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ORIGINAL PAPER

Two Improved Nonlinear Conjugate Gradient Methods with the Strong Wolfe Line Search

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Abstract

Two improved nonlinear conjugate gradient methods are proposed by using the second inequality of the strong Wolfe line search. Under usual assumptions, we proved that the improved methods possess the sufficient descent property and global convergence. By testing the unconstrained optimization problems which taken from the CUTE library and other usual test collections, some large-scale numerical experiments for the presented methods and their comparisons are executed. The detailed results are listed in tables and their corresponding performance profiles are reported in figures, which show that our improved methods are superior to their comparisons.

Keywords Unconstrained optimization · Conjugate gradient method · Strong Wolfe line search · Sufficient descent property · Global convergence

Mathematics Subject Classification 90C26 · 90C30 · 65K05

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1 Introduction

The conjugate gradient method (CGM) is an ideal method for solving the large-scale unconstrained optimization problems due to its simple structure and low storage capacity, and therefore, it is usually extended to obtain the solution of optimization arising in many fields of science and engineering. For example, sparse recovery [6], heat conduction problem [21], inverse blackbody radiation problem [15], power system [16,22,23] and so on.

In this paper, we consider the following unconstrained optimization problem:

$$\min\{f(x)|\ x \in R^n\},$$

where $f : R^n \rightarrow R$ is a continuously differentiable function. Denote $g_k = g(x_k) = \nabla f(x_k)$ for $x_k \in R^n$. For CGM, the classic formula for generating the new iterate x_{k+1} is

$$x_{k+1} = x_k + \alpha_k d_k, \tag{1.1}$$

$$d_k = \begin{cases} -g_k, & k = 0, \\ -g_k + \beta_k d_{k-1}, & k \geq 1, \end{cases} \tag{1.2}$$

where d_k is the search direction, and α_k is a step-length computed by a suitable inexact line search. Usually, the popular inexact line searches include the weak Wolfe line search

$$\begin{cases} f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^{\mathrm{T}} d_k, \\ g(x_k + \alpha_k d_k)^{\mathrm{T}} d_k \geq \sigma g_k^{\mathrm{T}} d_k, \end{cases} \tag{1.3}$$

and the strong Wolfe line search

$$\begin{cases} f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^{\mathrm{T}} d_k, \\ |g(x_k + \alpha_k d_k)^{\mathrm{T}} d_k| \leq \sigma |g_k^{\mathrm{T}} d_k|, \end{cases} \tag{1.4}$$

where the parameters δ and σ in (1.3) and (1.4) are required to satisfy $0 < \delta < \sigma < 1$. The scalar β_k in (1.2) is the so-called conjugate parameter, and different choices for β_k lead to different CGMs. The classical CGMs include the Hestenes–Stiefel (HS,1952) CGM [11], the Fletcher–Reeves (FR, 1964) CGM [7], the Polak–Ribière–Polyak (PRP,1969) CGM [18,19] and the Dai–Yuan (DY,1999) CGM [3], and their parameters β_k are specified as follows:

$$\beta_k^{\mathrm{HS}} = \frac{g_k^{\mathrm{T}} y_{k-1}}{d_{k-1}^{\mathrm{T}} y_{k-1}}, \quad \beta_k^{\mathrm{FR}} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2},$$

$$\beta_k^{\mathrm{PRP}} = \frac{g_k^{\mathrm{T}} y_{k-1}}{\|g_{k-1}\|^2}, \quad \beta_k^{\mathrm{DY}} = \frac{\|g_k\|^2}{d_{k-1}^{\mathrm{T}} y_{k-1}}.$$

Here $y_{k-1} = g_k - g_{k-1}$ and $\| \cdot \|$ stands for the Euclidean norm.

In the past few decades, based on the methods above, many CGMs with excellent theoretical properties and numerical results are proposed, see [9,10,12–14,20,24–26] for details.

Based on the PRP CGM, Huang et al. [12] proposed a new CGM, where β_k is written as

$$\beta_k^{\text{MPRP}} = \frac{\|g_k\|^2 - \frac{(g_k^{\mathrm{T}} g_{k-1})^2}{\|g_{k-1}\|^2}}{\|g_{k-1}\|^2}. \tag{1.5}$$

Zhou and Lu [25] modified HS formula as follows:

$$\beta_k^{\text{MHS}} = \frac{\|g_k\|^2 - \frac{(g_k^{\mathrm{T}} g_{k-1})^2}{\|g_{k-1}\|^2}}{d_{k-1}^{\mathrm{T}}(g_k - g_{k-1})}. \tag{1.6}$$

Recently, Jiang and Jian [13] improved the FR and DY CGMs, and proposed three CGMs with good convergence and high efficiency, in which the conjugate parameters β_k^{IFR}, β_k^{IDY} and β_k^{IFD} are computed by

$$\begin{aligned}
\beta_k^{\text{IFR}} &= \tau_k \cdot \beta_k^{\text{FR}} = \frac{|g_k^{\mathrm{T}} d_{k-1}|}{-g_{k-1}^{\mathrm{T}} d_{k-1}} \cdot \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, \\
\beta_k^{\text{IDY}} &= \tau_k \cdot \beta_k^{\text{DY}} = \frac{|g_k^{\mathrm{T}} d_{k-1}|}{-g_{k-1}^{\mathrm{T}} d_{k-1}} \cdot \frac{\|g_k\|^2}{d_{k-1}^{\mathrm{T}} y_{k-1}}, \\
\beta_k^{\text{IFD}} &= \frac{\|g_k\|^2}{\|g_{k-1}\|^2} - \frac{|g_k^{\mathrm{T}} d_{k-1}|}{-g_{k-1}^{\mathrm{T}} d_{k-1}} \cdot \frac{\|g_k\|^2}{d_{k-1}^{\mathrm{T}}(g_k - g_{k-1})},
\end{aligned}$$

where $\tau_k = \frac{|g_k^{\mathrm{T}} d_{k-1}|}{-g_{k-1}^{\mathrm{T}} d_{k-1}}$ is the disturbance parameter.

Based on the the numerical results in Ref. [13], we further test the $\tau_k \cdot \text{FR}$, $(1 - \tau_k) \cdot \text{FR}$, $\tau_k \cdot \text{PRP}$, $(1 - \tau_k) \cdot \text{PRP}$, $\tau_k \cdot \text{HS}$, $(1 - \tau_k) \cdot \text{HS}$, $\tau_k \cdot \text{DY}$, and $(1 - \tau_k) \cdot \text{DY}$ CGMs, where the test problems randomly are taken from Refs. [1,2,17]. And all the steplengthes are yielded by the strong Wolfe line search with $\sigma = 0.1$, $\delta = 0.01$. The numerical results are demonstrated by the following performance charts, and see Figs. 1 and 2:

We notice that there is an interesting phenomenon, that is the methods perform well when the disturbance parameter is yielded by τ_k if the β_k with the denominator $d_{k-1}^{\mathrm{T}}(g_k - g_{k-1})$ and the disturbance parameter is obtained from $1 - \tau_k$ while the denominator of β_k is $\|g_{k-1}\|^2$. Along these ideas and inspired by [12,13,25], we

