Bulletin of the Iranian Mathematical Society (2022) 48:1869–1893
https://doi.org/10.1007/s41980-021-00622-7

ORIGINAL PAPER

A Generalized Self-Adaptive Algorithm for the Split Feasibility Problem in Banach Spaces

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Received: 27 August 2020 / Revised: 4 May 2021 / Accepted: 19 July 2021 / Published online: 24 August 2021

Abstract

In this paper, we propose a generalized self-adaptive method for solving the multiple-set split feasibility problem in the framework of certain Banach spaces. Under some suitable conditions, we prove the strong convergence of the sequence generated by our method with a new way to select the step-sizes without prior knowledge of the operator norm. Several numerical experiments to illustrate the convergence behavior are presented. The results presented in this paper improve and extend the corresponding results in the literature.

Keywords Metric projection · Banach space · Strong convergence · Self-adaptive method · Multiple-set split feasibility problem

Mathematics Subject Classification 47H09 · 47H10 · 47J25 · 47J05

1 Introduction

Let E and F be two real p-uniformly convex Banach spaces which are also uniformly smooth. Let C_i, $i = 1, 2, \dots, M$ and Q_j, $j = 1, 2, \dots, N$ be nonempty, closed and convex subsets of E and F, respectively. Let $A : E \to F$ be a bounded linear operator with its adjoint $A^* : F^* \to E^*$. We consider the following so-called *multiple-set split*

Communicated by Xinmin Yang.

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feasibility problem (MSFP):

$$\text{find } \ x^* \in \bigcap_{i=1}^{M} C_i \ \text{ such that } \ Ax^* \in \bigcap_{j=1}^{N} Q_j. \tag{1.1}$$

We denote by $\Omega := \left(\bigcap_{i=1}^{M} C_i\right) \cap A^{-1}\left(\bigcap_{j=1}^{N} Q_j\right)$ the solution set of Problem (1.1). This problem was first introduced in finite-dimensional Hilbert spaces by Censor et al. [10]. The MSFP has broad applicability in many areas of mathematics and the physical and engineering sciences, for example, it can be applied in fields of image reconstruction and signal processing (see [33]) and in the inverse problem of intensity-modulated radiation therapy (IMRT) in the field of medical care (see [10,13,14]). Moreover, this problem is a generalization of convex feasibility problem (CFP) and as a generalization of the split feasibility problem. In particular, if $M = N = 1$, then the MSFP becomes the following well-known *split feasibility problem* (SFP) [12]:

$$\text{find } \ x^* \in C \ \text{ such that } \ Ax^* \in Q. \tag{1.2}$$

There are many modification methods have been proposed for solving the MSFP and the SFP in different styles (see for instance [6,9,16,19,29–32,34–46,50]).

A one efficient method for solving the SFP in Hilbert spaces is known as *Byrne's CQ algorithm* [9] which is defined in the following manner: for given $x_1 \in C$, compute the sequences $\{x_n\}$ generated iteratively by

$$x_{n+1} = P_C(x_n - \tau_n A^*(I - P_Q)Ax_n), \ \ \forall n \geq 1, \tag{1.3}$$

where P_C and P_Q are the metric projections onto C and Q, respectively. It was proved that the sequence $\{x_n\}$ defined by (1.3) converges weakly to a solution of the SFP provided the step-size $\tau_n \in (0, \frac{2}{\|A\|^2})$.

Note that the choice of the step-size τ_n of above work and other corresponding results depend on the operator norm $\|A\|$. In general, the implementation of such algorithms is not an easy work in practice. As a result the implementation of the iteration process inefficient when the computation of the operator norm is not explicit. To overcome this difficulty, López et al. [21] constructed a new choice to select the following step-size so that without prior knowledge of the operator norm:

$$\tau_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \tag{1.4}$$

where $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ with its gradient $\nabla f(x) = A^*(I - P_Q)Ax$ and $\{\rho_n\} \subset (0, 4)$ satisfies $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$. They established the weak convergence of the Byrne's CQ algorithm (1.3) to a solution of SFP with the step-size τ_n defined by (1.4).

Let C and Q be nonempty, closed and convex subsets of E and F, respectively. Schöpfer et al. [34] first introduced the following algorithm for solving SFP in Banach spaces: for given $x_1 \in E$ and

$$x_{n+1} = \Pi_C J_q^{E^*}(J_p^E(x_n) - \tau_n A^* J_p^F(I - P_Q)Ax_n), \quad \forall n \geq 1, \tag{1.5}$$

where Π_C is the generalized projection onto C, P_Q is the metric projection onto Q. They considered more general Bregman distance functions for its solution and proved that the sequence $\{x_n\}$ generated by (1.5) converges weakly to a solution of the SFP provided the duality mappings are weak-to-weak continuous and the step-size τ_n satisfies $0 < \tau_n < \left(\frac{q}{c_q\|A\|^q}\right)^{\frac{1}{q-1}}$, where $\frac{1}{p} + \frac{1}{q} = 1$ and c_q is the uniform smoothness coefficient of E (see [48]). Clearly, the algorithm (1.5) covers the Byrne's CQ algorithm as a special case.

To obtain the strong convergence result, Shehu [35] proposed the following algorithm for solving the SFP in p-uniformly convex Banach spaces which are also uniformly smooth: for given $u, x_1 \in E$ and

$$x_{n+1} = \Pi_C J_q^{E^*}(\alpha_n J_p^E(u) + (1 - \alpha_n)(J_p^E(x_n) - \tau_n A^* J_p^F(I - P_Q)Ax_n)), \quad \forall n \geq 1, \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and the step-size τ_n satisfies $0 < a \leq \tau_n \leq b < \left(\frac{q}{\kappa_q\|A\|^q}\right)^{\frac{1}{q-1}}$ for some $a, b > 0$. He proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a solution of the SFP under some mild conditions.

Very recently, Alsulami and Takahashi [6] introduced an algorithm for solving the SFP between Hilbert space and strictly convex, reflexive and smooth Banach space. To be more precise, they obtained the following result.

Theorem 1.1 *Let H be a Hilbert space and E be a strictly convex, reflexive and smooth Banach space. Let J_E be the duality mapping on E. Let C and Q be nonempty, closed and convex subsets of H and E, respectively. Let P_C and P_Q be the metric projections of H onto C and E onto Q, respectively. Let $A : H \to E$ be a bounded linear operator with its adjoint A^* such that $A \neq 0$. Suppose that the solution set Ω of the SFP (1.2) is nonempty. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For given $x_1 \in H$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)P_C(x_n - \tau A^* J_E(I - P_Q)Ax_n)), \quad \forall n \geq 1, \tag{1.7}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(ii) $0 < a \leq \beta_n \leq b < 1$ *for some* $a, b \in (0, 1)$*;*
(iii) $0 < \tau\|A\|^2 < 2$*, where* $\tau > 0$.

Then $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega u$.

There are some open questions which are posed as follows:

(1) Can we extend Theorem 1.1 for solving the MSFP in two Banach spaces?
(2) It is possible to remove the conditions $0 < \tau\|A\|^2 < 2$ and $0 < a \leq \beta_n$?

In this paper, we propose a new iterative method to answer two above open questions. We prove the strong convergence of the sequence generated by our method under some suitable conditions. Finally, we give some numerical examples to illustrate for the main result and showing its performance in finite and infinite dimensional spaces.

