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ORIGINAL PAPER

# Relative Derived Categories

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**Abstract**
In this paper, the relative derived categories with respect to $\mathcal{X}$-Gorenstein projective modules and $\mathcal{Y}$-Gorenstein injective modules are introduced to unify Gorestein derived categories and Ding derived categories in certain sense. A triangle-equivalence and description of morphisms in such relative derived categories are given. We also discuss a generalized Tate cohomology and obtain Avramov–Martsinkovsky exact sequences. Finally, some applications are obtained.

**Keywords** Relative derived categories · $\mathcal{X}$-Gorenstein projective modules · Generalized Tate cohomology · Avramov–Martsinkovsky exact sequence

**Mathematics Subject Classification** 16E05 · 16E30

## 1 Introduction

Derived categories were first introduced by Grothendieck in [17]. Until 1963, his student Verdier gave the definition of triangulated categories and developed the theory of localization in [26]. He obtained the construction of derived categories. An important generalization of classical homological algebra is Gorenstein homological algebra. In [13], Gao and Zhang introduced so-called Gorenstein derived categories. They gave the relation with the usual derived categories and found that the bounded Gorenstein derived categories of Gorenstein rings and of finite dimensional algebras can be explicitly described via the homotopy categories of Gorenstein projective modules.

The Quillen model structures over a Gorenstein ring could be generalized to the one over a so-called Ding–Chen ring, with respect to some special Gorenstein projective

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and Gorenstein injective modules [15,19] named Ding projective and injective modules by Gillespie. Because these modules are introduced by Ding and co-authors in [11] and [10]. Many researchers pay attention to these modules, such as [27,28,30]. Especially, in [24], Ren, Liu and Yang started from above conclusion of Quillen model structure and introduced the relative derived categories with respect to Ding modules. They gave the relation with derived and Gorenstein derived categories and a triangle-equivalence of Ding derived categories over Ding–Chen rings. Also the authors in [1] studied relative derived category with respect to a contravariantly finite subcategory of an abelian category as a generalization of the work in [13].

$\mathcal{X}$-Gorenstein projective modules and $\mathcal{Y}$-Gorenstein injective modules are introduced in [6] and [22], where $\mathcal{X}$ is a class of modules that contains all projective modules, $\mathcal{Y}$ is a class of modules that contains all injective modules. It has been proved that the principle results on Gorenstein projective and injective modules remain true for these modules. Now it is natural to ask whether the relative derived categories with respect to $\mathcal{X}$-Gorenstein projective and $\mathcal{Y}$-Gorenstein injective modules share nice analogous properties with corresponding Gorenstein derived categories and Ding derived categories. We study the triangle-equivalence and morphisms in current relative derived categories. Also a generalized Tate cohomology is discussed and we obtain Avramov–Martsinkovsky exact sequences. These results cover the related conclusions in Gorenstein derived categories and Ding derived categories. The triangle-equivalence of $\mathcal{D}^*_{\widetilde{\mathcal{C}}}(R)$ and $\mathcal{D}^*_{\widetilde{\mathcal{F}}}(R)$ is achieved, where $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{F}}$ denote the classes of Gorenstein AC-projective modules and Gorenstein AC-injective modules, resp.

The layout of this paper is as follows: In Sect. 2, we review some definitions and notations which are basic to the rest of the paper. Sect. 3 is devoted to define and study the relative derived categories with respect to $\mathcal{X}$-Gorenstein projective modules and $\mathcal{Y}$-Gorenstein injective modules. We show the triangle-equivalence and give a new characterization of relative derived functor of Hom with respect to $\mathcal{X}$-Gorenstein projective modules and $\mathcal{Y}$-Gorenstein injective modules as the morphisms in the corresponding relative derived categories. In Sect. 4, we discuss a generalized Tate cohomology in the sense of Iacob [20] with respect to $\mathcal{X}$-Gorenstein projective modules and $\mathcal{Y}$-Gorenstein injective modules and get Avramov–Martsinkovsky exact sequences. Finally, some applications are obtained in Sect. 5.

## 2 Preliminaries

Throughout this paper, $R$ denotes a ring with unity, and modules are left $R$-modules. We first review some notations and basic facts.

$\mathcal{X}$-Gorenstein modules Let $\mathcal{X}$(resp., $\mathcal{Y}$) be a class of modules that contains projective (resp., injective) $R$-modules. Recall that an $R$-module $M$ is $\mathcal{X}$-Gorenstein projective, if there exists an exact sequence of projective $R$-modules

$$\mathbf{P} = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and $\mathrm{Hom}_R(\mathbf{P}, Q)$ is exact whenever $Q \in \mathcal{X}$. Dually, an $R$-module $N$ is $\mathcal{Y}$-Gorenstein injective, if there exists an exact sequence of injective $R$-modules

$$\mathbf{I} = \cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

such that $M \cong \mathrm{Im}(I_0 \to I^0)$ and $\mathrm{Hom}_R(E, \mathbf{I})$ is exact whenever $E \in \mathcal{Y}$ [6].

We denote by $\mathcal{P}$(resp., $\mathcal{F}$, $\mathcal{GP}$, $\mathcal{DP}$, $\mathcal{X}$-$\mathcal{GP}$) the class of projective (resp., flat, Gorenstein projective, Ding projective, $\mathcal{X}$-Gorenstein projective) modules and $\mathcal{I}$(resp., $\mathcal{GI}$, $\mathcal{DI}$, $\mathcal{Y}$-$\mathcal{GI}$) the class of injective (resp., Gorenstein injective, Ding injective, $\mathcal{Y}$-Gorenstein injective) modules.

*Cotorsion pair* Let $\mathcal{A}$ be an abelian category, $\mathcal{C}$ and $\mathcal{D}$ be classes of objects in $\mathcal{A}$. $(\mathcal{C}, \mathcal{D})$ is called cotorsion pair if $\mathcal{C}^{\perp} = \mathcal{D}$ and $\mathcal{C} = {}^{\perp}\mathcal{D}$, where $\mathcal{C}^{\perp} = \{Y \in \mathrm{Ob}(\mathcal{A}) \mid \ \mathrm{Ext}^1_R(C, Y) = 0 \ for\ any \ \ C \in \mathcal{C}\}$, ${}^{\perp}\mathcal{C} = \{Y \in \mathrm{Ob}(\mathcal{A}) \mid \mathrm{Ext}^1_R(Y, C) = 0 \ for\ any \ \ C \ \in \mathcal{C}\}$. The cotorsion pair $(\mathcal{C}, \mathcal{D})$ is said to be complete if for any $A \in \mathcal{A}$,there are short exact sequence $0 \to D \to C \to A \to 0$ and $0 \to A \to C' \to D' \to 0$ with $C, C' \in \mathcal{C}$, $D, D' \in \mathcal{D}$. If $\mathcal{A}$ is an abelian category with enough projective and enough injective objects, the cotorsion pair $(\mathcal{C}, \mathcal{D})$ in $\mathcal{A}$ is called hereditary if one of the following equivalent conditions holds: (1) $\mathcal{C}$ is resolving, that is, $\mathcal{C}$ is closed under taking kernels of epics; (2) $\mathcal{D}$ is coresolving, that is, $\mathcal{D}$ is closed under taking cokernels of monics; (3) $\mathrm{Ext}^i_R(C, D) = 0$ for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$ and $i \geq 1$. In [22], it has been proved that $({}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}), \mathcal{Y}\text{-}\mathcal{GI})$ is a complete hereditary cotorsion pair if $l\mathcal{Y}\text{-GID}(R) < \infty$ and $(\mathcal{X}\text{-}\mathcal{GP}, (\mathcal{X}\text{-}\mathcal{GP})^{\perp})$ is a complete hereditary cotorsion pair if $l\mathcal{X}\text{-GPD}(R) < \infty$.

*Complex* An $R$-complex $X$ is a sequence of $R$-modules

$$\cdots \xrightarrow{\delta_2} X_1 \xrightarrow{\delta_1} X_0 \xrightarrow{\delta_0} X_{-1} \xrightarrow{\delta_{-1}} \cdots,$$

with $\delta_n \delta_{n+1} = 0$ for all $n \in \mathbb{Z}$. The $n$th homology module of $X$ is $\mathrm{H}_n(X) = \mathrm{Z}_n(X)/\mathrm{B}_n(X)$, where $\mathrm{Z}_n(X) = \mathrm{Ker}(\delta_n^X)$, $\ \mathrm{B}_n(X) = \mathrm{Im}(\delta_{n+1}^X)$. We set $\mathrm{C}_n(X) = \mathrm{Coker}(\delta_{n+1}^X)$. A complex $X$ is called acyclic or exact if $\mathrm{H}_n(X) = 0$ for any $n \in \mathbb{Z}$. For an integer $m$, $\Sigma^m X$ denotes the complex $X$ shifting $m$ degrees; it is given by $(\Sigma^m X)_l = X_{l-m}$, $\delta_l^{\Sigma^m X} = (-1)^m \delta_{l-m}^X$.

Denote by $\mathcal{C}(R)$ (resp., $\mathcal{K}(R)$, $\mathcal{D}(R)$) the category of complexes (resp., the homotopy category, derived category). The homotopy category $\mathcal{K}(R)$ has the same objects as $\mathcal{C}(R)$ and the morphisms are homotopy equivalence classes of morphisms in $\mathcal{C}(R)$. The derived category $\mathcal{D}(R)$ is the localization of $\mathcal{K}(R)$ with respect to the multiplication system formed by the quasi-isomorphisms in $\mathcal{K}(R)$. For a class of $R$-modules $\mathcal{B}$, $\mathcal{K}(\mathcal{B})$ is the homotopy category with each complex constructed by modules in $\mathcal{B}$. For $* \in \{-, +, b\}$, $\mathcal{K}^*(R)$ and $\mathcal{D}^*(R)$ stand for the corresponding homotopy category and the derived category.

