inequality* (1.3) *holds under the time step restriction* (1.4)*. Assume also that* $\alpha_{ij} \geq 0$ *and* $\beta_{ij} \geq 0$*. Then, the solution* $\{y_n\}$ *obtained by the RK method* (1.6) *or* (1.7) *satisfies the strong stability bound* (1.3) *under the time step restriction* (1.5) *with SSP coefficient* $\mathcal{C} = \mathcal{C}(\alpha, \beta)$ *given by*

$$\mathcal{C}(\alpha, \beta) = \min\Big\{\frac{\alpha_{ij}}{\beta_{ij}} : i = 2, 3, \ldots, s, \ j = 1, 2, \ldots, i-1\Big\}.$$

It is also possible to characterize SSP coefficient of (1.7) if some of the β_{ij} are negative, and we refer to the monograph [12] for such results.

A lot of work was devoted to finding explicit SSP RK methods with large SSP coefficients both by mathematical analysis, for low order methods, and numerical searches, for higher order schemes. Most of these numerical searches were based on formulation the appropriate optimization problems using Shu–Osher representation of RK methods, and the characterization of SSP coefficient formulated in Theorem 1.1.

In this paper, we will search for explicit and implicit SSP RK formulas using a different approach than that used in [12, 14], and which is based on monotonicity theory of general linear methods (GLMs) developed by Spijker [36]. In this approach we first reformulate RK methods in Butcher representation (1.6) as GLMs examined in [36], and then reformulate the appropriate optimization problems in terms of the abscissa vector **c**, the coefficient matrix **A**, and the weight vector **b**.

The organization of this paper is as follows. In Sect. 2 we review generalized Shu–Osher representation introduced by Ferracina and Spijker [10], which can be used to investigate implicit SSP RK methods. The monotonicity theory for GLMs developed by Spijker [36] is reviewed in Sect. 3, and the resulting monotonicity theory for RK methods, as well as the formulation of appropriate minimization problems for computing explicit and implicit SSP RK methods, is described in Sect. 4. In Sect. 5, we analyze explicit and in Sect. 6, implicit SSP RK methods obtained by solving the minimization problems described in Sect. 4. We present examples of explicit SSP RK methods of order $p = 1$, of order $p = 2$ with $s = 2, 3, \ldots, 10$ stages, of order $p = 3$ with $s = 3, 4, \ldots, 10$ stages, and of order $p = 4$ with $s = 4, 5, \ldots, 10$ stages, and implicit SSP RK methods up to the order $p = 4$. In Sect. 7, we analyze SSP LMMs, and examples of explicit and implicit SSP LMMs of order $p = 2$, $p = 3$, and $p = 4$, are presented in Sects. 8 and 9. Finally, in Sect. 10, some concluding remarks are given.

2 A Generalization of the Shu–Osher Representation of RK Methods

As discussed in Sect. 1, the SSP properties of explicit RK methods can be investigated using the Shu–Osher representation (1.9), and SSP coefficient of these methods is characterized in Theorem 1.1. It was proved in [14] that if implicit RK method has order $p > 1$, then the coefficients α_{ij} of its Shu–Osher representation (1.9) cannot be all nonnegative. Hence, this representation cannot be used to find implicit SSP methods of order greater than one. SSP properties of implicit RK methods are usually investigated using a generalization of Shu–Osher representation introduced by Ferracina and Spijker [10] (see also [12, 18]). For RK methods with s stages, this representation takes the form

$$\begin{cases} Y_i^{[n]} = \left(1 - \sum_{j=1}^{s} \lambda_{ij}\right) y_{n-1} \\ \qquad + \sum_{j=1}^{s} \left(\lambda_{ij} Y_j^{[n]} + h\mu_{ij} f\left(t_{n-1} + c_j h, Y_j^{[n]}\right)\right), \quad i = 1, 2, \ldots, s, \\ y_n = \left(1 - \sum_{j=1}^{s} \lambda_{s+1,j}\right) y_{n-1} \\ \qquad + \sum_{j=1}^{s} \left(\lambda_{s+1,j} Y_j^{[n]} + h\mu_{s+1,j} f\left(t_{n-1} + c_j h, Y_j^{[n]}\right)\right), \end{cases} \tag{2.1}$$

$n = 1, 2, \ldots, N$. Here, λ_{ij} and μ_{ij} are real coefficients. Following [10], we introduce the matrices

$$\mathbf{L} = \left[\frac{\mathbf{L}_0}{\mathbf{L}_1}\right], \quad \mathbf{L}_0 = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1s} \\ \vdots & \ddots & \vdots \\ \lambda_{s1} & \cdots & \lambda_{ss} \end{bmatrix}, \quad \mathbf{L}_1 = \begin{bmatrix} \lambda_{s+1,1} & \cdots & \lambda_{s+1,s} \end{bmatrix},$$

and

$$\mathbf{M} = \left[\frac{\mathbf{M}_0}{\mathbf{M}_1}\right], \quad \mathbf{M}_0 = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1s} \\ \vdots & \ddots & \vdots \\ \mu_{s1} & \cdots & \mu_{ss} \end{bmatrix}, \quad \mathbf{M}_1 = \begin{bmatrix} \mu_{s+1,1} & \cdots & \mu_{s+1,s} \end{bmatrix}.$$

The generalized Shu–Osher representation (2.1) of implicit RK method is not unique. Assuming that the coefficient matrices $\mathbf{L}$ and $\mathbf{M}$ are given and that the matrix $\mathbf{I} - \mathbf{L}_0$ is nonsingular, the Butcher representation of RK method

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} = \begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array} \tag{2.2}$$

can be computed from the formulas

$$\mathbf{A} = (\mathbf{I} - \mathbf{L}_0)^{-1}\mathbf{M}_0, \quad \mathbf{b}^T = \mathbf{M}_1 + \mathbf{L}_1\,\mathbf{A}, \quad \mathbf{c} = \mathbf{A}\,\mathbf{e}, \tag{2.3}$$

where $\mathbf{I}$ is the identity matrix of dimension s, and $\mathbf{e} = [1, \ldots, 1] \in \mathbb{R}^s$. Similarly, if Butcher representation $(\mathbf{c}, \mathbf{A}, \mathbf{b})$ of RK method is given, to compute the generalized Shu–Osher representation (2.1) we have to choose the coefficient matrices $\mathbf{L}_0$ and $\mathbf{L}_1$, and then compute the coefficient matrices $\mathbf{M}_0$ and $\mathbf{M}_1$ from the formulas

$$\mathbf{M}_0 = \mathbf{A} - \mathbf{L}_0\,\mathbf{A}, \quad \mathbf{M}_1 = \mathbf{b}^T - \mathbf{L}_1\,\mathbf{A}. \tag{2.4}$$

We have the following generalization of the Shu–Osher Theorem 1.1, which follows from the results of Ferracina and Spijker [11].

Theorem 2.1 *Assume that the forward Euler method* (1.2) *applied to* (1.1) *is strongly stable, i.e., the inequality* (1.3) *holds under the time step restriction* (1.4)*. Assume also that* $\mathbf{I} - \mathbf{L}_0$ *is invertible and*

$$\lambda_{ij} \geq 0 \quad and \quad \sum_{k=1}^{s} \lambda_{ik} \leq 1 \quad for \quad 1 \leq i \leq s+1, \quad 1 \leq j \leq s.$$

Then, the solution $\{y_n\}$ *obtained by the implicit RK method* (2.1) *or* (2.2) *satisfies the strong stability bound* (1.3) *under the time step restriction* (1.5) *with SSP coefficient* $\mathcal{C} = \mathcal{C}(\mathbf{A}, \mathbf{b}, \mathbf{L})$ *given by*

$$\mathcal{C}(\mathbf{A}, \mathbf{b}, \mathbf{L}) = \min\left\{\frac{\lambda_{ij}}{\mu_{ij}} : i = 1, 2, \ldots, s+1, \ j = 1, 2, \ldots, s\right\}.$$

3 Monotonicity Theory for GLMs

Consider a class of GLMs of the form

$$\begin{cases} Y_i^{[n]} = h \displaystyle\sum_{j=1}^{m} t_{ij} f\big(t_{n-1} + c_j h, Y_j^{[n]}\big) + \sum_{j=1}^{\ell} s_{ij} y_j^{[n-1]}, & i = 1, 2, \ldots, m, \\ y_i^{[n]} = Y_{m-\ell+i}^{[n]}, & i = 1, 2, \ldots, \ell, \end{cases} \tag{3.1}$$

$n = 1, 2, \ldots, N$, $1 \leq \ell \leq m$. In this formulation $Y_i^{[n]}$, $i = 1, 2, \ldots, m$, are internal approximations or stages, which are used to compute external approximations $y_i^{[n]}$, $i = 1, 2, \ldots, \ell$, which propagate from step to step. The formulation (3.1), which was considered by Spijker [36], is specified by the abscissa vector $\mathbf{c} = [c_1, \ldots, c_m]^T \in \mathbb{R}^m$, and two coefficient matrices $\mathbf{T} = [t_{ij}] \in \mathbb{R}^{m \times m}$ and $\mathbf{S} = [s_{ij}] \in \mathbb{R}^{m \times \ell}$, where it is assumed, without loss of generality, that

$$\sum_{j=1}^{\ell} s_{ij} = 1, \quad i = 1, 2, \ldots, m. \tag{3.2}$$

This representation is not the most common one, and different representations of GLMs are discussed by Burrage [2], Butcher [3, 4, 6], Hairer et al. [15], Hairer and Wanner [16], Jackiewicz [24], and Wright [37].

Following the definition in [36], the GLM (3.1) is said to be monotonic if

$$\left\|Y_i^{[n]}\right\| \leq \max\left\{\left\|y_j^{[n-1]}\right\| : j = 1, 2, \ldots \ell\right\}, \quad i = 1, 2, \ldots, m, \tag{3.3}$$

where $Y_i^{[n]}$ and $y_j^{[n-1]}$ satisfy (3.1). Denote by $\mathbf{I}$ the identity matrix of dimension s, and, as in [36], let $\big[\mathbf{S} \mid \gamma\mathbf{T}\big]$, $\gamma \in \mathbb{R}$, be the $s \times (\ell + s)$ matrix whose first ℓ columns are equal to $\mathbf{S}$, and the last s columns are equal to $\gamma\mathbf{T}$. Consider the condition

$$\det(\mathbf{I} + \gamma\mathbf{T}) \neq 0 \quad \text{and} \quad (\mathbf{I} + \gamma\mathbf{T})^{-1}\left[\mathbf{S}\middle|\gamma\,\mathbf{T}\right] \geq 0, \tag{3.4}$$

where the inequality in (3.4) should be interpreted componentwise. Then the following theorem, which characterizes the SSP coefficient of GLMs (3.1) can be deduced from the results of [36].

