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ORIGINAL PAPER

Inverse Problems for Sturm–Liouville-Type Differential Equation with a Constant Delay Under Dirichlet/Polynomial Boundary Conditions

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Abstract

The topic of this paper are non-self-adjoint second-order differential operators with constant delay generated by $-y'' + q(x)y(x - \tau)$ where potential q is complex-valued function, $q \in L^2[0, \pi]$. We study inverse problems of these operators for $\tau \in \left[\frac{2\pi}{5}, \pi\right)$. We investigate the inverse spectral problems of recovering operators from their two spectra, firstly under Dirichlet–Dirichlet and second under Dirichlet/Polynomial boundary conditions. We will prove theorem of uniqueness, and we will give procedure for constructing potential. In the first case, for $\tau \in \left[\frac{\pi}{2}, \pi\right)$: we will show that Fourier coefficients of a potential are uniquely0 determined by spectra. In the second case for $\tau \in \left[\frac{2\pi}{5}, \frac{\pi}{2}\right)$, we will construct integral equation under potential and we will prove that this integral equation has a unique solution. Also, we will show that other parameters are uniquely determined by spectra.

Keywords Differential operators with delay · Inverse problems · Fourier trigonometric coefficients · Integral equations

Mathematics Subject Classification 34B24 · 34A55

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1 Introduction

An intensive development of the spectral theory for various classes of differential and integral operators as well as for operators in abstract spaces took place in the second half of twentieth century and twenty-first century. Within this theory inverse spectral problems take a special place. Inverse spectral problems are problems studying operator from some of its spectral characteristics. The inverse spectral problem can be regarded from three aspects: existence, uniqueness and reconstruction of the operators with specific properties of eigenvalues and eigenfunctions. The Sturm–Liouville-type operators are generated by second-order differential expression and boundary conditions (see [7,9] and references therein). In this paper, we study Sturm–Liouville operators with one constant delay under Dirichlet–Dirichlet and Dirichlet/Polynomial boundary conditions. In the papers [1–3,6,8,13–16] authors study this differential expression under different types of boundary conditions. Also in many papers authors study boundary conditions with spectral parameter (see [10–12]). We will show uniqueness and we will recover the operators from two spectra.

In this paper we generalize the results in the paper [13] and [14].

In this paper, we study two boundary value problems $L_k, k = 0, 1$, L_0 generated by (1.1),(1.2),(1.3) and L_1 generated by (1.1),(1.2),(1.4)

$$-y''(x) + q(x)y(x - \tau) = \lambda y(x),\ x \in [0, \pi], \tag{1.1}$$

$$y(0) = 0, \tag{1.2}$$

$$y(\pi) = 0, \tag{1.3}$$

$$y'(\pi) + P(\lambda)y(\pi) = 0, \tag{1.4}$$

where λ is the spectral parameter, $\tau \in \left[\frac{2\pi}{5}, \pi\right)$, Function $q(x)$ is a complex-valued function which we call potential, such that $q \in L^2(0, \pi)$, and $q(x) \equiv 0$ for $x \in [0, \tau]$. Function $P(\lambda)$ is normalized polynomial with degree s, $s \in N$ and complex coefficients. We will separately study two cases, the first when $\tau \in \left[\frac{\pi}{2}, \pi\right)$ and the second when $\tau \in \left[\frac{2\pi}{5}, \frac{\pi}{2}\right)$.

The spectra of L_0 and L_1 are countable. We will prove that the potential q and polynomial P are uniquely determined from the spectra of L_0 and L_1. Let $(\lambda_n)_{n=1}^{\infty}$ be the eigenvalues of L_0 and $(\mu_n)_{n=1}^{\infty}$ be the eigenvalues of L_1.

The inverse problem is to prove that $q(x)$ and $P(\lambda)$ are uniquely determined from $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$, and determine $q(x)$ and $P(\lambda)$ from $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$.

2 Preliminaries

Let the function $Y(x)$ be the solution of the differential equation (1.1) under initial condition $Y(0) = 0$, $Y'(0) = 1$, then the function $Y(x)$ satisfy an integral equation

$$Y(x) = \frac{1}{z}\int_0^x q(t)Y(t-\tau)\sin z(x-t)\mathrm{d}t + \frac{\sin zx}{z}, \tag{2.1}$$

where $\lambda = z^2$. We will be solving equation (2.1) using $q(x) \equiv 0$ for $x \in [0, \tau]$.