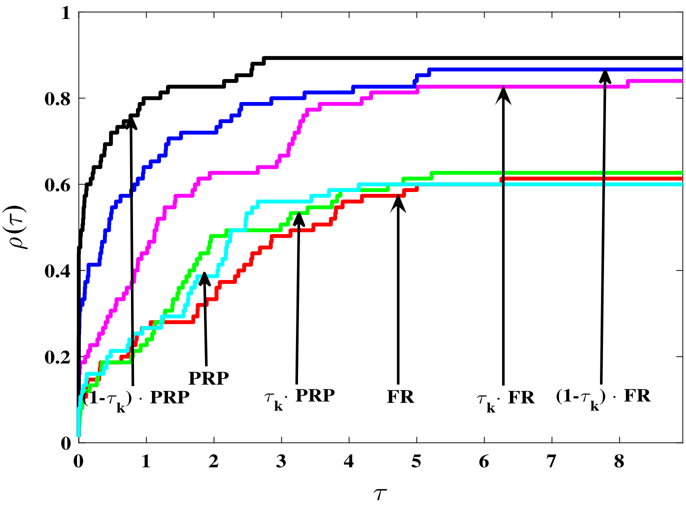

Fig. 1 Performance profiles on Tcpu

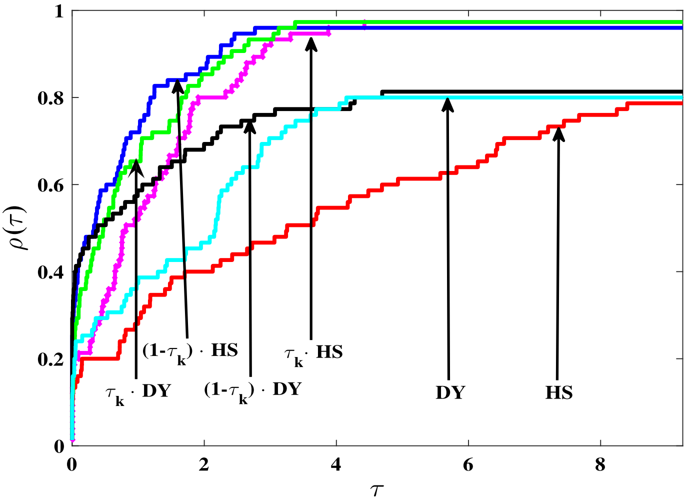

Fig. 2 Performance profiles on Tcpu

propose two improved conjugate parameters as follows:

$$\beta_k^{\mathrm{IMPRP}} = (1-\tau_k)\cdot\frac{\|g_k\|^2 - \frac{(g_k^{\mathrm{T}} g_{k-1})^2}{\|g_{k-1}\|^2}}{\|g_{k-1}\|^2}, \tag{1.7}$$

$$\beta_k^{\mathrm{IMHS}} = \tau_k\cdot\frac{\|g_k\|^2 - \frac{(g_k^{\mathrm{T}} g_{k-1})^2}{\|g_{k-1}\|^2}}{d_{k-1}^{\mathrm{T}}(g_k - g_{k-1})}. \tag{1.8}$$

In particular, by the second inequality of the strong Wolfe line search (1.4), we know that $0 \le \tau_k = \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \le \sigma$ if $g_{k-1}^T d_{k-1} < 0$.

This paper is organized as follows. In Sect. 2, the algorithms and relevant properties are demonstrated. The preliminary numerical results are reported in Sect. 3. The conclusion is presented in the final section.

2 Algorithms and Relevant Properties

In this section, based on the conjugate parameters (1.7) and (1.8), we first introduce corresponding algorithms with the strong Wolfe line search (1.4) as follows:

Step 0. Choose constants $\epsilon > 0$, $0 < \delta < \sigma < 1$, and initial point $x_0 \in R^n$. Let $d_0 = -g_0$, $k := 0$.

Step 1. If $\|g_k\| \le \epsilon$, then stop. Otherwise, go to Step 2.

Step 2. Determine a step-length α_k by the strong Wolfe line search (1.4).

Step 3. Let $x_{k+1} := x_k + \alpha_k d_k$, compute $g_{k+1} = g(x_{k+1})$ and parameter β_{k+1} by (1.7) or (1.8), i.e., $\beta_{k+1} = \beta_{k+1}^{\text{IMPRP}}$ or $\beta_{k+1}^{\text{IMHS}}$.

Step 4. Compute $d_{k+1} = -g_{k+1} + \beta_{k+1} d_k$. Let $k := k+1$, and go back to Step 1.

For the sake of convenience, we call these iteration methods which are decided by the algorithms with $\beta_{k+1}^{\text{IMPRP}}$ and $\beta_{k+1}^{\text{IMHS}}$ the IMPRP CGM and the IMHS CGM, respectively.

The following lemmas state that the search directions yielded by the above two CGMs are all sufficient descent, and possess some properties.

Lemma 2.1 *Let d_k be yielded by the IMPRP CGM. If $0 < \sigma < \frac{1}{2}$, then the relation*

$$-\frac{1}{1-\sigma} \le \frac{g_k^T d_k}{\|g_k\|^2} \le -\frac{1-2\sigma}{1-\sigma} \quad (\forall\, k \ge 0) \tag{2.1}$$

holds. In other words, the direction d_k satisfies the sufficient descent condition, that is

$$g_k^T d_k \le -c_1 \|g_k\|^2, \quad c_1 := \frac{1-2\sigma}{1-\sigma} > 0 \quad (\forall\, k \ge 0). \tag{2.2}$$

Moreover, the relation $0 \le \beta_k^{\text{IMPRP}} \le \beta_k^{\text{FR}}$ holds.

Proof Observe that $g_0^T d_0 = -\|g_0\|^2$ and the relation (2.1) holds for $k = 0$. Now, assume that (2.1) holds for $k-1$ ($k > 1$), and we will prove that (2.1) holds for k. Firstly, it follows from $g_{k-1}^T d_{k-1} < 0$ and the second inequality of (1.4) that

$$|g_k^T d_{k-1}| \le \sigma |g_{k-1}^T d_{k-1}| = -\sigma g_{k-1}^T d_{k-1},$$

and we further have $0 \le \tau_k \le \sigma$. This together with (1.7) implies that

$$0 \le \beta_k^{\text{IMPRP}} = (1-\tau_k) \cdot \frac{\|g_k\|^2 (1-\cos^2 \xi_k)}{\|g_{k-1}\|^2} \le \frac{\|g_k\|^2}{\|g_{k-1}\|^2} = \beta_k^{\text{FR}}, \tag{2.3}$$

where $\cos \xi_k = \frac{g_k^T g_{k-1}}{\|g_k\|\|g_{k-1}\|}$.

To proceed, based on the inequality (2.3), and with the help of [8, Lemma 3.1], the relations (2.1) and (2.2) hold, and the proof is completed. □

Lemma 2.2 *Let d_k be yielded by the IMHS CGM. If $0 < \sigma < 1$, then the direction d_k satisfies the sufficient descent condition*

$$g_k^{\mathrm{T}} d_k \leq -c_2\|g_k\|^2,\ c_2 := 1-\sigma > 0\ (\forall\, k \geq 0), \tag{2.4}$$

and the relation

$$0 \leq \beta_k^{\mathrm{IMHS}} \leq \frac{g_k^{\mathrm{T}} d_k}{g_{k-1}^{\mathrm{T}} d_{k-1}}\ (\forall\, k \geq 1) \tag{2.5}$$

holds.

