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ORIGINAL PAPER

Weighted Composition-Differentiation Operators on the Hardy and Bergman Spaces

Abbas Babaei[1] · **Ali Abkar**[1]

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Abstract
We study weighted composition-differentiation operators on the Hardy and Bergman spaces in the unit disk. We first find necessary and sufficient conditions for weighted composition-differentiation operators to be Hilbert–Schmidt. We then characterize convergence of sequences of composition-differentiation operators acting on the Hardy space both in weak operator topology, and in strong operator topology.

Keywords Hardy space · Bergman space · Hilbert–Schmidt operator · Composition-differentiation operator · Weak operator topology · Strong operator topology

Mamathematics Subject Classification 47B32 · 30H10 · 46E22

1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane, and let $\mathcal{H}$ denote certain functional Hilbert space of analytic functions on the unit disk. For instance, we may assume that $\mathcal{H}$ is the classical Hardy space, the Bergman space, the Dirichlet space, and so on. For an analytic self-mapping φ on the unit disk, the *composition operator* $C_\varphi : \mathcal{H} \to \mathcal{H}$ is defined by

$$C_\varphi(f) = f \circ \varphi.$$

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✉ Ali Abkar
abkar@sci.ikiu.ac.ir

Abbas Babaei
a.babaei@edu.ikiu.ac.ir

[1] Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin 34149, Iran

It is well-known [1, Corollary 3.7] that the composition operator is bounded on the Hardy space H^2 (see §2 for definition) and

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{1/2} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}.$$

For an analytic function ψ, the *weighted composition operator* $C_{\psi,\varphi} : \mathcal{H} \to \mathcal{H}$ is given by

$$C_{\psi,\varphi}(f) = \psi \cdot f \circ \varphi.$$

In the context of analytic functions, it is easy to verify that the differentiation operator $D(f) = f'$ is not bounded on the Hardy space H^2; since $\{z^n\}_{n\geq 1}$ is a sequence of unit vectors in the Hardy space satisfying $\|D(z^n)\| = n$. Nevertheless, for many analytic self-mappings φ on the unit disk, the operator $D_\varphi : H^2 \to H^2$ defined by

$$D_\varphi(f) = f' \circ \varphi$$

is bounded. We follow Fatehi and Hammond [4] to call D_φ a *composition-differentiation* operator. In [9], Shuichi Ohno established a set of sufficient conditions that guarantee when the operator D_φ is bounded or compact. In particular, S. Ohno proved that if $\|\varphi\|_\infty < 1$, then D_φ is a Hilbert–Schmidt operator; and hence bounded and compact; see [9, Theorem 3.3]. We recall that an operator T on a separable Hilbert space $\mathcal{X}$ is said to be Hilbert–Schmidt if for some orthonormal basis $e_n \subset \mathcal{X}$ we have

$$\sum_{n=0}^{\infty} \|Te_n\|^2 < \infty.$$

According to [9, Corollary 3.2], for a univalent self-map φ of the unit disk, the operator D_φ on the Hardy space H^2 is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{1-|z|}{(1-|\varphi(z)|)^3} < \infty.$$

Moreover, the operator D_φ on H^2 is compact if and only if

$$\lim_{|z|\to 1} \frac{1-|z|}{(1-|\varphi(z)|)^3} = 0.$$

Now, let ψ be an analytic function on the unit disk, and define the *weighted composition-differentiation* operator $D_{\psi,\varphi} : H^2 \to H^2$ by the following relation:

$$D_{\psi,\varphi}(f) = \psi \cdot f' \circ \varphi.$$

In Sect. 3, we characterize when the operator $D_{\psi,\varphi}$ is Hilbert–Schmidt. We prove that the composition-differentiation operator $D_{\psi,\varphi}$ is Hilbert–Schmidt if and only if

$$\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|\psi\left(re^{i\theta}\right)|^2}{(1-\left|\varphi\left(re^{i\theta}\right)\right|^2)^3}\mathrm{d}\theta\right\}<\infty.$$

This extends a result of S. Ohno [9, Theorem 3.3] to weighted composition-differentiation operators. We will also discuss the same problem for other types of composition-differentiation operators; in particular, the operators ($k\in\mathbb{N}$):

$$D_\varphi^k(f)=f^{(k)}\circ\varphi,\quad D_{\psi,\varphi}^k(f)=\psi\cdot(f^{(k)}\circ\varphi),$$
$$D_\varphi^{(k)}(f)=(f^{(k)}\circ\varphi)\varphi',\quad D_{\psi,\varphi}^{(k)}(f)=\psi\cdot(f^{(k)}\circ\varphi)\varphi'.$$

We will find necessary and sufficient conditions for these operators to become Hilbert–Schmidt. In particular, we shall see that the operator $D_{\psi,\varphi}^k$ is Hilbert–Schmidt on H^2 if and only if

$$\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|\psi\left(re^{i\theta}\right)|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^{2k+1}}\mathrm{d}\theta\right\}<\infty.$$

Moreover, the operator $D_{\psi,\varphi}^{(k)}$ is Hilbert–Schmidt on H^2 if and only if

$$\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|\psi\left(re^{i\theta}\right)\varphi'\left(re^{i\theta}\right)|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^{2k+1}}\mathrm{d}\theta\right\}<\infty.$$

Similar results are established in the Bergman space (see Sect. 2, for the definition) too. The analogs of the above conditions in the Bergman space are the following; the functions ψ and φ have to satisfy

$$\int_{\mathbb{D}}\frac{|\psi(z)|^2}{(1-|\varphi(z)|^2)^{2k+2}}\mathrm{d}A(z)<\infty,$$

(respectively),

$$\int_{\mathbb{D}}\frac{\left|\psi(z)\varphi'(z)\right|^2}{(1-|\varphi(z)|^2)^{2k+2}}\mathrm{d}A(z)<\infty,$$

where $\mathrm{d}A(z)=\mathrm{d}x\,\mathrm{d}y$ is the area measure on the unit disk.

Our next objective in Sect. 4 is to find the relationships between convergence of the sequence of operators D_{ψ_n,φ_n} in operator topologies from one hand, and the convergence of the sequences of functions ψ_n and φ_n on the other hand. From historical point of view, Howard Schwarz [11] studied the relationship between the convergence of sequence of composition operators C_{φ_n}, and the convergence of the sequence of self-maps φ_n. Valetine Matache [7] extended Schwarz's work by relating the convergence of C_{φ_n} in Hilbert–Schmidt norm to the convergence of the sequence φ_n. Then appeared Gunatillake's paper [6] on the relationship between convergence of weighted composition operators C_{ψ_n,φ_n}, and the convergence of $\{\psi_n\}$ and $\{\varphi_n\}$. This latter was extended by Mehrangiz and Khani-Robati [8] to generalized weighted composition operators on Bloch type spaces. Here we intend to generalize Gunatillake's result to weighted composition-differentiation operator $D_{\psi,\varphi}$ in the setting of classical Hardy spaces. More specifically, let $\mathcal{B}(H^2)$ denote the Banach algebra of all bounded linear operators on the Hilbert space H^2. It is rather well-known that the dual space of $\mathcal{B}(H^2)$ is too big, so that the weak and weak-star topology of this space is not so clear. For this reason, it is customary to equip this space with the weak operator topology, the strong operator topology, and the uniform operator topology. We intend to have a characterization of the convergence of D_{ψ_n,φ_n} to $D_{\psi,\varphi}$ with respect to operator topologies in terms of the convergence of $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in the weak and strong operator topologies of H^2.