For $x \in [0, \tau]$, the solution is

$$Y(x) = \frac{1}{z}\int_0^x q(t)Y(t-\tau)\sin z(x-t)\mathrm{d}t + \frac{\sin zx}{z} = \frac{\sin zx}{z}.$$

For $x \in (\tau, 2\tau]$, the solution is

$$Y(x) = \frac{1}{z^2}\int_\tau^x q(t)\sin z(t-\tau)\sin z(x-t)\mathrm{d}t + \frac{\sin zx}{z}. \tag{2.2}$$

For $x \in (2\tau, 3\tau]$, the solution is

$$Y(x) = \frac{\sin zx}{z} + \frac{1}{z^2}\int_\tau^x q(t)\sin z(t-\tau)\sin z(x-t)\mathrm{d}t$$
$$+\frac{1}{z^3}\int_{2\tau}^x\int_\tau^{t-\tau} q(t)q(t_1)\sin z(x-t)\sin z(t_1-\tau)\sin z(t-\tau-t_1)\mathrm{d}t_1\mathrm{d}t. \tag{2.3}$$

3 Main Results

3.1 Linear Case, $\tau \in \left[\frac{\pi}{2}, \pi\right)$

In the case, when $\tau \in \left[\frac{\pi}{2}, \pi\right)$ we have $\pi \in (\tau, 2\tau]$, and using (1.3) and (1.4) from Eq. (2.2), we get

$$\Delta_0(\lambda) = y(\pi) = \frac{\sin z\pi}{z} + \frac{1}{z^2}\int_\tau^\pi q(t)\sin z(t-\tau)\sin z(\pi-t)\mathrm{d}t,$$

$$\Delta_1(\lambda) = y'(\pi) + P(\lambda)y(\pi) = \cos z\pi + \frac{1}{z}\int_\tau^\pi q(t)\sin z(t-\tau)\cos z(\pi-t)\mathrm{d}t + \tag{3.1}$$

$$P(z^2)\left(\frac{\sin z\pi}{z} + \frac{1}{z^2}\int_\tau^\pi q(t)\sin z(t-\tau)\sin z(\pi-t)\mathrm{d}t\right). \tag{3.2}$$

The functions $\Delta_0(\lambda)$, $\Delta_1(\lambda)$ are entire in λ of order 1/2. It is clear that the set of zeros of functions $\Delta_0(\lambda)$, $\Delta_1(\lambda)$ is equivalent to the spectrum of boundary spectral problems L_0, L_1, respectively. For the spectrum $(\lambda_n)_{n=1}^{\infty}$ of boundary spectral problems L_0, we have the asymptotic formula (see [8]):

$$\lambda_n = n^2 + \frac{\cos \tau n}{\pi} \int_{\tau}^{\pi} q(t)\mathrm{d}t + o(1), \ (n \to \infty).$$

For the spectrum $(\mu_n)_{n=1}^{\infty}$ of boundary spectral problems L_1, using well-known method (see [8]), based on Rouche‘s theorem, we have asymptotic formula which depends on the degree of the polynomial $P(\lambda)$. When degree of the polynomial $P(\lambda)$ is equal to 1, $s = 1$ we have asymptotic formula

$$\mu_n = n^2 + \frac{\cos \tau n}{\pi} \int_{\tau}^{\pi} q(t)\mathrm{d}t - \frac{2}{\pi} + o(1), \ (n \to \infty).$$

When degree of the polynomial $P(\lambda)$ is different from 1, $s > 1$, we have asymptotic formula

$$\mu_n = n^2 + \frac{\cos \tau n}{\pi} \int_{\tau}^{\pi} q(t)\mathrm{d}t + o(1), \ (n \to \infty).$$

Using Hadamard’s factorization theorem we conclude that spectra uniquely determine functions $\Delta_0(\lambda)$, $\Delta_1(\lambda)$. We introduce notation

$$F_0(z) = z\Delta_0(\lambda), \ F_1(z) = z\Delta_1(\lambda).$$

The delay τ and the integral $\int_{\tau}^{\pi} q(t)\mathrm{d}t = I_1$ are uniquely determined from the spectrum $(\lambda_n)_{n=1}^{\infty}$(see [1]).

Lemma 3.1 *The spectra $(\mu_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$ of boundary value problem L_1 and L_0 uniquely determine polynomial $P(\lambda)$.*

Proof Function $F_1(z)$ is uniquely determined by spectrum $(\mu_n)_{n=1}^{\infty}$ and function $F_0(z)$ is uniquely determined by spectrum $(\lambda_n)_{n=1}^{\infty}$.

Let $P(\lambda) = \lambda^s + p_{s-1}\lambda^{s-1} + \cdots + p_0$, $p_i \in C$. We have

$$\begin{aligned} F_1(z) &= z \cos z\pi + \int_{\tau}^{\pi} q(t) \sin z(t-\tau) \cos z(\pi - t)\mathrm{d}t \\ &\quad + (z^{2s} + p_{s-1}z^{2s-2} + \cdots + p_0)F_0(z). \end{aligned}$$

Now, we put $z = 2m + \frac{1}{2}$, $m \in N$ and we have

$$F_1\left(2m + \frac{1}{2}\right) - \left(2m + \frac{1}{2}\right)^{2s} F_0\left(2m + \frac{1}{2}\right)$$

$$= \int_{\tau}^{\pi} q(t) \sin\left(\left(2m+\frac{1}{2}\right)(t-\tau)\right) \cos\left(\left(2m+\frac{1}{2}\right)(\pi-t)\right) \mathrm{d}t$$
$$+ \left(p_{s-1}\left(2m+\frac{1}{2}\right)^{2s-2} + \cdots + p_0\right) F_0\left(2m+\frac{1}{2}\right).$$