Proof Using mathematical induction, firstly, it is easy to see that (2.4) holds for $k = 0$ since $d_0 = -g_0$. And we assume that (2.4) holds for $k-1$ $(k > 1)$ which implies that $g_{k-1}^{\mathrm{T}} d_{k-1} < 0$. This together with the second inequality of (1.4) indicates that $0 \leq \tau_k \leq \sigma$ and

$$d_{k-1}^{\mathrm{T}}(g_k - g_{k-1}) \geq -(1-\sigma) g_{k-1}^{\mathrm{T}} d_{k-1} > 0, \tag{2.6}$$

which further gives

$$\beta_k^{\mathrm{IMHS}} = \tau_k \cdot \frac{\|g_k\|^2(1-\cos^2\phi_k)}{d_{k-1}^{\mathrm{T}}(g_k - g_{k-1})} \geq 0, \tag{2.7}$$

where $\cos\phi_k = \frac{g_k^{\mathrm{T}} g_{k-1}}{\|g_k\|\|g_{k-1}\|}$. Multiplying the both sides (1.2) by g_k^{T}, and then from (2.7), we have

$$\begin{aligned} g_k^T d_k =& - \|g_k\|^2 + \tau_k \cdot \frac{\|g_k\|^2(1-\cos^2\phi_k)}{d_{k-1}^{\mathrm{T}}(g_k - g_{k-1})} \cdot g_k^T d_{k-1} \\ =& - \|g_k\|^2 + \tau_k \cdot \|g_k\|^2 \cdot (1-\cos^2\phi_k) + \tau_k \cdot \frac{\|g_k\|^2(1-\cos^2\phi_k)}{d_{k-1}^{\mathrm{T}}(g_k - g_{k-1})} \cdot g_{k-1}^{\mathrm{T}} d_{k-1} \\ \leq& - \|g_k\|^2 + \tau_k \cdot \|g_k\|^2 + \beta_k^{\mathrm{IMHS}} \cdot g_{k-1}^{\mathrm{T}} d_{k-1} \\ \leq& - (1-\sigma)\|g_k\|^2 + \beta_k^{\mathrm{IMHS}} \cdot g_{k-1}^{\mathrm{T}} d_{k-1}. \end{aligned} \tag{2.8}$$

Finally, it follows from (2.7), (2.8) and $g_{k-1}^{\mathrm{T}} d_{k-1} < 0$ that relations (2.4), (2.5) hold, which ends the proof. □

By [8, Lemma 3.1] for the IMPRP CGM and [4, Theorem 2.3] for the IMHS CGM, we obtain the global convergence results for the IMPRP and IMHS CGMs obviously, respectively, and omit their for brevity.

3 Numerical Results

In this section, we report some preliminary numerical performance of our methods to illustrate their effectiveness and potentiality. The numerical experiments are shown in the following two subsections.

3.1 Experimental Set-up

To demonstrate more clearly the effectiveness of our methods, we have run two groups preliminary numerical experiments for the IMPRP CGM and the IMHS CGM, respectively.

- In Group A, we compare the IMPRP CGM with the MPRP CGM [12] and the KD CGM [14]. There are 102 problems are tested, in which problems 1–52 (from bard to vardim) come from CUTE library [2], and others come from [1,17].
- In Group B, the IMHS CGM is compared with the MHS CGM [25] and the NHS CGM [24]. Specifically, the test problems 1–62 (from bard to woods) are taken from the CUTE library [2], and the others (63–107) are taken from [1,17].

In the testing process, the dimensions of the test problems range from 2 to 1,000,000. The initial iterate points for all test problems are same as that given in [1,2,17]. All codes were written in Matlab R2014a, and run on a DELL laptop (Inter(R) Core(TM) i5-5200U CPU 2.20 GHz) with 4 GB RAM memory and Windows 10 operating system. For all methods, we adopt the strong Wolfe line search rule with $\sigma = 0.1,\ \delta = 0.01$. We stop the iteration when the gradient values $\|g_k\| \le 10^{-6}$. Also, when the number of iteration Itr > 2000, we stop the iteration to indicate that the test problem is not valid, and denote it as "NaN".

The comparison results are shown in Tables 1, 2, 3 and 4. Where "name" is the abbreviation of the test problem, and "n" is the dimension of the test problem. Moreover, denote the number of iteration, function evaluations, gradient evaluations, the computing time of CPU and gradient values the "Itr, NF, NG, Tcpu and $\|g_*\|$", respectively.

To show the test results clearly, we adopt the performance profiles introduced by Dolan and Moré [5] to illustrate and compare the performances on Tcpu and Itr, respectively. About the performance profile, see [5] for details. Generally speaking, the higher the curve $\rho(\tau)$, the better the numerical profile of the corresponding method.

3.2 Numerical Testing Reports

We observe from Tables 1, 2 and Figs. 3, 4 that the IMPRP CGM solves about 97% of test problems in Group A, while the MPRP CGM and the KD CGM solve about 85% and 89% of this group, respectively. Furthermore, the IMPRP CGM is fastest since it solves about 40% of the test problems with the least Itr and Tcpu, respectively.

Tables 3, 4 and Figs. 5, 6 show that the IMHS CGM exhibits the best performance since it can solve almost 42% of test problems with the smallest Itr and Tcpu in Group B. Meanwhile, the MHS CGM and the NHS CGM are superior for solving 25% and

Table 1 Numerical comparisons of Group A

Problems Name/n	IMPRP Itr/NF/NG/Tcpu/$\|g_*\|$	MPRP Itr/NF/NG/Tcpu/$\|g_*\|$	KD Itr/NF/NG/Tcpu/$\|g_*\|$
bard 3	689/20453/8275/2.617/5.74e−07	850/24940/10202/3.083/7.75e−07	1247/38342/15638/4.789/9.29e−07
beale 2	171/5006/2213/0.240/6.39e−07	448/13134/5877/0.556/6.40e−07	326/9668/4400/0.415/7.72e−07
box 3	745/21902/9160/1.257/9.37e−07	346/9993/4248/0.566/8.09e−07	330/10108/4264/0.587/5.98e−07
cosine 500	123/3211/1510/0.460/9.65e−07	NaN/NaN/NaN/NaN/3.84e+02	752/22826/9300/2.765/4.72e−07
cosine 600	56/1333/655/0.206/3.10e−08	NaN/NaN/NaN/NaN/1.32e+03	1119/34091/13777/4.588/7.34e−07
cosine 1000	1170/34669/15287/7.031/9.87e−07	NaN/NaN/NaN/NaN/7.66e+01	NaN/NaN/NaN/NaN/2.34e−01
dixmaana 9000	17/241/102/2.730/4.83e−07	20/329/147/3.710/1.22e−07	17/266/118/3.000/5.34e−07
dixmaanb 3000	12/150/50/0.588/1.85e−07	12/150/50/0.581/1.82e−07	12/150/49/0.571/3.26e−07
dixmaanc 3000	29/489/219/2.005/4.31e−07	31/746/324/3.000/9.00e−07	25/458/213/1.852/5.54e−07
dixmaand 9000	26/375/159/4.223/1.64e−07	NaN/NaN/NaN/NaN/2.73e+01	32/526/243/6.007/3.90e−07
dixmaanf 120	132/3562/1573/1.040/9.36e−07	139/3767/1735/1.072/7.05e−07	128/3439/1525/0.978/9.45e−07
dixmaang 300	226/6253/2761/3.195/8.88e−07	176/4792/2168/2.392/9.27e−07	151/4046/1865/2.056/6.21e−07
dixmaanh 120	111/2961/1348/0.876/5.12e−07	105/2601/1209/0.751/8.60e−07	125/3143/1482/1.017/3.87e−07
dixmaani 30	420/11994/5294/2.238/4.38e−07	467/13406/5882/2.383/2.73e−07	373/10139/4616/2.216/5.55e−07
dixmaanj 30	237/6796/2949/1.252/8.45e−07	179/4882/2243/0.856/6.95e−07	261/7204/3283/1.413/7.88e−07
dixmaank 60	861/25052/10961/5.925/2.29e−07	353/9829/4476/2.128/7.61e−07	1068/31593/14205/6.718/5.88e−07
dixmaanl 120	986/29071/12307/8.260/8.41e−07	763/21804/9442/6.246/9.13e−07	1142/33908/14499/10.449/9.63e−07
dixon3dq 10	151/4089/1844/0.116/6.50e−07	81/1943/908/0.052/8.67e−07	142/3734/1666/0.099/3.61e−07
dqdrtic 1000	343/10251/4460/0.907/7.49e−07	136/3534/1562/0.321/7.36e−07	420/12543/5495/1.133/6.19e−07
dqdrtic 2000	330/9896/4349/1.482/7.42e−07	213/6104/2726/0.975/8.09e−07	443/13221/5812/2.007/2.91e−07