*Gorenstein ring* A ring $R$ is called a Gorenstein ring if it is both left and right noetherian and has finite self-injective dimension on both the left and the right.

*Ding–Chen ring* A ring $R$ is called Ding–Chen ring or $n$-FC ring if it is both left and right coherent, and has both left and right self-FP-injective dimensions at most $n$ for some non-negative integer $n$ ([8] or [9]).

## 3 Relative Derived Categories

In this section, we introduce relative derived categories with respect to $\mathcal{X}$-Gorenstein projective and $\mathcal{Y}$-Gorenstein injective modules, and study corresponding triangle-equivalence.

**Definition 3.1** An $R$-complex $X$ is called $\mathcal{X}$-$\mathcal{GP}$-acyclic, if $\mathrm{Hom}_R(G, X)$ is acyclic for every $\mathcal{X}$-Gorenstein projective module $G$. A morphism $f : X \to Y$ of $R$-complexes is called an $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism, if $\mathrm{Hom}_R(G, f)$ is a quasi-isomorphism for every $\mathcal{X}$-Gorenstein projective module $G$.

Dually, an $R$-complex $X$ is called $\mathcal{X}$-$\mathcal{GI}$-acyclic, if $\mathrm{Hom}_R(X, E)$ is acyclic for every $\mathcal{Y}$-Gorenstein injective module $E$. A morphism $f : X \to Y$ of $R$-complexes is called a $\mathcal{Y}$-$\mathcal{GI}$-quasi-isomorphism, if $\mathrm{Hom}_R(f, E)$ is a quasi-isomorphism for every $\mathcal{Y}$-Gorenstein injective module $E$.

***Remark 3.2*** 1. Let $\mathcal{X} = \mathcal{P}$. Then $\mathcal{X}$-$\mathcal{GP} = \mathcal{GP}$. Let $\mathcal{X} = \mathcal{F}$. Then $\mathcal{X}$-$\mathcal{GP} = \mathcal{DP}$.

2. Since $\mathcal{P} \subseteq \mathcal{X}$, $\mathcal{P} \subseteq \mathcal{X}$-$\mathcal{GP} \subseteq \mathcal{GP}$, then every $\mathcal{GP}$-acyclic complex is $\mathcal{X}$-$\mathcal{GP}$-acyclic, every $\mathcal{X}$-$\mathcal{GP}$-acyclic complexes is acyclic. Moreover, every $\mathcal{GP}$-quasi-isomorphism is an $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism and every $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism is a quasi-isomorphism.

(3) By [7, Lemma 2.4] a complex is $\mathcal{X}$-$\mathcal{GP}$-acyclic if and only if $\mathrm{Hom}_R(G, X)$ is exact for each complex $G \in \mathcal{K}^+(\mathcal{X}\text{-}\mathcal{GP})$. By [7, Proposition 2.6], a morphism $f : X \to Y$ of $R$-complexes is $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism if and only if $\mathrm{Hom}_R(G, f)$ is quasi-isomorphism for each complex $G \in \mathcal{K}^+(\mathcal{X}\text{-}\mathcal{GP})$.

(4) Let $\mathcal{X} = \mathcal{GP}$. Then $\mathcal{X}$-$\mathcal{GP} = \mathcal{P}$.

(5) Dually, let $\mathcal{Y} = \mathcal{I}$. Then $\mathcal{Y}$-$\mathcal{GI} = \mathcal{GI}$. Let $\mathcal{X} = \mathcal{FI}$. Then $\mathcal{Y}$-$\mathcal{GI} = \mathcal{DI}$. Let $\mathcal{Y} = \mathcal{GI}$. Then $\mathcal{Y}$-$\mathcal{GI} = \mathcal{I}$. The dual conclusions on $\mathcal{Y}$-$\mathcal{GI}$-acyclic complexes and $\mathcal{Y}$-$\mathcal{GI}$–quasi-isomorphisms also hold.

Following Rickards criterion (see [25, Proposition 1.3]), we have the following lemma.

**Lemma 3.3** *For $* \in \{blank, -, +, b\}$, $\mathcal{K}^*_{\mathcal{XGP}}(R)$ and $\mathcal{K}^*_{\mathcal{YGI}}(R)$ are thick subcategories of $\mathcal{K}^*(R)$, where $\mathcal{K}^*_{\mathcal{XGP}}(R)$ is the subcategory of $\mathcal{K}^*(R)$ consisting of $\mathcal{X}$-$\mathcal{GP}$-acyclic complexes and $\mathcal{K}^*_{\mathcal{YGI}}(R)$ is the subcategory of $\mathcal{K}^*(R)$ consisting of $\mathcal{Y}$-$\mathcal{GI}$-acyclic complexes.*

***Proof*** It is obvious that $\mathcal{K}^*_{\mathcal{XGP}}(R)$ and $\mathcal{K}^*_{\mathcal{YGI}}(R)$ are full triangulated subcategories of $\mathcal{K}^*(R)$. Also direct summands of $\mathcal{X}$-$\mathcal{GP}$-acyclic complexes($\mathcal{Y}$-$\mathcal{GI}$-acyclic complexes) are $\mathcal{X}$-$\mathcal{GP}$-acyclic complexes($\mathcal{Y}$-acyclic complexes). The conclusion holds by Rickards criterion. □

Note that a morphism of complexes $f : X \to Y$ is a $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism (resp., $\mathcal{Y}$-$\mathcal{GI}$-quasi-isomorphism) if and only if its mapping cone $\mathrm{Cone}(f)$ is $\mathcal{X}$-$\mathcal{GP}$-acyclic (resp., $\mathcal{Y}$-$\mathcal{GI}$-acyclic). Denote the collection of $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism

($\mathcal{Y}$-$\mathcal{GI}$-quasi-isomorphism) by $S_{\mathcal{XGP}}$ and $S_{\mathcal{YGI}}$. Then $S_{\mathcal{XGP}}$ and $S_{\mathcal{YGI}}$ are saturated multiplicative systems corresponding to the subcategories $\mathcal{K}^*_{\mathcal{XGP}}(R)$ and $\mathcal{K}^*_{\mathcal{YGI}}(R)$.

**Definition 3.4** For $* \in \{blank, -, +, b\}$, the relative derived category $\mathcal{D}^*_{\mathcal{XGP}}(R)$ of the category of $R$-modules with respect to $\mathcal{X}$-Gorenstein projective modules is defined to be the Verdier quotient of $\mathcal{K}^*(R)$, i.e.

$$\mathcal{D}^*_{\mathcal{XGP}}(R) := \mathcal{K}^*(R)/\mathcal{K}^*_{\mathcal{XGP}}(R) = S^{-1}_{\mathcal{XGP}}\mathcal{K}^*(R).$$

Similarly, the relative derived category $\mathcal{D}^*_{\mathcal{YGI}}(R)$ of the category of $R$-modules with respect to $\mathcal{Y}$-Gorenstein injective modules is defined to be

$$\mathcal{D}^*_{\mathcal{YGI}}(R) := \mathcal{K}^*(R)/\mathcal{K}^*_{\mathcal{YGI}}(R) = S^{-1}_{\mathcal{YGI}}\mathcal{K}^*(R).$$

We call $\mathcal{D}^*_{\mathcal{XGP}}(R)$ and $\mathcal{D}^*_{\mathcal{YGI}}(R)$ $\mathcal{X}$-Gorenstein projective derived category and $\mathcal{Y}$-Gorenstein injective derived category, resp.

We know that $\mathcal{D}^*_{\mathcal{XGP}}(R)$ and $\mathcal{D}^*_{\mathcal{YGI}}(R)$ are the derived categories of the exact categories ($R$-Mod, $\mathcal{E}_{\mathcal{XGP}}$) and ($R$-Mod, $\mathcal{E}_{\mathcal{YGI}}$) in sense of [23], where $\mathcal{E}_{\mathcal{XGP}}$ and $\mathcal{E}_{\mathcal{YGI}}$ are the collections of all the short $\mathcal{X}$-$\mathcal{GP}$-acyclic and $\mathcal{Y}$-$\mathcal{GI}$-acyclic sequences, resp.

It has been proved that $\mathcal{D}^*_{\mathcal{GP}}(R)$ and $\mathcal{D}^*_{\mathcal{GI}}(R)$ are triangle-equivalent over a Gorenstein ring $R$ in [13]. When $R$ is a Ding–Chen ring, $\mathcal{D}^*_{\mathcal{DP}}(R)$ and $\mathcal{D}^*_{\mathcal{DI}}(R)$ are triangle-equivalent (see [24]). In the following, we investigate when $\mathcal{D}^*_{\mathcal{XGP}}(R)$ and $\mathcal{D}^*_{\mathcal{YGP}}(R)$ are triangle-equivalent.

In the following, we say that an $R$-module $M$ has a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution if there exists an $\mathcal{X}$-$\mathcal{GP}$-acyclic sequence $\cdots \to G_1 \to G_0 \to M \to 0$ with each $G_i \in \mathcal{X}$-$\mathcal{GP}$.