2 Preliminaries

Let E and E^* be real Banach spaces and the dual space of E, respectively. We write $\langle x, j\rangle$ for the value of a functional j in E^* at x in E. We shall use the notations $x_n \to x$ means that $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. Let $S_E = \{x \in E : \|x\| = 1\}$ and $B_E = \{x \in E : \|x\| \leq 1\}$. The *modulus of convexity* of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in B_E, \|x - y\| \geq \epsilon\right\}.$$

Let $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. The space E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$ and *p-uniformly convex* if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0, 2]$. The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup\left\{\tfrac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : x, y \in S_E\right\}.$$

The space E is called *uniformly smooth* if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$ and called *q-uniformly smooth* if there exists a $c_q > 0$ such that $\rho_E(\tau) \leq c_q\tau^q$ for all $\tau > 0$. It is known that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space and E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex) (see [1]). Furthermore, L_p (or ℓ_p) and the Sobolev spaces are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ while Hilbert space is uniformly smooth (see [48]).

Definition 2.1 A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *gauge function* if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.2 The mapping $J_\varphi : E \to 2^{E^*}$ associated with a gauge function φ defined by

$$J_\varphi(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\varphi(\|x\|),\ \|f\| = \varphi(\|x\|)\},\quad x \in E$$

is called the *duality mapping with gauge* φ, where $\langle\cdot, \cdot\rangle$ denotes the duality pairing between E and E^*.

In the particular case $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called the *normalized duality mapping*. In the case $\varphi(t) = t^{p-1}$, where $p > 1$, the duality mapping $J_\varphi = J_p$ is called the *generalized duality mapping* which is defined by

$$J_p(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}.$$

It follows from the definition that $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$ and $J_p(x) = \|x\|^{p-2} J(x)$, $p > 1$. It is well-known that if E is uniformly smooth, the generalized duality mapping J_p is norm-to-norm uniformly continuous on bounded subsets of E (see [27]). Furthermore, J_p is one-to-one, single-valued and satisfies $J_p = J_q^{-1}$, where J_q is the generalized duality mapping of E^* (see [15,26] for more details).

Lemma 2.3 [48] *Let E be a q-uniformly smooth Banach space. Then there exists a constant $c_q > 0$ which is called the q-uniform smoothness coefficient of E such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_q(x)\rangle + c_q\|y\|^q,$$

for all $x, y \in E$.

Let C be a nonempty, closed and convex subset of a strictly convex, smooth and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that

$$\|x - z\| \leq \inf_{y \in C} \|x - y\|.$$

The mapping $P_C : E \to C$ defined by $z = P_C x$ is called the *metric projection* of E onto C. It is well-known that $P_C x$ is the unique minimizer of the norm distance, which can be characterized by the variational inequality:

$$\langle y - P_C x, J_\varphi(x - P_C x)\rangle \leq 0, \quad \forall y \in C. \tag{2.1}$$

For a gauge function φ, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Phi(t) = \int_0^t \varphi(s)ds$$

is a continuous, convex and strictly increasing differentiable function on $\mathbb{R}^+$ with $\Phi'(t) = \varphi(t)$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. Therefore, Φ has a continuous inverse function Φ^{-1}.

Let E be a real smooth Banach space. The *Bregman distance* $D_\varphi : E \times E \to \mathbb{R}^+$ [7] is defined by

$$D_\varphi(x, y) = \Phi(\|y\|) - \Phi(\|x\|) - \langle y - x, J_\varphi(x)\rangle$$

for all $x, y \in E$. We note that $D_\varphi(x, y) \geq 0$ and $D_\varphi(x, y) = 0$ if and only of $x = y$. In general, the Bregman distance is not a metric due to the fact that it is not symmetric. The Bregman distance has the following important properties:

$$D_\varphi(x, y) + D_\varphi(y, x) = \langle x - y, J_\varphi(x) - J_\varphi(y)\rangle$$

and

$$D_\varphi(x, y) + D_\varphi(y, z) - D_\varphi(x, z) = \langle x - y, J_\varphi(z) - J_\varphi(y)\rangle$$

for all $x, y, z \in E$.

In the case $\varphi(t) = t^{p-1}$, $p > 1$, we have $\Phi(t) = \int_0^t \varphi(s)ds = \frac{t^p}{p}$. So we have the distance $D_\varphi = D_p$ is called the *p-Lyapunov function* which was studied in [8] and it is given by

$$D_p(x, y) = \frac{\|x\|^p}{p} - \langle x, J_p(y)\rangle + \frac{\|y\|^p}{q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 2$, then the Bregman distance becomes the Lyapunov function $\phi : E \times E \to \mathbb{R}^+$ [2,3] defined as

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2.$$

Let E be a strictly convex, smooth and reflexive Banach space. Following [2,11], we make use of the function $V_p : E \times E^* \to \mathbb{R}^+$ which is given by

$$V_p(x, \bar{x}) = \frac{\|x\|^p}{p} - \langle x, \bar{x}\rangle + \frac{\|\bar{x}\|^q}{q}$$

for all $x \in E$ and $\bar{x} \in E^*$. Then V_p is nonnegative and V_p satisfies the following properties:

$$V_p(x, \bar{x}) = D_p(x, J_q(\bar{x})), \quad \forall x \in E, \ \bar{x} \in E^* \tag{2.2}$$

and

$$V_p(x, \bar{x}) + \langle J_q(\bar{x}) - x, \bar{y}\rangle \le V_p(x, \bar{x} + \bar{y}), \quad \forall x \in E, \ \bar{x}, \bar{y} \in E^*. \tag{2.3}$$

Moreover, V_p is convex in the second variable. Then for all $z \in E$,

$$D_p\Big(z, J_q\Big(\sum_{i=1}^{M} t_i J_p(x_i)\Big)\Big) \le \sum_{i=1}^{M} t_i D_p(z, x_i),$$

where $\{x_i\}_{i=1}^M \subset E$ and $\{t_i\}_{i=1}^M \subset (0, 1)$ with $\sum_{i=1}^M t_i = 1$.

The *Bregman projection*, denoted by Π_C^φ, is defined as the unique solution of the following minimization problem:

$$\Pi_C^\varphi x = \operatorname{argmin}_{y \in C} D_\varphi(x, y), \quad x \in E.$$

It can be characterized by the variational inequality [20]:

$$\langle z - \Pi_C^\varphi x, J_\varphi(x) - J_\varphi(\Pi_C^\varphi x)\rangle \le 0, \quad \forall z \in C.$$

Moreover, we have

$$D_\varphi(y, \Pi_C^\varphi x) + D_\varphi(\Pi_C^\varphi x, x) \le D_\varphi(y, x), \quad \forall y \in C. \tag{2.4}$$

When $\varphi(t) = t$, we have Π_C^{φ} coincides with the generalized projection which studied in [2]. When $\varphi(t) = t^{p-1}$, where $p > 1$, we have Π_C^{φ} becomes the Bregman projection with respect to p and denoted by Π_C.