Table 1 continued

Problems Name/n	IMPRP Itr/NF/NG/Tcpu/$\|g_*\|$	MPRP Itr/NF/NG/Tcpu/$\|g_*\|$	KD Itr/NF/NG/Tcpu/$\|g_*\|$
dqdrtic 3000	335/10057/4410/2.143/7.33e−07	137/3632/1629/0.771/8.01e−07	430/12829/5616/2.761/3.77e−07
dqrtic 300	30/356/150/0.089/2.38e−07	29/407/173/0.074/4.70e−07	34/562/251/0.097/3.74e−07
dqrtic 400	32/487/214/0.115/4.32e−07	59/1348/622/0.318/5.52e−07	28/355/154/0.084/1.70e−07
edensch 10000	62/1533/662/5.349/1.01e−07	55/1365/640/7.615/4.22e−07	NaN/NaN/NaN/NaN/3.67e−05
edensch 15000	55/1131/521/9.929/6.18e−07	NaN/NaN/NaN/NaN/1.63e−05	74/1776/856/12.007/9.74e−07
edensch 20000	58/1324/631/12.152/6.75e−07	105/2749/1297/26.867/2.21e−07	NaN/NaN/NaN/NaN/1.80e−05
eg2 10	429/12366/5486/0.484/1.57e−07	NaN/NaN/NaN/NaN/7.77e+00	371/10373/4779/0.372/3.30e−07
eg2 100	154/4167/1820/0.228/1.87e−07	NaN/NaN/NaN/NaN/1.25e−05	225/6302/2884/0.405/4.77e−07
engval1 2	NaN/NaN/NaN/NaN/7.68e−02	NaN/NaN/NaN/NaN/8.58e−02	64/1784/850/0.060/9.15e−08
extrosnb 10000	1/1/1/0.003/0.00e+00	1/1/1/0.000/0.00e+00	1/1/1/0.000/0.00e+00
fletchcr 5	98/2442/1084/0.070/9.56e−07	96/2424/1100/0.068/9.83e−07	102/2596/1177/0.075/2.07e−07
fletchcr 30	85/2127/929/0.071/1.67e−07	107/2839/1323/0.086/9.41e−07	115/3024/1364/0.097/8.69e−07
freuroth 2	1579/47967/19758/1.553/5.52e−07	NaN/NaN/NaN/NaN/2.60e−04	409/12173/5427/0.419/9.79e−07
genrose 10000	298/8691/3704/10.831/8.17e−08	556/15679/7068/19.077/7.49e−07	327/9408/4101/11.345/9.34e−07
genrose 15000	224/6274/2693/11.248/2.67e−07	709/20691/8998/38.972/4.84e−07	372/10662/4716/18.990/9.53e−07
genrose 50000	339/9887/4412/60.977/9.50e−07	1015/29922/13083/183.687/3.62e−07	396/11361/5009/70.586/6.18e−07
gulf 3	2/1/2/0.001/0.00e+00	2/1/2/0.000/0.00e+00	2/1/2/0.000/0.00e+00
helix 3	456/13224/5661/0.959/8.03e−07	706/19919/8828/1.400/8.38e−07	274/7886/3342/0.467/7.79e−07
himmelbg 20000	3/6/7/0.060/7.11e−28	3/6/7/0.037/7.11e−28	3/6/7/0.037/5.93e−28
himmelbg 50000	3/6/7/0.094/1.12e−27	3/6/7/0.094/1.12e−27	3/6/7/0.095/9.38e−28

Table 1 continued

Problems Name/n	IMPRP Itr/NF/NG/Tcpu/$\|g_*\|$	MPRP Itr/NF/NG/Tcpu/$\|g_*\|$	KD Itr/NF/NG/Tcpu/$\|g_*\|$
himmelbg 100000	3/6/7/0.189/1.59e−27	3/6/7/0.186/1.59e−27	3/6/7/0.193/1.33e−27
kowosb 4	1033/30178/12655/1.879/7.61e−07	573/15442/7182/1.029/7.25e−07	626/18251/7752/1.042/8.78e−07
liarwhd 1000	1169/36094/14692/5.887/5.29e−07	852/26252/10890/4.300/6.89e−07	NaN/NaN/NaN/NaN/1.33e−04
liarwhd 2000	940/28866/11757/7.915/6.72e−07	1724/52569/21114/14.290/2.97e−07	NaN/NaN/NaN/NaN/8.02e−02
liarwhd 5000	NaN/NaN/NaN/NaN/3.55e−03	NaN/NaN/NaN/NaN/2.46e+03	1655/50241/20188/31.964/8.85e−07
penalty1 1000	15/253/94/1.421/1.21e−07	15/253/94/1.422/1.21e−07	15/253/94/1.436/1.21e−07
penalty1 5000	12/162/51/13.702/1.48e−07	12/162/51/13.687/1.48e−07	12/162/52/13.693/1.48e−07
quartc 100	19/209/86/0.017/5.77e−07	41/802/366/0.059/2.08e−07	23/326/144/0.025/6.36e−07
quartc 300	30/356/150/0.063/2.38e−07	29/407/173/0.070/4.70e−07	34/562/251/0.097/3.74e−07
tridia 100	855/24680/10612/0.942/7.88e−07	878/25441/11160/0.962/7.41e−07	730/21133/9363/0.816/7.45e−07
sinquad 3	1217/37038/14429/1.148/6.78e−07	103/2846/1212/0.087/3.72e−08	272/7998/3397/0.249/8.95e−07

Table 2 Numerical comparisons of Group A

Problems Name/n	IMPRP Itr/NF/NG/Tcpu/$\|g_*\|$	MPRP Itr/NF/NG/Tcpu/$\|g_*\|$	KD Itr/NF/NG/Tcpu/$\|g_*\|$
vardim 8	12/168/52/0.027/3.98e−08	12/168/52/0.012/3.98e−08	12/168/52/0.013/3.98e−08
bdexp 500000	3/2/3/1.067/3.63e−109	3/2/3/1.057/3.63e−109	3/2/3/1.084/3.62e−109
bdexp 800000	3/2/3/1.710/4.56e−109	3/2/3/1.693/4.56e−109	3/2/3/1.729/4.55e−109
bdexp 1000000	3/2/3/2.135/5.09e−109	3/2/3/2.119/5.09e−109	3/2/3/2.180/5.08e−109
exdenschnf 100000	27/563/237/11.637/3.76e−07	111/3118/1485/65.956/3.16e−07	69/1886/914/40.422/6.00e−07
exdenschnf 500000	27/564/246/63.072/8.40e−07	111/3157/1516/345.565/7.20e−07	72/1957/914/217.782/6.99e−08
exdenschnf 1000000	28/576/253/124.902/1.19e−07	111/3187/1549/708.854/9.71e−07	72/1974/926/435.194/9.90e−08
mccormak 2	22/451/200/0.018/3.09e−07	29/645/287/0.021/3.63e−07	19/364/163/0.011/8.11e−07
exdenschnb 500000	23/432/179/19.766/6.66e−07	52/1217/571/56.682/5.18e−09	30/632/279/29.972/5.69e−07
exdenschnb 800000	23/432/185/32.008/8.41e−07	49/1153/565/86.562/9.84e−07	30/632/295/47.907/7.00e−07
exdenschnb 1000000	23/432/174/39.091/9.41e−07	46/1054/482/100.309/5.14e−07	30/632/287/60.851/7.84e−07
genquartic 6000	23/396/170/0.281/2.03e−07	75/1803/870/1.171/2.58e−07	28/510/225/0.335/2.80e−08
genquartic 10000	26/426/201/0.445/8.95e−08	46/998/476/1.041/4.64e−07	33/615/266/0.639/1.61e−07
genquartic 20000	26/425/172/0.869/2.63e−07	60/1331/654/2.746/8.00e−07	43/971/417/1.929/6.37e−07
biggsb1 5	43/965/436/0.050/9.82e−07	46/1004/488/0.033/4.32e−07	42/917/408/0.026/6.58e−07
biggsb1 10	80/2023/903/0.059/2.37e−07	107/2779/1250/0.076/7.68e−07	143/3948/1748/0.115/6.03e−07
sine 120	63/1725/811/0.100/1.46e−07	65/1746/794/0.096/1.10e−08	66/1778/837/0.103/3.85e−07
sine 2000	64/1755/846/0.941/6.91e−07	62/1714/822/0.906/7.13e−07	66/1816/881/0.934/4.45e−07
fletcbv3 10	47/1098/542/0.078/4.98e−07	70/1478/745/0.060/9.11e−07	64/1441/731/0.061/7.36e−07
nonscomp 10000	201/5548/2536/4.844/7.57e−07	1673/51732/20663/43.224/7.45e−07	927/26358/11123/22.513/5.55e−07