2 Preliminaries

Let f be an analytic function in the unit disk $\mathbb{D}$. The function f is said to belong to the Hardy space H^2 if

$$\|f\|^2 = \sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} |f\left(re^{i\theta}\right)|^2 \mathrm{d}\theta < \infty.$$

It is easy to see that for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the norm of f in H^2 is given by

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

It is well-known (see for instance, [10]) that for $f \in H^2$, the radial limit

$$f^*(e^{i\theta}) := \lim_{r\to 1^-} f\left(re^{i\theta}\right) = \lim_{r\to 1^-} f_r\left(e^{i\theta}\right)$$

for almost every $\theta \in [0, 2\pi]$ exists. The function f^* is known as the radial function of f. The space H^2 is a functional Hilbert space, and its inner product is given by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f^*\left(e^{i\theta}\right)\overline{g^*\left(e^{i\theta}\right)}\mathrm{d}\theta.$$

Since the evaluation functionals are bounded, the Hardy space is a reproducing kernel Hilbert space; this means that for each $w \in \mathbb{D}$, there is a function

$$K_w(z) = \frac{1}{1-\overline{w}z} \in H^2$$

such that every $f \in H^2$ has the following representation

$$f(w) = \langle f, K_w \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta})\overline{K_w^*(e^{i\theta})}\mathrm{d}\theta.$$

It is also well-known that the functional $w \mapsto f'(w)$ is bounded on H^2 ([1, Theorem 2.16]). It then follows from the Riesz representation theorem that there is a function $K_w^{(1)} \in H^2$ such that

$$f'(w) = \langle f, K_w^{(1)} \rangle, \quad f \in H^2.$$

It turns out that (see §4)

$$K_w^{(1)}(z) = \frac{z}{(1-\overline{w}z)^2}, \quad (z, w) \in \mathbb{D} \times \mathbb{D}.$$

Another functional Hilbert space on the unit disk is the Bergman space L_a^2 consisting of all analytic functions f in the unit disk for which the integral

$$\frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \mathrm{d}A(z)$$

is finite; here $\mathrm{d}A(z) = \mathrm{d}x\mathrm{d}y$ is the usual area measure in the complex plane. The norm of f is defined by

$$\|f\|_{L_a^2} = \left(\frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \mathrm{d}A(z)\right)^{1/2}.$$

A computation reveals that for $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$\|f\|_{L_a^2}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1},$$

from which we conclude that $H^2 \subset L_a^2$.

We now turn to recall different notions of convergence in the space of bounded linear operators on a given Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. We consider a sequence $T_n \in \mathcal{B}(\mathcal{H})$. We say that T_n converges to T in *weak operator topology*, and write $T_n \to T$ (WOT) if for each $x \in \mathcal{H}$, $T_n x \to Tx$ weakly. Similarly, we say that T_n converges to T in *strong operator topology*, and write $T_n \to T$ (SOT) if for each $x \in \mathcal{H}$, $T_n x \to Tx$ (in norm).

In this paper, we are concerned with bounded operators of the form $D_{\psi,\varphi}$ on H^2. It should be emphasized that if ψ is a bounded analytic function on the open unit disk, and if φ is a non-constant self-map of the unit disk such that

$$\|\varphi\|_\infty = \sup\{|\varphi(z)| : z \in \mathbb{D}\} \leq r < 1,$$

then the composition-differentiation operator $D_{\psi,\varphi}$ is bounded; indeed, by a result due to Fatehi and Hammond ([4, Proposition 4]) we have

$$\|D_\varphi\| \leq \left(\frac{r + |\varphi(0)|}{r - |\varphi(0)|}\right)^{1/2} \left\lfloor \frac{1}{1-r} \right\rfloor r^{\left\lfloor \frac{1}{1-r} \right\rfloor - 1}, \tag{1}$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function. On the other hand, if ψ is bounded, then the operator of multiplication by ψ, i.e. $M_\psi(f) = \psi f$ is bounded, and its norm does not exceed $\|\psi\|_\infty$. Therefore,

$$D_{\psi,\varphi}(f) = M_\psi D_\varphi(f), \quad f \in H^2,$$

is bounded too. We mention that there are instances of bounded operators when $\|\varphi\|_\infty = 1$. Indeed, if we know that D_φ maps H^2 into itself, it follows from Closed Graph Theorem that the operator D_φ and hence the operator $D_{\psi,\varphi}$ is bounded (for a bounded analytic function ψ). To see this, let f_n be a sequence of analytic self-maps such that $f_n \to f$ in H^2, and $D_\varphi(f_n) \to g$ in H^2. But norm convergence implies pointwise convergence, so that $f_n'(\varphi(z)) \to g(z)$ pointwise. Since $f_n'(z) \to f'(z)$, it follows that $f_n'(\varphi(z)) \to f'(\varphi(z))$. By the uniqueness of limit, we conclude that $f'(\varphi(z)) = g(z)$ from which it follows that $D_\varphi(f_n) \to D_\varphi(f)$.

3 Hilbert–Schmidt operators

This section is devoted to the study of Hilbert–Schmidt composition-differentiation operators. We start by finding conditions on ψ and φ to guarantee that the weighted composition-differentiation operator $D_{\psi,\varphi}$ is a Hilbert–Schmidt operator. We recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is called Hilbert–Schmidt if for some orthonormal basis $e_n \subset \mathcal{X}$ we have $\sum_n \|Te_n\|^2 < \infty$ (here $\mathcal{X}$ is a separable Hilbert space). It is well-known that every Hilbert–Schmidt operator is compact (see [3, page 87]). Assume that φ is an analytic self-map of the unit disk, and ψ is an analytic function on the unit disk. In this section, in addition to $D_{\psi,\varphi}(f) = \psi \cdot f' \circ \varphi$, we shall consider the following weighted composition-differentiation operators on a given functional Hilbert space $\mathcal{H}$ (this functional Hilbert space is either the Hardy space H^2 or the Bergman space L_a^2, moreover $k \geq 1$ is an integer):

$$D_\varphi^k(f) = f^{(k)} \circ \varphi, \quad D_{\psi,\varphi}^k(f) = \psi \cdot (f^{(k)} \circ \varphi),$$

$$D_{\varphi}^{(k)}(f) = (f^{(k)} \circ \varphi)\varphi', \quad D_{\psi,\varphi}^{(k)}(f) = \psi \cdot (f^{(k)} \circ \varphi)\varphi'.$$

Note that in case $k = 1$ or $k = 2$, it is easier to use the notations

$$D'_{\varphi}(f) = (f' \circ \varphi)\varphi', \quad D'_{\psi,\varphi}(f) = \psi \cdot (f' \circ \varphi)\varphi',$$
$$D''_{\varphi}(f) = (f'' \circ \varphi)\varphi', \quad D''_{\psi,\varphi}(f) = \psi \cdot (f'' \circ \varphi)\varphi'.$$

Some authors (see for instance [9]) denote the operator $D'_{\varphi}(f)$ by

$$DC_{\varphi} f(z) = f'(\varphi(z))\varphi'(z).$$

Our first theorem generalizes a similar result proved in [9] for D_{φ}.