Theorem 3.1 *Assume that the forward Euler method* (1.2) *applied to* (1.1) *is strongly stable, i.e., the inequality* (1.3) *holds under the time step restriction* (1.4)*. Then the solution* $\{Y_i^{[n]}\}$ *and* $\{y_i^{[n-1]}\}$ *obtained by the GLM* (3.1) *satisfies* (3.3) *under the time step restriction* (1.5) *with SSP coefficient* $\mathcal{C} = \mathcal{C}(\mathbf{T}, \mathbf{S})$ *given by*

$$\mathcal{C}(\mathbf{T}, \mathbf{S}) = \sup\Big\{\gamma \in \mathbf{R}:\ \gamma \text{ satisfies } (3.4)\Big\}. \tag{3.5}$$

It follows from this theorem that the coefficients of GLMs (3.1), and the corresponding SSP coefficient, can be computed by solving the minimization problem

$$F(\mathbf{T}, \mathbf{S}) = -\gamma \longrightarrow \min, \tag{3.6}$$

with a very simple objective function $F(\mathbf{T}, \mathbf{S}) := -\gamma$, subject to the nonlinear inequality constrains (3.4), and to equality constrains

$$\Phi_{p,q}(\mathbf{T}, \mathbf{S}) = 0, \tag{3.7}$$

Here, $\Phi_{p,q}(\mathbf{T}, \mathbf{S}) = 0$ represents conditions for order p and stage order q. Such order and stage order conditions for GLMs, in different representations, were investigated in [3–5, 8, 15]. The approach based on solving the minimization problem (3.6), subject to constrains (3.4) and (3.7), was used in [7, 21–23] to investigate some classes of SSP GLMs up to the order $p = 4$. SSP GLMs were also investigated in [9, 19, 29]. SSP two-step RK methods were investigated in [27], and multistep RK methods in [1].

4 Monotonicity Theory for RK Methods

In this section, we apply the monotonicity theory of GLMs (3.1) presented in [36], and summarized in Theorem 3.1, to the special case of explicit RK methods in Butcher representation. It can be verified that the RK method (1.6) given by the abscissa vector $\mathbf{c} = [c_1, \ldots, c_s]^T \in \mathbb{R}^s$, coefficient matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{s\times s}$, and weight vector $\mathbf{b} = [b_1, \ldots, b_s]^T \in \mathbb{R}^s$, can be written as GLM (3.1) with $m = s + 1$, $\ell = 1$, and the coefficient matrices $\mathbf{T}$ and $\mathbf{S}$ defined by

$$\mathbf{T} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{b}^T & 0 \end{array}\right] \in \mathbb{R}^{(s+1)\times(s+1)}, \quad \mathbf{S} = \left[\begin{array}{c} \mathbf{e} \\ \hline 1 \end{array}\right] \in \mathbb{R}^{s+1}, \tag{4.1}$$

where $\mathbf{e} = [1, \ldots, 1]^T \in \mathbb{R}^s$. Observe that for this representation the condition (3.2) is automatically satisfied. Putting $y_1^{[n-1]} = y_{n-1}$ and $y_1^{n]} = y_n$, the RK method (1.6) takes now the form

$$\begin{cases} Y_i^{[n]} = y_{n-1} + h \sum_{j=1}^{m} t_{ij} f\big(t_{n-1} + c_j h, Y_j^{[n]}\big) \quad i = 1, 2, \ldots, s+1, \\ y_n = Y_{s+1}^{[n]}, \end{cases} \tag{4.2}$$

$n = 1, 2, \ldots, N$. For the representation (4.2), the monotonicity condition (3.3) takes the form

$$\big\| Y_i^{n]} \big\| \le \big\| y_{n-1} \big\|, \quad i = 1, 2, \ldots, s+1. \tag{4.3}$$

In particular, for $i = s+1$, we have $Y_{s+1}^{[n]} = y_n$, and the condition (4.3) implies the condition (1.3).

We will reformulate next the nonlinear inequality constrains (3.4), the characterization of SSP coefficient given by the condition (3.5) in Theorem 3.1, and the minimization problem (3.6) in terms of the coefficients $\mathbf{c}$, $\mathbf{A}$, and $\mathbf{b}$, of RK method (1.6). We have

$$\mathbf{I} + \gamma \mathbf{T} = \left[\begin{array}{c|c} \mathbf{I} + \gamma \mathbf{A} & \mathbf{0} \\ \hline \gamma \mathbf{b}^T & 1 \end{array}\right]$$

and it follows that $\det(\mathbf{I} + \gamma \mathbf{T}) \ne 0$ if $\mathbf{A}$ is strictly lower triangular, which is the case for explicit RK methods, or if $\mathbf{A}$ is lower triangular with nonzero entries on the diagonal, which is the case for diagonally implicit RK methods. We have also

$$(\mathbf{I} + \gamma \mathbf{T})^{-1} = \left[\begin{array}{c|c} \mathbf{I} + \gamma \mathbf{A} & \mathbf{0} \\ \hline \gamma \mathbf{b}^T & 1 \end{array}\right]^{-1} = \left[\begin{array}{c|c} (\mathbf{I} + \gamma \mathbf{A})^{-1} & \mathbf{0} \\ \hline -\gamma \mathbf{b}^T (\mathbf{I} + \gamma \mathbf{A})^{-1} & 1 \end{array}\right],$$

and it follows that

$$\begin{aligned} (\mathbf{I} + \gamma \mathbf{T})^{-1} \big[\mathbf{S} \mid \gamma \mathbf{T}\big] &= \left[\begin{array}{c|c} (\mathbf{I} + \gamma \mathbf{T})^{-1} & \mathbf{0} \\ \hline -\gamma \mathbf{b}^T (\mathbf{I} + \gamma \mathbf{A})^{-1} & 1 \end{array}\right] \left[\begin{array}{c|c|c} \mathbf{e} & \gamma \mathbf{A} & \mathbf{0} \\ \hline 1 & \gamma \mathbf{b}^T & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c|c} \gamma (\mathbf{I} + \gamma \mathbf{A})^{-1} \mathbf{e} & \gamma (\mathbf{I} + \gamma \mathbf{A})^{-1} \mathbf{A} & \mathbf{0} \\ \hline 1 - \gamma \mathbf{b}^T (\mathbf{I} + \gamma \mathbf{A})^{-1} \mathbf{e} & \gamma (\mathbf{b}^T - \gamma \mathbf{b}^T (\mathbf{I} + \gamma \mathbf{A})^{-1} \mathbf{A}) & 0 \end{array}\right]. \end{aligned}$$

Since

$$\gamma (\mathbf{b}^T - \gamma \mathbf{b}^T (\mathbf{I} + \gamma \mathbf{A})^{-1} \mathbf{A}) = \gamma \mathbf{b}^T (\mathbf{I} - \gamma (\mathbf{I} + \gamma \mathbf{A})^{-1} \mathbf{A}) = \gamma \mathbf{b}^T (\mathbf{I} + \gamma \mathbf{A})^{-1},$$

we obtain

$$(\mathbf{I} + \gamma \mathbf{T})^{-1} \big[\mathbf{S} \mid \gamma \mathbf{T}\big] = \left[\begin{array}{c|c|c} \gamma (\mathbf{I} + \gamma \mathbf{A})^{-1} \mathbf{e} & \gamma (\mathbf{I} + \gamma \mathbf{A})^{-1} \mathbf{A} & \mathbf{0} \\ \hline 1 - \gamma \mathbf{b}^T (\mathbf{I} + \gamma \mathbf{A})^{-1} \mathbf{e} & \gamma \mathbf{b}^T (\mathbf{I} + \gamma \mathbf{A})^{-1} & 0 \end{array}\right].$$

It follows from $\det(\mathbf{I} + \gamma\mathbf{T}) \neq 0$ and the above relation that the condition (3.4) is equivalent to

$$\begin{array}{ll} \gamma(\mathbf{I}+\gamma\mathbf{A})^{-1}\mathbf{e} \geq 0, & \gamma(\mathbf{I}+\gamma\mathbf{A})^{-1}\mathbf{A} \geq 0, \\ 1-\gamma\mathbf{b}^T(\mathbf{I}+\gamma\mathbf{A})^{-1}\mathbf{e} \geq 0, & \gamma\mathbf{b}^T(\mathbf{I}+\gamma\mathbf{A})^{-1} \geq 0. \end{array} \tag{4.4}$$

We summarize the above discussion in the following theorem.

Theorem 4.1 *Assume that the forward Euler method* (1.2) *applied to* (1.1) *is strongly stable, i.e., the inequality* (1.3) *holds under the time step restriction* (1.4)*. Then the solution* $\{Y_i^{[n]}\}$ *and* $\{y_{n-1}\}$ *obtained by the RK method* (4.2) *satisfies* (4.3) *(hence also* (1.3)*) under the time step restriction* (1.5) *with SSP coefficient* $\mathcal{C} = \mathcal{C}(\mathbf{c}, \mathbf{A}, \mathbf{b})$ *given by*

$$\mathcal{C}(\mathbf{c}, \mathbf{A}, \mathbf{b}) = \sup\Big\{\gamma \in \mathbf{R} : \ \gamma \text{ satisfies } (4.4)\Big\}. \tag{4.5}$$

It follows from this theorem that, similarly as for GLMs (3.1), the coefficients of RK methods (1.6) or (4.2), and the corresponding SSP coefficient, can be computed by solving the minimization problem

$$F(\mathbf{c}, \mathbf{A}, \mathbf{b}) = -\gamma \longrightarrow \min, \tag{4.6}$$

subject to the nonlinear inequality constrains (4.4), and the equality constrains

$$\Phi_p(\mathbf{c}, \mathbf{A}, \mathbf{b}) = 0. \tag{4.7}$$

Here, $\Phi_p(\mathbf{c}, \mathbf{A}, \mathbf{b}) = 0$ stands for order conditions for RK methods up to the order p. Such conditions are discussed in the monographs [3, 4, 15].

5 Explicit SSP RK Methods

In this section, we analyze explicit SSP RK methods. Examples of such methods, up to the order $p = 4$ with $s \leq 10$ stages, are obtained by solving the minimization problem (4.6), subject to the inequality constrains (4.4), and the equality constrains (4.7).

5.1 Explicit SSP RK Methods of Order $p = 1$

The explicit RK method of order $p = 1$ with $s = 1$ stages corresponds to the forward Euler method (1.2), for which $\mathbf{c} = 0$, $\mathbf{A} = 0$, $\mathbf{b} = 1$,

$$\mathbf{T} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right], \quad \mathbf{S} = \left[\begin{array}{c} 1 \\ \hline 1 \end{array}\right].$$

For this method the condition $\det(\mathbf{I} + \gamma\mathbf{T}) \neq 0$ is automatically satisfied, and the inequalities (4.4) reduce to

$$\gamma \geq 0, \quad 1 - \gamma \geq 0.$$

This leads, as expected, to $\mathcal{C} = \mathcal{C}_{\text{eff}} = 1$.