Since

$$\lim_{m\to\infty} F_0\left(2m+\frac{1}{2}\right)$$
$$= \lim_{m\to\infty}\left(1+\frac{1}{2m+\frac{1}{2}}\int_{\tau}^{\pi} q(t)\sin\left(\frac{(4m+1)(t-\tau)}{2}\right)\sin\left(\frac{(4m+1)(\pi-t)}{2}\right)\mathrm{d}t\right) = 1,$$

we conclude $\lim_{m\to\infty} \frac{\left(p_{s-2}\left(2m+\frac{1}{2}\right)^{2s-4}+\cdots+p_0\right)F_0\left(2m+\frac{1}{2}\right)}{\left(2m+\frac{1}{2}\right)^{2s-2}} = 0$, and from this equation, we have

$$p_{s-1} = \lim_{m\to\infty} \frac{F_1\left(2m+\frac{1}{2}\right) - \left(2m+\frac{1}{2}\right)^{2s} F_0\left(2m+\frac{1}{2}\right)}{\left(2m+\frac{1}{2}\right)^{2s-2}}.$$

Now, when we showed that coefficient p_{s-1} is known from spectra $(\mu_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$, we have

$$p_{s-2} = \lim_{m\to\infty} \frac{F_1\left(2m+\frac{1}{2}\right) - \left(\left(2m+\frac{1}{2}\right)^{2s} + p_{s-1}\left(2m+\frac{1}{2}\right)^{2s-2}\right) F_0\left(2m+\frac{1}{2}\right)}{\left(2m+\frac{1}{2}\right)^{2s-4}}.$$

We repeat this procedure and we have all coefficients p_{s-1}, ..., p_1 are determined by spectra $(\mu_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$.

Now, we will show that coefficient p_0 is determined by spectra $(\mu_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$.

First we transform product trigonometric functions to sum and we have

$$\int_{\tau}^{\pi} q(t)\sin z(t-\tau)\cos z(\pi-t)\mathrm{d}t = \frac{\sin z(\pi-\tau)}{2} I_1$$
$$+\frac{\cos z(\pi+\tau)}{2}\int_{\tau}^{\pi} q(t)\sin 2zt\mathrm{d}t - \frac{\sin z(\pi+\tau)}{2}\int_{\tau}^{\pi} q(t)\cos 2zt\mathrm{d}t,$$

if we put $z = 2m + \frac{1}{2}$, $m \in N$, from Riemann–Lebesgue lemma we have

$$\lim_{m\to\infty}\left(\int_{\tau}^{\pi} q(t)\sin\frac{(4m+1)(t-\tau)}{2}\cos\frac{(4m+1)(\pi-t)}{2}\mathrm{d}t - \frac{\sin\frac{(4m+1)(\pi-\tau)}{2}}{2}I_1\right) = 0.$$

Finally, we have

$$p_0 = \lim_{m\to\infty}\frac{F_1\left(2m+\frac{1}{2}\right) - \frac{\sin\frac{(4m+1)(\pi-\tau)}{2}}{2}I_1}{F_0\left(2m+\frac{1}{2}\right)} - \left(\left(2m+\frac{1}{2}\right)^{2s} + p_{s-1}\left(2m+\frac{1}{2}\right)^{2s-2} + \cdots + p_1\left(2m+\frac{1}{2}\right)^{2}\right),$$

and polynomial function $P(\lambda)$ is ordered by spectra $(\mu_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$. □

Since $q(x) = 0$ for $x \in [0,\tau)$, we have $a_0 = \int_0^{\pi} q(t)\mathrm{d}t = \int_{\tau}^{\pi} q(t)\mathrm{d}t = I_1$ and coefficient a_0 is ordered by spectrum $(\lambda_n)_{n=1}^{\infty}$. Now we will prove that the other coefficients $a_m = \int_0^{\pi} q(t)\cos 2mt\mathrm{d}t$ and $b_m = \int_0^{\pi} q(t)\sin 2mt\mathrm{d}t$ of the potential q are also uniquely determined by spectra.