Theorem 3.1 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D_{\psi,\varphi}$ is a Hilbert–Schmidt operator on H^2 if and only if*

$$\sup_{0 \le r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{|\psi\left(re^{i\theta}\right)|^2}{(1 - \left|\varphi\left(re^{i\theta}\right)\right|^2)^3} \mathrm{d}\theta \right\} < \infty.$$

Proof For the orthonormal basis $\{z^n\}$ for H^2, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \|D_{\psi,\varphi}(z^n)\|^2 &= \sum_{n=1}^{\infty} \sup_{0 \le r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \psi\left(re^{i\theta}\right) n\varphi\left(re^{i\theta}\right)^{n-1} \right|^2 \mathrm{d}\theta \right\} \\
&= \sup_{0 \le r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\psi\left(re^{i\theta}\right)|^2 \sum_{n=0}^{\infty} (n+1)^2 \left|\varphi(re^{i\theta})\right|^{2n} \mathrm{d}\theta \right\} \\
&\le \sup_{0 \le r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{2|\psi(re^{i\theta})|^2}{(1 - \left|\varphi(re^{i\theta})\right|^2)^3} \mathrm{d}\theta \right\}.
\end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} \|D_{\psi,\varphi}(z^n)\|^2$ is finite if

$$\sup_{0 \le r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{|\psi\left(re^{i\theta}\right)|^2}{(1 - \left|\varphi\left(re^{i\theta}\right)\right|^2)^3} \mathrm{d}\theta \right\} < \infty.$$

For the reverse implication note that

$$\begin{aligned}
\infty > \sum_{n=0}^{\infty} \|D_{\psi,\varphi}(z^n)\|^2 &= \sum_{n=1}^{\infty} \sup_{0 \le r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \psi\left(re^{i\theta}\right) n\varphi\left(re^{i\theta}\right)^{n-1} \right|^2 \mathrm{d}\theta \right\} \\
&\ge \sup_{0 \le r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{|\psi\left(re^{i\theta}\right)|^2}{\left(1 - \left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3} \mathrm{d}\theta \right\}.
\end{aligned}$$

□

Corollary 3.2 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D'_{\psi,\varphi}$ is a Hilbert–Schmidt operator on H^2 if and only if*

$$\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{\left|\psi\left(re^{i\theta}\right)\varphi'\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3}\mathrm{d}\theta\right\}<\infty.$$

Proof We just note that in this case

$$D'_{\psi,\varphi}(z^n)\left(re^{i\theta}\right)=\psi\left(re^{i\theta}\right)\cdot n\varphi\left(re^{i\theta}\right)^{n-1}\cdot\varphi'\left(re^{i\theta}\right).$$

The rest calculations remain unchanged. □

Theorem 3.3 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D^2_{\psi,\varphi}$ is a Hilbert–Schmidt operator on H^2 if and only if*

$$\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|\psi\left(re^{i\theta}\right)|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^5}\mathrm{d}\theta\right\}<\infty. \tag{2}$$

Proof First, assume that (2) holds true. We note that

$$D^2_{\psi,\varphi}(z^n)\left(re^{i\theta}\right)=\psi\left(re^{i\theta}\right)\cdot n(n-1)\varphi\left(re^{i\theta}\right)^{n-2}.$$

Using the series expansion

$$\sum_{n=0}^{\infty}(n+1)^2(n+2)^2x^n=\frac{4x^2+16x+4}{(1-x)^5},\quad |x|<1,$$

we conclude that

$$\begin{aligned}\sum_{n=0}^{\infty}\|D^2_{\psi,\varphi}(z^n)\|^2&=\sum_{n=2}^{\infty}\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\psi\left(re^{i\theta}\right)n(n-1)\varphi\left(re^{i\theta}\right)^{n-2}\right|^2\mathrm{d}\theta\right\}\\&=\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\psi\left(re^{i\theta}\right)\right|^2\sum_{n=0}^{\infty}(n+2)^2(n+1)^2\left|\varphi\left(re^{i\theta}\right)\right|^{2n}\mathrm{d}\theta\right\}\\&=\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|\psi\left(re^{i\theta}\right)|^2\left(4|\varphi\left(re^{i\theta}\right)|^4+16|\varphi\left(re^{i\theta}\right)|^2+4\right)}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^5}\mathrm{d}\theta\right\}\end{aligned}$$

$$\leq \sup_{0\leq r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{24|\psi\left(re^{i\theta}\right)|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^5}\mathrm{d}\theta\right\}<\infty.$$

Conversely, if $D^2_{\psi,\varphi}$ is a Hilbert–Schmidt operator, then

$$\begin{aligned}\infty > \sum_{n=0}^{\infty}\|D^2_{\psi,\varphi}(z^n)\|^2 &= \sum_{n=2}^{\infty}\sup_{0\leq r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\psi\left(re^{i\theta}\right)n(n-1)\varphi\left(re^{i\theta}\right)^{n-2}\right|^2\mathrm{d}\theta\right\}\\ &= \sup_{0\leq r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|\psi\left(re^{i\theta}\right)|^2\left(4|\varphi\left(re^{i\theta}\right)|^4+16|\varphi\left(re^{i\theta}\right)|^2+4\right)}{(1-\left|\varphi\left(re^{i\theta}\right)\right|^2)^5}\mathrm{d}\theta\right\}\\ &\geq \sup_{0\leq r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{4|\psi\left(re^{i\theta}\right)|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^5}\mathrm{d}\theta\right\},\end{aligned}$$

which implies that (2) is satisfied. □

Corollary 3.4 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D''_{\psi,\varphi}$ is a Hilbert–Schmidt operator on H^2 if and only if*

$$\sup_{0\leq r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{\left|\psi\left(re^{i\theta}\right)\varphi'\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^5}\mathrm{d}\theta\right\}<\infty.$$

Proof In this case note that

$$D''_{\psi,\varphi}(z^n)(re^{i\theta}) = \psi(re^{i\theta})\cdot n(n-1)\varphi(re^{i\theta})^{n-2}\varphi'(re^{i\theta}).$$