Lemma 2.4 [28] *Let E be a smooth and uniformly convex real Banach space. Suppose that $x \in E$, if $\{D_p(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.5 [25] *Let E be a smooth and uniformly convex Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E. Then $\lim_{n\to\infty} D_p(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.6 [22] *Let $\{a_n\}$ and $\{c_n\}$ be nonnegative real sequences such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad \forall n \geq 1,$$

where $\{\delta_n\}$ is a sequence in (0,1) and $\{b_n\}$ is a real sequence. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) *If $\frac{b_n}{\delta_n} \leq M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
(ii) *If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.7 [23] *Let $\{\Gamma_n\}$ be a nonnegative real sequence that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_k} < \Gamma_{n_k+1}$ for all $k \in \mathbb{N}$. For each $n \geq n_0$, define an integer sequence $\{\tau(n)\}$ as follows:*

$$\tau(n) = \max\{n_0 \leq k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then the following results hold:

(i) *$\tau(n) \to \infty$ as $n \to \infty$;*
(ii) *$\max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}$ for all $n \geq n_0$.*

3 Main Result

In this section, we propose a new self-adaptive algorithm to solve the multiple-set split feasibility problem in Banach spaces E and prove a convergence theorem of the generated sequences by the proposed method. Throughout this paper, we denote by J_p^E and $J_q^{E^*}$ the duality mappings of E and its dual space, respectively, where $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3.1 *Let E be a p-uniformly convex and uniformly smooth Banach space and F be a reflexive, strictly convex and smooth Banach space. Let C_i, $i = 1, 2, \ldots, M$ and Q_j, $j = 1, 2, \ldots, N$ be nonempty, closed and convex subsets of E and F, respectively. Let $A : E \to F$ be a bounded linear operator and $A^* : F^* \to E^*$ be an adjoint of A.*

Suppose that the solution set Ω *of the MSFP* (1.1) *is nonempty. Let* $\{u_n\}$ *be a sequence in* E *such that* $u_n \to u$*. For given* $x_1 \in E$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} v_{n,1} = J_q^{E^*}(J_p^E(x_n) - \tau_{n,1}\nabla f(x_n)), \\ v_{n,2} = J_q^{E^*}(J_p^E(v_{n,1}) - \tau_{n,2}\nabla f(v_{n,1})), \\ \quad \vdots \\ v_{n,N} = J_q^{E^*}(J_p^E(v_{n,N-1}) - \tau_{n,N}\nabla f(v_{n,N-1})), \\ y_n = J_q^{E^*}(a_{n,0}J_p^E(v_{n,N}) + \sum_{i=1}^M a_{n,i}J_p^E(\Pi_{C_i}v_{n,N})), \\ x_{n+1} = J_q^{E^*}(\beta_n J_p^E(x_n) + (1-\beta_n)(\alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n))), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$*,* $\{a_{n,i}\}_{i=1}^M \subset (0,1)$*,* $\{\beta_n\} \subset [0,1)$*,* $f(v_{n,j}) = \frac{1}{p}\|(I - P_{Q_{j+1}})Av_{n,j}\|^p$ *for* $j = 1, 2, \ldots, N-1$ *and* $f(x_n) = \frac{1}{p}\|(I - P_{Q_1})Ax_n\|^p$ *with the step-sizes* $\tau_{n,1}$ *and* $\tau_{n,j}$*,* $j = 1, 2, \ldots, N-1$ *are chosen self-adaptively as*

$$\tau_{n,1} = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tau_{n,j+1} = \begin{cases} \frac{\rho_n f^{p-1}(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}, & \text{if } f(v_{n,j}) \neq 0; \\ 0, & \text{otherwise}, \end{cases}$$

respectively, where $\{\rho_n\} \subset \left(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\right)$*. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$*;*
(C2) $\liminf_{n\to\infty} \rho_n\left(p - \frac{\rho_n^{q-1}c_q}{q}\right) > 0$*;*
(C3) $\sum_{i=0}^M a_{n,i} = 1$ *and* $\liminf_{n\to\infty} a_{n,i} > 0$ *for* $i = 1, 2, \ldots, M$*;*
(C4) $\limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $x^* = \Pi_\Omega u$*, where* Π_Ω *is the Bregman projection from* E *onto* Ω*.*

Proof For each $j = 1, 2, \ldots, N-1$, we note that $\nabla f(v_{n,j}) = A^*J_p^F(I - P_{Q_{j+1}})Av_{n,j}$ (see [17, Proposition 5.7]). Let $z \in \Omega$, that is, $z \in \bigcap_{i=1}^M C_i$ and $Az \in \bigcap_{j=1}^N Q_j$. Then for each $j = 1, 2, \ldots, N-1$, we have from (2.1) that

$$\begin{aligned} \|v_{n,j} - z\|\|\nabla f(v_{n,j})\| &\geq \langle v_{n,j} - z, \nabla f(v_{n,j})\rangle \\ &= \langle v_{n,j} - z, A^*J_p^E(I - P_{Q_{j+1}})Av_{n,j}\rangle \\ &= \langle Av_{n,j} - Az, J_p^E(I - P_{Q_{j+1}})Av_{n,j}\rangle \\ &\geq \langle Av_{n,j} - Az, J_p^E(I - P_{Q_{j+1}})Av_{n,j}\rangle \end{aligned}$$

$$
\begin{aligned}
&+\langle Az - P_{Q_{j+1}} Av_{n,j}, J_p^E (I - P_{Q_{j+1}}) Av_{n,j} \rangle \\
&= \langle Av_{n,j} - P_{Q_{j+1}} Av_{n,j}, J_p^E (I - P_{Q_{j+1}}) Av_{n,j} \rangle \\
&= \|(I - P_{Q_{j+1}}) Av_{n,j}\|^p = pf(v_{n,j}).
\end{aligned} \tag{3.1}
$$

We see that $\|\nabla f(v_{n,j})\| > 0$, when $f(v_{n,j}) \neq 0$. This implies that $\|\nabla f(v_{n,j})\| \neq 0$ for each $j = 1, 2, \ldots, N-1$. Hence, $\tau_{n,j+1}$ is well defined. In the same manner, we also have $\tau_{n,1}$ is well defined. For each $j = 1, 2, \ldots, N-1$, it follows from Lemma 2.3 and (3.1) that

$$
\begin{aligned}
D_p(z, v_{n,j+1}) &= D_p(z, J_q^{E^*}(J_p^E(v_{n,j}) - \tau_{n,j+1} \nabla f(v_{n,j}))) \\
&= V_p(z, J_p^E(v_{n,j}) - \tau_{n,j+1} \nabla f(v_{n,j})) \\
&= \frac{\|z\|^p}{p} - \langle z, J_p^E(v_{n,j}) \rangle + \tau_{n,j+1} \langle z, \nabla f(v_{n,j}) \rangle \\
&\quad + \frac{1}{q} \|J_p^E(v_{n,j}) - \tau_{n,j+1} \nabla f(v_{n,j})\|^q \\
&\leq \frac{\|z\|^p}{p} - \langle z, J_p^E(v_{n,j}) \rangle + \tau_{n,j+1} \langle z, \nabla f(v_{n,j}) \rangle \\
&\quad + \frac{1}{q} \|J_p^E(v_{n,j})\|^q - \tau_{n,j+1} \langle v_{n,j}, \nabla f(v_{n,j}) \rangle \\
&\quad + \frac{c_q \tau_{n,j+1}^q}{q} \|\nabla f(v_{n,j})\|^q \\
&= \frac{\|z\|^p}{p} - \langle z, J_p^E(v_{n,j}) \rangle + \frac{1}{q} \|v_{n,j}\|^p - \tau_{n,j+1} \langle v_{n,j} - z, \nabla f(v_{n,j}) \rangle \\
&\quad + \frac{c_q \tau_{n,j+1}^q}{q} \|\nabla f(v_{n,j})\|^q \\
&= D_p(z, v_{n,j}) - \tau_{n,j+1} pf(v_{n,j}) + \frac{c_q \tau_{n,j+1}^q}{q} \|\nabla f(v_{n,j})\|^q \\
&= D_p(z, v_{n,j}) - \frac{\rho_n p f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p} + \frac{\rho_n^q c_q}{q} \frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p} \\
&= D_p(z, v_{n,j}) - \rho_n \Big(p - \frac{\rho_n^{q-1} c_q}{q} \Big) \frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}.
\end{aligned} \tag{3.2}
$$

In the same manner, we can see that

$$
D_p(z, v_{n,1}) \leq D_p(z, x_n) - \rho_n \Big(p - \frac{\rho_n^{q-1} c_q}{q} \Big) \frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}. \tag{3.3}
$$