Theorem 3.2 *Let $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ be the spectra of boundary spectral problems L_k, $k = 0, 1$, respectively, then potential q is uniquely determined by $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ if $\frac{\pi}{2} \le \tau < \pi$.*

Proof From (3.1) and (3.2), we have

$$F_0(z) = \sin z\pi + \frac{1}{z}\int_{\tau}^{\pi} q(t)\sin z(t-\tau)\sin z(\pi-t)\mathrm{d}t$$

$$F_1(z) = z\cos z\pi + P(z^2)\sin z\pi + \int_{\tau}^{\pi} q(t)\sin z(t-\tau)\cos z(\pi-t)\mathrm{d}t + \frac{P(z^2)}{z}\int_{\tau}^{\pi} q(t)\sin z(t-\tau)\sin z(\pi-t)\mathrm{d}t.$$

Using transformations product of trigonometric functions to sum and addition formulas, we have

$$F_0(z) = \sin z\pi - \frac{\cos z(\pi-\tau)}{2z}\int_{\tau}^{\pi} q(t)\mathrm{d}t + \frac{\cos z(\pi+\tau)}{2z}\int_{\tau}^{\pi} q(t)\cos 2zt\mathrm{d}t$$

$$+\frac{\sin z(\pi+\tau)}{2z}\int\limits_{\tau}^{\pi} q(t)\sin 2zt\mathrm{d}t,$$

and

$$\begin{aligned}F_1(z) &= z\cos z\pi + P(z^2)\sin z\pi + \frac{\sin z(\pi-\tau)}{2}\int\limits_{\tau}^{\pi} q(t)\mathrm{d}t - \frac{\sin z(\pi+\tau)}{2}\int\limits_{\tau}^{\pi} q(t)\cos 2zt\mathrm{d}t \\ &+\frac{\cos z(\pi+\tau)}{2}\int\limits_{\tau}^{\pi} q(t)\sin 2zt\mathrm{d}t - \frac{P(z^2)\cos z(\pi-\tau)}{2z}\int\limits_{\tau}^{\pi} q(t)\mathrm{d}t \\ &+\frac{P(z^2)\cos z(\pi+\tau)}{2z}\int\limits_{\tau}^{\pi} q(t)\cos 2zt\mathrm{d}t + \frac{P(z^2)\sin z(\pi+\tau)}{2z}\int\limits_{\tau}^{\pi} q(t)\sin 2zt\mathrm{d}t.\end{aligned}$$

Now we put $z = m$, $m \in N$ and using $q(x) = 0$ for $x \in [0, \tau)$, we have

$$F_0(m) + \frac{\cos m(\pi-\tau)}{2m}\int\limits_{\tau}^{\pi} q(t)\mathrm{d}t = \frac{\cos m(\pi+\tau)}{2m}a_m + \frac{\sin m(\pi+\tau)}{2m}b_m,$$

$$\begin{aligned}&F_1(m) - m(-1)^m - \frac{\sin m(\pi-\tau)}{2}\int_{\tau}^{\pi} q(t)\mathrm{d}t + \frac{P(m^2)\cos m(\pi-\tau)}{2m}\int\limits_{\tau}^{\pi} q(t)\mathrm{d}t \\ &= \left(\frac{P(m^2)\cos m(\pi+\tau)}{2m} - \frac{\sin m(\pi+\tau)}{2}\right)a_m \\ &+\left(\frac{P(m^2)\sin m(\pi+\tau)}{2m} + \frac{\cos m(\pi+\tau)}{2}\right)b_m.\end{aligned}$$

This is linear system of two variables a_m and b_m with determinant $D = \frac{1}{4m} \neq 0$, and which has unique solution

$$\begin{aligned}a_m &= \left((-1)^m F_0(m) + \frac{\cos m\tau}{2m}\int\limits_{\tau}^{\pi} q(t)\mathrm{d}t\right)\left(2m\cos m\tau + P(m^2)\sin m\tau\right) \\ &-2\left((-1)^m F_1(m) - m + \frac{\sin m\tau}{2}\int_{\tau}^{\pi} q(t)\mathrm{d}t + \frac{P(m^2)\cos m\tau}{2m}\int\limits_{\tau}^{\pi} q(t)\mathrm{d}t\right)\sin m\tau \\ b_m &= 2\cos m\tau\left((-1)^m F_1(m) - m + \frac{\sin m\tau}{2}\int\limits_{\tau}^{\pi} q(t)\mathrm{d}t + \frac{P(m^2)\cos m\tau}{2m}\int\limits_{\tau}^{\pi} q(t)\mathrm{d}t\right) \\ &-2\left(P(m^2)\cos m\tau - m\sin m\tau\right)\left((-1)^m F_0(m) + \frac{\cos m\tau}{2m}\int\limits_{\tau}^{\pi} q(t)\mathrm{d}t\right).\end{aligned}$$

Since τ, $\int_\tau^\pi q(t)\mathrm{d}t$, polynomial $P(\lambda)$ and functions F_0, F_1 are determined from spectra $(\lambda_n)_{n=1}^\infty$ and $(\mu_n)_{n=1}^\infty$, coefficients a_m and b_m are also determined from this spectra. Since $q \in L^2[0, \pi]$, we have

$$q(x) = \sum_{m=-\infty}^{m=+\infty} c_m \mathrm{e}^{2imx},$$

where is $c_m = \frac{1}{\pi} a_m - \frac{i}{\pi} b_m$, and we finally conclude that potential q is uniquely determine from spectra $(\lambda_n)_{n=1}^\infty$ and $(\mu_n)_{n=1}^\infty$. □