Table 2 continued

Problems Name/n	IMPRP Itr/NF/NG/Tcpu/$\|g_*\|$	MPRP Itr/NF/NG/Tcpu/$\|g_*\|$	KD Itr/NF/NG/Tcpu/$\|g_*\|$
nonscomp 30000	185/5126/2340/13.154/8.11e−07	NaN/NaN/NaN/NaN/3.32e−03	146/3541/1611/9.149/8.09e−07
nonscomp 50000	424/12254/5431/52.616/7.24e−07	379/10616/4609/45.512/8.05e−07	1695/51643/21087/217.190/3.60e−07
power1 5	68/1663/766/0.068/5.84e−07	79/1914/914/0.051/3.81e−07	65/1605/746/0.044/4.32e−07
raydan1 120	129/3197/1458/0.121/3.92e−07	191/5296/2463/0.187/3.85e−07	217/6362/2876/0.223/8.61e−07
raydan1 150	183/4829/2279/0.191/7.75e−07	162/4382/2070/0.172/8.82e−07	204/5915/2766/0.282/4.92e−07
raydan2 100000	14/238/87/2.883/8.10e−07	49/1256/635/14.711/7.10e−07	NaN/NaN/NaN/NaN/7.47e−05
raydan2 200000	29/676/322/16.980/8.61e−07	NaN/NaN/NaN/NaN/9.77e-05	NaN/NaN/NaN/NaN/1.48e−04
raydan2 500000	29/693/329/42.753/1.38e−07	NaN/NaN/NaN/NaN/2.23e−04	NaN/NaN/NaN/NaN/1.55e−04
diagonal1 10	76/2187/1040/0.092/1.66e−07	126/3762/1807/0.105/1.42e−07	108/3194/1526/0.091/1.63e−07
diagonal2 1000	571/16519/7762/3.303/7.87e−07	466/13383/6312/2.615/9.44e−07	508/14838/7084/2.935/6.31e−07
diagonal3 60	135/3600/1623/0.161/9.85e−07	119/3154/1473/0.126/2.88e−07	162/4546/2073/0.182/8.36e−08
diagonal3 120	186/5187/2344/0.268/2.82e−07	196/5345/2483/0.274/5.01e−07	NaN/NaN/NaN/NaN/5.00e−06
bv 3000	1/1/1/0.001/5.55e−07	1/1/1/0.000/5.55e−07	1/1/1/0.000/5.55e−07
ie 4	13/198/86/0.016/3.61e−08	12/172/70/0.013/2.82e−08	15/230/97/0.017/3.31e−07
ie 100	16/257/105/1.358/5.08e−07	16/257/106/1.378/4.14e−07	13/183/76/0.975/6.17e−07
singx 4	NaN/NaN/NaN/NaN/8.57e−05	1993/60529/23025/3.171/4.18e−07	394/11364/4834/0.604/3.55e−07
band 3	20/307/131/0.045/7.20e−07	22/369/149/0.023/3.53e−07	67/1838/865/0.097/6.29e−07
froth 2	1967/59664/24392/2.551/9.94e−07	NaN/NaN/NaN/NaN/3.20e−05	382/11264/5056/0.497/1.77e−07
gauss 3	40/1067/479/0.091/1.69e−07	20/475/218/0.040/8.19e−07	47/1263/533/0.106/6.39e−07
jensam 2	170/5020/2379/0.243/6.64e−07	153/4454/2180/0.215/5.26e−07	119/3545/1695/0.172/6.79e−07

Table 2 continued

Problems Name/n	IMPRP Itr/NF/NG/Tcpu/$\|g_*\|$	MPRP Itr/NF/NG/Tcpu/$\|g_*\|$	KD Itr/NF/NG/Tcpu/$\|g_*\|$
lin0 2	1/1/1/0.000/0.00e+00	1/1/1/0.000/0.00e+00	1/1/1/0.000/0.00e+00
lin 300	2/2/2/0.051/2.52e−13	2/2/2/0.036/2.52e−13	2/2/2/0.038/2.52e−13
lin 500	2/2/2/0.038/1.10e−13	2/2/2/0.038/1.10e−13	2/2/2/0.037/1.10e−13
osb2 11	1576/45126/20048/5.677/4.92e−07	1572/44130/19664/5.527/9.47e−07	NaN/NaN/NaN/NaN/1.09e−05
pen1 5	1032/31647/12732/3.024/3.38e−07	728/22254/8849/2.091/9.13e−07	399/12180/4932/1.165/6.68e−07
pen2 2	31/679/315/0.050/2.29e−07	37/868/421/0.062/3.43e−07	93/2716/1341/0.196/8.11e−07
rose 2	616/18191/7627/0.750/1.51e−07	609/17347/7408/0.709/7.38e−07	911/28228/11815/1.180/8.56e−07
rosex 2	616/18191/7627/0.871/1.51e−07	609/17347/7408/0.790/7.38e−07	911/28228/11815/1.282/8.56e−07
trid 40	94/2325/1034/0.280/5.12e−07	158/4212/1982/0.462/9.55e−07	148/3949/1794/0.443/5.43e−07
trid 120	98/2447/1098/0.611/7.87e−07	279/7953/3536/1.978/4.11e−07	NaN/NaN/NaN/NaN/2.11e−02
wood 4	1908/55141/24295/2.527/9.64e−07	NaN/NaN/NaN/NaN/6.98e−06	874/25147/11212/1.109/3.75e−07