It follows from (3.2) and (3.3) that

$$
\begin{aligned}
&D_p(z, v_{n,N}) \\
&\le D_p(z, v_{n,N-1}) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,N-1})}{\|\nabla f(v_{n,N-1})\|^p} \\
&\vdots \\
&\le D_p(z, v_{n,1}) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,1})}{\|\nabla f(v_{n,1})\|^p} - \dots \\
&\quad -\rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,N-1})}{\|\nabla f(v_{n,N-1})\|^p} \\
&\le D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,1})}{\|\nabla f(v_{n,1})\|^p} \\
&\quad - \dots - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,N-1})}{\|\nabla f(v_{n,N-1})\|^p} \\
&= D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\Big[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + \sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\Big]. \qquad (3.4)
\end{aligned}
$$

From (2.4) and (3.4), we see that

$$
\begin{aligned}
D_p(z, y_n) &= D_p(z, J_q^{E^*}(a_{n,0}J_p^E(v_{n,N}) + \sum_{i=1}^{M} a_{n,i}J_p^E(\Pi_{C_i}v_{n,N}))) \\
&\le a_{n,0}D_p(z, v_{n,N}) + \sum_{i=1}^{M} a_{n,i}D_p(z, \Pi_{C_i}v_{n,N}) \\
&\le a_{n,0}D_p(z, v_{n,N}) + \sum_{i=1}^{M} a_{n,i}D_p(z, v_{n,N}) - \sum_{i=1}^{M} a_{n,i}D_p(\Pi_{C_i}v_{n,N}, v_{n,N}) \\
&= D_p(z, v_{n,N}) - \sum_{i=1}^{M} a_{n,i}D_p(\Pi_{C_i}v_{n,N}, v_{n,N}) \\
&\le D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\Big[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + \sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\Big] \\
&\quad - \sum_{i=1}^{M} a_{n,i}D_p(\Pi_{C_i}v_{n,N}, v_{n,N}), \qquad (3.5)
\end{aligned}
$$

which implies by the assumption of $\{\rho_n\}$ that

$$D_p(z, y_n) \le D_p(z, x_n).$$

Put $w_n = J_q^{E^*}(\alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n))$ for all $n \geq 1$, we have

$$\begin{aligned} D_p(z, w_n) &= D_p(z, J_q^{E^*}(\alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n))) \\ &\leq \alpha_n D_p(z, u_n) + (1-\alpha_n)D_p(z, y_n) \\ &\leq \alpha_n D_p(z, u_n) + (1-\alpha_n)D_p(z, x_n). \end{aligned}$$

It follows that

$$\begin{aligned} D_p(z, x_{n+1}) &= D_p(z, J_q^{E^*}(\beta_n J_p^E(x_n) + (1-\beta_n)J_p^E(w_n))) \\ &\leq \beta_n D_p(z, x_n) + (1-\beta_n)D_p(z, w_n) \\ &\leq \beta_n D_p(z, x_n) + (1-\beta_n)(\alpha_n D_p(z, u_n) + (1-\alpha_n)D_p(z, x_n)) \\ &= (1-(1-\beta_n)\alpha_n)D_p(z, x_n) + (1-\beta_n)\alpha_n D_p(z, u_n). \end{aligned}$$

Since $\{u_n\}$ is bounded, we also have $\{D_p(z, u_n)\}$ is bounded. By induction, we have $\{D_p(z, x_n)\}$ is bounded. Hence, by Lemma 2.6, we have $\{x_n\}$ is bounded, so are $\{v_{n,j}\}$ and $\{y_n\}$ for each $j = 1, 2, \ldots, N-1$. Let $x^* = \Pi_\Omega u$. From (2.3) and (3.5), we have

$$\begin{aligned} D_p(x^*, w_n) &= D_p(x^*, J_q^{E^*}(\alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n))) \\ &= V_p(x^*, \alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n)) \\ &\leq V_p(x^*, \alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n) - \alpha_n(J_p^E(u_n) - J_p^E(x^*)) \\ &\quad +\alpha_n\langle w_n - x^*, J_p^E(u_n) - J_p^E(x^*)\rangle \\ &= V_p(x^*, \alpha_n J_p^E(x^*) + (1-\alpha_n)J_p^E(y_n)) \\ &\quad +\alpha_n\langle w_n - x^*, J_p^E(u_n) - J_p^E(x^*)\rangle \\ &= \alpha_n D_p(x^*, x^*) + (1-\alpha_n)D_p(x^*, y_n) \\ &\quad\quad +\alpha_n\langle w_n - x^*, J_p^E(u_n) - J_p^E(x^*)\rangle \\ &\leq (1-\alpha_n)\left\{D_p(x^*, x_n) - \rho_n\left(p - \frac{\rho_n^{q-1}c_q}{q}\right)\left[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}\right.\right. \\ &\quad\quad \left. + \sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\right] \\ &\quad\quad \left. - \sum_{i=1}^{M} a_{n,i} D_p(\Pi_{C_i} v_{n,N}, v_{n,N})\right\} + \alpha_n\langle w_n - x^*, J_p^E(u_n) - J_p^E(x^*)\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} &D_p(x^*, x_{n+1}) \\ &\quad \leq \beta_n D_p(x^*, x_n) + (1-\beta_n)D_p(x^*, w_n) \end{aligned}$$

$$
\begin{aligned}
&\leq (1-(1-\beta_n)\alpha_n)D_p(x^*, x_n) \\
&\quad -(1-\alpha_n)(1-\beta_n)\rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\Big[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + \sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\Big] \\
&\quad -(1-\alpha_n)(1-\beta_n)\sum_{i=1}^{M} a_{n,i} D_p(\Pi_{C_i} v_{n,N}, v_{n,N}) \\
&\quad +\alpha_n(1-\beta_n)\langle w_n - x^*, J_p^E(u_n) - J_p^E(u)\rangle \\
&\quad +\alpha_n(1-\beta_n)\langle w_n - x^*, J_p^E(u) - J_p^E(x^*)\rangle.
\end{aligned} \tag{3.6}
$$

Put $\Gamma_n = D_p(x^*, x_n)$ for all $n \geq 1$. From (3.6), we have

$$
\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\Big[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + \sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\Big] \\
&\quad +(1-\alpha_n)(1-\beta_n)\sum_{i=1}^{M} a_{n,i} D_p(\Pi_{C_i} v_{n,N}, v_{n,N}) \\
&\quad \leq \Gamma_n - \Gamma_{n+1} + \alpha_n(1-\beta_n)\langle w_n - x^*, J_p^E(u_n) \\
&\quad -J_p^E(u)\rangle + \alpha_n(1-\beta_n)\langle w_n - x^*, J_p^E(u) - J_p^E(x^*)\rangle.
\end{aligned} \tag{3.7}
$$

We now show that $\Gamma_n \to 0$ as $n \to \infty$ by the following two possible cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq n_0$. Then we have

$$\Gamma_n - \Gamma_{n+1} \to 0.$$

By our assumptions, we have

$$\lim_{n\to\infty}\Big[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + \sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\Big] = 0$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{M} a_{n,i} D_p(\Pi_{C_i} v_{n,N}, v_{n,N}) = 0.$$

Since $\{\|\nabla f(x_n)\|^p\}$ and $\{\|\nabla f(v_{n,j})\|^p\}$ for all $j = 1, 2, \ldots, N-1$ are bounded, we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}\|(I - P_{Q_1})Ax_n\| = 0$$

and

$$\lim_{n\to\infty} f(v_{n,j}) = \lim_{n\to\infty} \|(I - P_{Q_{j+1}})Av_{n,j}\| = 0 \text{ for each } j = 1, 2, \ldots, N-1. \tag{3.8}$$