3.2 Nonlinear Case, $\tau \in \left[\frac{2\pi}{5}, \frac{\pi}{2}\right)$

In the case when $\tau \in \left[\frac{2\pi}{5}, \frac{\pi}{2}\right)$ we have $\pi \in (2\tau, 3\tau]$, and using (1.3) and (1.4) from Eq. (2.3), we get

$$\begin{aligned}\Delta_0(\lambda) = y(\pi) &= \frac{\sin z\pi}{z} + \frac{1}{z^2}\int_\tau^\pi q(t)\sin z(t-\tau)\sin z(\pi-t)\mathrm{d}t \\ &+\frac{1}{z^3}\int_{2\tau}^{\pi}\int_\tau^{t-\tau} q(t)q(t_1)\sin z(\pi-t)\sin z(t_1-\tau)\sin z(t-\tau-t_1)\mathrm{d}t_1\mathrm{d}t.\end{aligned} \tag{3.3}$$

$$\begin{aligned}\Delta_1(\lambda) = y'(\pi) + P(\lambda)y(\pi) &= \cos z\pi + \frac{1}{z}\int_\tau^\pi q(t)\cos z(\pi-t)\sin z(t-\tau)\mathrm{d}t \\ &+\frac{1}{z^2}\int_{2\tau}^{\pi}\int_\tau^{t-\tau} q(t)q(t_1)\cos z(\pi-t)\sin z(t_1-\tau)\sin z(t-\tau-t_1)\mathrm{d}t_1\mathrm{d}t \\ &+P(z^2)\frac{\sin z\pi}{z} + \frac{P(z^2)}{z^2}\int_\tau^\pi q(t)\sin z(t-\tau)\sin z(\pi-t)\mathrm{d}t \\ &+\frac{P(z^2)}{z^3}\int_{2\tau}^{\pi}\int_\tau^{t-\tau} q(t)q(t_1)\sin z(\pi-t)\sin z(t_1-\tau)\sin z(t-\tau-t_1)\mathrm{d}t_1\mathrm{d}t.\end{aligned} \tag{3.4}$$

The functions $\Delta_0(\lambda)$, $\Delta_1(\lambda)$ are entire in λ of order $1/2$. It is clear that the set of zeros of functions $\Delta_0(\lambda)$, $\Delta_1(\lambda)$ is equivalent to the spectrum of boundary spectral problems L_0, L_1, respectively. Like in previous case specification of the spectra uniquely determines functions $\Delta_0(\lambda)$, $\Delta_1(\lambda)$. We introduce notation

$$F_0(z) = z\Delta_0(\lambda),\ F_1(z) = z\Delta_1(\lambda)$$

and

$$I_1 = \int_{\tau}^{\pi} q(t)\mathrm{d}t, \quad I_2 = \int_{2\tau}^{\pi} q(t) \int_{\tau}^{t-\tau} q(t_1)\mathrm{d}t_1 \mathrm{d}t.$$

Similar to the first case we have that delay τ and the integral I_1 are uniquely ordered from the spectrum $(\lambda_n)_{n=1}^{\infty}$(see [1]).

Lemma 3.3 *The spectra $(\mu_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$ of boundary value problem L_1 and L_0 uniquely determine polynomial $P(\lambda)$.*

Proof Similar like in first case. □

Now we transform the products of trigonometric functions into sums/differences and we have

$$\begin{aligned} F_0(z) &= \sin z\pi - \frac{\cos z(\pi-\tau)}{2z} I_1 + \frac{1}{2z}\int_{\tau}^{\pi} q(t)\cos z(\pi+\tau-2t)\mathrm{d}t - \frac{\sin z(\pi-2\tau)}{4z^2} I_2 \\ &\quad + \frac{1}{4z^2}\int_{2\tau}^{\pi}\int_{\tau}^{t-\tau} q(t)q(u)\left(\sin z(\pi-2u) - \sin z(\pi-2t+2\tau) + \sin z(\pi-2t+2u)\right)\mathrm{d}u\mathrm{d}t, \\ F_1(z) &= z\cos z\pi + \frac{\sin z(\pi-\tau)}{2} I_1 - \frac{1}{2}\int_{\tau}^{\pi} q(t)\sin z(\pi+\tau-2t)\mathrm{d}t \\ &\quad + \frac{1}{4z}\int_{2\tau}^{\pi}\int_{\tau}^{t-\tau} q(t)q(u)\left(\cos z(\pi-2u) - \cos z(\pi-2t+2\tau) + \cos z(\pi-2t+2u)\right)\mathrm{d}u\mathrm{d}t \\ &\quad - \frac{\cos z(\pi-2\tau)}{4z} I_2 + P(z^2)\sin z\pi + \frac{P(z^2)}{2z}\int_{\tau}^{\pi} q(t)\cos z(\pi+\tau-2t)\mathrm{d}t \\ &\quad - \frac{P(z^2)\cos z(\pi-\tau)}{2z} I_1 - \frac{P(z^2)\sin z(\pi-2\tau)}{4z^2} I_2 \\ &\quad + \frac{P(z^2)}{4z^2}\int_{2\tau}^{\pi}\int_{\tau}^{t-\tau} q(t)q(u)(\sin z(\pi-2u) - \sin z(\pi-2t+2\tau) \\ &\quad + \sin z(\pi-2t+2u))\mathrm{d}u\mathrm{d}t. \end{aligned}$$