**Lemma 3.5** *Let $R$ be a ring and $M$ an $R$-module. If $l\mathcal{Y}$-GID$(R) < \infty$, $l\mathcal{X}$-GPD$(R) < \infty$, and ${}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}) = (\mathcal{X}\text{-}\mathcal{GP})^{\perp}$, then $\mathrm{Hom}_R(\mathbf{G}, E)$ is exact for a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution $\mathbf{G} : \cdots \to G_1 \to G_0 \to M \to 0$ and an arbitrary $\mathcal{Y}$-Gorenstein injective module $E$.*

***Proof*** Let $E$ be a $\mathcal{Y}$-Gorenstein injective module. Since $\mathcal{P} \subseteq \mathcal{X}$-$\mathcal{GP}$, then $\mathbf{G}$ is exact. Break it up into short exact sequences. We just show that

$$\mathbf{T} : 0 \to L \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

is $\mathrm{Hom}_R(-, E)$-exact, where $G_0 \to M$ is a $\mathcal{X}$-Gorenstein projective precover of $M$. By [22, Theorems 2.20 and 3.19], there exists a special short exact sequence

$$\mathbf{T}' : 0 \longrightarrow L' \stackrel{u}{\longrightarrow} G \longrightarrow M \to 0,$$

where $G \to M$ is an $\mathcal{X}$-Gorenstein projective precover of $M$ and $L' \in (\mathcal{X}\text{-}\mathcal{GP})^{\perp}$. It is easy to prove that $\mathbf{T}$ and $\mathbf{T}'$ are homotopic equivalent. So
$\mathrm{Hom}_R(\mathbf{T}, E)$ and $\mathrm{Hom}_R(\mathbf{T}', E)$ are homotopic equivalent for $\mathcal{Y}$-Gorenstein injective module $E$. We only need to show that $\mathrm{Hom}_R(\mathbf{T}', E)$ is exact.

We show $\mathrm{Hom}_R(G, E) \overset{\mathrm{Hom}_R(u,E)}{\longrightarrow} \mathrm{Hom}_R(L', E) \longrightarrow 0$ is exact for arbitrary $\mathcal{Y}$-Gorenstein injective module $E$, that is, for arbitrary morphism $f : L' \to E$, there exists $g : G \to E$ such that $gu = f$. Since $E$ is $\mathcal{Y}$-Gorenstein injective module, then $0 \to E' \to I \overset{\alpha}{\to} E \longrightarrow 0$, where $I$ is injective module and $E'$ is $\mathcal{Y}$-Gorenstein injective module. By assumption $\mathrm{Ext}^1_R(L', E') = 0$. So there exists $\beta : L' \to I$ such that $\alpha\beta = f$. Since $I$ is injective, there exists $\widetilde{\beta}$ such that $\widetilde{\beta}u = \beta$. Then $g = \alpha\widetilde{\beta}$ is the needed morphism. □

**Theorem 3.6** *Let $R$ be a ring. If $l\mathcal{Y}$-$\mathrm{GID}(R) < \infty$, $l\mathcal{X}$-$\mathrm{GPD}(R) < \infty$, and ${}^{\perp}(\mathcal{Y}$-$\mathcal{GI}) = (\mathcal{X}$-$\mathcal{GP})^{\perp}$, then $\mathcal{D}^*_{\mathcal{XGP}}(R)$ is triangle-equivalent to $\mathcal{D}^*_{\mathcal{YGI}}(R)$, where $* \in \{blank, -, +, b\}$.*

***Proof*** It is sufficient to prove that $\mathcal{E}_{\mathcal{XGP}}$ coincides with $\mathcal{E}_{\mathcal{YGI}}$. By assumption and [22, Theorems 2.20 and 3.19], $({}^{\perp}(\mathcal{Y}$-$\mathcal{GI}), \mathcal{Y}$-$\mathcal{GI})$ and $(\mathcal{X}$-$\mathcal{GP}, (\mathcal{X}$-$\mathcal{GP})^{\perp})$ are complete hereditary cotorsion pairs. Hence, every $R$-module $M$ has a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution $\cdots \to G_1 \to G_0 \to M \to 0$, denoted by $\mathbf{G} \to M \to 0$, where $\mathbf{G}$ is the deleted complex.

Let $0 \to K \to N \to M \to 0$ be a $\mathcal{X}$-$\mathcal{GP}$-acyclic sequence of $R$-modules. We need to prove that it is $\mathcal{Y}$-$\mathcal{GI}$-acyclic. Assume that $\mathbf{G} \to M \to 0$ and $\mathbf{G}' \to K \to 0$ are proper left $\mathcal{X}$-$\mathcal{GP}$-resolutions of $M$ and $K$ respectively. By the Horseshoe Lemma [12, Lemma 8.2.1], there is a a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{G}' & \longrightarrow & \mathbf{G}' \oplus \mathbf{G} & \longrightarrow & \mathbf{G} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

where $\mathbf{G}' \oplus \mathbf{G} \to N \to 0$ is a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution of $N$.

For any $\mathcal{Y}$-Gorenstein injective module $E$, by applying $\mathrm{Hom}_R(-, E)$ to the above commutative diagram, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(\mathbf{G}, E) & \longrightarrow & \mathrm{Hom}_R(\mathbf{G}' \oplus \mathbf{G}, E) & \longrightarrow & \mathrm{Hom}_R(\mathbf{G}', E) & \longrightarrow & 0\ . \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(M, E) & \longrightarrow & \mathrm{Hom}_R(N, E) & \longrightarrow & \mathrm{Hom}_R(K, E) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Each column is exact by Lemma 3.5, and the upper row is a sequence of complexes with each degree split-exact. We infer that each column is a quasi-isomorphism. Thus, it follows from the long homology exact sequence induced by the upper row that the second row is exact. Therefore, we deduce that $0 \to K \to N \to M \to 0$ is $\mathcal{Y}$-$\mathcal{GI}$-acyclic, as required.

Dually, we can show that every short $\mathcal{Y}$-$\mathcal{GI}$-acyclic sequence of $R$-modules is also $\mathcal{X}$-$\mathcal{GP}$-acyclic. This completes the proof. □

If ring $R$ satisfies the conditions in Theorem 3.6, then we call $\mathcal{D}^*_{\mathcal{XGP}}(R)$ and $\mathcal{D}^*_{\mathcal{YGI}}(R)$ the relative derived categories.

***Remark 3.7*** The known rings, including Gorenstein rings and Ding–Chen rings, satisfy the assumption of Theorem 3.6 (see more details in Sect. 5).

The relations between $\mathcal{D}^*_{\mathcal{GP}}(R)$, $\mathcal{D}^*_{\mathcal{XGP}}(R)$ and $\mathcal{D}^*(R)$ are as follows.

**Proposition 3.8** *For $* \in \{blank, -, +, b\}$, there are isomorphisms of triangulated categories*

$$\mathcal{D}^*(R) \cong \mathcal{D}^*_{\mathcal{XGP}}(R)/\left(\mathcal{K}^*_{\mathcal{P}}(R)/\mathcal{K}^*_{\mathcal{XGP}}(R)\right)$$

*and*

$$\mathcal{D}^*_{\mathcal{XGP}}(R) \cong \mathcal{D}^*_{\mathcal{GP}}(R)/\left(\mathcal{K}^*_{\mathcal{XGP}}(R)/\mathcal{K}^*_{\mathcal{GP}}(R)\right).$$

***Proof*** It follows by [13, Lemma 2.4]. □

It is well known that Ext functor has a tight connection with morphisms in derived categories. That is, for any $R$-module $M$ and $N$, $\mathrm{Ext}^n_R(M, N) = \mathrm{Hom}_{\mathcal{D}^b(R)}(M, \Sigma^n N)$. If $M$ has a proper left $\mathcal{GP}$-resolution $\mathbf{G} \to M \to 0$, then for any $R$-module $N$, Holm [18] defined $\mathrm{Ext}^n_{\mathcal{GP}}(M, N) = \mathrm{H}^n\mathrm{Hom}_R(\mathbf{G}, N)$. By [13, Theorem 3.12], $\mathrm{Ext}^n_{\mathcal{GP}}(M, N) = \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{GP}}(R)}(M, \Sigma^n N)$. For Ding derived categories, there is a similar conclusion. If $R$-module $M$ admits a proper left Ding projective resolution, $N$ is an arbitrary $R$-module, then $\mathrm{Ext}^n_{\mathcal{DP}}(M, N) = \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{DP}}(R)}(M, \Sigma^n N)$ [24, Theorem 4.4]. In the following, we will show that corresponding results also hold in the current relative derived categories.

**Lemma 3.9** *Let $X$ be an $R$-complex. Suppose that there is a complex $G \in \mathcal{K}^+(\mathcal{X}$-$\mathcal{GP})$ and an $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism $f : X \to G$. Then there exists a morphism $g : G \to X$ such that $fg$ is homotopic to $1_G$.*

***Proof*** Since $f : X \to G$ is an $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism and $G \in \mathcal{K}^+(\mathcal{X}$-$\mathcal{GP})$, then $\mathrm{Hom}_R(G, f) : \mathrm{Hom}_R(G, X) \to \mathrm{Hom}_R(G, G)$ is a quasi-isomorphism by [7, Proposition 2.6]. So it follows from [2, (1.1)], for the morphism $1_G$, there is a morphism $g : G \to X$ such that $fg \sim 1_G$. □

The following result gives another version of morphisms in $\mathcal{D}_{\mathcal{XGP}}(R)$.

**Proposition 3.10** *Let $X$ be an $R$-complex, $G \in \mathcal{K}^+(\mathcal{XGP})$. Then $\varphi : f \to f/1_G$ gives an isomorphism of abelian groups* $\mathrm{Hom}_{\mathcal{K}(R)}(G, X) \cong \mathrm{Hom}_{\mathcal{D}_{\mathcal{XGP}}(R)}(G, X)$.

***Proof*** If $f/1_G = 0$, then by the calculus of right fractions there is a $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism $t : Y \to G$ for some complex $Y$ such that $ft \sim 0$. It follows from Lemma 3.9 that there is a morphism $g : G \to Y$ such that $tg \sim 1_G$. Thus, $f \sim ftg \sim 0$, that is, $\varphi : f \to f/1_G$ is injective.