5.2 Explicit SSP RK Methods of Order $p = 2$

In this section, we will search for explicit SSP RK methods of order $p = 2$ with $s \geq 2$ stages. Solving the minimization problem (4.6) with inequality constrains (4.4) and equality constrains (4.7) corresponding to $p = 2$ leads to RK methods with

$$\mathbf{A} = \begin{bmatrix} 0 & & & & & \\ \frac{1}{s-1} & 0 & & & & \\ \frac{1}{s-1} & \frac{1}{s-1} & 0 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{s-1} & \frac{1}{s-1} & \frac{1}{s-1} & \cdots & 0 & \\ \frac{1}{s-1} & \frac{1}{s-1} & \frac{1}{s-1} & \cdots & \frac{1}{s-1} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s} \\ \frac{1}{s} \\ \vdots \\ \frac{1}{s} \\ \frac{1}{s} \end{bmatrix}, \quad \mathbf{c} = \mathbf{A}\,\mathbf{e}.$$

For these methods $\mathcal{C} = s - 1$ and $\mathcal{C}_{\text{eff}} = (s-1)/s$. We have listed in Table 1 SSP coefficients $\mathcal{C}$ and $\mathcal{C}_{\text{eff}}$, intervals of absolute stability, and areas of regions of absolute stability for RK methods of order $p = 2$ with $s = 2, 3, \ldots, 10$, stages. We have also plotted on Fig. 1 stability regions of these methods and on Fig. 2 scaled stability regions obtained by multiplying the points on the boundary of regions of absolute stability by p/s, where p is the order and s is the number of stages. These regions increase in size as s ranges from 2 to 10. On these figures stability regions of SSP RK method with $p = s = 2$ are plotted by a thick line.

Table 1 SSP coefficients $\mathcal{C}$ and $\mathcal{C}_{\text{eff}}$, intervals of absolute stability, and areas of the regions of absolute stability for RK methods of order $p = 2$ with $s = 2, 3, \ldots, 10$, stages

s	$\mathcal{C}$	$\mathcal{C}_{\text{eff}}$	Interval	Area
2	1	1/2	$(-2, 0)$	5.83
3	2	2/3	$(-4.5, 0)$	15.87
4	3	3/4	$(-6, 0)$	31.94
5	4	4/5	$(-8.3, 0)$	54.00
6	5	5/6	$(-10, 0)$	82.26
7	6	6/7	$(-12.3, 0)$	116.58
8	7	7/8	$(-14, 0)$	157.17
9	8	8/9	$(-16.2, 0)$	203.85
10	9	9/10	$(-18, 0)$	256.82

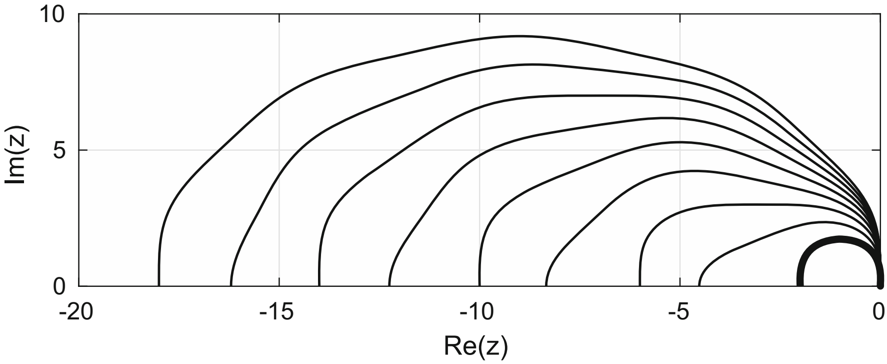

Fig. 1 Stability region of SSP RK method with $p = s = 2$ (thick line) and stability regions of SSP RK methods of order $p = 2$ with $s = 3, 4, \ldots, 10$ stages (thin lines). These regions increase in size as s ranges from 2 to 10

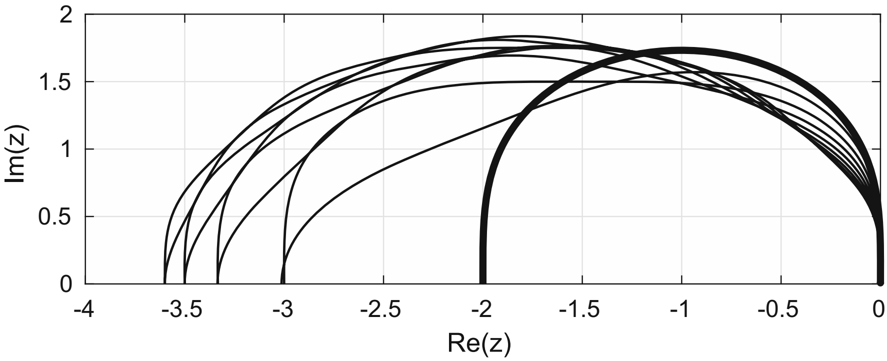

Fig. 2 Stability region of SSP RK method with $p = s = 2$ (thick line) and scaled stability regions of SSP RK methods of order $p = 2$ with $s = 3, 4, \ldots, 10$ stages (thin lines). These regions increase in size as s ranges from 2 to 10

5.3 Explicit SSP RK Methods of Order $p = 3$

We will search for explicit SSP RK methods of order $p = 3$ with $s = 3, 4, \ldots, 10$, stages. Solving the minimization problem (4.6) with constrains (4.4) and (4.7) corresponding to $p = s = 3$, we obtain the method whose Butcher representation is

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} = \begin{array}{c|ccc} 0 & & & \\ 1 & 1 & & \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array},$$

Table 2 SSP coefficients $\mathcal{C}$ and $\mathcal{C}_{\text{eff}}$, intervals of absolute stability, and areas of the regions of absolute stability for RK methods of order $p = 3$ with $s = 3, 4, \ldots, 10$, stages

s	$\mathcal{C}$	$\mathcal{C}_{\text{eff}}$	Interval	Area
3	1	1/3	$(-2.5, 0)$	9.03
4	2	1/2	$(-5.1, 0)$	19.32
5	2.65	0.53	$(-6.2, 0)$	33.03
6	3.52	0.59	$(-7.2, 0)$	51.21
7	4.29	0.61	$(-8.9, 0)$	73.42
8	5.11	0.64	$(-11, 0)$	100.77
9	6.00	0.67	$(-13.3, 0)$	131.18
10	6.79	0.68	$(-14.6, 0)$	168.25

and for which $\mathcal{C} = 1$ and $\mathcal{C}_{\text{eff}} = 1/3$. It can be verified that this method is equivalent to the Shu–Osher method SSPRK(3,3)

$$\begin{cases} Y_1^{[n]} = y_{n-1}, \\ Y_2^{[n]} = y_{n-1} + hf\big(t_{n-1}, Y_1^{[n]}\big), \\ Y_3^{[n]} = \dfrac{3}{4}y_{n-1} + \dfrac{1}{4}Y_2^{[n]} + \dfrac{1}{4}hf\big(t_{n-1} + h, Y_2^{[n]}\big), \\ y_n = \dfrac{1}{3}y_{n-1} + \dfrac{2}{3}Y_3^{[n]} + \dfrac{2}{3}hf\big(t_{n-1} + h/2, Y_3^{[n]}\big), \end{cases}$$

described in Theorem 2.3 in [12].

Solving the minimization problem (4.6) with constrains (4.4) and (4.7) corresponding to $p = 3$ and $s = 4$, we obtain the method whose Butcher representation is

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} = \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ 1 & \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \end{array}.$$

For this method $\mathcal{C} = 2$ and $\mathcal{C}_{\text{eff}} = 1/2$.

The coefficients of methods obtained by solving the minimization problem (4.6) with constrains (4.4) and (4.7) corresponding to $p = 3$ and $s = 5, 6, \ldots, 10$, are not listed here, but can be obtained from the authors. We have listed in Table 2 SSP coefficients $\mathcal{C}$ and $\mathcal{C}_{\text{eff}}$, interval of absolute stability, and areas of regions of absolute stability for RK methods of order $p = 3$ with $s = 3, 4, \ldots, 10$, stages. We have also plotted on Fig. 3 stability regions of these methods and on Fig. 4 scaled stability regions obtained by multiplying the points on the boundary of regions of absolute stability by p/s, where p is the order and s is the number of stages. As for methods

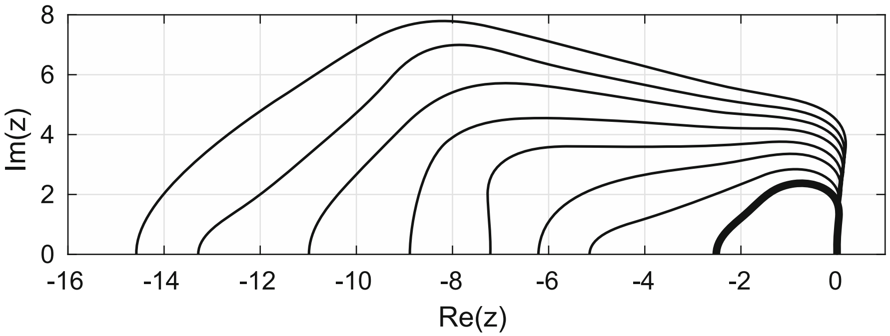

Fig. 3 Stability region of SSP RK method with $p = s = 3$ (thick line) and stability regions of SSP RK methods of order $p = 3$ with $s = 4, 5, \ldots, 10$ stages (thin lines). These regions increase in size as s ranges from 3 to 10

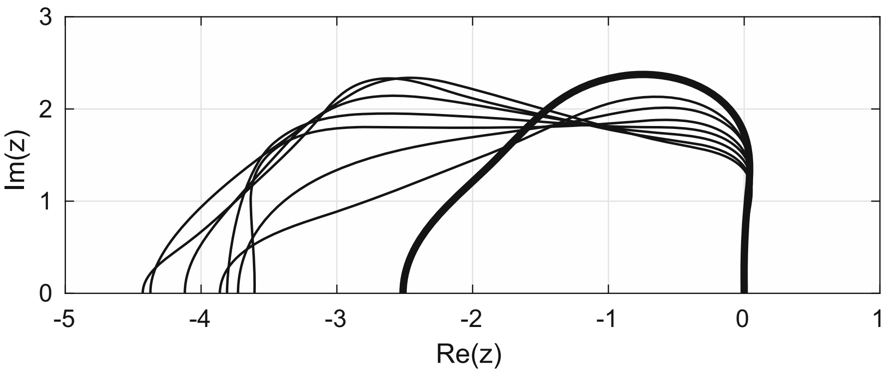

Fig. 4 Stability region of SSP RK method with $p = s = 3$ (thick line) and scaled stability regions of SSP RK methods of order $p = 3$ with $s = 4, 5, \ldots, 10$ stages (thin lines). These regions increase in size as s ranges from 3 to 10

of order two these regions increase in size as s ranges from 3 to 10. On these figures stability regions of SSP RK method with $p = s = 3$ are plotted by a thick line.