Moreover, we also have

$$\lim_{n\to\infty} D_p(\Pi_{C_i} v_{n,N}, v_{n,N}) = 0 \text{ for each } i = 1, 2, \ldots, M$$

and hence

$$\begin{aligned} D_p(y_n, v_{n,N}) &\le a_{n,0} D_p(v_{n,N}, v_{n,N}) + \sum_{i=1}^{M} a_{n,i} D_p(\Pi_{C_i} v_{n,N}, v_{n,N}) \\ &\to 0. \end{aligned}$$

By Lemma 2.5, we have

$$\lim_{n\to\infty} \|v_{n,N} - \Pi_{C_i} v_{n,N}\| = 0 \text{ for each } i = 1, 2, \ldots, M \tag{3.9}$$

and

$$\lim_{n\to\infty} \|y_n - v_{n,N}\| = 0.$$

From (3.8), we see that

$$\begin{aligned} \|J_p^E(v_{n,j+1}) - J_p^E(v_{n,j})\| &= \tau_{n,j+1} \|\nabla f(v_{n,j})\| \\ &\le \tau_{n,j+1} \|A^*\| \|(I - P_{Q_{j+1}})Av_{n,j}\|^{p-1} \\ &\to 0 \end{aligned}$$

for each $j = 1, 2, \ldots, N-1$. In a similar way, we can see that

$$\begin{aligned} \|J_p^E(v_{n,1}) - J_p^E(x_n)\| &= \tau_{n,1} \|\nabla f(x_n)\| \\ &\le \tau_{n,1} \|A^*\| \|(I - P_{Q_1})Ax_n\|^{p-1} \\ &\to 0. \end{aligned}$$

Since $J_q^{E^*}$ is norm-to-norm uniformly continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} \|v_{n,j+1} - v_{n,j}\| = 0 \text{ for each } j = 1, 2, \ldots, N-1 \tag{3.10}$$

and

$$\lim_{n\to\infty} \|v_{n,1} - x_n\| = 0. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\begin{aligned}\|y_n - x_n\| &\leq \|y_n - v_{n,N}\| + \|v_{n,N} - v_{n,N-1}\| + \ldots + \|v_{n,1} - x_n\| \\ &\to 0. \end{aligned} \tag{3.12}$$

It follows that

$$\begin{aligned}\|x_n - v_{n,N}\| &\leq \|x_n - y_n\| + \|y_n - v_{n,N}\| \\ &\to 0. \end{aligned} \tag{3.13}$$

From (3.12), we see that

$$\begin{aligned}D_p(w_n, x_n) &\leq \alpha_n D_p(u_n, x_n) + (1-\alpha_n) D_p(y_n, x_n) \\ &\to 0\end{aligned}$$

and hence

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.14}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v \in E$ as $k \to \infty$. Also, we have a subsequence $\{v_{n_k,N}\}$ of $\{v_{n,N}\}$ such that $v_{n_k,N} \rightharpoonup v \in E$ as $k \to \infty$.

We next show that $v \in \Omega$. From (2.1) and (3.9), we have

$$\begin{aligned}D_p(v, \Pi_{C_i} v) &\leq \langle v - \Pi_{C_i} v, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &= \langle v - v_{n_k,N}, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &\quad + \langle v_{n_k,N} - \Pi_{C_i} v_{n_k,N}, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &\quad + \langle \Pi_{C_i} v_{n_k,N} - \Pi_{C_i} v, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &\leq \langle v - v_{n_k,N}, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &\quad + \langle v_{n_k,N} - \Pi_{C_i} v_{n_k,N}, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &\to 0.\end{aligned}$$

This gives $v \in C_i$ for $i = 1, 2, \ldots, M$ and so $v \in \bigcap_{i=1}^M C_i$. Form (3.10) and (3.13), for each $j = 1, 2, \ldots, N-1$, we have

$$\begin{aligned}\|x_n - v_{n,j}\| &\leq \|x_n - v_{n,N}\| + \|v_{n,N} - v_{n,N-1}\| + \ldots + \|v_{n,j+1} - v_{n,j}\| \\ &\to 0.\end{aligned}$$

Since $x_{n_k} \rightharpoonup v$, we also have $v_{n_k,j} \rightharpoonup v$ as $k \to \infty$. For each $j = 1, 2, \dots, N-1$, we note that

$$\begin{aligned}
&\|Av - P_{Q_{j+1}} Av\|^p \\
&\quad = \langle Av - P_{Q_{j+1}} Av, J_p^F(Av - P_{Q_{j+1}} Av)\rangle \\
&\quad = \langle Av - Av_{n_k,j}, J_p^F(Av - P_{Q_{j+1}} Av)\rangle \\
&\qquad + \langle Av_{n_k,j} - P_{Q_{j+1}} Av_{n_k,j}, J_p^F(Av - P_{Q_{j+1}} Av)\rangle \\
&\qquad + \langle P_{Q_{j+1}} Av_{n_k,j} - P_{Q_{j+1}} Av, J_p^F(Av - P_{Q_{j+1}} Av)\rangle \\
&\quad \le \langle Av - Av_{n_k,j}, J_p^F(Av - P_{Q_{j+1}} Av)\rangle \\
&\qquad + \langle Av_{n_k,j} - P_{Q_{j+1}} Av_{n_k,j}, J_p^F(Av - P_{Q_{j+1}} Av)\rangle. \qquad (3.15)
\end{aligned}$$

By the continuity of A, we have $Av_{n_k,j} \rightharpoonup Av$ and $Av_{n_k,j} - P_{Q_{j+1}} v_{n_k,j} \to 0$. Letting $k \to \infty$ in (3.15), we have $\|Av - P_{Q_{j+1}} Av\| = 0$ for each $j = 1, 2, \dots, N-1$. In a similar way, we can see that $\|Av - P_{Q_1} Av\| = 0$. Hence, we have $Av \in Q_j$ for $j = 1, 2, \dots, N$ and so $Av \in \bigcap_{j=1}^N Q_j$. Therefore, $v \in \Omega$.

We next show that

$$\limsup_{n\to\infty} \langle w_n - x^*, J_p^E(u) - J_p^E(x^*)\rangle \le 0.$$

To get this inequality, we can choose a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty} \langle w_n - x^*, J_p^E(u) - J_p^E(x^*)\rangle = \lim_{k\to\infty} \langle w_{n_k} - x^*, J_p^E(u) - J_p^E(x^*)\rangle.$$

Since $x_{n_k} \rightharpoonup v$ and by (3.14), we also have $w_{n_k} \rightharpoonup v$. Then we have

$$\limsup_{n\to\infty} \langle w_n - x^*, J_p^E(u) - J_p^E(x^*)\rangle = \langle v - x^*, J_p^E(u) - J_p^E(x^*)\rangle \le 0. \qquad (3.16)$$

Since $u_n \to u$, it follows that $\lim_{n\to\infty} \langle w_n - x^*, J_p^E(u_n) - J_p^E(u)\rangle = 0$. This together with (3.6) and (3.16), we conclude by Lemma 2.6 that $\Gamma_n \to 0$ as $n \to \infty$. Therefore, $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Then by Lemma 2.7, we can define an integer sequence $\{\tau(n)\}$ for all $n \ge n_0$ by

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Moreover, $\{\tau(n)\}$ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for all $n \geq n_0$. From (3.7), we can show that

$$\lim_{n\to\infty} \|(I - P_{Q_1})Ax_{\tau(n)}\| = 0,$$
$$\lim_{n\to\infty} \|(I - P_{Q_{j+1}})Av_{\tau(n),j}\| = 0 \text{ for each } j = 1, 2, \ldots, N-1$$

and

$$\lim_{n\to\infty} \|v_{\tau(n),N} - \Pi_{C_i} v_{\tau(n),N}\| = 0 \text{ for each } i = 1, 2, \ldots, M.$$