Next we show that $\varphi : f \to f/1_G$ is surjective. For each morphism $f/s \in \mathrm{Hom}_{\mathcal{D}_{\mathcal{XGP}}(R)}(G, X)$, $f/s$ presented by $G \overset{s}{\Longleftarrow} Y \overset{f}{\longrightarrow} X$, by Lemma 3.9, there is a morphism $g : G \to Y$ such that $sg \sim 1_G$. So $f/s = fg/1_G = \varphi(fg)$, that is, $\varphi : f \to f/1_G$ is surjective. Hence, $\varphi$ is an isomorphism as desired. □

**Definition 3.11** Suppose that $R$-module $M$ has a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution $\mathbf{G} \to M \to 0$. For an arbitrary $R$-module $N$ and every $n \in \mathbb{Z}$, define a relative cohomology group

$$\mathrm{Ext}^n_{\mathcal{XGP}}(M, N) = \mathrm{H}_{-n}\mathrm{Hom}_R(\mathbf{G}, N).$$

Dually, suppose that $R$-module $N$ has a coproper right $\mathcal{Y}$-$\mathcal{GI}$-resolution $0 \to N \to \mathbf{E}$. For an arbitrary $R$-module $M$ and every $n \in \mathbb{Z}$, define a relative cohomology group

$$\mathrm{Ext}^n_{\mathcal{YGI}}(M, N) = \mathrm{H}_{-n}\mathrm{Hom}_R\,(M, \mathbf{E})\,.$$

It follows from Comparison Lemma that every two proper left $\mathcal{X}$-$\mathcal{GP}$-resolution of $M$ are homotopy equivalent. Hence, $\mathrm{Ext}^n_{\mathcal{XGP}}(M, N)$ has no dependence with the chose of proper left $\mathcal{X}$-$\mathcal{GP}$-resolutions of $M$. Therefore, $\mathrm{Ext}^n_{\mathcal{XGP}}(M, N)$ is well defined. Similarly, $\mathrm{Ext}^n_{\mathcal{YGI}}(M, N)$ is well defined.

For arbitrary homology group $\mathrm{Ext}^n_{\mathcal{XGP}}(M, N)$ and $\mathrm{Ext}^n_{\mathcal{YGI}}(M, N)$, we have the description in current relative derived categories.

**Theorem 3.12** 1. *Assume that R-module M has a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution, N is an arbitrary R-module. Then*

$$\mathrm{Ext}^n_{\mathcal{XGP}}(M, N) = \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}(M, \Sigma^n N).$$

2. *Assume that R-module N has a coproper right $\mathcal{Y}$-$\mathcal{GI}$-resolution, M is an arbitrary R-module. Then*

$$\mathrm{Ext}^n_{\mathcal{GI}}(M, N) = \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{YGI}}(R)}\left(M, \Sigma^n N\right).$$

To prove the theorem, we firstly show the following lemma.

**Lemma 3.13** *$\mathcal{D}^+_{\mathcal{XGP}}(R)$ is a full triangulated subcategory of $\mathcal{D}_{\mathcal{XGP}}(R)$; $\mathcal{D}^b_{\mathcal{XGP}}(R)$ is a full triangulated subcategory of $\mathcal{D}^+_{\mathcal{XGP}}(R)$, and hence of $\mathcal{D}_{\mathcal{XGP}}(R)$.*

***Proof*** Let $S$ be the collection of all $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism in $\mathcal{K}(R)$. Then $S$ is the compatible multiplicative system determined by the thick subcategory $\mathcal{K}_{\mathcal{XGP}}(R)$ of $\mathcal{K}(R)$. Hence $\mathcal{D}_{\mathcal{XGP}}(R) = S^{-1}\mathcal{K}(R)$, $\mathcal{D}^+_{\mathcal{XGP}}(R) = (S \cap \mathcal{K}^+(R))^{-1}\mathcal{K}^+(R)$. By [14, Proposition 2.10(III)], it is enough to prove that for any $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism $f : M \to N$ with $N \in \mathcal{K}^+(R)$, there is a morphism $g : M' \to$

$M, M' \in \mathcal{K}^+(R)$, such that $fg$ is $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism. Then the canonical functor $(S \cap \mathcal{K}^+(R))^{-1}\mathcal{K}^+(R) \to S^{-1}\mathcal{K}(R)$ is fully faithful. Hence, $\mathcal{D}^+_{\mathcal{XGP}}(R)$ is a full triangulated subcategory of $\mathcal{D}_{\mathcal{XGP}}(R)$.

Since $N \in \mathcal{K}^+(R)$, there exists an integer $i$, such that $N_k = 0$ for any $k < i$. Let $M'$ be the soft truncation $M_{\supset i}$ of $M$. Then there is a commutative diagram

$$\begin{array}{cccccccccccc}
M_{\supset i} & & \cdots \longrightarrow & M_{i+2} & \longrightarrow & M_{i+1} & \longrightarrow & \mathrm{Ker}d_i & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow g & & & \| & & \| & & \hookdownarrow & & \downarrow & & \\
M & & \cdots \longrightarrow & M_{i+2} & \longrightarrow & M_{i+1} & \longrightarrow & M_i & \longrightarrow & M_{i-1} & \longrightarrow & \cdots \\
\downarrow f & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
N & & \cdots \longrightarrow & N_{i+2} & \longrightarrow & N_{i+1} & \longrightarrow & N_i & \longrightarrow & 0 & \longrightarrow & \cdots,
\end{array}$$

where $f$ is $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism. It is easy to see that $g$ is also a $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism. So is $fg$.

Similarly, we can prove that $\mathcal{D}^b_{\mathcal{XGP}}(R)$ is a full triangulated subcategory of $\mathcal{D}^+_{\mathcal{XGP}}(R)$. Obviously, $\mathcal{D}^b_{\mathcal{XGP}}(R)$ is a full triangulated subcategory of $\mathcal{D}_{\mathcal{XGP}}(R)$. □

Now we prove Theorem 3.12.

***Proof*** Let $\mathbf{G} \to M \to 0$ be a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution of $M$. Regard $R$-module $M$ as a complex, then $\mathbf{G} \to M$ is an $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism. Hence, in the relative derived category $\mathcal{D}^b_{\mathcal{XGP}}(R)$, $\mathbf{G} \cong M$. Therefore,

$$\begin{aligned}
\mathrm{Ext}^n_{\mathcal{XGP}}(M, N) &= \mathrm{H}_{-n}\mathrm{Hom}_R\,(\mathbf{G}, N) \\
&= \mathrm{Hom}_{\mathcal{K}(R)}\left(\mathbf{G}, \Sigma^n N\right) \\
&\cong \mathrm{Hom}_{\mathcal{D}_{\mathcal{XGP}}(R)}\left(\mathbf{G}, \Sigma^n N\right) \\
&\cong \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}\left(M, \Sigma^n N\right),
\end{aligned}$$

where the first isomorphism follows from Proposition 3.10 and the second isomorphism is from Lemma 3.13. □

Balance problem is an important issue in homological algebra. It is well known that the classical cohomology functor $\mathrm{Ext}^n_R(-,-)$ can be defined by projective resolution of the first module or injective resolution of the second module. However, the balance of the relative cohomology functors is not obvious. For example, the authors in [18] have proved that for Gorenstein derived functors, when $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gid}_R N < \infty$, $\mathrm{Ext}^n_{\mathcal{GP}}(M, N) \cong \mathrm{Ext}^n_{\mathcal{GI}}(M, N)$. Especially, when $R$ is an $n$-Gorenstein ring, Gorenstein derived functors are balanced (see [12]). For relative cohomology functors with respect to Ding modules, even $\mathrm{Ext}^n_{\mathcal{DP}}(M, N)$ and $\mathrm{Ext}^n_{\mathcal{DI}}(M, N)$ are well defined for $M$, $N$, they are not balance. But in [27], it has been proved that if $R$ is a Ding–Chen ring, $\mathrm{Hom}_R(-,-)$ is right balanced by $\mathcal{DP} \times \mathcal{DI}$ on $R$-Mod $\times$ $R$-Mod. Hence $\mathrm{Ext}^n_{\mathcal{DP}}(M, N) \cong \mathrm{Ext}^n_{\mathcal{DI}}(M, N)$. On the other hand, in [24], the balanced result has been reproved when $R$ is a Ding–Chen ring, $\mathcal{D}^b_{\mathcal{DP}}(R)$ and $\mathcal{D}^b_{\mathcal{DI}}(R)$ are triangulated equivalent. We can get the balancedness of the relative cohomology functors defined here. Based on Theorem 3.12 and Theorem 3.6, we get the following result.

**Corollary 3.14** *Let $R$ be a ring. If $l\mathcal{Y}$-GID$(R) < \infty$, $l\mathcal{X}$-GPD$(R) < \infty$, and ${}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}) = (\mathcal{X}\text{-}\mathcal{GP})^{\perp}$, then there is an isomorphism*

$$\mathrm{Ext}^n_{\mathcal{XGP}}(M, N) \cong \mathrm{Ext}^n_{\mathcal{YGI}}(M, N)$$

*for arbitrary $R$-module $M$, $N$.*

Recall that for $\mathrm{Ext}_{\mathcal{GP}}(-, -)$ and $\mathrm{Ext}_{\mathcal{DP}}(-, -)$, there are long exact sequences, see [2, Propositions 4.4 and 4.6] and [24, Proposition 4.6]. Next we give long exact sequences concerning these relative cohomology functions.