5.4 Explicit SSP RK Methods of Order $p = 4$

In this section we will search for explicit SSP RK methods of order $p = 4$ with $s = 4, 5, \ldots, 10$, stages. It was proved in [13, 30] that explicit SSP RK methods with $s = 4$ stages do not exist, and this was also confirmed by our numerical searches. Solving the minimization problem (4.6) with constrains (4.4) and (4.7) corresponding to $p = 4$ and $s = 5$, we obtain the method with coefficients

Table 3 SSP coefficients $\mathcal{C}$ and $\mathcal{C}_{\text{eff}}$, intervals of absolute stability, and areas of the regions of absolute stability for RK methods of order $p = 4$ with $s = 5, 6, \ldots, 10$, stages

s	$\mathcal{C}$	$\mathcal{C}_{\text{eff}}$	Interval	Area
5	1.51	0.30	(−5.3, 0)	22.59
6	2.29	0.38	(−5.9, 0)	34.53
7	3.32	0.47	(−7.1, 0)	53.33
8	4.15	0.52	(−8.7, 0)	76.56
9	4.91	0.55	(−11.3, 0)	102.37
10	6.00	0.60	(−13.9, 0)	134.06

$$\mathbf{A} = \begin{bmatrix} 0 & & & & \\ 0.391752226571889 & 0 & & & \\ 0.217669096261169 & 0.368410593050372 & 0 & & \\ 0.082692086657811 & 0.139958502191896 & 0.251891774271693 & 0 & \\ 0.067966283637115 & 0.115034698504632 & 0.207034898597385 & 0.544974750228520 & 0 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 0.146811876084786 \\ 0.248482909444976 \\ 0.104258830331980 \\ 0.274438900901350 \\ 0.226007483236907 \end{bmatrix}, \quad \mathbf{c} = \mathbf{A}\,\mathbf{e}.$$

For this method $\mathcal{C} = 1.51$ and $\mathcal{C}_{\text{eff}} = 0.30$. This methods is equivalent to SSPRK(5,4) scheme, whose Shu–Osher representation is listed in [12].

The coefficients of methods obtained by solving the minimization problem (4.6) with constrains (4.4) and (4.7) corresponding to $p = 4$ and $s = 6, 7, 8, 9$, are not listed here, but can be obtained from the authors. The coefficients of method corresponding to $p = 4$ and $s = 10$ are given by

$$\mathbf{A} = \begin{bmatrix} 0 & & & & & & & & & \\ \frac{1}{6} & 0 & & & & & & & & \\ \frac{1}{6} & \frac{1}{6} & 0 & & & & & & & \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & & & & & & \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & & & & & \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & 0 & & & & \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{6} & 0 & & & \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{6} & \frac{1}{6} & 0 & & \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \end{bmatrix}, \quad \mathbf{c} = \mathbf{A}\,\mathbf{e}.$$

For this method $\mathcal{C} = 6$ and $\mathcal{C}_{\text{eff}} = 3/5$. This method is equivalent to the SSPRK(10,4) scheme, whose Shu–Osher representation is listed in [12]. We have listed in Table 3 SSP coefficients $\mathcal{C}$ and $\mathcal{C}_{\text{eff}}$, interval of absolute stability, and areas of regions of

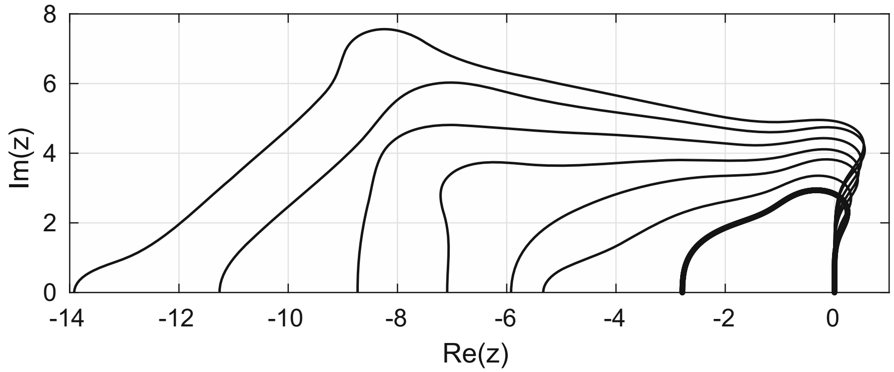

Fig. 5 Stability region of RK method with $p = s = 4$ (thick line) and stability regions of SSP RK methods of order $p = 4$ with $s = 5, 6, \ldots, 10$ stages (thin lines). These regions increase in size as s ranges from 4 to 10

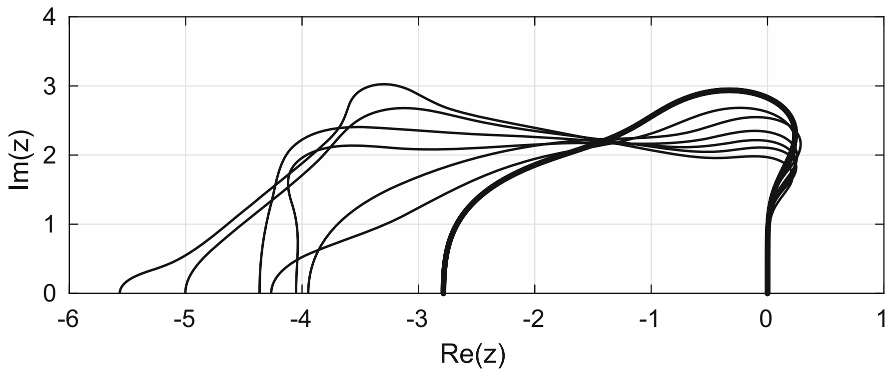

Fig. 6 Stability region of RK method with $p = s = 4$ (thick line) and scaled stability regions of SSP RK methods of order $p = 4$ with $s = 5, 6, \ldots, 10$ stages (thin lines). These regions increase in size as s ranges from 4 to 10

absolute stability for RK methods of order $p = 4$ with $s = 5, 6, \ldots, 10$, stages. We have also plotted on Fig. 5 stability regions of these methods and on Fig. 6 scaled stability regions obtained by multiplying the points on the boundary of regions of absolute stability by p/s, where p is the order and s is the number of stages. We have also plotted on these figures by a thick line stability region of RK method with $p = s = 4$. As for methods of order two and three these regions increase in size as s ranges from 4 to 10.

6 Implicit SSP RK Methods

In this section, we analyze implicit SSP RK methods. Examples of such methods, up to the order $p = 4$, are obtained by solving the minimization problem (4.6), subject to the inequality constrains (4.4), and the equality constrains (4.7).

6.1 Implicit SSP RK Methods of Order $p = 1$

The implicit RK method of order $p = 1$ with $s = 1$ stages corresponds to the backward Euler method

$$y_n = y_{n-1} + f(t_n, y_n),$$

$n = 1, 2, \ldots, N$, for which $\mathbf{c} = 1$, $\mathbf{A} = 1$, $\mathbf{b} = 1$,

$$\mathbf{T} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 1 & 0 \end{array}\right], \quad \mathbf{S} = \left[\begin{array}{c} 1 \\ \hline 1 \end{array}\right].$$

For this method, the condition $\det(\mathbf{I} + \gamma\mathbf{T}) = 1 + \gamma \neq 0$ is satisfied for $\gamma \geq 0$, and the inequalities (4.4) reduce to

$$\gamma(1+\gamma)^{-1} \geq 0, \quad 1 - \gamma(1+\gamma)^{-1} \geq 0.$$

These conditions lead to $\mathcal{C} = \mathcal{C}_{\text{eff}} = \infty$, compare [12, 28].

6.2 Implicit SSP RK Methods of Order $p = 2$

In this section, we will search for implicit SSP RK methods of order $p = 2$ with $s \geq 2$ stages. Solving the minimization problem (4.6) with inequality constrains (4.4) and equality constrains (4.7) corresponding to $p = 2$ leads to singly diagonally implicit RK methods with

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2s} & & & & & \\ \frac{1}{s} & \frac{1}{2s} & & & & \\ \frac{1}{s} & \frac{1}{s} & \frac{1}{2s} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{s} & \frac{1}{s} & \frac{1}{s} & \cdots & \frac{1}{2s} & \\ \frac{1}{s} & \frac{1}{s} & \frac{1}{s} & \cdots & \frac{1}{s} & \frac{1}{2s} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s} \\ \frac{1}{s} \\ \vdots \\ \frac{1}{s} \\ \frac{1}{s} \end{bmatrix}, \quad \mathbf{c} = \mathbf{A}\,\mathbf{e}.$$

For these methods $\mathcal{C} = 2s$ and $\mathcal{C}_{\text{eff}} = 2$. The stability function

$$R(z) = 1 + z\mathbf{b}^T(\mathbf{I} - z\mathbf{A})^{-1}\mathbf{e}$$

of these methods takes the form $R(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials of degree s equal to

$$P(z) = \det(\mathbf{I} - z\,\mathbf{A} + z\,\mathbf{e}\,\mathbf{b}^T),$$

and

$$Q(z) = \det(\mathbf{I} - z\,\mathbf{A}) = (1 - \lambda z)^s.$$

Here, $\lambda = 1/(2s)$ is the diagonal element of the coefficient matrix $\mathbf{A}$. It can be verified that the Nørsett polynomial $E(y)$ defined by [16]

$$E(y) = Q(iy)Q(-iy) - P(iy)P(-iy), \quad y \in \mathbf{R},$$

is identically equal to zero. This proves that all these methods are A-stable.

Choosing the matrix $\mathbf{L}$ to be equal to

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_0 \\ \hline \mathbf{L}_1 \end{bmatrix} = \left[\begin{array}{cccccc} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \\ \hline 0 & 0 & \cdots & \cdots & 0 & 1 \end{array}\right] \in \mathbb{R}^{(s+1)\times s},$$

the coefficient matrix $\mathbf{M}$ of the generalized Shu–Osher representation (2.1) takes the form

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_0 \\ \hline \mathbf{M}_1 \end{bmatrix} = \left[\begin{array}{cccccc} \frac{1}{2s} & & & & & \\ \frac{1}{2s} & \frac{1}{2s} & & & & \\ & \frac{1}{2s} & \frac{1}{2s} & & & \\ & & \ddots & \ddots & & \\ & & & \frac{1}{2s} & \frac{1}{2s} & \\ & & & & \frac{1}{2s} & \frac{1}{2s} \\ \hline 0 & 0 & \cdots & \cdots & 0 & \frac{1}{2s} \end{array}\right] \in \mathbb{R}^{(s+1)\times s}.$$

This representation was presented in [12]. As already observed, the generalized Shu–Osher representation is not unique, and for RK method of order $p = 2$ with $s = 2$ stages given by

$$\mathbf{c} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix},$$

Ferracina and Spijker [10] have chosen

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_0 \\ \hline \mathbf{L}_1 \end{bmatrix} = \left[\begin{array}{cc} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \\ \hline 0 & 1 \end{array}\right],$$

which leads to

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_0 \\ \hline \mathbf{M}_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{8} \\ \hline 0 & \frac{1}{4} \end{bmatrix}.$$

6.3 Implicit SSP RK Methods of Order $p = 3$

In this section, we will search for implicit SSP RK methods of order $p = 3$ with $s \geq 2$ stages. Solving the minimization problem (4.6) with inequality constrains (4.4) and equality constrains (4.7) corresponding to $p = 3$ leads to singly diagonally implicit RK methods with

$$\mathbf{A} = \begin{bmatrix} \frac{s+1-\sqrt{s^2-1}}{2(s+1)} & & & & & \\ \frac{\sqrt{s^2-1}}{s^2-1} & \frac{s+1-\sqrt{s^2-1}}{2(s+1)} & & & & \\ \frac{\sqrt{s^2-1}}{s^2-1} & \frac{\sqrt{s^2-1}}{s^2-1} & \frac{s+1-\sqrt{s^2-1}}{2(s+1)} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{\sqrt{s^2-1}}{s^2-1} & \frac{\sqrt{s^2-1}}{s^2-1} & \frac{\sqrt{s^2-1}}{s^2-1} & \cdots & \frac{s+1-\sqrt{s^2-1}}{2(s+1)} & \\ \frac{\sqrt{s^2-1}}{s^2-1} & \frac{\sqrt{s^2-1}}{s^2-1} & \frac{\sqrt{s^2-1}}{s^2-1} & \cdots & \frac{\sqrt{s^2-1}}{s^2-1} & \frac{s+1-\sqrt{s^2-1}}{2(s+1)} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s} & \cdots & \frac{1}{s} \end{bmatrix}^T, \quad \mathbf{c} = \mathbf{A}\,\mathbf{e}.$$

For these methods $\mathcal{C} = s - 1 + \sqrt{s^2 - 1}$ and $\mathcal{C}_{\text{eff}} = (s - 1 + \sqrt{s^2 - 1})/s$.