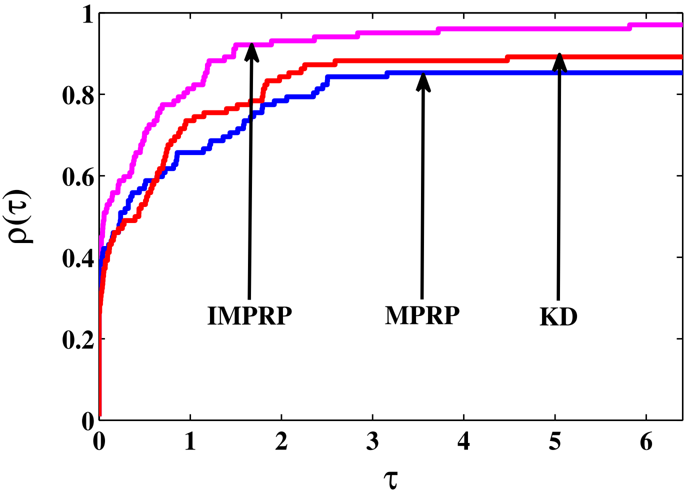

Fig. 3 Performance profiles on Tcpu of Group A

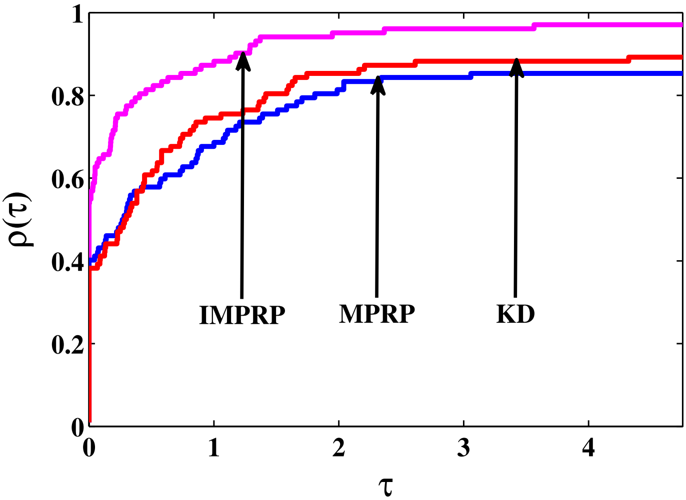

Fig. 4 Performance profiles on Itr of Group A

30% of test problems, respectively. Moreover, the IMHS CGM solves about 97% of the test problems successfully, while the MHS CGM and the NHS CGM solve about 86% and 90% of this group, respectively.

In summary, from the performance profiles in Figs. 3, 4, 5 and 6, the curves corresponding to the IMPRP CGM and the IMHS CGM are all at the top, respectively, which indicates that the proposed methods perform well, at least for these collections of experiments.

Table 3 Numerical comparisons of Group B

Problems Name/n	IMHS Itr/NF/NG/Tcpu/$\|g_*\|$	MHS Itr/NF/NG/Tcpu/$\|g_*\|$	NHS Itr/NF/NG/Tcpu/$\|g_*\|$
bard 3	802/24022/9995/3.081/9.72e−07	283/7390/3363/0.939/7.39e−07	736/21499/8748/2.738/3.61e−07
beale 2	245/6730/3033/0.291/2.71e−07	190/5440/2419/0.232/4.52e−07	289/8220/3791/0.354/7.95e−07
box 3	1116/32978/13579/1.881/4.07e−07	390/11132/4551/0.629/8.99e−07	683/19187/8209/1.097/8.52e−07
cosine 200	26/502/220/0.065/6.61e−08	31/650/300/0.050/1.57e−09	27/570/287/0.048/2.73e−07
cosine 2700	430/11502/5340/6.406/3.18e−07	NaN/NaN/NaN/NaN/4.07e−01	NaN/NaN/NaN/NaN/1.69e−03
dixmaana 3000	18/233/85/0.994/2.84e−07	16/267/116/1.081/7.75e−07	15/216/68/0.835/2.50e−07
dixmaanb 3000	12/150/57/0.669/2.43e−07	12/150/47/0.619/1.86e−07	12/150/52/0.622/2.26e−07
dixmaanb 9000	11/126/37/1.328/4.44e−07	12/150/53/1.676/3.18e−07	12/150/49/1.606/3.93e−07
dixmaanc 9000	19/368/164/4.363/5.25e−07	86/2320/990/28.385/8.53e−08	23/389/179/4.540/2.66e−07
dixmaanc 10500	51/1396/647/18.799/8.17e−07	NaN/NaN/NaN/NaN/1.47e+02	38/708/341/9.471/7.57e−08
dixmaand 1200	19/265/111/0.480/6.88e−07	23/363/161/0.613/1.26e−07	19/271/98/0.447/7.23e−08
dixmaand 1500	21/331/135/0.727/1.94e−07	42/966/456/2.030/4.17e−07	19/303/129/0.607/6.80e−07
dixmaane 120	143/3348/1597/1.005/6.07e−07	165/4177/1973/1.218/6.78e−07	148/3742/1741/1.075/5.50e−07
dixmaane 600	254/6350/2978/5.666/8.94e−07	270/6863/3221/5.965/7.51e−07	306/7640/3528/6.785/8.49e−07
dixmaanf 210	109/2739/1306/1.133/5.92e−07	136/3211/1521/1.276/8.22e−07	175/4486/2080/1.772/9.93e−07
dixmaanf 300	151/3846/1805/1.933/5.44e−07	198/5222/2423/2.626/4.65e−07	220/5936/2732/3.021/7.00e−07
dixmaang 30	76/2101/973/0.406/8.27e−07	108/3057/1438/0.544/9.75e−07	121/3411/1566/0.604/5.87e−07
dixmaanh 30	62/1385/638/0.276/3.56e−07	70/1707/826/0.311/8.38e−07	88/2238/1014/0.399/4.11e−07
dixmaani 15	154/4215/1886/0.690/5.00e−07	185/4691/2222/0.746/3.50e−07	186/4923/2253/0.781/4.02e−07
dixmaanj 15	134/3519/1549/0.582/9.88e−07	135/3452/1551/0.544/8.36e−07	120/3077/1409/0.486/9.69e−07

Table 3 continued

Problems Name/n	IMHS Itr/NF/NG/Tcpu/$\|g_*\|$	MHS Itr/NF/NG/Tcpu/$\|g_*\|$	NHS Itr/NF/NG/Tcpu/$\|g_*\|$
dixmaank 900	1042/29188/12634/34.884/9.99e−07	1247/33641/15084/42.755/9.34e−07	919/25419/11222/35.761/6.46e−07
dixmaank 1500	740/21223/9184/44.962/9.25e−07	1746/48045/21045/94.232/8.80e−07	808/23148/10004/46.237/8.98e−07
dixmaanl 30	191/5046/2275/1.007/6.98e−07	208/5434/2594/1.081/8.32e−07	207/5271/2337/1.078/7.77e−07
dixon3dq 5	57/1424/632/0.037/1.92e−07	72/1708/795/0.044/5.04e−07	65/1505/697/0.041/4.02e−07
dqdrtic 100000	195/5552/2403/37.247/5.93e−07	241/6781/2957/45.118/6.40e−07	936/28010/12270/192.862/7.47e−07
dqdrtic 800000	104/2547/1109/148.359/9.17e−07	175/4940/2183/290.098/2.40e−07	796/23764/10388/1369.431/6.03e−07
dqdrtic 1000000	116/2598/1181/196.649/3.27e−07	118/3172/1375/225.992/9.05e−07	877/26169/11562/1931.796/9.37e−07
dqrtic 200	24/336/136/0.161/5.08e−07	32/524/213/0.141/7.20e−07	27/373/154/0.054/7.27e−07
dqrtic 320	37/649/299/0.194/4.08e−08	52/1077/504/0.199/7.13e−07	33/647/300/0.119/8.19e−07
dqrtic 400	34/616/253/0.163/3.67e−07	57/1202/538/0.269/2.19e−07	41/762/348/0.169/4.72e−07
edensch 1000	79/1915/887/1.186/3.65e−07	NaN/NaN/NaN/NaN/1.40e−06	58/1404/682/0.728/3.70e−07
edensch 2000	73/1880/923/2.231/5.48e−07	NaN/NaN/NaN/NaN/2.72e−06	65/1580/756/3.229/2.64e−07
edensch 6000	61/1533/735/4.360/4.05e−07	NaN/NaN/NaN/NaN/1.42e−06	NaN/NaN/NaN/NaN/3.59e−06
eg2 2	12/161/64/0.015/2.40e−07	59/1625/802/0.058/3.08e−07	120/3267/1429/0.090/9.69e−07
engval1 4	96/2777/1307/0.151/8.80e−07	NaN/NaN/NaN/NaN/1.50e−02	NaN/NaN/NaN/NaN/1.17e−06
errinros 3	NaN/NaN/NaN/NaN/1.65e−02	1866/54283/21501/2.088/9.68e−07	NaN/NaN/NaN/NaN/1.22e−01
extrosnb 10000	1/1/1/0.047/0.00e+00	1/1/1/0.000/0.00e+00	1/1/1/0.000/0.00e+00
fletchcr 10	113/3034/1359/0.106/9.32e−07	140/3752/1770/0.107/8.07e−07	153/4119/1836/0.121/5.81e−07
fletchcr 60	53/1270/589/0.045/9.29e−07	130/3298/1570/0.112/5.79e−07	217/6076/2844/0.206/8.84e−07
fletchcr 110	84/2139/1019/0.125/9.01e−07	181/4898/2322/0.210/7.51e−07	83/1947/979/0.080/8.78e−07