**Proposition 3.15** *Let $M$ be an $R$-module, $\mathbf{N} = 0 \to N \xrightarrow{f} N' \xrightarrow{g} N'' \to 0$ be an $\mathcal{X}$-$\mathcal{GP}$-acyclic sequence of $R$-modules.*

1. *If $M$ has proper left $\mathcal{X}$-$\mathcal{GP}$-resolution, then there is a natural morphism $\vartheta^n_{\mathcal{XGP}}(M, \mathbf{N})$ such that the following sequence is exact.*

$$\cdots \longrightarrow \mathrm{Ext}^n_{\mathcal{XGP}}(M, N) \xrightarrow{\mathrm{Ext}^n_{\mathcal{XGP}}(M,f)} \mathrm{Ext}^n_{\mathcal{XGP}}(M, N') \xrightarrow{\mathrm{Ext}^n_{\mathcal{XGP}}(M,g)} \mathrm{Ext}^n_{\mathcal{XGP}}(M, N'')$$

$$\xrightarrow{\vartheta^n_{\mathcal{XGP}}(M,\mathbf{N})} \mathrm{Ext}^{n+1}_{\mathcal{XGP}}(M, N) \xrightarrow{\mathrm{Ext}^{n+1}_{\mathcal{XGP}}(M,f)} \mathrm{Ext}^{n+1}_{\mathcal{XGP}}(M, N') \xrightarrow{\mathrm{Ext}^{n+1}_{\mathcal{XGP}}(M,g)} \cdots.$$

2. *Assume that $N, N'$ and $N''$ have proper left $\mathcal{X}$-$\mathcal{GP}$-resolutions. Then there is a natural morphism $\vartheta^n_{\mathcal{XGP}}(\mathbf{N}, M)$, such that the following sequence is exact.*

$$\cdots \longrightarrow \mathrm{Ext}^n_{\mathcal{XGP}}(N'', M) \xrightarrow{\mathrm{Ext}^n_{\mathcal{XGP}}(g,M)} \mathrm{Ext}^n_{\mathcal{XGP}}(N', M) \xrightarrow{\mathrm{Ext}^n_{\mathcal{XGP}}(f,M)} \mathrm{Ext}^n_{\mathcal{XGP}}(N, M)$$

$$\xrightarrow{\vartheta^n_{\mathcal{XGP}}(\mathbf{N},M)} \mathrm{Ext}^{n+1}_{\mathcal{XGP}}(N'', M) \xrightarrow{\mathrm{Ext}^{n+1}_{\mathcal{XGP}}(g,M)} \mathrm{Ext}^{n+1}_{\mathcal{XGP}}(N', M) \xrightarrow{\mathrm{Ext}^{n+1}_{\mathcal{XGP}}(f,M)} \cdots.$$

***Proof*** Since $\mathbf{N}$ is $\mathcal{X}$-$\mathcal{GP}$-acyclic, then $g$ induces a $\mathcal{X}$-$\mathcal{GP}$-quasi-isomorphism $\mathrm{Cone}(f) \to N''$ in the derived category $\mathcal{D}^b(R)$, where the mapping cone of $f$ is $\mathrm{Cone}(f) = 0 \to N \xrightarrow{f} N' \to 0$. So in $\mathcal{D}^b_{\mathcal{XGP}}(R)$, $\mathrm{Cone}(f) \cong N''$.

We consider the commutative diagram in $\mathcal{D}^b_{\mathcal{XGP}}(R)$

$$\begin{array}{ccccccc} N & \xrightarrow{f} & N' & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & \Sigma N \\ \| & & \| & & \downarrow & & \| \\ N & \xrightarrow{f} & N' & \xrightarrow{g} & N'' & \longrightarrow & \Sigma N. \end{array}$$

Since $N \xrightarrow{f} N' \to \mathrm{Cone}(f) \to \Sigma N$ is a distinguished triangle in $\mathcal{D}^b_{\mathcal{XGP}}(R)$, so is $N \xrightarrow{f} N' \xrightarrow{g} N'' \to \Sigma N$. Applying $\mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}(M, -)$ to the distinguished triangle, we get the exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}\left(M, \Sigma^n N\right) \to \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}\left(M, \Sigma^n N'\right) \to \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}\left(M, \Sigma^n N''\right) \to \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}\left(M, \Sigma^{n+1} N\right) \to \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}(M, \Sigma^{n+1} N') \to \cdots .$$

Similarly, applying $\mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}(-, M)$ to the distinguished triangle $N \xrightarrow{f} N' \xrightarrow{g} N'' \to \Sigma N$, we get the long exact sequence in (2). □

Dually, we can get long exact sequences concerning $\mathrm{Ext}_{\mathcal{YGI}}(-,-)$.

In [22], the $\mathcal{X}$-Gorenstein projective and $\mathcal{Y}$-Gorenstein injective dimension are discussed. Now we investigate them further by conclusions established in the current relative derived categories.

**Proposition 3.16** *Let $M$ be an $R$-module admitting a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution, $n$ be a non-negative integer. Then the following are equivalent.*

1. $\mathcal{X}$-$\mathrm{Gpd}_R M \leq n$.
2. *For every $R$-module $N$ and $i > 0$,* $\mathrm{Ext}^{n+i}_{\mathcal{XGP}}(M, N) = 0$.
3. *For every $R$-module $N$,* $\mathrm{Ext}^{n+1}_{\mathcal{XGP}}(M, N) = 0$.
4. *For any projective resolution of $M$, $\mathbf{P} \to M \to 0$,* $\mathrm{Ker}(P^{-n+1} \to P^{-n+2}) \in \mathcal{X}$-$\mathcal{GP}$.
5. *For any $\mathcal{X}$-$\mathcal{GP}$-resolution of $M$, $\mathbf{G} \to M \to 0$,* $\mathrm{Ker}(G^{-n+1} \to G^{-n+2}) \in \mathcal{X}$-$\mathcal{GP}$.
6. *For any proper left $\mathcal{X}$-$\mathcal{GP}$-resolution of $M$, $\mathbf{G} \to M \to 0$,* $\mathrm{Ker}(G^{-n+1} \to G^{-n+2}) \in \mathcal{X}$-$\mathcal{GP}$.

***Proof*** (5)⇒(4)⇒(1) and (5)⇒(6)⇒(2)⇒(3) are obvious.

(1)⇒(6) Suppose that $0 \to G^{-n} \to G^{-n+1} \to \cdots \to G^{-1} \to G^0 \to M \to 0$ is an $\mathcal{X}$-$\mathcal{GP}$-resolution of $M$ with length $n$. Let $\mathbf{G}' \to M \to 0$ be any proper left $\mathcal{X}$-$\mathcal{GP}$-resolution of $M$. By the Comparison Lemma we get the following commutative diagram:

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & G^{-n} & \longrightarrow & G^{-n+1} & \longrightarrow \cdots \longrightarrow & G^{-1} & \longrightarrow & G^0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ \cdots & \longrightarrow & G'^{-n-1} & \longrightarrow & G'^{-n} & \longrightarrow & G'^{-n+1} & \longrightarrow \cdots \longrightarrow & G'^{-1} & \longrightarrow & G'^0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Take mapping cone and get the exact sequence

$$\cdots \to G'^{-n-2} \to G^{-n} \oplus G'^{-n-1} \to G^{-n+1} \oplus G'^{-n} \to \cdots \to G^0 \oplus G'^{-1} \to G'^0 \to 0.$$

Since $\mathcal{X}$-$\mathcal{GP}$ is closed under taking kernels of epics and direct summands [22, Lemma 3.2], then by the following exact sequence

$$0 \to G^{-n} \to G^{-n+1} \oplus \mathrm{Ker}\left(G'^{-n+1} \to G'^{-n+2}\right) \to G^{-n+2} \oplus G'^{-n+1} \to \\ \cdots \to G^0 \oplus G'^{-1} \to G'^0 \to 0,$$

we get $\mathrm{Ker}(G'^{-n+1} \to G'^{-n+2}) \in \mathcal{X}$-$\mathcal{GP}$.

(6)⇒(5) Assume that $0 \to G^{-n} \to G^{-n+1} \to \cdots \to G^{-1} \to G^0 \to M \to 0$ be a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution of $M$ with length $n$. Let $\mathbf{G}' \to M \to 0$ be any $\mathcal{X}$-$\mathcal{GP}$-resolution of $M$. By the Comparison Lemma we get the following commutative diagram:

$$\begin{array}{ccccccccccccccc} \cdots \to & G'^{-n-1} & \to & G'^{-n} & \to & G'^{-n+1} & \to \cdots \to & G'^{-1} & \to & G'^0 & \to & M & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ \cdots \to & 0 & \to & G^{-n} & \to & G^{-n+1} & \to \cdots \to & G^{-1} & \to & G^0 & \to & M & \to & 0. \end{array}$$

Take mapping cone and get the exact sequence

$$\cdots \to G'^{-n-1} \to G'^{-n} \to G^{-n} \oplus G'^{-n+1} \to \\ G^{-n+1} \oplus G'^{-n+2} \to \cdots \to G^{-1} \oplus G'^0 \to G^0 \to 0.$$

Since $\mathcal{X}$-$\mathcal{GP}$ is closed under taking kernels of epics [22, Lemma 3.2],

$$\mathrm{Coker}\left(G'^{-n-1} \to G'^{-n}\right) \\ \cong \mathrm{Ker}\left(G^{-n} \oplus G'^{-n+1} \to G^{-n+1} \oplus G'^{-n+2}\right) \in \mathcal{X} - \mathcal{GP}.$$

So $\mathrm{Ker}(G'^{-n+1} \to G'^{-n+2}) \cong \mathrm{Coker}(G'^{-n-1} \to G'^{-n}) \in \mathcal{X}$-$\mathcal{GP}$.