Choosing the matrix $\mathbf{L}$ to be equal to

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_0 \\ \hline \mathbf{L}_1 \end{bmatrix} = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \\ \hline 0 & 0 & \cdots & \cdots & 0 & \lambda_{s+1,s} \end{bmatrix} \in \mathbb{R}^{(s+1)\times s},$$

where

$$\lambda_{s+1,s} = \frac{(s+1)\left(s - 1 + \sqrt{s^2-1}\right)}{s\left(s + 1 + \sqrt{s^2-1}\right)},$$

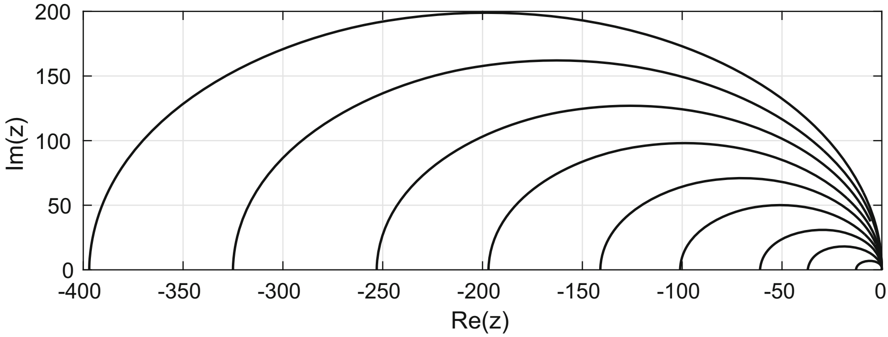

Fig. 7 Stability regions of implicit SSP RK methods of order $p = 3$ with $s = 2, 3, \dots, 10$ stages. These regions increase in size as s ranges from 2 to 10

the coefficient matrix $\mathbf{M}$ of the generalized Shu–Osher representation (2.1) takes the form

$$\mathbf{M} = \left[\frac{\mathbf{M}_0}{\mathbf{M}_1}\right] = \left[\begin{array}{cccccc} \mu_{11} & & & & & \\ \mu_{21} & \mu_{11} & & & & \\ & \mu_{21} & \mu_{11} & & & \\ & & \ddots & \ddots & & \\ & & & \mu_{21} & \mu_{11} & \\ & & & & \mu_{21} & \mu_{11} \\ \hline 0 & 0 & \cdots & \cdots & 0 & \mu_{s+1,s} \end{array}\right] \in \mathbb{R}^{(s+1)\times s},$$

where

$$\mu_{11} = \frac{1}{2}\left(1 - \sqrt{\frac{s-1}{s+1}}\right), \quad \mu_{21} = \frac{1}{2}\left(\sqrt{\frac{s+1}{s-1}} - 1\right),$$

and

$$\mu_{s+1,s} = \frac{s+1}{s\left(s + 1 + \sqrt{s^2 - 1}\right)}.$$

This generalized Shu–Osher representation was presented in [12]. All these methods have bounded stability regions. These regions are plotted on Fig. 7 for $p = 3$ and $s = 2, 3, \dots, 10$, and the scaled stability regions, obtained by multiplying points on the boundary of these regions by p/s are plotted on Fig. 8. These regions increase in size as s ranges from 2 to 10.

6.4 Implicit SSP RK Methods of Order $p = 4$

In this section, we will search for implicit SSP RK methods of order $p = 4$ with $s \geq 3$ stages. Solving the minimization problem (4.6) with inequality constrains (4.4)

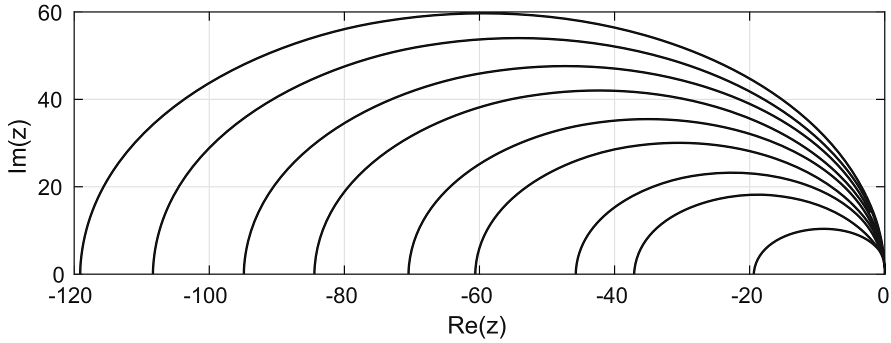

Fig. 8 Scaled stability regions of implicit SSP RK methods of order $p = 3$ with $s = 2, 3, \ldots, 10$ stages. These regions increase in size as s ranges from 2 to 10

and equality constrains (4.7) corresponding to $p = 4$ and $s = 3$ leads to diagonally implicit RK methods with coefficients given by

$$\mathbf{A} = \begin{bmatrix} 0.1572288784923564 & 0 & 0 \\ 0.4522707327651950 & 0.0450020385343502 & 0 \\ 0.3140166706802460 & 0.3693145682602409 & 0.1583343805297566 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 0.3540519556894777 & 0.2884480711749641 & 0.3574999731355583 \end{bmatrix}^T,$$

$$\mathbf{c} = \mathbf{A}\,\mathbf{e}.$$

For this method $\mathcal{C} = 2.05$ and $\mathcal{C}_{\text{eff}} = 0.68$. The generalized Shu–Osher representation of this method is presented in [12].

Solving the minimization problem (4.6) with inequality constrains (4.4) and equality constrains (4.7) corresponding to $p = 4$ and $s = 4$ leads to diagonally implicit RK methods with coefficients given by

$$\mathbf{A} = \begin{bmatrix} 0.1193097641665687 & 0 & 0 & 0 \\ 0.3454513963186752 & 0.0706060213251719 & 0 & 0 \\ 0.2761337244300779 & 0.2372027892889927 & 0.0706060424397012 & 0 \\ 0.2595320654029659 & 0.2229417285069986 & 0.2789066675911974 & 0.1193097692494192 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 0.2766452587403324 & 0.2233547210615690 & 0.2233547484281947 & 0.2766452717699039 \end{bmatrix}^T,$$

$$\mathbf{c} = \mathbf{A}\,\mathbf{e}.$$

For this method $\mathcal{C} = 4.42$ and $\mathcal{C}_{\text{eff}} = 1.11$. The generalized Shu–Osher representation of this method is presented in [12].

We have also solved the minimization problem (4.6) with inequality constrains (4.4) and equality constrains (4.7) corresponding to $p = 4$ and $s = 5, 6, \ldots, 10$. The coefficients of these methods in Butcher form are not listed here, but can be obtained from the authors, and the generalized Shu–Osher representations of these methods are presented in [12].

The stability regions of these SSP RK methods of order $p = 4$ with $s = 3, 4, \ldots, 10$ stages are plotted on Fig. 9 and the scaled stability regions, obtained by multiplying

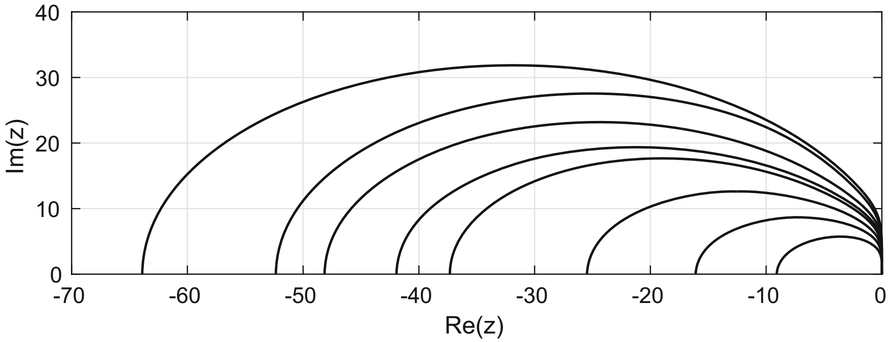

Fig. 9 Stability regions of implicit SSP RK methods of order $p = 4$ with $s = 3, 4, \ldots, 10$ stages. These regions increase in size as s ranges from 3 to 10

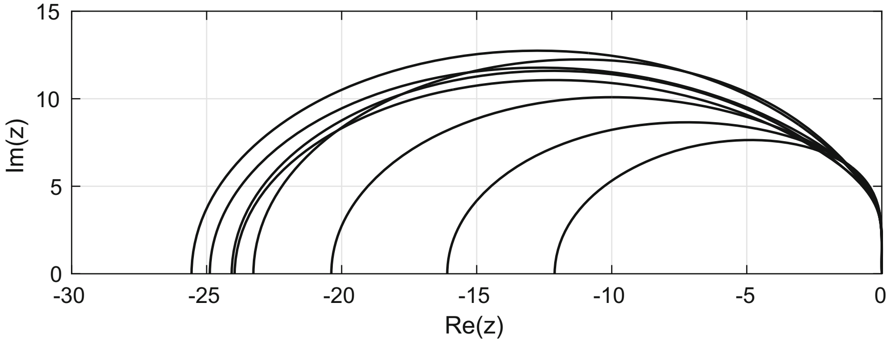

Fig. 10 Scaled stability regions of implicit SSP RK methods of order $p = 4$ with $s = 3, 4, \ldots, 10$ stages. These regions increase in size as s ranges from 3 to 10

points on the boundary of these regions by p/s are plotted on Fig. 10. Similarly as for SSP RK methods of order $p = 3$, these regions increase in size as s ranges from 3 to 10.

Coefficients of optimal implicit SSP RK methods up to order $p = 6$ are listed in [12].

7 Linear Multistep Methods

In this section, we will analyze LMMs specified by the coefficients

$$\alpha = \left[\, \alpha_k \ \alpha_{k-1} \ \cdots \ \alpha_1 \,\right] \in \mathbb{R}^k, \quad \left[\, \beta \middle| \beta_0 \,\right] \in \mathbb{R}^{k+1},$$

where

$$\beta = \left[\, \beta_k \ \beta_{k-1} \ \cdots \ \beta_1 \,\right] \in \mathbb{R}^k.$$

These methods are defined by

$$y_n = \sum_{j=1}^{k} \alpha_j \, y_{n-j} + h \sum_{j=0}^{k} \beta_j \, f(t_{n-j}, y_{n-j}), \tag{7.1}$$

$n = k, k+1, \ldots, N$, $Nh = t_{\text{end}} - t_0$, $t_n = t_0 + nh$, where $y_0 = y(t_0)$, $y_1, \ldots, y_{k-1}$ are the given starting values.