The rest calculations remain unchanged. □

Theorem 3.5 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D^3_{\psi,\varphi}$ is a Hilbert–Schmidt operator on H^2 if and only if*

$$\sup_{0\leq r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|\psi(re^{i\theta})|^2}{(1-\left|\varphi(re^{i\theta})\right|^2)^7}\mathrm{d}\theta\right\}<\infty. \tag{3}$$

Proof For the sake of simplicity, let us assume that $\psi\equiv 1$. We first assume that (3) holds true. Since

$$D^3_{\varphi}(z^n)(re^{i\theta}) = n(n-1)(n-2)\varphi(re^{i\theta})^{n-3},$$

we can write

$$\sum_{n=0}^{\infty}\|D_{\varphi}^{3}(z^n)\|^2=\sum_{n=3}^{\infty}\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|n(n-1)(n-2)\varphi(re^{i\theta})^{n-3}\right|^2\mathrm{d}\theta\right\}$$
$$=\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty}(n+1)^2(n+2)^2(n+3)^2\left|\varphi(re^{i\theta})\right|^{2n}\mathrm{d}\theta\right\}.$$

Note that

$$\sum_{n=0}^{\infty}(n+1)^2(n+2)^2(n+3)^2x^n=\frac{36(x^3+9x^2+9x+3)}{(1-x)^7},\quad |x|<1, \tag{4}$$

so that

$$\sum_{n=0}^{\infty}\|D_{\varphi}^{3}(z^n)\|^2=\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{36\left(|\varphi(re^{i\theta})|^6+9|\varphi(re^{i\theta})|^4+9|\varphi(re^{i\theta})|^2+3\right)}{\left(1-\left|\varphi(re^{i\theta})\right|^2\right)^7}\mathrm{d}\theta\right\}$$
$$\le\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{(36)(22)}{\left(1-\left|\varphi(re^{i\theta})\right|^2\right)^7}\mathrm{d}\theta\right\}<\infty.$$

On the other hand,

$$\infty>\sum_{n=0}^{\infty}\|D_{\varphi}^{3}(z^n)\|^2=\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{36\left(|\varphi(re^{i\theta})|^6+9|\varphi(re^{i\theta})|^4+9|\varphi(re^{i\theta})|^2+3\right)}{(1-\left|\varphi(re^{i\theta})\right|^2)^7}\mathrm{d}\theta\right\}$$
$$\ge\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{(36)(3)}{(1-\left|\varphi(re^{i\theta})\right|^2)^7}\mathrm{d}\theta\right\}.$$

□

Corollary 3.6 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D_{\psi,\varphi}^{(3)}$ is a Hilbert–Schmidt operator on H^2 if and only if*

$$\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|\psi(re^{i\theta})\varphi'(re^{i\theta})|^2}{(1-\left|\varphi(re^{i\theta})\right|^2)^7}\mathrm{d}\theta\right\}<\infty.$$

Proof We just note that

$$D_{\varphi}^{(3)}(z^n)(re^{i\theta})=n(n-1)(n-2)\varphi(re^{i\theta})^{n-3}\varphi'(re^{i\theta}).$$

□

The above pattern can be extended to higher order weighted composition-differentiation operators. We just state the following two results.

Theorem 3.7 *Let φ be an analytic self-map of the unit disk, ψ be an analytic function on the unit disk, and k be a non-negative integer. Then $D^k_{\psi,\varphi}$ is a Hilbert–Schmidt operator on H^2 if and only if*

$$\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|\psi(re^{i\theta})|^2}{(1-\left|\varphi(re^{i\theta})\right|^2)^{2k+1}}\mathrm{d}\theta\right\}<\infty.$$

Corollary 3.8 *Let φ be an analytic self-map of the unit disk, ψ be an analytic function on the unit disk, and k be a non-negative integer. Then $D^{(k)}_{\psi,\varphi}$ is a Hilbert–Schmidt operator on H^2 if and only if*

$$\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|\psi(re^{i\theta})\varphi'(re^{i\theta})|^2}{(1-\left|\varphi(re^{i\theta})\right|^2)^{2k+1}}\mathrm{d}\theta\right\}<\infty.$$

Remark 3.9 If $k=0$ and ψ is the constant function 1, then $D^k_{\psi,\varphi}=C_\varphi$, so that C_φ is a Hilbert–Schmidt operator on H^2 if and only if

$$\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-\left|\varphi(re^{i\theta})\right|^2}\mathrm{d}\theta\right\}<\infty.$$

This result is of course known (see [12, page 227, Ex. 4]).

The Bergman space

As is well-known, the Bergman spaces are special cases of weighted Hardy spaces; we mean by a *weighted Hardy space*, a space of analytic functions on the unit disk such that the monomials $\{z^n\}_{n\ge0}$ constitute an orthonormal basis for the space. If $\|z^n\|=\beta(n)$, then

$$\|f\|^2=\left\|\sum_{n=0}^{\infty}a_nz^n\right\|^2=\sum_{n=0}^{\infty}|a_n|^2\beta(n)^2.$$

If $\beta(n)=1$ for each n, we get the Hardy space, and if $\beta(n)=(n+1)^{-1/2}$ we get the Bergman space consisting of functions f that are analytic in the unit disk and satisfy

$$\|f\|^2_{L^2_a}=\frac{1}{\pi}\int_{\mathbb{D}}|f(z)|^2\mathrm{d}A(z)<\infty,$$

where $\mathrm{d}A(z) = \mathrm{d}x\mathrm{d}y$ is the usual area measure. It is easy to see that for $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$\|f\|^2_{L^2_a} = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

An easy computation shows that the sequence $\{\sqrt{n+1}\, z^n\}_{n=0}^{\infty}$ constitutes an orthonormal basis for the Bergman space L^2_a. It should be mentioned that the operator $C_\varphi(f) = f \circ \varphi$ is bounded on L^2_a and its norm satisfies (see [12, Theorem 10.3.2])

$$\|C_\varphi\|_{L^2_a} \le \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}.$$

The subscript L^2_a reflects the norm of operator acting on L^2_a. Therefore, for $\psi \in H^\infty$, the operator $D_{\psi,\varphi}$ is bounded as long as D_φ maps the Bergman space into itself; however there are instances in which this operator is unbounded.

We are now in a position to characterize weighted composition-differentiation operators on the Bergman space that are Hilbert–Schmidt.