Table 3 continued

Problems Name/n	IMHS Itr/NF/NG/Tcpu/$\|g_*\|$	MHS Itr/NF/NG/Tcpu/$\|g_*\|$	NHS Itr/NF/NG/Tcpu/$\|g_*\|$
freuroth 12	903/25758/11329/1.092/8.43e−07	NaN/NaN/NaN/NaN/3.42e−05	1231/36262/15220/1.437/8.30e−07
genrose 5	832/25279/11451/0.720/9.20e−07	904/27056/12179/0.710/6.46e−07	885/26194/11684/0.711/3.24e−07
gulf 3	2/1/2/0.002/0.00e+00	2/1/2/0.001/0.00e+00	2/1/2/0.001/0.00e+00
helix 3	553/15533/6765/0.986/4.86e−07	177/4380/2114/0.258/8.36e−07	290/7767/3489/0.498/7.39e−07
himmelbg 200000	3/6/7/0.402/2.26e−27	3/6/7/0.400/2.25e−27	3/6/7/0.403/2.25e−27
himmelbg 500000	3/6/7/0.999/3.57e−27	3/6/7/0.984/3.55e−27	3/6/7/0.989/3.56e−27
himmelbg 1000000	3/6/7/2.014/5.05e−27	3/6/7/2.005/5.03e−27	3/6/7/1.999/5.04e−27
kowosb 4	408/11007/5092/0.706/7.70e−07	395/11484/4918/0.696/6.61e−07	491/13633/5942/0.792/3.66e−07
liarwhd 2000	554/16156/6960/4.826/6.64e−07	NaN/NaN/NaN/NaN/4.75e−04	523/16010/6614/4.693/3.79e−07
liarwhd 4000	1837/56396/22132/30.444/6.84e−07	NaN/NaN/NaN/NaN/9.66e−03	NaN/NaN/NaN/NaN/1.60e−03
liarwhd 6000	1688/50843/20190/39.497/3.88e−07	1939/59809/23427/48.731/8.46e−07	1395/42675/16482/34.113/6.16e−07
nondquar 4	NaN/NaN/NaN/NaN/4.99e−05	NaN/NaN/NaN/NaN/9.33e−05	NaN/NaN/NaN/NaN/9.84e−05
penalty1 800	23/503/203/2.316/1.78e−07	23/503/205/2.332/1.79e−07	23/503/203/2.229/1.79e−07
penalty1 1000	15/253/94/1.494/1.21e−07	15/253/94/1.466/1.21e−07	15/253/94/1.498/1.21e−07

Table 4 Numerical comparisons of Group B

Problems Name/n	IMHS Itr/NF/NG/Tcpu/$\|g_*\|$	MHS Itr/NF/NG/Tcpu/$\|g_*\|$	NHS Itr/NF/NG/Tcpu/$\|g_*\|$
penalty1 5000	12/162/51/14.066/1.48e−07	12/162/51/14.599/1.48e−07	12/162/51/14.518/1.48e−07
quartc 100	30/462/187/0.083/6.51e−08	33/534/221/0.043/7.68e−08	25/362/134/0.027/8.11e−07
quartc 200	24/336/136/0.044/5.08e−07	32/524/213/0.068/7.20e−07	27/373/154/0.058/7.27e−07
tridia 50	321/8545/3915/0.359/4.83e−07	452/12092/5577/0.450/7.71e−07	378/10380/4538/0.362/2.63e−07
sinquad 2	160/4587/1859/0.173/4.12e−07	183/5501/2073/0.171/7.08e−07	295/8797/3397/0.279/9.32e−07
vardim 8	12/168/52/0.025/3.98e−08	12/168/52/0.013/3.98e−08	12/168/52/0.013/3.98e−08
watson 2	20/368/143/0.095/5.43e−08	108/3193/1528/0.275/1.51e−07	107/3161/1500/0.335/3.46e−07
woods 10000	744/19826/9066/23.638/9.39e−07	687/19100/8828/23.125/8.96e−07	1007/28618/12682/33.644/7.32e−07
bdexp 500000	3/2/3/1.183/3.63e−109	3/2/3/1.151/3.63e−109	3/2/3/1.090/3.63e−109
bdexp 1000000	3/2/3/2.192/5.09e−109	3/2/3/2.168/5.09e−109	3/2/3/2.189/5.09e−109
exdenschnf 100000	27/522/210/11.001/4.69e−07	45/971/430/22.367/4.11e−08	103/2949/1462/66.099/1.48e−07
exdenschnf 500000	33/631/301/71.228/3.51e−07	44/917/421/101.533/3.08e−08	103/2997/1426/330.187/3.50e−07
exdenschnf 1000000	35/743/344/165.420/4.24e−07	39/833/372/183.359/9.26e−08	103/3034/1437/674.229/5.06e−07
mccormak 2	18/329/137/0.041/5.55e−07	20/373/169/0.015/3.79e−07	17/315/137/0.013/6.81e−07
exdenschnb 100	53/1410/653/0.097/9.64e−08	29/587/267/0.023/3.27e−07	48/1237/595/0.048/7.15e−07
genquartic 1000000	77/1969/934/238.282/1.89e−07	93/2250/1085/264.857/9.25e−08	64/1344/610/150.102/1.48e−07
biggsb1 20	194/5070/2316/0.198/9.94e−07	234/6542/2997/0.186/5.52e−07	297/8291/3750/0.243/5.84e−07
sine 100	48/1299/616/0.095/3.49e−07	62/1719/799/0.092/5.02e−07	71/1969/936/0.100/3.72e−09
sine 800	62/1714/821/0.384/8.86e−07	62/1714/828/0.380/9.18e−07	64/1778/824/0.391/1.87e−07
fletcbv3 5	8/157/98/0.019/9.99e−07	21/540/284/0.026/8.60e−07	12/286/158/0.009/7.28e−07