The first and second characteristic polynomials of LMM (7.1) are defined by

$$\rho(\xi) = \xi^k - \sum_{j=1}^{k} \alpha_j \xi^{k-j}, \quad \sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^{k-j}.$$

The LMM (7.1) is said to be zero-stable if no root of the first characteristic polynomial $\rho(\xi)$ has modulus greater than one, and if every root with modulus one is simple.

The method (7.1) has order p if and only if the following order conditions are satisfied

$$\sum_{j=1}^{k} \alpha_j = 1, \quad \sum_{j=1}^{k} j^{\ell} \alpha_j = \ell \sum_{j=0}^{k} j^{\ell-1} \beta_j, \quad \ell = 1, 2, \ldots, p, \tag{7.2}$$

compare [4, 6, 15, 17, 24, 31, 32]. The relations corresponding to $p = 1$,

$$\sum_{j=1}^{k} \alpha_j = 1, \quad \sum_{j=1}^{k} j \, \alpha_j = \sum_{j=0}^{k} \beta_j,$$

are usually referred to as consistency conditions. These conditions can be written in the more compact form

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1).$$

If $\beta_0 = 0$ the method (7.1) is explicit. For such methods the following result was formulated in [12], compare also [20, 34].

Theorem 7.1 *Assume that the forward Euler method* (1.2) *applied to* (1.1) *is strongly stable, i.e., the inequality* (1.3) *holds under the time step restriction* (1.4). *Then, the solution y_n and obtained by the LMM* (7.1) *satisfies* (1.3) *under the time step restriction* (1.5) *with SSP coefficient $\mathcal{C} = \mathcal{C}(\alpha, \beta)$ given by*

$$\mathcal{C}(\alpha, \beta) = \begin{cases} \min\left\{\dfrac{\alpha_i}{\beta_i}, \ \alpha_i, \beta_i \geq 0, \ i = 0, 1, \ldots, k\right\}, \\ 0, \quad \textit{otherwise}. \end{cases} \tag{7.3}$$

We have also the following bound on the SSP coefficients of explicit LMMs.

Theorem 7.2 (Lenferink [33]) *The SSP coefficient* $\mathcal{C}(\alpha, \beta)$ *of an k step explicit LMM* (7.1) *of order* $p > 1$ *satisfies the bound*

$$\mathcal{C}(\alpha, \beta) \leq \frac{k-p}{k-1}. \tag{7.4}$$

In what follows we will investigate SSP properties of LMMs (7.1) by reformulating these methods as GLMs (3.1), and then using monotonicity theory of GLMs developed by Spijker [36]. Putting

$$\mathbf{c} = \begin{bmatrix} -k & -k+1 & \cdots & 0 \end{bmatrix}^T, \quad t_{n+\mathbf{c}} = \begin{bmatrix} t_{n-k} & t_{n-k+1} & \cdots & t_n \end{bmatrix}^T,$$
$$Y^{[n]} = \begin{bmatrix} y_{n-k} \\ y_{n-k+1} \\ \vdots \\ y_n \end{bmatrix}, \quad y^{[n]} = \begin{bmatrix} y_{n-k+1} \\ y_{n-k+2} \\ \vdots \\ y_n \end{bmatrix},$$
$$f(t_{n+\mathbf{c}}, Y^{[n]}) = \begin{bmatrix} f(t_{n-k}, y_{n-k}) \\ f(t_{n-k+1}, y_{n-k+1}) \\ \vdots \\ f(t_n, y_n) \end{bmatrix},$$

the LMM (7.1) can be written as GLM (3.1) with $m = k+1$, $\ell = k$, abscissa vector $\mathbf{c}$ and coefficient matrices $\mathbf{S}$ and $\mathbf{T}$ defined by

$$\mathbf{S} = \left[\begin{array}{c} \mathbf{I} \\ \hline \alpha \end{array}\right] \in \mathbb{R}^{(k+1)\times k}, \quad \mathbf{T} = \left[\begin{array}{c|c} \mathbf{0} & 0 \\ \hline \beta & \beta_0 \end{array}\right] \in \mathbb{R}^{(k+1)\times(k+1)},$$

where $\mathbf{0}$ is $k \times k$ zero matrix, 0 is $k \times 1$ zero vector, and $\mathbf{I}$ is $k \times k$ identity matrix. Then, $\det(\mathbf{I} + \gamma\,\mathbf{T}) = 1 + \gamma\beta_0 \neq 0$ for $\gamma \neq -1/\beta_0$,

$$(\mathbf{I} + \gamma\,\mathbf{T})^{-1} = \left[\begin{array}{c|c} \mathbf{I} & 0 \\ \hline -\dfrac{\gamma\,\beta}{1+\gamma\,\beta_0} & \dfrac{1}{1+\gamma\,\beta_0} \end{array}\right],$$
$$(\mathbf{I} + \gamma\,\mathbf{T})^{-1}\left[\mathbf{S}\middle|\gamma\,\mathbf{T}\right] = \left[\begin{array}{c|c|c} \mathbf{I} & \mathbf{0} & 0 \\ \hline \dfrac{\alpha - \gamma\,\beta}{1+\gamma\,\beta_0} & \dfrac{\gamma\,\beta}{1+\gamma\,\beta_0} & \dfrac{\gamma\,\beta_0}{1+\gamma\,\beta_0} \end{array}\right],$$

and it follows that the conditions (3.4) take the form

$$\gamma \neq -\frac{1}{\beta_0}, \quad \frac{\alpha - \gamma\,\beta}{1+\gamma\,\beta_0} \geq 0, \quad \frac{\gamma\,\beta}{1+\gamma\,\beta_0} \geq 0, \quad \frac{\gamma\,\beta_0}{1+\gamma\,\beta_0} \geq 0. \tag{7.5}$$

We summarize the above discussion in the following theorem.

Theorem 7.3 *Assume that the forward Euler method (1.2) applied to (1.1) is strongly stable, i.e., the inequality (1.3) holds under the time step restriction (1.4). Then the solution y_n and obtained by the LMM (7.1) satisfies (1.3) under the time step restriction (1.5) with SSP coefficient $\mathcal{C} = \mathcal{C}(\alpha, \beta, \beta_0)$ given by*

$$\mathcal{C}(\alpha, \beta, \beta_0) = \sup \Big\{ \gamma \in \mathbf{R} : \ \gamma \text{ satisfies } (7.5) \Big\}. \tag{7.6}$$

We have also the following bound on SSP coefficient of implicit LMMs.

Theorem 7.4 ([20, 34]) *The SSP coefficient $\mathcal{C}(\alpha, \beta, \beta_0)$ of implicit LMM (7.1) of order $p > 1$ satisfies the bound*

$$\mathcal{C}(\alpha, \beta, \beta_0) \leq 2. \tag{7.7}$$

It follows from Theorem 7.3 that the coefficients α, β, and β_0 of LMM (7.1), and the corresponding SSP coefficient $\mathcal{C}(\alpha, \beta, \beta_0)$, can be computed by solving the minimization problem

$$F(\alpha, \beta, \beta_0) = -\gamma \longrightarrow \min, \tag{7.8}$$

subject to the nonlinear inequality constrains (7.5), and the equality constrains (7.2) corresponding to the order conditions up to the order p.

8 Explicit SSP LMMs

In this section, we investigate explicit SSP LMMs up to the order $p = 4$, i.e., methods (7.1) corresponding to $\beta_0 = 0$. The explicit method with $p = 1$ and $k = 1$ corresponds to the forward Euler method, and was analyzed in Sect. 5.1. The explicit methods of order $p = 2$, $p = 3$, and $p = 4$, are analyzed in the sections below.

8.1 Explicit SSP LMMs of Order $p = 2$

In this section, we will search for explicit LMMs of order $p = 2$ with $k \geq 3$ stages. Solving the minimization problem (7.8) with inequality constrains (7.5) and equality constrains (7.2) corresponding to $\beta_0 = 0$ and the order $p = 2$ leads to LMMs with coefficients given by

$$\alpha_1 = \frac{(k-1)^2 - 1}{(k-1)^2}, \quad \alpha_i = 0, \ i = 2, 3, \ldots, k-1, \quad \alpha_k = \frac{1}{(k-1)^2},$$
$$\beta_1 = \frac{k}{k-1}, \quad \beta_i = 0, \ i = 2, 3, \ldots, k,$$

and SSP coefficient $\mathcal{C}_{\text{eff}} = (k-2)/(k-1)$. These methods were derived in [33], and were also reproduced in [12]. We have listed in Table 4 SSP coefficients $\mathcal{C} = \mathcal{C}_{\text{eff}}$, intervals of absolute stability, areas of regions of absolute stability and $\max\{|r_j|, \ j =$

Table 4 SSP coefficients $\mathcal{C} = \mathcal{C}_{\text{eff}}$, intervals of absolute stability, areas of the regions of absolute stability, and $\max\{| r_j |\}$ for LMMs of order $p = 2$ with $k = 3, 4, \ldots, 10$, steps

k	$\mathcal{C} = \mathcal{C}_{\text{eff}}$	Interval	Area	$\max\{\|r_j\|\}$
3	1/2	(−1.33, 0)	1.22	0.50
4	2/3	(−1.33, 0)	1.70	0.50
5	3/4	(−1.60, 0)	1.98	0.54
6	4/5	(−1.60, 0)	2.17	0.57
7	5/6	(−1.71, 0)	2.30	0.60
8	6/7	(−1.71, 0)	2.40	0.63
9	7/8	(−1.78, 0)	2.48	0.65
10	8/9	(−1.78, 0)	2.55	0.67

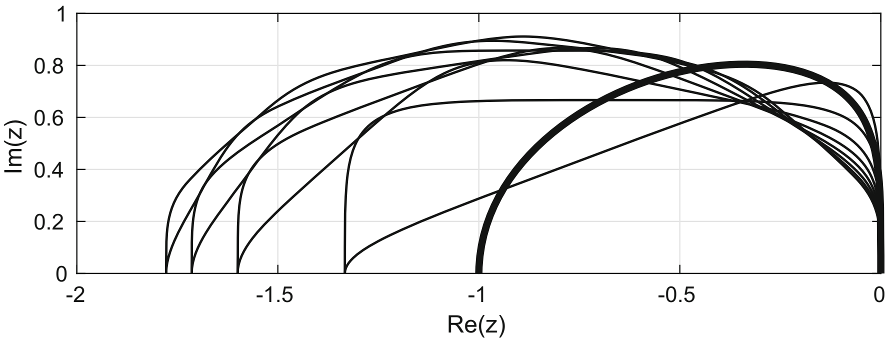

Fig. 11 Stability region of Adams–Bashforth method of order $p = 2$ (thick line) and of SSP LMMs of order $p = 2$ with $k = 3, 4, \ldots, 10$, steps (thin lines). These regions increase in size as k ranges from 3 to 10

$2, 3, \ldots, k\}$, where r_j, $j = 2, 3, \ldots, k$, are roots of the first characteristic polynomial $\rho(\xi)$, which are less than one, for LMMs of order $p = 2$ with $k = 3, 4, \ldots, 10$, steps. The regions of absolute stability for these methods corresponding to $k = 3, 4, \ldots, 10$, are plotted on Fig. 11. These regions increase in size as k ranges from 3 to 10. We have also plotted on this figure by a thick line the stability region of Adams–Bashforth methods of order $p = 2$. This method is defined by

$$y_n = y_{n-1} + h\left(\frac{3}{2} f(t_{n-1}, y_{n-1}) - \frac{1}{2} f(t_{n-2}, y_{n-2})\right),$$

$n = 2, 3, \ldots, N$, with given starting values $y_0 = y(t_0)$ and y_1.