Theorem 3.10 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D_{\psi,\varphi}$ is a Hilbert–Schmidt operator on L^2_a if and only if*

$$\int_{\mathbb{D}} \frac{|\psi(z)|^2}{(1-|\varphi(z)|^2)^4} \mathrm{d}A(z) < \infty.$$

Proof Let φ satisfy the above condition, and $\{\sqrt{n+1}\, z^n\}$ be an orthonormal basis for L^2_a. Using the elementary expansion

$$\sum_{n=0}^{\infty} (n+1)^2(n+2)x^n = \frac{4x+2}{(1-x)^4}, \quad |x| < 1,$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \|D_{\psi,\varphi}(\sqrt{n+1}\, z^n)\|^2_{L^2_a} &= \sum_{n=1}^{\infty} \int_{\mathbb{D}} \left|\psi(z) n\sqrt{n+1}\, \varphi(z)^{n-1}\right|^2 \mathrm{d}A(z) \\
&= \int_{\mathbb{D}} |\psi(z)|^2 \sum_{n=0}^{\infty} (n+1)^2(n+2)\, |\varphi(z)|^{2n}\, \mathrm{d}A(z) \\
&= \int_{\mathbb{D}} \frac{|\psi(z)|^2 \left(4|\varphi(z)|^2+2\right)}{(1-|\varphi(z)|^2)^4} \mathrm{d}A(z) \\
&\le 6 \int_{\mathbb{D}} \frac{|\psi(z)|^2}{(1-|\varphi(z)|^2)^4} \mathrm{d}A(z) < \infty.
\end{aligned}$$

On the other hand, if $D_{\psi,\varphi}$ is Hilbert–Schmidt on L_a^2, then we have

$$\begin{aligned}\infty > \sum_{n=0}^{\infty}\left\|D_{\psi,\varphi}(\sqrt{n+1}\,z^n)\right\|_{L_a^2}^2 &= \sum_{n=1}^{\infty}\int_{\mathbb{D}}\left|\psi(z)n\sqrt{n+1}\,\varphi(z)^{n-1}\right|^2 \mathrm{d}A(z)\\ &= \int_{\mathbb{D}}\frac{|\psi(z)|^2\left(4|\varphi(z)|^2+2\right)}{(1-|\varphi(z)|^2)^4}\mathrm{d}A(z)\\ &\geq 2\int_{\mathbb{D}}\frac{|\psi(z)|^2}{(1-|\varphi(z)|^2)^4}\mathrm{d}A(z).\end{aligned}$$

□

Corollary 3.11 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D'_{\psi,\varphi}$ is a Hilbert–Schmidt operator on L_a^2 if and only if*

$$\int_{\mathbb{D}}\frac{|\psi(z)\varphi'(z)|^2}{(1-|\varphi(z)|^2)^4}\mathrm{d}A(z) < \infty.$$

Proof It is enough to notice that

$$D'_{\psi,\varphi}(\sqrt{n+1}\,z^n) = \psi(z)n\sqrt{n+1}\,\varphi(z)^{n-1}\varphi'(z).$$

□

Theorem 3.12 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D^2_{\psi,\varphi}$ is a Hilbert–Schmidt operator on L_a^2 if and only if*

$$\int_{\mathbb{D}}\frac{|\psi(z)|^2}{(1-|\varphi(z)|^2)^6}\mathrm{d}A(z) < \infty. \tag{5}$$

Proof Let ψ and φ satisfy the condition (5). Then, using

$$\sum_{n=0}^{\infty}(n+1)^2(n+2)^2(n+3)x^n = \frac{36x^2+72x+12}{(1-x)^6}, \quad |x|<1, \tag{6}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left\|D^2_{\psi,\varphi}(\sqrt{n+1}\,z^n)\right\|^2 &= \sum_{n=2}^{\infty}\int_{\mathbb{D}}\left|\psi(z)n(n-1)\sqrt{n+1}\,\varphi(z)^{n-2}\right|^2\\ &= \int_{\mathbb{D}}|\psi(z)|^2\sum_{n=0}^{\infty}(n+1)^2(n+2)^2(n+3)|\varphi(z)|^{2n}\end{aligned}$$

$$= \int_{\mathbb{D}} |\psi(z)|^2 \frac{36|\varphi(z)|^4 + 72|\varphi(z)|^2 + 12}{(1-|\varphi(z)|^2)^6}.$$

This implies that

$$\int_{\mathbb{D}} \frac{12|\psi(z)|^2}{(1-|\varphi(z)|^2)^6} \mathrm{d}A(z) \le \sum_{n=0}^{\infty} \left\| D^2_{\psi,\varphi}(\sqrt{n+1}\, z^n) \right\|^2 \le \int_{\mathbb{D}} \frac{120|\psi(z)|^2}{(1-|\varphi(z)|^2)^6} \mathrm{d}A(z),$$

from which the result follows. □

Corollary 3.13 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D^{(2)}_{\psi,\varphi}$ is a Hilbert–Schmidt operator on L^2_a if and only if*

$$\int_{\mathbb{D}} \frac{|\psi(z)\varphi'(z)|^2}{(1-|\varphi(z)|^2)^6} \mathrm{d}A(z) < \infty.$$

Theorem 3.14 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D^3_{\psi,\varphi}$ is a Hilbert–Schmidt operator on L^2_a if and only if*

$$\int_{\mathbb{D}} \frac{|\psi(z)|^2}{(1-|\varphi(z)|^2)^8} \mathrm{d}A(z) < \infty. \tag{7}$$

Proof We write

$$\sum_{n=0}^{\infty} \left\| D^3_{\psi,\varphi}(\sqrt{n+1}\, z^n) \right\|^2 = \sum_{n=3}^{\infty} \int_{\mathbb{D}} \left| \psi(z) n(n-1)(n-2)\sqrt{n+1}\, \varphi(z)^{n-3} \right|^2$$
$$= \int_{\mathbb{D}} |\psi(z)|^2 \sum_{n=0}^{\infty} (n+1)^2(n+2)^2(n+3)^2(n+4)|\varphi(z)|^{2n}.$$

Multiplying (4) through x^4, differentiating with respect to x, and dividing both sides by x^3, we obtain

$$\sum_{n=0}^{\infty} (n+1)^2(n+2)^2(n+3)^2(n+4)x^n = \frac{36(110x^3 + 72x^2 + 54x + 12)}{(1-x)^8}, \quad |x| < 1.$$

This implies that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left\|D^{3}_{\psi,\varphi}(\sqrt{n+1}\,z^{n})\right\|^{2}\\
&=\int_{\mathbb{D}}|\psi(z)|^{2}\sum_{n=0}^{\infty}\frac{36(110|\varphi(z)|^{6}+72|\varphi(z)|^{4}+54|\varphi(z)|^{2}+12)}{(1-|\varphi(z)|^{2})^{8}}.
\end{aligned}
$$

Since $0\le|\varphi(z)|\le 1$, it follows that the operator $D^{3}_{\psi,\varphi}$ is Hilbert–Schmidt if and only if (7) holds true. □

Corollary 3.15 *Let φ be an analytic self-map of the unit disk, and ψ be an analytic function on the unit disk. Then $D^{(3)}_{\psi,\varphi}$ is a Hilbert–Schmidt operator on L^2_a if and only if*

$$
\int_{\mathbb{D}}\frac{\left|\psi(z)\varphi'(z)\right|^{2}}{(1-|\varphi(z)|^{2})^{8}}\mathrm{d}A(z)<\infty.
$$

The above pattern can be used repeatedly to establish the following results.