By the similar argument as in Case 1, we can show that

$$\limsup_{n\to\infty} \langle w_{\tau(n)} - x^*, J_p^E(u) - J_p^E(x^*)\rangle \leq 0.$$

Also, from (3.6) and the assumptions of $\{\alpha_{\tau(n)}\}$ and $\{\beta_{\tau(n)}\}$, we have

$$\Gamma_{\tau(n)} \leq \langle w_{\tau(n)} - x^*, J_p^E(u_{\tau(n)}) - J_p^E(u)\rangle + \langle w_{\tau(n)} - x^*, J_p^E(u) - J_p^E(x^*)\rangle. \tag{3.17}$$

Hence, $\limsup_{n\to\infty} \Gamma_{\tau(n)} \leq 0$ and so $\lim_{n\to\infty} \Gamma_{\tau(n)} = 0$. Again from (3.6), we see that

$$\begin{aligned}\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} \leq\ & \alpha_{\tau(n)}(1 - \beta_{\tau(n)})\langle w_{\tau(n)} - x^*, J_p^E(u_{\tau(n)}) - J_p^E(u)\rangle \\ & + \alpha_{\tau(n)}(1 - \beta_{\tau(n)})\langle w_{\tau(n)} - x^*, J_p^E(u) - J_p^E(x^*)\rangle \\ \to\ & 0.\end{aligned}$$

This together with (3.17) implies that $\lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0$. Thus, we have

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1} \to 0,$$

which implies that $D_p(x^*, x_n) \to 0$. Therefore, $x_n \to x^* \in \Omega$. We thus complete the proof. □

Remark 3.2 We note that Theorem 3.1 improves and extends the main results of López et al. [21] and Alsulami and Takahashi [6] in the following ways:

(i) Our result extends the result of López et al. [21] (from SFP in Hilbert spaces to MSFP in Banach spaces) and Alsulami and Takahashi [6] (from SFP between Hilbert and Banach spaces to MSFP in two Banach spaces).

(ii) The step-sizes of our method are very different from Alsulami and Takahashi [6] because they do not depend on the operator norm of the bounded linear operators, while the step-size of those work depends on the operator norm.

(iii) Our result is proved with a new assumption on the control condition $\{\beta_n\}$. However, the assumption that $\liminf_{n\to\infty} \beta_n > 0$ of our result can be removed.

Taking $\beta_n = 0$ for all $n \geq 1$, we obtain the following Halpern-type iteration process in Banach spaces immediately.

Corollary 3.3 *Let E be a p-uniformly convex and uniformly smooth Banach space and F be a reflexive, strictly convex and smooth Banach space. Let $C_i, i = 1, 2, \ldots, M$ and Q_j, $j = 1, 2, \ldots, N$ be nonempty, closed and convex subsets of E and F, respectively. Let $A : E \to F$ be a bounded linear operator and $A^* : F^* \to E^*$ be the adjoint of A. Suppose that $\Omega \neq \emptyset$. Let $\{u_n\}$ be a sequence in E such that $u_n \to u$. For given $x_1 \in E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} v_{n,1} = J_q^{E^*}(J_p^E(x_n) - \tau_{n,1}\nabla f(x_n)), \\ v_{n,2} = J_q^{E^*}(J_p^E(v_{n,1}) - \tau_{n,2}\nabla f(v_{n,1})), \\ \quad \vdots \\ v_{n,N} = J_q^{E^*}(J_p^E(v_{n,N-1}) - \tau_{n,N}\nabla f(v_{n,N-1})), \\ y_n = J_q^{E^*}(a_{n,0}J_p^E(v_{n,N}) + \sum_{i=1}^M a_{n,i}J_p^E(\Pi_{C_i}v_{n,N})), \\ x_{n+1} = J_q^{E^*}(\alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n)), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{a_{n,i}\}_{i=1}^M \subset (0, 1)$, $f(v_{n,j}) = \frac{1}{p}\|(I - P_{Q_{j+1}})Av_{n,j}\|^p$ for $j = 1, 2, \ldots, N-1$ and $f(x_n) = \frac{1}{p}\|(I - P_{Q_1})Ax_n\|^p$ with the step-sizes $\tau_{n,1}$ and $\tau_{n,j}$, $j = 1, 2, \ldots, N-1$ are chosen self-adaptively as

$$\tau_{n,1} = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tau_{n,j+1} = \begin{cases} \frac{\rho_n f^{p-1}(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}, & \text{if } f(v_{n,j}) \neq 0; \\ 0, & \text{otherwise}, \end{cases}$$

respectively, where $\{\rho_n\} \subset \big(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\big)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$*;*

(C2) $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big) > 0$*;*

(C3) $\sum_{i=0}^M a_{n,i} = 1$ *and* $\liminf_{n\to\infty} a_{n,i} > 0$ *for* $i = 1, 2, \ldots, M$.

Then $\{x_n\}$ converges strongly to $x^* = \Pi_\Omega u$, where Π_Ω is the Bregman projection from E onto Ω.

We consequently obtain the following result in Hilbert spaces.

Corollary 3.4 *Let H_1 and H_2 be two real Hilbert spaces. Let C_i, $i = 1, 2, \ldots, M$ and Q_j, $j = 1, 2, \ldots, N$ be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be* *the adjoint of A. Suppose that* $\Omega \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence in* H_1 *such that* $u_n \to u$. *For given* $x_1 \in H_1$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} v_{n,1} = x_n - \tau_{n,1} \nabla f(x_n), \\ v_{n,2} = v_{n,1} - \tau_{n,2} \nabla f(v_{n,1}), \\ \ \vdots \\ v_{n,N} = v_{n,N-1} - \tau_{n,N} \nabla f(v_{n,N-1}), \\ y_n = a_{n,0} v_{n,N} + \sum_{i=1}^{M} a_{n,i} P_{C_i} v_{n,N}, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n) y_n), \quad \forall n \geq 1, \end{cases} \tag{3.18}$$

where $\{\alpha_n\} \subset (0,1)$, $\{a_{n,i}\}_{i=1}^{M} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$, $f(v_{n,j}) = \frac{1}{2}\|(I - P_{Q_{j+1}})Av_{n,j}\|^2$ *for* $j = 1, 2, \ldots, N-1$ *and* $f(x_n) = \frac{1}{2}\|(I - P_{Q_1})Ax_n\|^2$ *with the step-sizes* $\tau_{n,1}$ *and* $\tau_{n,j}$, $j = 1, 2, \ldots, N-1$ *are chosen self-adaptively as*

$$\tau_{n,1} = \begin{cases} \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tau_{n,j+1} = \begin{cases} \frac{\rho_n f(v_{n,j})}{\|\nabla f(v_{n,j})\|^2}, & \text{if } f(v_{n,j}) \neq 0; \\ 0, & \text{otherwise}, \end{cases}$$

respectively, where $\{\rho_n\} \subset (0,4)$. *Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$;
(C3) $\sum_{i=0}^{M} a_{n,i} = 1$ *and* $\liminf_{n\to\infty} a_{n,i} > 0$ *for* $i = 1, 2, \ldots, M$;
(C4) $\limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $x^* = P_\Omega u$, *where* P_Ω *is the metric projection from* H_1 *onto* Ω.

We obtain the following result for the SFP in Banach spaces.