8.2 Explicit SSP LMMs of Order $p = 3$

Solving the minimization problem (7.8) with inequality constrains (7.5) and equality constrains (7.2) corresponding to $\beta_0 = 0$, $p = 3$, and $k = 4$, $k = 5$, and $k = 6$, lead

Table 5 SSP coefficients $\mathcal{C} = \mathcal{C}_{\text{eff}}$, intervals of absolute stability, areas of the regions of absolute stability, and $\max\{|r_j|\}$ for LMMs of order $p = 3$ with $k = 4$, $k = 5$, and $k \geq 6$, steps

k	$\mathcal{C} = \mathcal{C}_{\text{eff}}$	Interval	Area	$\max\{\|r_j\|\}$
4	1/3	$(-0.89, 0)$	0.73	0.77
5	1/2	$(-1.07, 0)$	1.12	0.73
≥ 6	0.5828	$(-1.35, 0)$	1.38	0.73

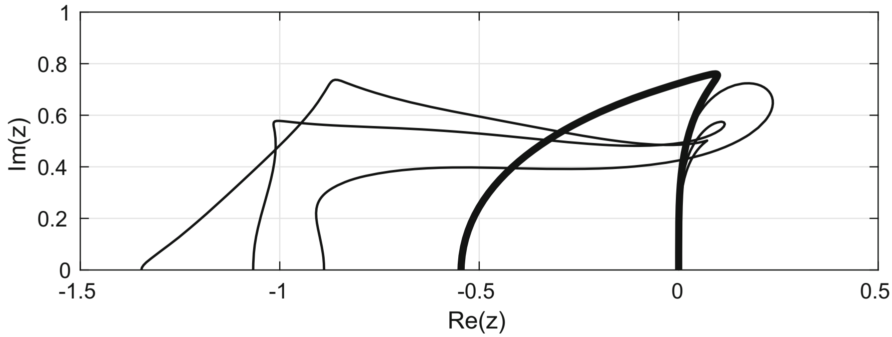

Fig. 12 Stability region of Adams–Bashforth method of order $p = 3$ (thick line) and of SSP LMMs of order $p = 3$ with $k = 4$, $k = 5$, and $k \geq 6$, steps (thin lines). These regions increase in size as k ranges from 4 to 6

to the methods

$$y_n = \frac{16}{27} y_{n-1} + \frac{11}{27} y_{n-4} + h\left(\frac{16}{9} f(t_{n-1}, y_{n-1}) + \frac{4}{9} f(t_{n-4}, y_{n-4})\right), \tag{8.1}$$

with optimal SSP coefficient $\mathcal{C} = \mathcal{C}_{\text{eff}} = 1/3$,

$$y_n = \frac{25}{16} y_{n-1} + \frac{7}{32} y_{n-5} + h\left(\frac{25}{16} f(t_{n-1}, y_{n-1}) + \frac{5}{16} f(t_{n-5}, y_{n-5})\right), \tag{8.2}$$

with optimal SSP coefficient $\mathcal{C} = \mathcal{C}_{\text{eff}} = 1/2$,

$$\begin{aligned} y_n = {} & \frac{1020}{1199} y_{n-1} + \frac{113}{3685} y_{n-5} + \frac{791}{6668} y_{n-6} \\ & + h\left(\frac{886}{607} f(t_{n-1}, y_{n-1}) + \frac{162}{3079} f(t_{n-5}, y_{n-5}) + \frac{909}{4466} f(t_{n-6}, y_{n-6})\right), \end{aligned} \tag{8.3}$$

with SSP coefficient $\mathcal{C} = \mathcal{C}_{\text{eff}} \approx 0.5828$. The coefficients of (8.3) are listed in Matlab rational format. The methods (8.1), (8.2), and (8.3), are equivalent to SSPMS(4, 3), SSPMS(5, 3), and SSPMS$(6, 3)_2$ formulas listed in [12]. The search for methods of order $p = 3$ and $k > 6$ leads to the method (8.3) corresponding to $k = 6$.

We have listed in Table 5 SSP coefficients $\mathcal{C} = \mathcal{C}_{\text{eff}}$, intervals of absolute stability, areas of regions of absolute stability and $\max\{|r_j|,\ j = 2, 3, \ldots, k\}$, where r_j, $j = 2, 3, \ldots, k$, are roots of the first characteristic polynomial $\rho(\xi)$, which are less than one, for LMMs of order $p = 3$ with $k = 4$, $k = 5$, and $k \geq 6$, steps. The regions of absolute stability for these methods corresponding to $k = 4$, $k = 5$, and $k \geq 6$, are plotted on Fig. 12. These regions increase in size as k ranges from 4 to 6. We have also plotted on this figure by a thick line the stability region of Adams–Bashforth methods of order $p = 3$. This method is defined by

$$y_n = y_{n-1} + h\left(\frac{23}{12} f(t_{n-1}, y_{n-1}) - \frac{4}{3} f(t_{n-2}, y_{n-2}) + \frac{5}{12} f(t_{n-3}, y_{n-3})\right),$$

$n = 3, 4, \ldots, N$, with given starting values $y_0 = y(t_0)$, y_1, and y_2.

8.3 Explicit SSP LMMs of Order $p = 4$

Solving the minimization problem (7.8) with inequality constrains (7.5) and equality constrains (7.2) corresponding to $\beta_0 = 0$, $p = 4$, and $k = 5$, leads to the method with rather small SSP coefficient and very small region of absolute stability. The coefficients of this method are not listed here. Solving this minimization problem corresponding to $p = 4$ and $k = 6$ we obtain the method, which in Matlab rational format takes the form

$$\begin{aligned} y_n = {} & \frac{963}{2812} y_{n-1} + \frac{304}{1585} y_{n-4} + \frac{747}{7984} y_{n-5} + \frac{808}{2171} y_{n-6} \\ & + h\left(\frac{1667}{802} f(t_{n-1}, y_{n-1}) + \frac{1823}{1566} f(t_{n-4}, y_{n-4}) + \frac{707}{1245} f(t_{n-5}, y_{n-5})\right). \end{aligned} \tag{8.4}$$

The SSP coefficient of this method is $\mathcal{C} = \mathcal{C}_{\text{eff}} \approx 0.1648$. This method is equivalent to SSPMS(4, 6) method, which was found numerically in [25, 26], and reproduced in [12].

We have listed in Table 6 SSP coefficients $\mathcal{C} = \mathcal{C}_{\text{eff}}$, intervals of absolute stability, areas of regions of absolute stability and $\max\{|r_j|,\ j = 2, 3, \ldots, k\}$, where r_j, $j = 2, 3, \ldots, k$, are roots of the first characteristic polynomial $\rho(\xi)$, which are less than one, for explicit LMMs of order $p = 4$ with $k = 5, 6, \ldots, 10$, steps. The regions of absolute stability for these methods corresponding to $k = 5, 6, \ldots, 10$, are plotted on Fig. 13. These regions increase in size as k ranges from 5 to 10. We have also plotted on this figure by a thick line the stability region of Adams–Bashforth methods of order $p = 4$. This method is defined by

$$\begin{aligned} y_n = y_{n-1} + h\Bigg(& \frac{55}{24} f(t_{n-1}, y_{n-1}) - \frac{59}{24} f(t_{n-2}, y_{n-2}) \\ & + \frac{37}{24} f(t_{n-3}, y_{n-3}) - \frac{9}{24} f(t_{n-4}, y_{n-4})\Bigg), \end{aligned}$$

Table 6 SSP coefficients $\mathcal{C} = \mathcal{C}_{\text{eff}}$, intervals of absolute stability, areas of the regions of absolute stability, and $\max\{| r_j |\}$ for LMMs of order $p = 4$ with $k = 5, 6, \ldots, 10$, steps

k	$\mathcal{C} = \mathcal{C}_{\text{eff}}$	Interval	Area	$\max\{\|r_j\|\}$
5	0.0212	$(-0.05, 0)$	0.01	0.98
6	0.1648	$(-0.59, 0)$	0.22	0.85
7	0.2816	$(-0.76, 0)$	0.47	0.78
8	0.3586	$(-0.72, 0)$	0.52	0.75
9	0.3925	$(-0.85, 0)$	0.71	0.75
10	0.4208	$(-0.95, 0)$	0.84	0.78

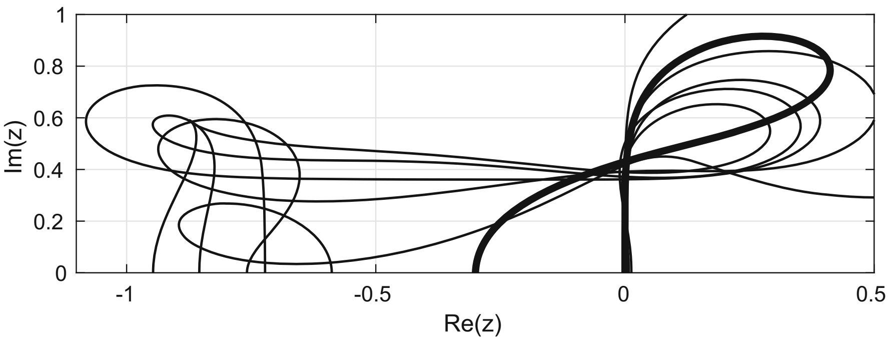

Fig. 13 Stability region of Adams–Bashforth method of order $p = 4$ (thick line) and of SSP LMMs of order $p = 4$ with $k = 5, 6, \ldots, 10$, steps (thin lines). These regions increase in size as k ranges from 5 to 10

$n = 4, 5, \ldots, N$, with given starting values $y_0 = y(t_0)$, y_1, y_2, and y_3.

The SSP coefficients of explicit LMMs of order up tp $p = 15$ with up to $k = 50$ steps are listed in [12].

9 Implicit SSP LMMs

In this section, we investigate implicit SSP LMMs up to the order $p = 4$, i.e., methods (7.1) with $\beta_0 \neq 0$. The implicit method with $p = 1$ and $k = 1$ corresponds to the backward Euler method, and was analyzed in Sect. 6.1. The implicit methods of order $p = 2$, $p = 3$, and $p = 4$, are analyzed in the sections below.