(3)⇒(1) Assume that

$$\cdots \to G^{-n-2} \xrightarrow{d^{-n-2}} G^{-n-1} \xrightarrow{d^{-n-1}} G^{-n} \xrightarrow{d^{-n}} G^{-n+1} \to \cdots \to G^0 \xrightarrow{d^0} M \to 0$$

is a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution of $M$. Let $K^{-n} = \mathrm{Ker} d^{-n+1}$, $K^{-n-1} = \mathrm{Ker} d^{-n}$. Especially, in (3) set $N = K^{-n-1}$, so $\mathrm{Ext}^{n+1}_{\mathcal{XGP}}(M, K^{-n-1}) = 0$, that is, the sequence

$$\mathrm{Hom}_R(G^{-n}, K^{-n-1}) \to \mathrm{Hom}_R(G^{-n-1}, K^{-n-1}) \to \mathrm{Hom}_R(G^{-n-2}, K^{-n-1})$$

is exact. For $f \in \mathrm{Hom}_R(G^{-n-1}, K^{-n-1})$, $\mathrm{Hom}_R(d^{-n-2}, K^{-n-1})(f) = 0$. Hence, there exists $g \in \mathrm{Hom}_R(G^{-n}, K^{-n-1})$ such that $f = \mathrm{Hom}_R(d^{-n-1}, K^{-n-1})(g) = gd^{-n-1}$. Let $h : K^{-n-1} \to G^{-n}$. Since $d^{-n-1} = hf$ and $f$ is epic, by $f = gd^{-n-1} = ghf$, we have $gh = 1^{K^{-n-1}}$, that is, the sequence

$$0 \longrightarrow K^{-n-1} \longrightarrow G^{-n} \longrightarrow K^{-n} \longrightarrow 0$$

splits. So $K^{-n} \in \mathcal{X}\text{-}\mathcal{GP}$. □

By Theorem 3.12 we get the following conclusions.

**Corollary 3.17** *Let $M$ be an $R$-module admitting a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution, $n$ be a non-negative integer. Then the followings are equivalent.*

1. $\mathcal{X}\text{-}\mathrm{Gpd}_R M \leq n$.
2. *For every $R$-module $N$ and $i > 0$,* $\mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}(M, \Sigma^{n+i} N) = 0$.
3. *For every $R$-module $N$,* $\mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}(M, \Sigma^{n+1} N) = 0$.

By Theorem 3.12, Corollary 3.14 and Corollary 3.17, we get the following proposition.

**Proposition 3.18** *Let $R$ be a ring such that $l\mathcal{Y}$-GID$(R) < \infty$, $l\mathcal{X}$-GPD$(R) < \infty$, and ${}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}) = (\mathcal{X}\text{-}\mathcal{GP})^{\perp}$, $n$ be a non-negative integer. Then for $R$-module $M$, the following are equivalent.*

1. $\mathcal{X}\text{-}\mathrm{Gpd}_R M \leq n$.
2. *For every $R$-module $N$ and $i > 0$, morphisms in the relative derived category*

$$\mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}(R)\left(M, \Sigma^{n+i} N\right) = 0.$$

3. *For every $R$-module $N$, morphisms in the relative derived category*

$$\mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}\left(M, \Sigma^{n+1} N\right) = 0.$$

4. *For every $R$-module $N$ and $i > 0$, morphisms in the relative derived category*

$$\mathrm{Hom}_{\mathcal{D}^b_{\mathcal{YGI}}(R)}\left(M, \Sigma^{n+i} N\right) = 0.$$

5. *For every $R$-module $N$, morphisms in the relative derived category* $\mathrm{Hom}_{\mathcal{D}^b_{\mathcal{YGI}}(R)}$ $(M, \Sigma^{n+1} N) = 0$.

Dually, we can get the characterizations of $\mathcal{Y}$-Gorenstein injective dimension.

## 4 Generalized Tate Cohomology

In the following, we consider the generalized Tate cohomology respect to $\mathcal{X}$-Gorenstein projective and $\mathcal{Y}$-Gorenstein injective modules.

**Definition 4.1** Assume that $R$-module $M$ has a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution, $N$ is an arbitrary $R$-module. Let $\mathbf{P} \to M \to 0$ and $\mathbf{G} \to M \to 0$ be a projective resolution and a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution, resp. Then there is a morphism $f : \mathbf{P} \to \mathbf{G}$. Define the generalized Tate cohomology with respect to $\mathcal{X}$-Gorenstein projective modules as

$$\widehat{\mathrm{Ext}}^n_{\mathcal{X}\mathcal{GP}}(M, N) := \mathrm{H}_{-n-1}\mathrm{Hom}(\mathrm{Cone}(f), N).$$

We discuss the existence of Avramov–Martsinkovsky exact sequence by the methods in derived categories.

**Lemma 4.2** *Given proper left $\mathcal{X}$-$\mathcal{GP}$-resolutions $\gamma : \mathbf{G} \to M$ and $\gamma' : \mathbf{G}' \to M'$, and projective resolutions $\pi : \mathbf{P} \to M$ and $\pi : \mathbf{P}' \to M'$, there exist unique up to homotopy morphisms of complexes $\varphi : \mathbf{P} \to \mathbf{G}$ and $\varphi' : \mathbf{P}' \to \mathbf{G}'$ with $\pi = \gamma\varphi$ and $\pi' = \gamma'\varphi'$. For each homomorphism of modules $\mu : M \to M'$, there exists a unique up to homotopy morphism $\widetilde{\mu}$, making the right-hand square of the diagram*

$$\begin{array}{ccccc} \mathbf{P} & \xrightarrow{\varphi} & \mathbf{G} & \xrightarrow{\gamma} & M \\ \downarrow{\scriptstyle\overline{\mu}} & & \downarrow{\scriptstyle\widetilde{\mu}} & & \downarrow{\scriptstyle\mu} \\ \mathbf{P}' & \xrightarrow{\varphi'} & \mathbf{G}' & \xrightarrow{\gamma'} & M, \end{array}$$

*commute; for each choice of $\widetilde{\mu}$, there exists a unique up to homotopy morphism $\overline{\mu}$, making the left-hand square commute up to homotopy. If $\mu = \mathrm{id}_M$, then $\overline{\mu}$ and $\widetilde{\mu}$ are homotopy equivalences, that is, isomorphisms in homotopy category.*

***Proof*** Denote

$$\mathbf{G} : \cdots \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to 0,$$

$$\mathbf{G}'^{+} : \cdots \to G'_n \to G'_{n-1} \to \cdots \to G'_1 \to G'_0 \xrightarrow{\gamma'} M' \to 0.$$

$\mathbf{G}'^{+}$ is $\mathcal{X}$-$\mathcal{GP}$-acyclic; hence, $\mathrm{HHom}_R(G_i, \mathbf{G}'^{+}) = 0$ for all $i \in \mathbb{Z}$. Note that $\mathbf{G} = \varinjlim \mathbf{G}_{\leq i}$, where $\mathbf{G}_{\leq i}$ is the subcomplex of $\mathbf{G}$ with $n$th component equal to $G_n$ for $n \leq i$ and to 0 for $n > i$. Consider the exact sequence of complexes

$$0 \longrightarrow \mathbf{G}_{\leq i-1} \longrightarrow \mathbf{G}_{\leq i} \longrightarrow \Sigma^i G_i \longrightarrow 0.$$

As $\mathbf{G}_{\leq -1} = 0$, using the induced exact sequence of complexes of abelian groups

$$0 \to \mathrm{Hom}_R\left(\Sigma^i G_i, \mathbf{G}'^{+}\right) \to \mathrm{Hom}_R\left(\mathbf{G}_{\leq i}, \mathbf{G}'^{+}\right) \to \mathrm{Hom}_R\left(\mathbf{G}_{\leq i-1}, \mathbf{G}'^{+}\right) \longrightarrow 0$$

and induction on $i$, we get $\mathrm{HHom}_R(\mathbf{G}_{\leq i}, \mathbf{G}'^{+}) = 0$ for all $i \in \mathbb{Z}$. Thus, we have

$$\mathrm{HHom}_R\left(\mathbf{G}, \mathbf{G}'^{+}\right) = \mathrm{HHom}_R\left(\varinjlim \mathbf{G}_{\leq i}, \mathbf{G}'^{+}\right) = \mathrm{H}\left(\varprojlim \mathrm{Hom}_R\left(\mathbf{G}_{\leq i}, \mathbf{G}'^{+}\right)\right) = 0$$

with the last equality coming from the Mittag-Leffler criterion.

The exact sequence $0 \to \Sigma^{-1}M^{'} \to \mathbf{G}^{'+} \to \mathbf{G}^{'} \to 0$ induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R\left(\mathbf{G}, \Sigma^{-1}M^{'}\right) \longrightarrow \mathrm{Hom}_R\left(\mathbf{G}, \mathbf{G}^{'+}\right) \longrightarrow \mathrm{Hom}_R\left(\mathbf{G}, \mathbf{G}^{'}\right) \longrightarrow 0.$$

For each $n$, the connecting map in the homology exact sequence is an isomorphism

$$\mathrm{H}_n\mathrm{Hom}_R\left(\mathbf{G}, \mathbf{G}^{'}\right) \cong \mathrm{H}_{n-1}\mathrm{Hom}_R\left(\mathbf{G}, \Sigma^{-1}M^{'}\right) \cong \mathrm{H}_n\mathrm{Hom}_R\left(\mathbf{G}, M^{'}\right).$$

So $\mathrm{Hom}_R(\mathbf{G}, \gamma^{'})$ is a quasi-isomorphism. The Lifting Lemma [2, 1.1(1)] yields a unique up to homotopy morphism $\widetilde{\mu}$ with $\gamma^{'}\widetilde{\mu} = \mu\gamma$.