We define functions $K : [0,\pi] \to R$, $\widetilde{q} : [0,\pi] \to R$

$$K(t) = \begin{cases} q(t+\tau)\int_{\tau}^{t} q(u)\mathrm{d}u - q(t)\int_{t+\tau}^{\pi} q(u)\mathrm{d}u - \int_{t+\tau}^{\pi} q(u-t)q(u)\mathrm{d}u, & t \in [\tau, \pi-\tau] \\ 0, & \text{else} \end{cases}$$

$$\widetilde{q}(t) = \begin{cases} q\left(t+\frac{\tau}{2}\right), & t \in \left[\frac{\tau}{2}, \pi-\frac{\tau}{2}\right] \\ 0, & \text{else} \end{cases}$$

It is obvious that $K(t) \in L^2[0,\pi]$ and $\widetilde{q}(t) \in L^2[0,\pi]$.

Using the following formulas as well as analogous formulas with cosine function function

$$\int_{2\tau}^{\pi}\int_{\tau}^{t-\tau} q(t)q(u)\sin z(\pi-2u)\mathrm{d}u\mathrm{d}t = \int_{\tau}^{\pi-\tau}\sin z(\pi-2t)q(t)\int_{t+\tau}^{\pi} q(u)\mathrm{d}u\mathrm{d}t,$$
$$\int_{2\tau}^{\pi}\int_{\tau}^{t-\tau} q(t)q(u)\sin z(\pi-2t+2\tau)\mathrm{d}u\mathrm{d}t = \int_{\tau}^{\pi-\tau}\sin z(\pi-2t)q(t+\tau)\int_{\tau}^{t} q(u)\mathrm{d}u\mathrm{d}t,$$
$$\int_{2\tau}^{\pi}\int_{\tau}^{t-\tau} q(t)q(u)\sin z(\pi-2t+2u)\mathrm{d}u\mathrm{d}t = \int_{\tau}^{\pi-\tau}\sin z(\pi-2t)\int_{t+\tau}^{\pi} q(u)q(u-t)\mathrm{d}u\mathrm{d}t,$$

and notation

$$\widetilde{a_c}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\widetilde{q}(t)\cos z(\pi-2t)\mathrm{d}t, \quad \widetilde{a_s}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\widetilde{q}(t)\sin z(\pi-2t)\mathrm{d}t,$$
$$k_c(z) = \int_{\tau}^{\pi-\tau} K(t)\cos z(\pi-2t)\mathrm{d}t, \quad k_s(z) = \int_{\tau}^{\pi-\tau} K(t)\sin z(\pi-2t)\mathrm{d}t,$$

we have

$$F_0(z) = \sin z\pi + \frac{1}{2z}\Big(\widetilde{a_c}(z) - I_1\cos z(\pi-\tau)\Big) - \frac{1}{4z^2}\Big(k_s(z) + I_2\sin z(\pi-2\tau)\Big), \tag{3.5}$$

$$\begin{aligned} F_1(z) = {} & z\cos z\pi - \frac{1}{2}\Big(\widetilde{a_s}(z) - I_1\sin z(\pi-\tau)\Big) - \frac{1}{4z}\Big(k_c(z) + I_2\cos z(\pi-2\tau)\Big) \\ & + P(z^2)\sin z\pi + \frac{P(z^2)}{2z}\Big(\widetilde{a_c}(z) - I_1\cos z(\pi-\tau)\Big) \\ & - \frac{P(z^2)}{4z^2}\Big(k_s(z) + I_2\sin z(\pi-2\tau)\Big). \end{aligned} \tag{3.6}$$