9.1 Implicit SSP LMMs of Order $p = 2$

Solving the minimization problem (7.8) with inequality constrains (7.5) and equality constrains (7.2) corresponding to $\beta_0 \neq 0$, $p = 2$, and $k \geq 1$, leads to the trapezoidal method

$$y_n = y_{n-1} + \frac{1}{2}\Big(f(t_n, y_n) + f(t_{n-1}, y_{n-1}) \Big), \tag{9.1}$$

$n = 1, 2, \ldots, N$. This method is A-stable. The coefficient matrices $\mathbf{S}$ and $\mathbf{T}$ of Spijker representation of this method are given by

$$\mathbf{S} = \left[\frac{1}{\alpha_1}\right] = \left[\frac{1}{1}\right], \quad \mathbf{T} = \left[\begin{array}{c|c} 0 & 0 \\ \hline \beta_1 & \beta_0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline \frac{1}{2} & \frac{1}{2} \end{array}\right],$$

and it can be verified that relations (7.5) for this method reduce to

$$\gamma \neq -2, \quad 0 \leq \gamma \leq 2.$$

This leads to the optimal SSP coefficient $\mathcal{C} = \mathcal{C}_{\text{eff}} = 2$, compare Theorem 7.4.

The method (9.1) can be also written as implicit RK method with FSAL (First Same As Last) property, with coefficients

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} = \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

or

$$\begin{cases} Y_1 = y_{n-1}, \\ Y_2 = y_{n-1} + \dfrac{1}{2}\Big(f(t_{n-1}, Y_1) + f(t_n, Y_2) \Big), \\ y_n = Y_2, \end{cases}$$

$n = 1, 2, \ldots, N$.

9.2 Implicit SSP LMMs of Order $p = 3$

Solving the minimization problem (7.8) with inequality constrains (7.5) and equality constrains (7.2) corresponding to $\beta_0 \neq 0$, $p = 3$, and $k \geq 2$, leads to the methods with coefficients

$$\alpha_1 = \frac{k^2(2k-3)}{(k-1)^2(2k+1)}, \quad \alpha_j = 0, \ j = 2, 3, \ldots, k-1, \quad \alpha_k = \frac{1}{(2k+1)(k-1)^2},$$
$$\beta_0 = \frac{k}{2k+1}, \quad \beta_1 = \frac{k^2}{(2k+1)(k-1)}, \quad \beta_j = 0, \ j = 2, 3, \ldots, k,$$

and optimal SSP coefficients

$$\mathcal{C} = \mathcal{C}_{\text{eff}} = \frac{2k-1}{k-1},$$

compare [34].

Table 7 SSP coefficients $\mathcal{C} = \mathcal{C}_{\text{eff}}$, intervals of absolute stability, areas of the regions of absolute stability, and $\max\{| r_j |\}$ for LMMs of order $p = 3$ with $k = 2, 3, \ldots, 10$, steps

k	$\mathcal{C} = \mathcal{C}_{\text{eff}}$	Interval	Area	$\max\{\|r_j\|\}$
2	1.0000	$(-4.00, 0)$	15.76	0.20
3	1.5000	$(-9.33, 0)$	67.76	0.19
4	1.6667	$(-13.33, 0)$	142.23	0.24
5	1.7500	$(-17.60, 0)$	245.30	0.29
6	1.8000	$(-21.60, 0)$	372.28	0.33
7	1.8333	$(-25.71, 0)$	527.17	0.37
8	1.8571	$(-29.71, 0)$	706.90	0.41
9	1.8750	$(-33.78, 0)$	914.44	0.44
10	1.8889	$(-37.78, 0)$	1147.51	0.46

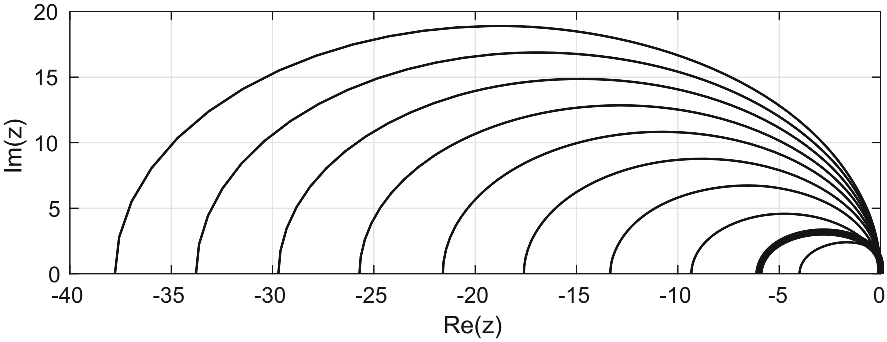

Fig. 14 Stability region of Adams–Moulton method of order $p = 3$ (thick line) and of SSP LMMs of order $p = 4$ with $k = 5, 6, \ldots, 10$, steps (thin lines). These regions increase in size as k ranges from 5 to 10

We have listed in Table 7 SSP coefficients $\mathcal{C} = \mathcal{C}_{\text{eff}}$, intervals of absolute stability, areas of regions of absolute stability and $\max\{|r_j|, \ j = 2, 3, \ldots, k\}$, where r_j, $j = 2, 3, \ldots, k$, are roots of the first characteristic polynomial $\rho(\xi)$, which are less than one, for implicit LMMs of order $p = 3$ with $k = 2, 3, \ldots, 10$, steps. The regions of absolute stability for these methods corresponding to $k = 2, 3, \ldots, 10$, are plotted on Fig. 14. These regions increase in size as k ranges from 2 to 10. We have also plotted on this figure by a thick line the stability region of Adams–Moulton methods of order $p = 3$. This method is defined by

$$
\begin{aligned}
y_n &= y_{n-1} \\
&\quad + h\left(\frac{5}{12} f(t_n, y_n) + \frac{2}{3} f(t_{n-1}, y_{n-1}) - \frac{1}{12} f(t_{n-2}, y_{n-2})\right),
\end{aligned}
$$

$n = 2, 3, \ldots, N$, with given starting values $y_0 = y(t_0)$, and y_1.

Table 8 SSP coefficients $\mathcal{C} = \mathcal{C}_{\text{eff}}$, intervals of absolute stability, areas of the regions of absolute stability, and $\max\{|\ r_j\ |\}$ for LMMs of order $p = 4$ with $k = 3$, and $k \geq 4$, steps

k	$\mathcal{C} = \mathcal{C}_{\text{eff}}$	Interval	Area	$\max\{\|r_j\|\}$
3	1.0000	$(-3.56, 0)$	11.25	0.40
≥ 4	1.2432	$(-5.65, 0)$	24.57	0.38

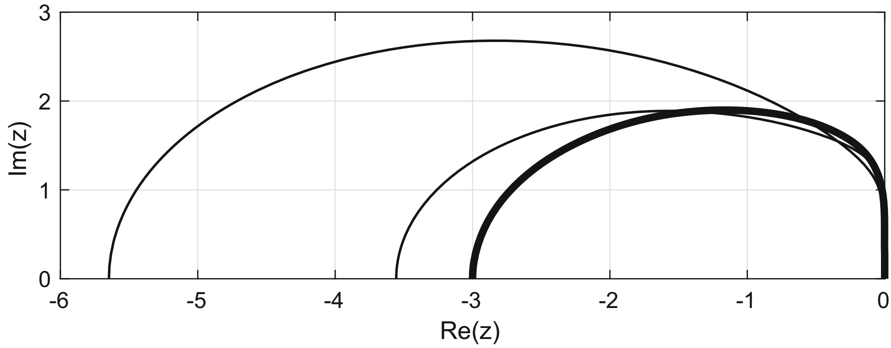

Fig. 15 Stability region of Adams–Moulton method of order $p = 4$ (thick line) and of SSP LMMs of order $p = 4$ with $k = 3$, and $k \geq 4$, steps (thin lines). These regions increase in size as k ranges from 3 to 4

9.3 Implicit SSP LMMs of Order $p = 4$

Solving the minimization problem (7.8) with inequality constrains (7.5) and equality constrains (7.2) corresponding to $\beta_0 \neq 0$, $p = 4$, and $k = 3$, leads to the method with coefficients

$$\begin{aligned} y_n &= \frac{27}{32} y_{n-1} + \frac{5}{32} y_{n-3} \\ &\quad + h\left(\frac{3}{8} f(t_n, y_n) + \frac{27}{32} f(t_{n-1}, y_{n-1}) + \frac{3}{32} f(t_{n-3}, y_{n-3})\right), \end{aligned} \tag{9.2}$$

$n = 3, 4, \ldots, N$, with starting values $y_0 = y(t_0)$, y_1, and y_2. Solving the minimization problem (7.8) with inequality constrains (7.5) and equality constrains (7.2) corresponding to $\beta_0 \neq 0$, $p = 4$, and $k \geq 4$, leads to the method with coefficients, which in Matlab rational format, are given by

$$\begin{aligned} &\alpha_1 = \tfrac{3407}{3691}, \quad \alpha_2 = 0, \quad \alpha_3 = \tfrac{177}{5482}, \quad \alpha_4 = \tfrac{323}{7233}, \\ &\alpha_j = 0, \ \ j = 5, 6, \ldots, k, \\ &\beta_0 = \tfrac{1272}{3277}, \quad \beta_1 = \tfrac{617}{831}, \quad \beta_2 = 0, \quad \beta_3 = \tfrac{115}{4428}, \quad \beta_4 = \tfrac{137}{3814}, \\ &\beta_j = 0, \ \ j = 5, 6, \ldots, k. \end{aligned} \tag{9.3}$$

The methods (9.2) and (9.3) are equivalent to the methods derived in [34]. We have listed in Table 8 SSP coefficients $\mathcal{C} = \mathcal{C}_{\text{eff}}$, intervals of absolute stability, areas of regions of absolute stability and $\max\{|r_j|,\ j = 2, 3, \dots, k\}$, where r_j, $j = 2, 3, \dots, k$, are roots of the first characteristic polynomial $\rho(\xi)$, which are less than one, for implicit LMMs of order $p = 3$ with $k = 3$, and $k \geq 4$, steps. The regions of absolute stability for these methods corresponding to $k = 3$, and $k \geq 4$, are plotted on Fig. 15. These regions increase in size as k ranges from 3 to 4. We have also plotted on this figure by a thick line the stability region of Adams–Moulton methods of order $p = 4$. This method is defined by

$$y_n = y_{n-1} + h\bigg(\frac{9}{24}f(t_n, y_n) + \frac{19}{24}f(t_{n-1}, y_{n-1}) - \frac{5}{24}f(t_{n-2}, y_{n-2}) + \frac{1}{24}f(t_{n-3}, y_{n-3}))\bigg),$$

$n = 3, 4, \dots, N$, with given starting values $y_0 = y(t_0)$, y_1, and y_2.

The SSP coefficients of implicit LMMs of order up tp $p = 15$ with up to $k = 50$ steps are listed in [12].

10 Concluding Remarks

SSP properties of explicit or implicit RK methods are usually investigated using Shu–Osher or generalized Shu–Osher representations. In this paper, we discussed the construction of explicit and implicit SSP RK methods using a different point of view. This different approach is based of reformulating first RK methods as GLMs, and then using monotonicity theory of GLMs developed by Spijker [36] to construct explicit or implicit SSP RK methods. Both approaches, i.e., approach based on Shu–Osher or generalized Shu–Osher representation, or monotonicity theory of GLMs, lead to the same classes of explicit or implicit RK methods, but expressed in different representations (Shu–Osher, generalized Shu–Osher, or Butcher). The SSP LMMs are also investigated using monotonicity theory of GLMs.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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