If $\mu = \mathrm{id}_M$, then reversing the roles of $M$ and $M^{'}$, we get a morphism $\widetilde{\mu}^{'} : \mathbf{G}^{'} \to \mathbf{G}$ inducing $\mathrm{id}_M$. Thus, $\widetilde{\mu}\widetilde{\mu}^{'} : \mathbf{G}^{'} \to \mathbf{G}^{'}$ induces $\mathrm{id}_M$, and hence is homotopic to $\mathrm{id}_{\mathbf{G}'}$. By symmetry, $\widetilde{\mu}^{'}\widetilde{\mu} \sim \mathrm{id}_{\mathbf{G}}$, so $\widetilde{\mu}$ is a homotopy equivalence.

By [2, 1.2(1) and 1.1(1)], we get morphisms $\varphi, \varphi^{'}$ with $\pi = \gamma\varphi, \pi^{'} = \gamma^{'}\varphi^{'}$, then a morphism $\widetilde{\mu}^{'}$ with $\varphi\overline{\mu}^{'} \sim \widetilde{\mu}^{'}\varphi^{'}$, all unique up to homotopy. If $\mu = \mathrm{id}_M$, then all these maps are quasi-isomorphisms, so $\overline{\mu}$ is a homotopy equivalence. □

**Proposition 4.3** *Let $M$ be an $R$-module admitting a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution and $N$ be an arbitrary $R$-module. Then $\widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(M, N)$ is well defined, and there is an* Avramov-Martsinkovsky *type exact sequence*

$$\begin{aligned}
&0 \longrightarrow \mathrm{Ext}^1_{\mathcal{XGP}}(M, N) \longrightarrow \mathrm{Ext}^1_R(M, N) \longrightarrow \widehat{\mathrm{Ext}}^1_{\mathcal{XGP}}(M, N) \\
&\quad \longrightarrow \mathrm{Ext}^2_{\mathcal{XGP}}(M, N) \longrightarrow \cdots \\
&\quad \longrightarrow \mathrm{Ext}^i_R(M, N) \longrightarrow \widehat{\mathrm{Ext}}^i_{\mathcal{XGP}}(M, N) \longrightarrow \\
&\mathrm{Ext}^{i+1}_{\mathcal{XGP}}(M, N) \longrightarrow \mathrm{Ext}^{i+1}_R(M, N) \longrightarrow \cdots .
\end{aligned}$$

***Proof*** Let $\mathbf{P} \to M$ and $\mathbf{G} \to M$ be a projective resolution and a proper left Gorenstein projective resolution of $M$, resp. For morphism $f : \mathbf{P} \to \mathbf{G}$, consider the exact triangle in $\mathcal{K}(R)$

$$\mathbf{P} \xrightarrow{f} \mathbf{G} \longrightarrow \mathrm{Cone}(f) \longrightarrow \Sigma\mathbf{P}.$$

Applying $\mathrm{Hom}_{\mathcal{K}(R)}(-, N)$ to the above triangle, we get the exact sequence

$$\begin{aligned}
&\cdots \to \mathrm{Hom}_{\mathcal{K}(R)}(\Sigma^{-n+1}\mathbf{P}, N) \\
&\quad \to \mathrm{Hom}_{\mathcal{K}(R)}(\Sigma^{-n}\mathrm{Cone}(f), N) \to \\
&\mathrm{Hom}_{\mathcal{K}(R)}\left(\Sigma^{-n}\mathbf{G}, N\right) \to \mathrm{Hom}_{\mathcal{K}(R)}\left(\Sigma^{-n}\mathbf{P}, N\right) \to \\
&\mathrm{Hom}_{\mathcal{K}(R)}(\Sigma^{-n-1}\mathrm{Cone}(f), N) \to \mathrm{Hom}_{\mathcal{K}(R)}(\Sigma^{-n-1}\mathbf{G}, N) \to \cdots ,
\end{aligned}$$

where $\mathrm{Hom}_{\mathcal{K}(R)}(\Sigma^{-n}\mathbf{P}, N) = \mathrm{Ext}^n_R(M, N)$. By Proposition 3.10 and Theorem 3.12, we get

$$\mathrm{Hom}_{\mathcal{K}(R)}(\Sigma^{-n}\mathbf{G}, N) \cong \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{XGP}}(R)}(M, \Sigma^n N) = \mathrm{Ext}^n_{\mathcal{XGP}}(M, N).$$

Since Hom is left exact, we get that $\widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(M, N) = 0$ whenever $n < 1$. Then the above Avramov–Martsinkovsky type exact sequence holds.

Take another projective resolution $\mathbf{Q} \to M \to 0$ and another proper left $\mathcal{X}$-$\mathcal{GP}$-resolution $\mathbf{X} \to M \to 0$. Then $1_M$ induces a morphism $g : \mathbf{Q} \to \mathbf{X}$, $u : \mathbf{P} \to \mathbf{Q}$ and $v : \mathbf{G} \to \mathbf{X}$, such that the following diagram commutes in $\mathcal{K}(R)$ by Lemma 4.2

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{f} & \mathbf{G} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \mathbf{Q} & \xrightarrow{g} & \mathbf{X}. \end{array}$$

Hence, we get the commutative diagram

$$\begin{array}{ccccccc} \mathbf{P} & \xrightarrow{f} & \mathbf{G} & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & \Sigma\mathbf{P} \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\downarrow & & {\scriptstyle \Sigma u}\downarrow \\ \mathbf{Q} & \xrightarrow{g} & \mathbf{X} & \longrightarrow & \mathrm{Cone}(g) & \longrightarrow & \Sigma\mathbf{Q}, \end{array}$$

where $w = \begin{pmatrix} \Sigma u & 0 \\ 0 & v \end{pmatrix}$ by [29, Lemma 2.2.3]. Since $u$ and $v$ are isomorphisms in $\mathcal{K}(R)$ by Lemma 4.2, $w$ is so in $\mathcal{K}(R)$. So

$$\mathrm{Hom}_{\mathcal{K}(R)}(\Sigma^{-n-1}\mathrm{Cone}(f), N) \cong \mathrm{Hom}_{\mathcal{K}(R)}(\Sigma^{-n-1}\mathrm{Cone}(g), N),$$

that is,

$$\mathrm{H}_{-n-1}\mathrm{Hom}_R(\mathrm{Cone}(f), N) \cong \mathrm{H}_{-n-1}\mathrm{Hom}_R(\mathrm{Cone}(g), N).$$

Hence, $\widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(M, N)$ is well defined. □

Dually,we have the following.

**Definition 4.4** Assume that $R$-module $N$ has a coproper right $\mathcal{Y}$-$\mathcal{GI}$-resolution, $M$ is an arbitrary $R$-module. Let $0 \to N \to \mathbf{I}$ and $0 \to N \to \mathbf{E}$ be an injective resolution and a coproper right $\mathcal{Y}$-$\mathcal{GI}$-resolution of $N$, resp. Then there exists a morphism $f : \mathbf{E} \to \mathbf{I}$. Define the Tate cohomology with respect to $\mathcal{Y}$-Gorenstein injective modules as

$$\widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(M, N) := \mathrm{H}_{-n-1}\mathrm{Hom}\left(M, \mathrm{Cone}(f)\right).$$

**Proposition 4.5** *Assume that $R$-module $N$ has a coproper right $\mathcal{Y}$-$\mathcal{GI}$-resolution, $M$ is an arbitrary $R$-module. Then $\widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(M, N)$ is well defined, and there exists an* Avramov-Martsinkovsky *exact sequence*

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^1_{\mathcal{YGI}}(M, N) \longrightarrow \mathrm{Ext}^1_R(M, N) \\ &\longrightarrow \widehat{\mathrm{Ext}}^1_{\mathcal{YGI}}(M, N) \longrightarrow \mathrm{Ext}^2_{\mathcal{YGI}}(M, N) \longrightarrow \cdots \\ &\longrightarrow \mathrm{Ext}^i_R(M, N) \longrightarrow \widehat{\mathrm{Ext}}^i_{\mathcal{YGI}}(M, N) \\ &\longrightarrow \mathrm{Ext}^{i+1}_{\mathcal{YGI}}(M, N) \longrightarrow \mathrm{Ext}^{i+1}_R(M, N) \longrightarrow \cdots . \end{aligned}$$

It is easy to get the balance of generalized Tate cohomology with respect to $\mathcal{X}$-Gorenstein projective and $\mathcal{Y}$-Gorenstein injective modules holds when $l\mathcal{Y}$-GID$(R) < \infty$, $l\mathcal{X}$-GPD$(R) < \infty$, and ${}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}) = (\mathcal{X}\text{-}\mathcal{GP})^{\perp}$. In addition, we discuss the long exact sequences of Tate cohomology with respect to $\mathcal{X}$-Gorenstein projective modules (resp., $\mathcal{Y}$-Gorenstein injective modules) $\widehat{\mathrm{Ext}}_{\mathcal{XGP}}$(resp., $\widehat{\mathrm{Ext}}_{\mathcal{YGI}}$).