Corollary 3.5 *Let E be a p-uniformly convex and uniformly smooth Banach space and F be a reflexive, strictly convex and smooth Banach space. Let C and Q be nonempty, closed and convex subsets of E and F, respectively. Let* $A : E \to F$ *be a bounded linear operator and* $A^* : F^* \to E^*$ *be the adjoint of A. Suppose that* $\Omega \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence in E such that* $u_n \to u$. *For given* $x_1 \in E$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = \Pi_C J_q^{E^*}(J_p^E(x_n) - \tau_n \nabla f(x_n)), \\ x_{n+1} = J_q^{E^*}(\beta_n J_p^E(x_n) + (1-\beta_n)(\alpha_n J_p^E(u_n) + (1-\alpha_n) J_p^E(y_n))), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$*,* $\{\beta_n\} \subset [0, 1)$ *and* $f(x_n) = \frac{1}{p}\|(I - P_Q)Ax_n\|^p$ *with the step-size* τ_n *is chosen self-adaptively as*

$$\tau_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise}, \end{cases}$$

where $\{\rho_n\} \subset \big(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\big)$*. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1} c_q}{q}\Big) > 0$*;*
(C3) $\limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $x^* = \Pi_\Omega u$*, where* Π_Ω *is the Bregman projection from* E *onto* Ω.

4 Numerical Examples

In this section, we give some numerical examples to support our main result.

4.1 Numerical Example in Finite Dimensional Spaces

Example 4.1 We consider MSFP (1.1) with $C_i \subset \mathbb{R}^{\mathcal{N}}$ and $Q_j \subset \mathbb{R}^{\mathcal{M}}$, which are defined by

$$C_i = \{x \in \mathbb{R}^{\mathcal{N}} : \langle a_i^C, x\rangle \leq b_i^C\},$$
$$Q_j = \{x \in \mathbb{R}^{\mathcal{M}} : \langle a_j^Q, x\rangle \leq b_j^Q\},$$

where $a_i^C \in \mathbb{R}^{\mathcal{N}}, a_j^Q \in \mathbb{R}^{\mathcal{M}}$, $b_i^C, b_j^Q \in \mathbb{R}$ for all $i = 1, 2, \ldots, M$ and all $j = 1, 2, \ldots, N$, and A is a bounded linear operator from $\mathbb{R}^{\mathcal{N}}$ into $\mathbb{R}^{\mathcal{M}}$ the elements of the representing matrix of which are randomly generated in the closed interval [5, 10]. Next, we use randomly generated values of the coordinates of a_i^C, a_j^Q in the closed interval [3, 5] and of b_i^C, b_j^Q in the closed interval [1, 10], respectively. It is clear that $\Omega := \Big(\bigcap_{i=1}^M C_i\Big) \cap A^{-1}\Big(\bigcap_{j=1}^N Q_j\Big) \neq \emptyset$ because $0 \in \Omega$.

Remark 4.2 In this example, we define the function TOL_n by

$$\text{TOL}_n = \frac{1}{M}\sum_{i=1}^{M} \|x_n - P_{C_i}x_n\|^2 + \frac{1}{N}\sum_{j=1}^{N} \|Ax_n - P_{Q_j}Ax_n\|^2, \quad \forall n \geq 1.$$

We use the stopping rule $\text{TOL}_n < \text{err}$ to stop the iterative process. Note that if at the nth step $\text{TOL}_n = 0$, then $x_n \in \Omega$, that is, x_n is a solution to this problem.

Table 1 Table of numerical results for Example 4.1

Stop condition: $\text{TOL}_n < \text{err}$			
err	TOL_n	n	Time (s)
10^{-5}	9.996495e − 06	251	0.907
10^{-6}	9.931760e − 07	411	1.383
10^{-7}	9.923342e − 08	541	1.803
10^{-8}	9.528326e − 09	603	1.947
10^{-9}	9.759029e − 010	625	2.099

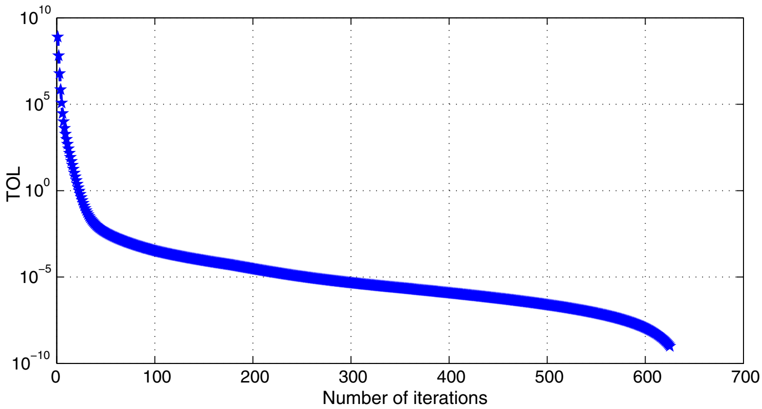

Fig. 1 The behavior of TOL_n with the stop condition $\text{TOL}_n < 10^{-9}$

Applying iterative method (3.18) in Corollary 3.4 with $\mathcal{N} = 40$, $\mathcal{M} = 50$, $M = 30$, $N = 40$, $\beta_n = \frac{3}{4}$, $\alpha_n = \frac{1}{n+1}$, $\rho_n = 0.25$ and $u_n = u$ for all $n \geq 1$. Take the initial values $u, x_1 \in \mathbb{R}^{\mathcal{N}}$ where its coordinates are also randomly generated in the closed interval [10, 50], we arrive at the following table of numerical results (Table 1).

The behavior of TOL_n in the case $\text{TOL}_n < 10^{-9}$ is described in Fig. 1.

4.2 Numerical Examples in Infinite Dimensional Spaces

Example 4.3 In this example, we take $E = F = L_2([0, \pi])$ with the inner product

$$\langle f, g \rangle = \int_0^{\pi} f(t)g(t)\mathrm{d}t$$

and the norm

$$\|f\| = \left(\int_0^{\pi} f^2(t)\mathrm{d}t \right)^{1/2},$$

for all $f, g \in L_2([0, \pi])$.

Now, let

$$C_i = \{x \in L_2([0,\pi]) : \langle a_i, x\rangle = b_i\},$$

where $a_i(t) = \sin(2it)$, $b_i = \dfrac{4i}{4i^2 - 1}$ for all $i = 1, 2, \ldots, M$ and $t \in [0,\pi]$,

$$Q_j = \{x \in L_2([0,\pi]) : \langle c_j, x\rangle \le d_j\},$$

in which $c_j(t) = \exp(jt)$, $d_j = \dfrac{\exp(j\pi) - 1}{j}$ for all $j = 1, 2, \ldots, N$ and $t \in [0,\pi]$.

Let us assume that

$$A : L_2([0,\pi]) \to L_2([0,\pi]),\ (Ax)(t) = \frac{x(t)}{2}.$$

We consider the Problem (1.1) with C_i, Q_j and A are defined as the above. It is easy to check that $x(t) = \cos t + c \in \bigcap_{i=1}^{M} C_i$, with c is an arbitrary real number. Moreover, if the constant $c \in [0, 1]$, then we have

$$\int_0^\pi \exp(jt)\frac{\cos t + c}{2}\mathrm{d}t \le \int_0^\pi \exp(jt)\mathrm{d}t = \frac{\exp(j\pi) - 1}{j},$$

for all $j = 1, 2, \ldots, N$. So, we obtain that $A(\cos t + c) \in \bigcap_{j=1}^{N} Q_j$. Thus, we arrive that

$$x(t) = \cos t + c \in \left(\bigcap_{i=1}^{M} C_i\right) \cap A^{-1}\left(\bigcap_{j=1}^{N} Q_j\right),\ \forall c \in [0, 1].$$

So, the set of the solutions of the Problem (1.1) is a nonempty set.