Theorem 3.16 *Let φ be an analytic self-map of the unit disk, ψ be an analytic function on the unit disk, and k be a non-negative integer. Then $D^{k}_{\psi,\varphi}$ is a Hilbert–Schmidt operator on L^2_a if and only if*

$$
\int_{\mathbb{D}}\frac{|\psi(z)|^{2}}{(1-|\varphi(z)|^{2})^{2k+2}}\mathrm{d}A(z)<\infty.
$$

Corollary 3.17 *Let φ be an analytic self-map of the unit disk, ψ be an analytic function on the unit disk, and k be a non-negative integer. Then $D^{(k)}_{\psi,\varphi}$ is a Hilbert–Schmidt operator on L^2_a if and only if*

$$
\int_{\mathbb{D}}\frac{\left|\psi(z)\varphi'(z)\right|^{2}}{(1-|\varphi(z)|^{2})^{2k+2}}\mathrm{d}A(z)<\infty.
$$

Remark 3.18 If $k=0$ and ψ is the constant function 1, then $D^{k}_{\psi,\varphi}=C_{\varphi}$, so that C_{φ} is a Hilbert–Schmidt operator on L^2_a if and only if

$$
\int_{\mathbb{D}}\frac{1}{(1-|\varphi(z)|^{2})^{2}}\mathrm{d}A(z)<\infty.
$$

This result is of course known (see [12, page 227, Ex. 5]).

4 Convergence in operator topologies

We begin by computing the adjoint of the composition-differentiation operator $D_{\psi,\varphi}$. The following lemma is known; but for the convenience of reader, we include a proof here.

Lemma 4.1 [5]. *Let $D_{\psi,\varphi}$ be a bounded operator on H^2. Then $D^*_{\psi,\varphi}(K_w) = \overline{\psi(w)}K^{(1)}_{\varphi(w)}$, where $K^{(1)}_w(z)$ is the reproducing kernel corresponding to differentiation functional $w \mapsto f'(w)$ on H^2.*

Proof Let $w \in \mathbb{D}$ be fixed. It is well-known that the functional $w \mapsto f'(w)$ is bounded on H^2 (see [1, Theorem 2.16]). It then follows from Riesz representation theorem that there is a function $K^{(1)}_w \in H^2$ such that

$$f'(w) = \langle f, K^{(1)}_w \rangle, \quad f \in H^2.$$

On the other hand, we have already seen that

$$f(w) = \langle f, K_w \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta}) \frac{1}{1 - we^{-i\theta}} \mathrm{d}\theta.$$

This implies that

$$f'(w) = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta}) \frac{e^{-i\theta}}{(1 - we^{-i\theta})^2} \mathrm{d}\theta,$$

from which, by the uniqueness of kernel function, we obtain

$$K^{(1)}_w(z) = \frac{z}{(1 - \overline{w}z)^2}, \quad (z, w) \in \mathbb{D} \times \mathbb{D}. \tag{8}$$

We now assume that $f \in H^2$, and write

$$\begin{aligned} \langle f, D^*_{\psi,\varphi}(K_w) \rangle &= \langle D_{\psi,\varphi}(f), K_w \rangle \\ &= \langle \psi \cdot f' \circ \varphi, K_w \rangle \\ &= \psi(w) \langle f' \circ \varphi, K_w \rangle \\ &= \psi(w) \langle f, K^{(1)}_{\varphi(w)} \rangle \\ &= \langle f, \overline{\psi(w)} K^{(1)}_{\varphi(w)} \rangle, \end{aligned}$$

from which the result follows. □

Assume that a sequence of self-maps φ_n on the unit disk satisfies $\|\varphi_n\|_\infty < r < 1$, and that ψ_n is a sequence of bounded analytic functions on the unit disk satisfying

$$\lim_{n \to \infty} \frac{\|\psi_n\|_\infty}{(r - |\varphi_n(0)|)^{1/2}} = 0.$$

It then follows from $\|D_{\psi_n,\varphi_n}\| \leq \|\psi_n\|_\infty \|D_{\varphi_n}\|$ and (1) that

$$\|D_{\psi_n,\varphi_n}\| \to 0, \quad n \to \infty,$$

while φ_n may not be convergent. To avoid this, we often consider sequences of operators D_{ψ_n,φ_n} with nonzero limits.

Lemma 4.2 *Let φ_n be a sequence of analytic self-mappings of the unit disk such that for each n we have $\|\varphi_n\|_\infty < 1$, and let ψ_n be a sequence in H^2 such that D_{ψ_n,φ_n} is bounded. Assume that D_{ψ_n,φ_n} converges in the weak operator topology to a nonzero operator T. Then we have*
(a) *ψ_n converges weakly to a nonzero function ψ,*
(b) *φ_n converges weakly to an analytic self-mapping of $\mathbb{D}$,*
(c) *T is a weighted composition-differentiation operator on H^2.*

Proof For part (a), consider $g(z) = z$. By assumption, $D_{\psi_n,\varphi_n}(g) \to T(g)$ weakly (in H^2). This means that $\psi_n g'(\varphi_n) \to T(g)$ weakly; or $\psi_n \to T(g)$ weakly. To prove that $T(g)$ is a nonzero function, assume on the contrary that $T(g) = 0$. Let f be a polynomial, then $D_{\psi_n,\varphi_n}(f) \to T(f)$ weakly, which implies that for every $w \in \mathbb{D}$,

$$D_{\psi_n,\varphi_n}(f)(w) \to T(f)(w),$$

or equivalently,

$$\lim_{n\to\infty} \psi_n(w) f'(\varphi_n(w)) = T(f)(w).$$

Since $\psi_n(w) \to 0$, and the sequence $f'(\varphi_n(w))$ is bounded, it follows that $T(f)(w) = 0$. Since the polynomials are dense in H^2, we conclude that for every $h \in H^2$, $T(h) = 0$ which contradicts our assumption on T. Thus (a) is proved.
As for part (b), we note that for each nonzero z and each w in the unit disk we have

$$\langle K_z, T^*(K_w)\rangle = \lim_{n\to\infty} \langle K_z, D^*_{\psi_n,\varphi_n}(K_w)\rangle.$$

We now use Lemma 4.1 together with (1) to obtain

$$T^*(K_w)(z) = \lim_{n\to\infty} \overline{\psi_n(w)} K^{(1)}_{\varphi_n(w)}(z) = \lim_{n\to\infty} \overline{\psi(w)} \frac{z}{(1-\overline{\varphi_n(w)}z)^2}. \tag{9}$$

Let $E = \{\zeta \in \mathbb{D} : \psi(\zeta) \neq 0\}$. Since ψ is a nonzero analytic function, it follows that E is topologically "big" in $\mathbb{D}$; meaning that it has accumulation points in the unit disk. Since the sequence

$$\overline{\psi(w)} \frac{z}{(1-\overline{\varphi_n(w)}z)^2}$$

is convergent, it follows that for each $w \in E$, the sequence $\varphi_n(w)$ converges, say to φ. On the other hand, $\|\varphi_n\|_\infty < 1$, so that φ_n forms a normal family. This implies that a subsequence of φ_n converges to an analytic function h on every compact subset of the unit disk. Thus h and φ must agree on the whole disk, that is, $\varphi_n \to \varphi$ uniformly on compact subsets of $\mathbb{D}$. It is clear that $|\varphi(w)| \leq 1$ for each $w \in \mathbb{D}$. Note also that uniform boundedness of φ_n implies that this sequence is norm bounded in H^2.