One can easily show that $\int_{\tau}^{\pi-\tau} K(t)\mathrm{d}t = -I_2$. Using this formula and integration by parts from (3.5) and (3.6), we have

$$F_0(z) = \sin z\pi + \frac{1}{2z}\Big(\widetilde{a_c}(z) - I_1\cos z(\pi-\tau)\Big) - \frac{1}{2z}K_c^{\circ}(z) - \frac{1}{2z^2}I_2\sin z(\pi-2\tau), \tag{3.7}$$

and

$$\begin{aligned} F_1(z) = {} & z\cos z\pi - \frac{1}{2}\Big(\widetilde{a_s}(z) - I_1\sin z(\pi-\tau)\Big) + \frac{1}{2}K_s^{\circ}(z) + P(z^2)\sin z\pi \\ & + \frac{P(z^2)}{2z}\Big(\widetilde{a_c}(z) - I_1\cos z(\pi-\tau)\Big) - \frac{P(z^2)}{2z^2}I_2\sin z(\pi-2\tau) - \frac{P(z^2)}{2z}K_c^{\circ}(z), \end{aligned} \tag{3.8}$$

where is

$$K_c^{\circ}(z) = \int\limits_{\tau}^{\pi-\tau} \int_{\tau}^{t} K(s)\mathrm{d}s \cos z(\pi - 2t)\mathrm{d}t,$$

$$K_s^{\circ}(z) = \int\limits_{\tau}^{\pi-\tau} \int_{\tau}^{t} K(s)\mathrm{d}s \sin z(\pi - 2t)\mathrm{d}t.$$

Using $\int_{\tau}^{\pi-\tau} K(t)\mathrm{d}t = -I_2$, we have

$$\int\limits_{\tau}^{t} K(s)\mathrm{d}s = -I_2 - \int\limits_{t}^{\pi-\tau} K(s)\mathrm{d}s,$$

and

$$\begin{aligned} K_c^{\circ}(z) &= \int\limits_{\tau}^{\pi-\tau} \int\limits_{\tau}^{t} K(s)\mathrm{d}s \cos z(\pi - 2t)\mathrm{d}t = \int\limits_{\tau}^{\pi-\tau} \left(-I_2 - \int\limits_{t}^{\pi-\tau} K(s) \right) \mathrm{d}s \cos z(\pi - 2t)\mathrm{d}t \\ &= -I_2 \int\limits_{\tau}^{\pi-\tau} \cos z(\pi - 2t)\mathrm{d}t - \int\limits_{\tau}^{\pi-\tau} \int\limits_{t}^{\pi-\tau} K(s)\mathrm{d}s \cos z(\pi - 2t)\mathrm{d}t \\ &= -\frac{I_2 \sin z(\pi - 2\tau)}{z} + K_c^{*}(z), \\ K_s^{\circ}(z) &= K_s^{*}(z), \end{aligned}$$

where is

$$K_c^{*}(z) = -\int\limits_{\tau}^{\pi-\tau} \int\limits_{t}^{\pi-\tau} K(s)\mathrm{d}s \cos z(\pi - 2t)\mathrm{d}t,$$

$$K_s^{*}(z) = -\int\limits_{\tau}^{\pi-\tau} \int\limits_{t}^{\pi-\tau} K(s)\mathrm{d}s \sin z(\pi - 2t)\mathrm{d}t.$$

Now we transform (3.7) and (3.8) and we have

$$F_0(z) = \sin z\pi + \frac{1}{2z}\left(\widetilde{a_c}(z) - I_1 \cos z(\pi - \tau) \right) - \frac{1}{2z} K_c^{*}(z), \tag{3.9}$$

and

$$F_1(z) = z \cos z\pi - \frac{1}{2}\left(\widetilde{a_s}(z) - I_1 \sin z(\pi - \tau) \right) + \frac{1}{2} K_s^{*}(z) + P(z^2) \sin z\pi$$

$$+\frac{P(z^2)}{2z}\left(\widetilde{a_c}(z) - I_1 \cos z(\pi - \tau)\right) - \frac{P(z^2)}{2z} K_c^*(z). \tag{3.10}$$

We define function $A(z)$

$$A(z) = 2zF_0(z) - 2z \sin z\pi + I_1 \cos z(\pi - \tau).$$

Function $A(z)$ is determined by spectrum $(\lambda_n)_{n=1}^{\infty}$ and from (3.10), we have

$$A(z) = \widetilde{a_c}(z) - K_c^*(z). \tag{3.11}$$

Also, we define function $B_1(z)$

$$B_1(z) = -2F_1(z) + 2z \cos z\pi + I_1 \sin z(\pi - \tau) + 2P(z^2) \sin z\pi - \frac{P(z^2)}{z} I_1 \cos z(\pi - \tau),$$

this function is ordered by spectrum $(\mu_n)_{n=1}^{\infty}$ and we have

$$B_1(z) = \widetilde{a_s}(z) - K_s^*(z) - \frac{P(z^2)}{z} A(z).$$

Finally, we define function

$$B(z) = B_1(z) + \frac{P(z^2)}{z} A(z),$$

which is determined by spectra $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$, and

$$B(z) = \widetilde{a_s}(z) - K_s^*(z). \tag{3.12}$$

We define function $K^* : [0, \pi] \to R$

$$K^*(t) = \begin{cases} \int_t^{\pi-\tau} K(s)\mathrm{d}s, & t \in (\tau, \pi - \tau) \\ 0, & \text{else} \end{cases}$$