When $M = 50$, $N = 100$, with the same initial guess elements $x_1(t) = t^2 + 1$ and $u_n(t) = u(t) = t$ for all $n \ge 1$ and $t \in [0,\pi]$, we now consider the convergence of iterative method (3.18) with $\rho_n = 0.05$, $\beta_n = 0.25$, $\alpha_n = 1/n$, $a_{n,i} = 1/(M + 1)$ for all $n \ge 1$, $i = 0, 1, \ldots, M$, and iterative method (27) in [38, Theorem 4.1] with $\rho_n = 0.05$, $\beta_{i,n} = 1.5$, $\lambda_{j,n} = 0.5$, $\alpha_n = 1/n$ for all $n \ge 1$, $i = 1, 2, \ldots, M$, and $j = 1, 2, \ldots, N$. Note that, we define the function TOL_n as in Example 4.1 and use the stopping rule $\mathrm{TOL}_n <$ err to stop the iterative process.

The behaviors of the approximation solution $x_n(t)$ in Table 2 (with $\mathrm{TOL}_n < 10^{-3}$ and $\mathrm{TOL}_n < 10^{-4}$) are presented in Figs. 2 and 3.

Finally, we provide some connection between the MSFP and the Fredholm integral equations.

Example 4.4 Let us consider the Fredholm integral equation of the first kind as considered in [4],

$$\int_a^b K(s,t)x(t)\mathrm{d}t = g(s),\quad a \le s \le b, \tag{4.1}$$

Table 2 Table of numerical results for Example 4.3

Stop condition: $\text{TOL}_n < \text{err}$							
Iterative method (3.18)				Iterative method (27) in [38]			
err	TOL_n	n	Time(s)	err	TOL_n	n	Time (s)
10^{-3}	$9.983193e-04$	904	2.112	10^{-3}	$9.973428e-04$	1183	2.629
10^{-4}	$9.995264e-05$	2856	6.482	10^{-4}	$9.956379e-05$	3737	8.328
10^{-5}	$9.999619e-06$	9028	20.195	10^{-5}	$9.976772e-06$	11744	25.908

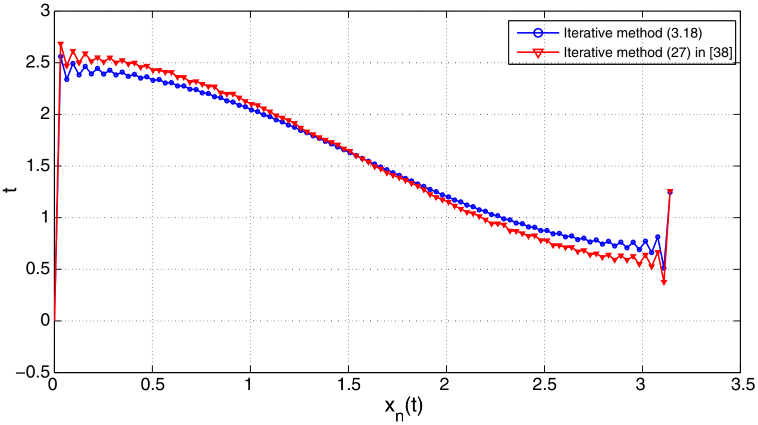

Fig. 2 The behavior of $x_n(t)$ with the stop condition $\text{TOL}_n < 10^{-3}$

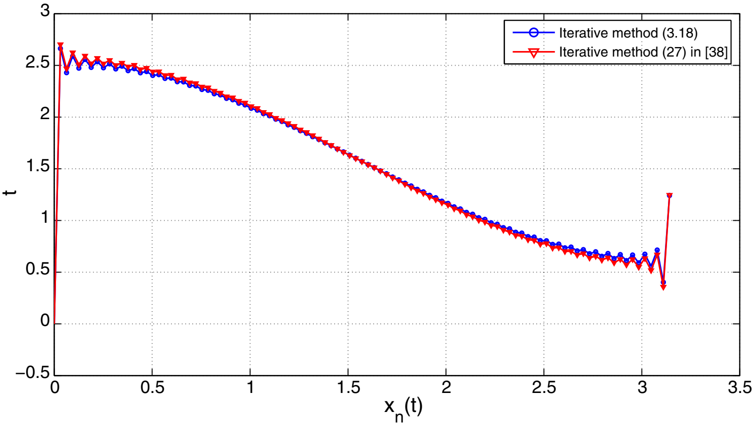

Fig. 3 The behavior of $x_n(t)$ with the stop condition $\text{TOL}_n < 10^{-4}$

where $K : [a, b]^2 \to \mathbb{R}$ is the continuous kernel and $g : [a, b] \to \mathbb{R}$ is the continuous free term. Consider the computing L_p-solutions of the Problem (4.1): find $x^* \in \bigcap_{i=1}^{M} C_i$, where

Table 3 Table of numerical results for Example 4.4

Stop condition: TOL$_n = \|x_n - x^*(t)\| <$ err			
err	TOL$_n$	n	Time (s)
10^{-3}	9.996388e − 04	2065	2.036
10^{-4}	9.999633e − 05	20722	20.108
10^{-5}	9.999961e − 06	208106	200.647

$$C_i = \{x \in L_p([a, b]) : \langle a_i, x\rangle = b_i\},$$

with $a_i(t) = K(s_i, t) \in L_q([a, b])$ and $b_i = g(s_i) \in \mathbb{R}$ for $i = 1, 2, .., M$, while $a = s_1 < s_2 < \cdots < s_M = b$ (see [18,49]). Under some hypothesis, (4.1) has solutions [24], then approximating an L_p-solution of (4.1) equivalent to solving the MSFP with $E = F = L_p([a, b])$, $A = I$ and $Q_j = L_p([a, b])$ for all $j = 1, 2, \ldots, N$.

We consider the following the Fredholm integral equations of the first kind [47, Example 2]:

$$\frac{\pi}{2}\cos s = \int_0^\pi \cos(t - s)x(t)\mathrm{d}t,\ 0 \le s \le \pi. \tag{4.2}$$

It follows from [47, Example 2] that the set of solutions of the Problem (4.2) is a nonempty set. Moreover, $x(t) = \cos t$ or $x(t) = \cos t + \sin(2n + 1)t$, $n = 1, 2, \ldots$ are solutions of this problem.

We now approximate the solution of the Problem (4.2) in $L_2([0, \pi])$ by solving the MSFP, that is, find $x^* \in \bigcap_{i=1}^M C_i$, where

$$C_i = \{x \in L_2([0, \pi]) : \langle a_i, x\rangle = b_i\},$$

with $a_i(t) = \cos(t - s_i)$ and $b_i = \dfrac{\pi}{2}\cos s_i$ for $i = 1, 2, .., M$, while $0 = s_1 < s_2 < \cdots < s_M = \pi$.

In this case, the sequence $\{x_n\}$ is defined by (see, iterative method (3.18) in Corollary 3.4) $x_1, u \in L_2([0, \pi])$, and

$$\begin{cases} y_n = a_{n,0}x_n + \sum_{i=1}^M a_{n,i}P_{C_i}x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)y_n),\ \forall n \ge 1. \end{cases} \tag{4.3}$$

Applying iterative method (4.3) with $a_{n,i} = 1/(M + 1)$, $\beta_n = 0.05$, $\alpha_n = 1/n$ for all $n \ge 1$ and for all $i = 0, 1, \ldots, M$. Take the initial values $x_1(t) = 1$, $u_n(t) = u(t) = \sin 3t$ for all $n \ge 1$ and $t \in [0, \pi]$, we obtain the following table of numerical results (Table 3).

Remark 4.5 Note that, in this example when $u_n(t) = u(t) = \sin 3t$ for all $n \ge 1$, then $x^*(t) = \cos t + \sin 3t$ is the projection of u onto the set of solutions of Problem (4.2).

Acknowledgements The first author was supported by RMUTT Research Grant for New Scholar under Grant NSF62D0602. The second author was supported by the Science and Technology Fund of the Vietnam Ministry of Education and Training (B2022). Both authors are grateful to two anonymous referees for their useful comments and helpful suggestions.

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