Table 4 continued

Problems Name/n	IMHS Itr/NF/NG/Tcpu/$\|g_*\|$	MHS Itr/NF/NG/Tcpu/$\|g_*\|$	NHS Itr/NF/NG/Tcpu/$\|g_*\|$
nonscomp 20000	236/6405/3030/10.825/7.39e−07	NaN/NaN/NaN/NaN/2.76e−03	NaN/NaN/NaN/NaN/8.65e−04
nonscomp 30000	214/5831/2781/14.264/2.68e−07	NaN/NaN/NaN/NaN/5.63e−05	165/4177/1951/10.458/2.07e−07
nonscomp 40000	81/1750/870/5.991/2.57e−07	NaN/NaN/NaN/NaN/5.18e−05	79/1941/922/6.490/9.80e−07
power1 2	26/582/273/0.038/3.34e−07	23/460/214/0.012/3.15e−07	31/670/300/0.017/3.27e−07
raydan1 5	19/384/187/0.013/4.94e−07	21/416/184/0.011/5.21e−07	19/294/129/0.008/5.79e−07
raydan2 20000	20/426/204/0.981/6.09e−07	23/508/246/1.177/2.37e−07	30/712/356/1.624/7.56e−07
raydan2 50000	57/1472/744/8.446/4.55e−07	63/1693/809/9.530/9.76e−07	40/1004/510/5.795/1.66e−07
diagonal1 10	112/3328/1595/0.118/1.51e−07	118/3509/1717/0.097/1.63e−07	110/3257/1587/0.090/1.58e−07
diagonal1 70	190/5565/2709/0.216/9.34e−07	NaN/NaN/NaN/NaN/1.51e−06	216/6400/3164/0.241/1.99e−07
diagonal1 150	294/8365/3987/0.425/5.80e−07	NaN/NaN/NaN/NaN/6.72e−05	NaN/NaN/NaN/NaN/8.57e−06
diagonal2 5	58/1630/785/0.071/5.29e−07	85/2445/1207/0.069/5.35e−07	61/1750/815/0.047/5.30e−07
diagonal3 10	34/714/340/0.023/1.48e−07	41/812/351/0.023/1.57e−07	36/867/431/0.027/6.67e−07
diagonal3 80	NaN/NaN/NaN/NaN/3.17e−06	NaN/NaN/NaN/NaN/3.60e−06	160/4168/1899/0.179/9.45e−07
bv 1500	8/193/91/3.057/6.20e−07	8/193/93/2.994/6.21e−07	9/225/89/3.321/6.24e−07
bv 2500	1/1/1/0.000/7.99e−07	1/1/1/0.000/7.99e−07	1/1/1/0.000/7.99e−07
ie 100	16/257/118/1.372/1.87e−07	14/205/84/1.067/2.05e−07	12/176/76/0.926/2.30e−07
singx 8	782/22950/9329/1.191/7.87e−07	771/22159/8993/1.147/5.41e−07	NaN/NaN/NaN/NaN/9.52e−06
expenalty 4	84/2423/1146/0.094/1.02e−07	93/2698/1268/0.075/9.34e−08	NaN/NaN/NaN/NaN/5.49e−02
band 3	19/265/110/0.031/5.12e−08	19/302/135/0.016/5.47e−07	16/223/80/0.012/2.75e−07
froth 2	289/8153/3787/0.373/7.80e−07	975/28482/11695/1.168/6.10e−07	NaN/NaN/NaN/NaN/2.01e−02

Table 4 continued

Problems Name/n	IMHS Itr/NF/NG/Tcpu/$\|g_*\|$	MHS Itr/NF/NG/Tcpu/$\|g_*\|$	NHS Itr/NF/NG/Tcpu/$\|g_*\|$
gauss 3	24/563/245/0.055/5.07e−07	18/399/187/0.034/6.73e−07	41/1099/487/0.091/9.58e−07
jensam 2	178/5253/2522/0.253/8.76e−07	141/4135/1983/0.193/3.59e−07	136/4026/1927/0.194/6.92e−07
lin0 2	1/1/1/0.000/0.00e+00	1/1/1/0.000/0.00e+00	1/1/1/0.000/0.00e+00
lin 100	2/2/2/0.020/4.02e−13	2/2/2/0.020/4.02e−13	2/2/2/0.019/4.02e−13
lin 500	2/2/2/0.037/1.10e−13	2/2/2/0.036/1.10e−13	2/2/2/0.038/1.10e−13
osb2 11	1709/48421/21551/5.876/7.21e−07	944/24800/11790/3.061/7.68e−07	860/23045/10190/2.848/7.55e−07
pen1 4	1035/31674/12588/2.716/7.33e−07	913/28135/10874/2.354/6.76e−07	737/22926/9110/1.936/8.15e−07
pen2 2	105/3034/1438/0.231/3.17e−07	106/3126/1492/0.219/9.34e−07	87/2504/1217/0.174/9.06e−07
rose 2	957/28770/12181/1.177/4.17e−07	701/21112/8658/0.833/4.51e−07	702/21055/8825/0.840/2.94e−07
rosex 1000	421/12055/5159/118.958/9.60e−07	738/21705/9030/220.205/9.42e−07	609/17951/7634/176.828/5.84e−07
sing 4	907/27002/10720/1.184/9.02e−07	647/17936/7628/0.776/3.54e−07	1590/49139/17893/2.052/9.33e−07
trid 15	67/1588/726/0.155/7.35e−07	140/3562/1745/0.270/7.59e−07	149/3906/1836/0.296/2.10e−07
wood 4	870/24278/10750/1.079/3.25e−07	540/14599/6528/0.620/9.85e−07	726/20208/9099/0.880/5.95e−07

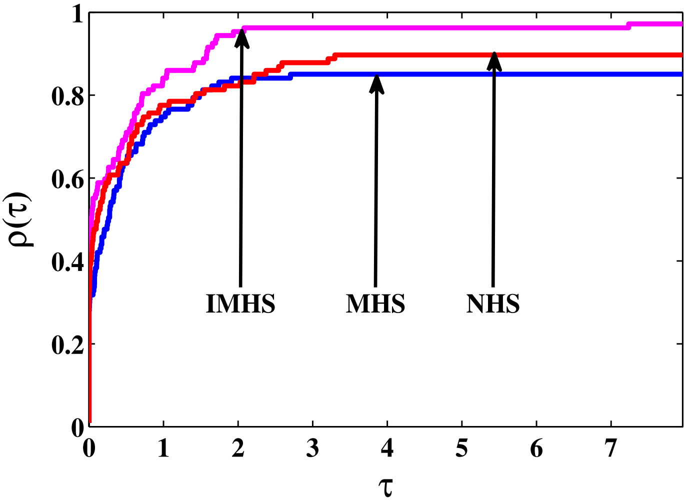

Fig. 5 Performance profiles on Tcpu of Group B

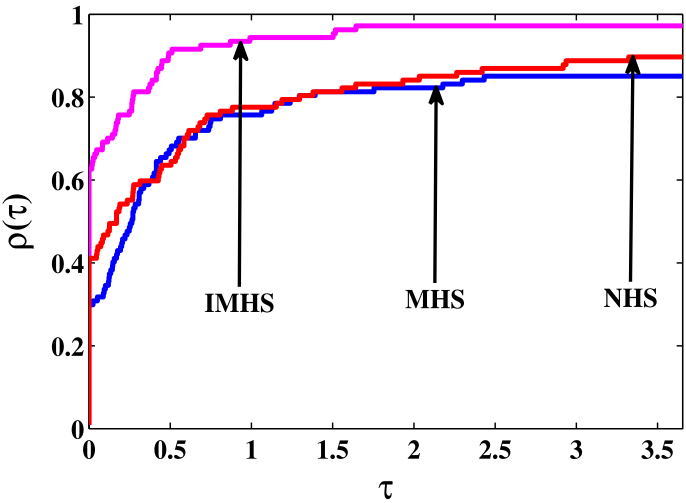

Fig. 6 Performance profiles on Itr of Group B

4 Conclusion

In this work, according to the interesting phenomenon in experiments, we have constructed two new conjugate parameters with the second inequality in the strong Wolfe line search. Based on general assumptions, all relevant methods satisfy the sufficient descent and global convergence. Finally, numerical experiments were done and reported, which show that the improved CGMs have good performance and application prospect.

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Author Contributions All authors read and approved the final manuscript. JJ mainly contributed to the algorithm design; PL and XJ mainly contributed to the convergence analysis, numerical results and drafted the manuscript; BH mainly contributed to the convergence analysis.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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