Put $z = m, m \in N$ into (3.11) and (3.12), we have

$$(-1)^m A(m) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \widetilde{q}(t) \cos 2mt\mathrm{d}t - \int_{\tau}^{\pi-\tau} \int_{t}^{\pi-\tau} K(s)\mathrm{d}s \cos 2mt\mathrm{d}t \tag{3.13}$$

$$(-1)^{m+1} B(m) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \widetilde{q}(t) \sin 2mt\mathrm{d}t - \int_{\tau}^{\pi-\tau} \int_{t}^{\pi-\tau} K(s)\mathrm{d}s \sin 2mt\mathrm{d}t. \tag{3.14}$$

From (3.11), we have

$$\lim_{z\to 0} A(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \widetilde{q}(t)\mathrm{d}t - \int_{\tau}^{\pi-\tau}\int_{t}^{\pi-\tau} K(s)\mathrm{d}s\mathrm{d}t. \tag{3.15}$$

We multiply Eq. (3.13) by $\frac{1}{\pi}\mathrm{e}^{2imt}$, Eq. (3.14) by $\frac{-i}{\pi}\mathrm{e}^{2imt}$ and Eq. (3.15) by $\frac{1}{\pi}$ and then sum them, using definition of function $\widetilde{q}(t)$ and $K^*(t)$ we get the integral equation

$$\widetilde{q}(t) - K^*(t) = f(t), t \in [0, \pi], \tag{3.16}$$

where

$$f(t) = \frac{1}{\pi}\lim_{z\to 0} A(z) + \frac{1}{\pi}\sum_{m\in Z\setminus 0}((-1)^m A(m) + i(-1)^m B(m))\mathrm{e}^{2imt}.$$

Theorem 3.4 *Let $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ be the spectra of boundary spectral problems $L_k, k = 0, 1$, respectively, then potential q is uniquely determined by $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ if $\frac{2\pi}{5} \leq \tau < \frac{\pi}{2}$.*

Proof The potential q satisfies integral equation (3.16), we will show uniqueness of solution of this equation.

- For $t \in \left(\pi - \tau, \pi - \frac{\tau}{2}]\right)$, since $K^*(t) = 0, t \in \left(\pi - \tau, \pi - \frac{\tau}{2}\right]$, integral equation (3.16) have a form:

  $$\widetilde{q}(t) = f(t)$$

  Function f is determined by $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$, then potential $q(x)$ is determined for $x \in (\pi - \frac{\tau}{2}, \pi]$.
- For $t \in \left(\frac{\tau}{2}, \tau\right]$, since $K^*(t) = 0, t \in \left(\frac{\tau}{2}, \tau\right]$, integral equation (3.16) have a form:

  $$\widetilde{q}(t) = f(t).$$

  Function f is determined by $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$, then potential q is determined for $x \in [\tau, \frac{3\tau}{2}]$.
- For $t \in (\tau, \pi - \tau]$, from (3.16), we have equation

  $$\widetilde{q}(t) - \int_{t}^{\pi-\tau} K(s)\mathrm{d}s = f(t).$$

One can easily show that arguments of the potential q appearing in the function

$$\int_t^{\pi-\tau} K(s)\mathrm{d}s = \int_t^{\pi-\tau}\left(q(s+\tau)\int_\tau^s q(u)\mathrm{d}u - q(s)\int_{s+\tau}^{\pi} q(u)\mathrm{d}u - \int_{s+\tau}^{\pi} q(u-s)q(u)\mathrm{d}u\right)\mathrm{d}s$$

belong to the intervals $[2\tau, \pi] \subset [\pi - \frac{\tau}{2}, \pi]$ and $[\tau, \pi - \tau] \subset [\tau, \frac{3\tau}{2}]$. Then the function $\int_t^{\pi-\tau} K(s)\mathrm{d}s$ is known. Therefore from (3.16) for $t \in (\tau, \pi - \tau]$, we get

$$\widetilde{q}(t) = \int_t^{\pi-\tau} K(s)\mathrm{d}s + f(t).$$

Function f is determined by $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$, then potential $q(x)$ is determined for $x \in \left(\frac{3\tau}{2}, \pi - \frac{\tau}{2}\right]$.

□

When $\frac{\pi}{3} \leq \tau < \frac{\pi}{2}$, we have same integral equation like 3.13, but in the case when $\frac{\pi}{3} \leq \tau < \frac{2\pi}{5}$ not satisfied $[2\tau, \pi] \subset [\pi - \frac{\tau}{2}, \pi]$ and $[\tau, \pi - \tau] \subset [\tau, \frac{3\tau}{2}]$, and we cannot prove Theorem 3.4 on this way. Moreover, in the case when $\frac{\pi}{3} \leq \tau < \frac{2\pi}{5}$ theorem of uniqueness not true. For this conclusion, the main arguments are the results published in the papers [4] and [5]. In the case when $\frac{\pi}{3} \leq \tau < \frac{2\pi}{5}$ then critical interval $\left(\frac{3\tau}{2}, \pi - \tau\right)$ not equal to $\varnothing$.

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