Therefore, $\varphi_n \to \varphi$ weakly in H^2. The last thing to be proved is that φ is a self mapping of the unit disk. To this end, let $w \in E$. Since we have from (9)

$$T^*(K_w)(z) = \overline{\psi(w)} \frac{z}{(1 - \overline{\varphi(w)}z)^2}, \tag{10}$$

and $T^*(K_w) \in H^2$, we get $|\varphi(w)| < 1$. In case that $w \notin E$, the function ψ does not vanish in a punctured closed disk around w, so that on the boundary of this small disk we have again $|\varphi| < 1$, and by the maximum principle $|\varphi(w)| < 1$.
Finally, for part (c), it follows from (10) that

$$\langle T^*(K_w), K_z\rangle = \overline{\psi(w)}\langle K^{(1)}_{\varphi(w)}, K_z\rangle,$$

and hence

$$T(K_z)(w) = \langle T(K_z), K_w\rangle = \langle \psi(w)K_z, K^{(1)}_{\varphi(w)}\rangle = \psi(w)K'_z(\varphi(w)).$$

Since the span of the reproducing kernel functions is dense in H^2, we have $T(f)(w) = \psi(w)f'(\varphi(w))$, or $T(f) = D_{\psi,\varphi}(f)$, which is the desired result. □

The following theorem describes the conditions under which the sequence of bounded operators D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in the weak operator topology.

Theorem 4.3 *Let $\{\varphi_n\}_{n\geq 1}$ and φ be analytic self-maps of the unit disk such that $\|\varphi_n\|_\infty < 1$, and let $\{\psi_n\}_{n\geq 1}$ and ψ be elements in H^2. Assume that each D_{ψ_n,φ_n} is bounded, and that $D_{\psi,\varphi}$ is a bounded nonzero operator on H^2. Then D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in weak operator topology if and only if*
(a) *ψ_n converges weakly to ψ in H^2,*
(b) *φ_n converges weakly to φ in H^2,*
(c) $\sup_n \|D_{\psi_n,\varphi_n}\| < \infty$.

Proof Assume that D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in weak operator topology. For (a), put $f(z) = z$. It follows that $D_{\psi_n,\varphi_n}(f) \to D_{\psi,\varphi}(f)$ weakly; or $\psi_n \to \psi$ weakly.
For (b), let $E \subset \mathbb{D}$ be the set on which ψ does not vanish. Consider $g(z) = z^2/2$. Since $D_{\psi_n,\varphi_n}(g) \to D_{\psi,\varphi}(g)$ weakly, it follows that $\psi_n\varphi_n \to \psi\varphi$ weakly. In particular, this convergence is pointwise. Therefore, on E where ψ is nonzero, $\varphi_n \to \varphi$ pointwise. On the other hand, by part (b) of the preceding lemma, $\varphi_n \to h$ weakly, where h is an analytic self-map of the unit disk. Therefore, $\varphi_n(z) \to h(z)$ for each $z \in \mathbb{D}$. This implies that $h = \varphi$ on E, and hence on the whole disk (E is a big set!).
The last part is clear from the Banach-Steinhaus theorem; for each $f \in H^2$, $D_{\psi_n,\varphi_n}(f)$ is weakly convergent, hence pointwise bounded (the bound depends on f). Banach-Steinhaus theorem now implies that D_{ψ_n,φ_n} is uniformly bounded.
To prove the converse statement, assume that parts (a), (b), and (c) hold true. Clearly, $\varphi_n \to \varphi$ pointwise, and $\psi_n \to \psi$ pointwise. For $z, w \in \mathbb{D}$, we have

$$\overline{\psi(w)} \frac{z}{(1 - \overline{\varphi(w)}z)^2} = \lim_{n\to\infty} \overline{\psi_n(w)} \frac{z}{(1 - \overline{\varphi_n(w)}z)^2},$$

and hence

$$\langle D_{\psi,\varphi}(K_z), K_w\rangle = \lim_{n\to\infty}\langle D_{\psi_n,\varphi_n}(K_z), K_w\rangle.$$

We recall that a sequence $T_n \in \mathcal{B}(\mathcal{H})$ converges to T in weak operator topology if $\sup_n \|T_n\| < \infty$ and for every x, y in a dense subset of $\mathcal{H}$ we have $\langle T_n x, y\rangle \to \langle Tx, y\rangle$ (see [2, Chap. IX, Proposition 1.3(b)]). Invoking this statement, and noting that the reproducing kernel functions $\{K_z : z \in \mathbb{D}\}$ is dense in H^2, and the sequence of operators D_{ψ_n,φ_n} is bounded, by part (c), we conclude that $D_{\psi_n,\varphi_n} \to D_{\psi,\varphi}$ in weak operator topology. □

Remark 4.4 If we consider the sequence D_{φ_n} in which $\sup_n \|\varphi_n\|_\infty \le r < 1$, and if we assume that $\varphi_n \to \varphi$ weakly, then from the estimate (1)

$$\|D_{\varphi_n}\| \le \left(\frac{r + |\varphi_n(0)|}{r - |\varphi_n(0)|}\right)^{1/2} \lfloor\frac{1}{1-r}\rfloor r^{\lfloor\frac{1}{1-r}\rfloor - 1},$$

we conclude that $\sup_n \|D_{\varphi_n}\|$ is bounded, so that the condition (c) is redundant.

In the following we turn to the convergence of D_{ψ_n,φ_n} in strong operator topology. Note that the weak convergence of $\varphi_n \to \varphi$ and the weak convergence of $\psi_n \to \psi$ are not sufficient for the convergence of $D_{\psi_n,\varphi_n} \to D_{\psi,\varphi}$ in the strong operator topology. For example, let $\psi_n = \varphi_n = z^n/2$. It is clear that both of this sequences converge to the zero function weakly (elements of an orthonormal basis in a Hilbert space converge to zero weakly). Therefore, by Theorem 4.3, $D_{\psi,\varphi} \to 0$ (WOT). Now let $f(z) = z$, then for each $m \ne n$, we have

$$\|D_{\psi_n,\varphi_n}(f) - D_{\psi_m,\varphi_m}(f)\| = \sqrt{2}/2,$$

which means that D_{ψ_n,φ_n} is not convergent in strong operator topology.

Lemma 4.5 *Let $\{\varphi_n\}_{n\ge1}$ and φ be analytic self-maps of the unit disk such that $\|\varphi_n\|_\infty < 1$, and let $\{\psi_n\}_{n\ge1}$ and ψ be elements in H^2 where ψ is nonzero. Assume that each D_{ψ_n,φ_n} and $D_{\psi,\varphi}$ are bounded operators on H^2. If D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in strong operator topology, then we have*
(a) *ψ_n converges to ψ in H^2,*
(b) *φ_n converges to φ in H^2.*

Proof For part (a), set $f(z) = z$, then by assumption

$$\|D_{\psi_n,\varphi_n}(f) - D_{\psi,\varphi}(f)\| \to 0,$$

which is the same as saying $\psi_n \to \psi$ in H^2.

For (b), we have to prove that $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$. Let $g(z) = z^2/2$, it then follows from the assumption that

$$\|\psi_n\varphi_n - \psi\varphi\| \to 0.$$

From part (a) and the fact that $\|\varphi_n\|_\infty < 1$ we conclude that

$$\|(\psi_n - \psi)\varphi_n\| \to 0.$$

Since

$$\|(\varphi_n - \varphi)\psi\| \leq \|(\psi - \psi_n)\varphi_n\| + \|(\psi_n\varphi_n - \psi\varphi)\|,$$

it follows that

$$\|(\varphi_n - \varphi)\psi\| \to 0, \quad n \to \infty. \tag{11}$$

Put $x_n := \|\varphi_n - \varphi\|$, and let x_{n_k} be an arbitrary subsequence of x_n. By (11) we have

$$\|(\varphi_{n_k} - \varphi)\psi\| \to 0,$$

from which it follows that a subsequence of $(\varphi_{n_k} - \varphi)\psi$ converges to zero at almost every point of the unit disk:

$$(\varphi_{n_{k_j}}(z) - \varphi(z))\psi(z) \to 0, \quad \text{a.e.}$$

Since the analytic function ψ is not identically zero, it can not vanish on a set with accumulation point in $\mathbb{D}$, so that it does not vanish almost everywhere in $\mathbb{D}$, implying that

$$\varphi_{n_{k_j}}(z) - \varphi(z) \to 0, \quad \text{a.e.}$$

Since φ_n, and its limit function, are bounded, the Bounded Convergence Theorem applies;

$$x_{n_{k_j}} = \|\varphi_{n_{k_j}} - \varphi\| \to 0, \quad j \to \infty.$$

We have proved that every subsequence of x_n (here x_{n_k}), in its turn has a subsequence that converges to zero, thus x_n converges to zero. □

Theorem 4.6 *Let $\{\varphi_n\}_{n\geq 1}$ and φ be analytic self-maps of the unit disk such that $\|\varphi_n\|_\infty < 1$, and let $\{\psi_n\}_{n\geq 1}$ and ψ be elements in H^2 where ψ is nonzero. Assume that each D_{ψ_n,φ_n} and $D_{\psi,\varphi}$ are bounded operators on H^2 where $D_{\psi,\varphi}$ is nonzero. Then D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in strong operator topology if and only if*
(a) *ψ_n converges to ψ in H^2,*
(b) *φ_n converges to φ in H^2,*
(c) $\sup_n \|D_{\psi_n,\varphi_n}\| < \infty$.

Proof Assume that D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in strong operator topology. Parts (a) and (b) follow from Lemma 4.5. Part (c) is a consequence of Theorem 4.3.

For the converse, assume that parts (a), (b) and (c) hold true. We first verify that for every polynomial f,

$$\|D_{\psi_n,\varphi_n}(f) - D_{\psi,\varphi}(f)\| \to 0, \quad n \to \infty. \tag{12}$$

Note that on the line segment joining $\varphi_n(z)$ to $\varphi(z)$ we have

$$f'(\varphi_n(z)) - f'(\varphi(z)) = \int f''(w)dw,$$

from which it follows that for each $z \in \mathbb{D}$,

$$|f'(\varphi_n(z)) - f'(\varphi(z))| \leq \|f''\|_\infty |\varphi_n(z) - \varphi(z)|,$$

or

$$\|f' \circ \varphi_n - f' \circ \varphi\| \leq \|f''\|_\infty \|\varphi_n - \varphi\|. \tag{13}$$

Again we set

$$x_n = \|D_{\psi_n,\varphi_n}(f) - D_{\psi,\varphi}(f)\|,$$

and assume that

$$x_{n_k} = \|D_{\psi_{n_k},\varphi_{n_k}}(f) - D_{\psi,\varphi}(f)\|$$

is an arbitrary subsequence of x_n. We prove that there is a subsequence $x_{n_{k_j}}$ that converges to 0. To this end, we see from part (b),

$$\|\varphi_{n_k} - \varphi\| \to 0,$$

so that (13) implies

$$\|f' \circ \varphi_{n_k} - f' \circ \varphi\| \to 0.$$

Thus we can find a subsequence, say $f' \circ \varphi_{n_{k_j}} - f' \circ \varphi$, that converges to zero at almost every point of the unit disk. Moreover, we have

$$\left|\psi(z)\big(f' \circ \varphi_{n_{k_j}}(z) - f' \circ \varphi(z)\big)\right| \leq 2|\psi(z)|\|f'\|_\infty.$$

The Dominated Convergence Theorem now implies that

$$\|\psi(f' \circ \varphi_{n_{k_j}} - f' \circ \varphi)\| \to 0, \quad j \to \infty. \tag{14}$$

On the other hand, by part (a),

$$\|(\psi_{n_{k_j}} - \psi) f' \circ \varphi_{n_{k_j}}\|^2 \le \|f'\|_\infty^2 \|\psi_{n_{k_j}} - \psi\|^2 \to 0, \quad j \to \infty. \tag{15}$$

Finally, we write

$$\begin{aligned}\|D_{\psi_n,\varphi_n}(f) - D_{\psi,\varphi}(f)\| &= \|\psi_n f' \circ \varphi_n - \psi f' \circ \varphi\| \\ &\le \|(\psi_n - \psi) f' \circ \varphi_n\| + \|\psi(f' \circ \varphi_n - f' \circ \varphi)\|,\end{aligned}$$

from which, using (14) and (15) we conclude that

$$\|D_{\psi_{n_{k_j}},\varphi_{n_{k_j}}}(f) - D_{\psi,\varphi}(f)\| \to 0, \quad j \to \infty.$$

We recall that by [2, Chap. IX, Proposition 1.3(d)]), a sequence $T_n \in \mathcal{B}(\mathcal{H})$ converges to T in strong operator topology if on a dense subset of $\mathcal{H}$ we have $\|T_n x - Tx\| \to 0$. This fact together with (12) proves that $D_{\psi_n,\varphi_n} \to D_{\psi,\varphi}$ in strong operator topology. □

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