**Lemma 4.6** *1. Let $\mathbf{N} = 0 \to N \stackrel{f}{\to} N' \stackrel{g}{\to} N'' \to 0$ be an $\mathcal{X}$-$\mathcal{GP}$-acyclic sequence of $R$-modules. Then we have the following commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{P}_N & \xrightarrow{\alpha} & \mathbf{P}_{N'} & \xrightarrow{\beta} & \mathbf{P}_{N''} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s''} & & \\ 0 & \longrightarrow & \mathbf{G}_N & \xrightarrow{\gamma} & \mathbf{G}_{N'} & \xrightarrow{\delta} & \mathbf{G}_{N''} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle t''} & & \\ 0 & \longrightarrow & N & \xrightarrow{f} & N' & \xrightarrow{g} & N'' & \longrightarrow & 0, \end{array}$$

*where the upper two rows are degreewise split exact sequences of complexes, $\mathbf{P}_N \stackrel{\pi}{\to} N \to 0$, $\mathbf{P}_{N'} \stackrel{\pi'}{\to} N' \to 0$ and $\mathbf{P}_{N''} \stackrel{\pi''}{\to} N'' \to 0$ are projective resolutions of $N$, $N'$ and $N''$, resp., $\mathbf{G}_N \stackrel{t}{\to} N \to 0$, $\mathbf{G}_{N'} \stackrel{t'}{\to} N' \to 0$ and $\mathbf{G}_{N''} \stackrel{t''}{\to} N'' \to 0$ are proper left $\mathcal{X}$-$\mathcal{GP}$-resolutions of $N$, $N'$ and $N''$ resp., and $\pi = ts$, $\pi' = t's'$, $\pi'' = t''s''$.*

*2. Let $\mathbf{M} = 0 \to M \stackrel{f}{\to} M' \stackrel{g}{\to} M'' \to 0$ be a $\mathcal{Y}$-$\mathcal{GI}$-acyclic sequence of $R$-modules. Then we have the following commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{I}_M & \xrightarrow{\alpha} & \mathbf{I}_{M'} & \xrightarrow{\beta} & \mathbf{I}_{M''} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle t'} & & \uparrow{\scriptstyle t''} & & \\ 0 & \longrightarrow & \mathbf{E}_M & \xrightarrow{\gamma} & \mathbf{E}_{M'} & \xrightarrow{\delta} & \mathbf{E}_{M''} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s'} & & \uparrow{\scriptstyle s''} & & \\ 0 & \longrightarrow & M & \xrightarrow{f} & M' & \xrightarrow{g} & M'' & \longrightarrow & 0, \end{array}$$

*where the upper two rows are degreewise split exact sequences of complexes, and* $0 \to M \xrightarrow{\varepsilon} \mathbf{I}_M$, $0 \to M' \xrightarrow{\varepsilon'} \mathbf{I}_{M'}$ *and* $0 \to M'' \xrightarrow{\varepsilon''} \mathbf{I}_{M''}$ *are injective resolutions of* $M$, $M'$ *and* $M''$ *resp.,* $0 \to M \xrightarrow{s} \mathbf{E}_M$, $0 \to M' \xrightarrow{s'} \mathbf{E}_{M'}$ *and* $0 \to M'' \xrightarrow{s''} \mathbf{E}_{M''}$ *are coproper right* $\mathcal{Y}$*-*$\mathcal{GI}$*-resolutions of* $M$, $M'$ *and* $M''$ *resp., and* $\varepsilon = ts$, $\varepsilon' = t's'$, $\varepsilon'' = t''s''$.

***Proof*** It is enough to prove (1). (2) is dually. Let $\mathbf{P}_N \xrightarrow{\pi} N \to 0$ and $\mathbf{P}_{N''} \xrightarrow{\pi''} N'' \to 0$ be projective resolutions of $N$ and $N''$ resp., then we have the following commutative diagram by Horseshoe Lemma:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{P}_N & \xrightarrow{\alpha} & \mathbf{P}_{N'} & \xrightarrow{\beta} & \mathbf{P}_{N''} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi''} & & \\
0 & \longrightarrow & N & \xrightarrow{f} & N' & \xrightarrow{g} & N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Set $(\mathbf{P}_{N'})_i = (\mathbf{P}_N)_i \oplus (\mathbf{P}_{N''})_i$. Then $\pi' : \mathbf{P}_{N'} \to N' \to 0$ is a projective resolution of $N'$.

Let $\mathbf{G}_N \xrightarrow{t} N \to 0$ and $\mathbf{G}_{N''} \xrightarrow{t} N'' \to 0$ be proper left $\mathcal{X}$-$\mathcal{GP}$-resolutions of $N$ and $N''$, resp. Since $\mathbf{N} = 0 \to N \xrightarrow{f} N' \xrightarrow{g} N'' \to 0$ is $\mathcal{X}$-$\mathcal{GP}$-acyclic sequence of $R$-modules, then we get the following commutative diagram by Horseshoe Lemma similarly:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{G}_N & \xrightarrow{\gamma} & \mathbf{G}_{N'} & \xrightarrow{\delta} & \mathbf{G}_{N''} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle t''} & & \\
0 & \longrightarrow & N & \xrightarrow{f} & N' & \xrightarrow{g} & N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Set $(\mathbf{G}_{N'})_i = (\mathbf{G}_N)_i \oplus (\mathbf{G}_{N''})_i$. Then $t' : \mathbf{G}_{N'} \to N' \to 0$ is a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution of $N'$.

For $\mathbf{G}_N \xrightarrow{t} N \to 0$, since $\mathbf{P}_N$ is dg projective, we have an epic quasi-isomorphism

$$\mathrm{Hom}_R(\mathbf{P}_N, \mathbf{G}_N) \xrightarrow{\mathrm{Hom}_R(\mathbf{P},t)} \mathrm{Hom}_R(\mathbf{P}_N, N) \longrightarrow 0.$$

Hence, by [2, (1.1)], for $\pi : \mathbf{P}_N \to N$, there exists $s : \mathbf{P}_N \to \mathbf{G}_N$, such that $\pi = ts$. Similarly for $\pi' : \mathbf{P}_{N'} \to N'$, $\pi'' : \mathbf{P}_{N''} \to N''$, there exists $s' : \mathbf{P}_{N'} \to \mathbf{G}_{N'}$ and $s'' : \mathbf{P}_{N''} \to \mathbf{G}_{N''}$ resp., such that $\pi' = t's'$, $\pi'' = t''s''$.

Consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{P}_N & \xrightarrow{\alpha} & \mathbf{P}_{N'} & \xrightarrow{\beta} & \mathbf{P}_{N''} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s''} & & \\
0 & \longrightarrow & \mathbf{G}_N & \xrightarrow{\gamma} & \mathbf{G}_{N'} & \xrightarrow{\delta} & \mathbf{G}_{N''} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle t''} & & \\
0 & \longrightarrow & N & \xrightarrow{f} & N^{'} & \xrightarrow{g} & N^{''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Since $t's'\alpha = \pi'\alpha = f\pi = fts = t'\gamma s$ and $t'$ is an epic morphism, so $s'\alpha = \gamma s$. Similarly, since $t''s''\beta = \pi''\beta = g\pi' = gt's' = t''\delta s'$ and $t''$ are epic morphisms, so $s''\beta = \delta s'$, that is the above diagram is commutative. □

**Proposition 4.7** *Let $M$ be an $R$-module and $\mathbf{N} = 0 \rightarrow N \overset{f}{\rightarrow} N^{'} \overset{g}{\rightarrow} N^{''} \rightarrow 0$ be an $\mathcal{X}$-$\mathcal{GP}$-acyclic sequence of $R$-modules.*

1. *If $M$ has a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution, then there exists a natural morphism $\widehat{\vartheta}^n_{\mathcal{XGP}}(M, \mathbf{N})$, such that the following sequence is exact.*

$$
\cdots \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(M, N) \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(M,f)} \widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(M, N^{'}) \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(M,g)} \widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(M, N^{''})
$$

$$
\xrightarrow{\widehat{\vartheta}^n_{\mathcal{XGP}}(M,\mathbf{N})} \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{XGP}}(M, N) \xrightarrow{\widehat{\mathrm{Ext}}^{n+1}_{\mathcal{XGP}}(M,f)} \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{XGP}}(M, N^{'}) \xrightarrow{\widehat{\mathrm{Ext}}^{n+1}_{\mathcal{XGP}}(M,g)} \cdots .
$$

2. *If $N,\ N^{'}$ and $N^{''}$ have proper left $\mathcal{X}$-$\mathcal{GP}$-resolutions, then there exists a natural morphism $\vartheta^n_{\mathcal{XGP}}(\mathbf{N}, M)$, such that the following sequence is exact.*

$$
\cdots \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(N^{''}, M) \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(g,M)} \widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(N^{'}, M) \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(f,M)} \widehat{\mathrm{Ext}}^n_{\mathcal{XGP}}(N, M)
$$

$$
\xrightarrow{\widehat{\vartheta}^n_{\mathcal{XGP}}(\mathbf{N},M)} \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{XGP}}(N^{''}, M) \xrightarrow{\widehat{\mathrm{Ext}}^{n+1}_{\mathcal{XGP}}(g,M)} \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{XGP}}(N^{'}, M) \xrightarrow{\widehat{\mathrm{Ext}}^{n+1}_{\mathcal{XGP}}(f,M)} \cdots .
$$

***Proof*** 1. Let $\mathbf{P} \rightarrow M \rightarrow 0$ and $\mathbf{G} \rightarrow M \rightarrow 0$ be a projective resolution and a proper left $\mathcal{X}$-$\mathcal{GP}$-resolution of $M$, resp. Then there exists a morphism $f : \mathbf{P} \rightarrow \mathbf{G}$. For the mapping cone $\mathrm{Cone}(f)$, $\mathrm{Cone}(f)_i = G_i \oplus P_{i-1}$ is $\mathcal{X}$-Gorenstein projective. Since the sequence of $R$-modules $\mathbf{N} = 0 \rightarrow N \overset{f}{\rightarrow} N^{'} \overset{g}{\rightarrow} N^{''} \rightarrow 0$ is $\mathcal{X}$-$\mathcal{GP}$-acyclic, so there is an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}_R\left(\mathrm{Cone}(f), N^{''}\right) \longrightarrow \mathrm{Hom}_R\left(\mathrm{Cone}(f), N^{'}\right) \longrightarrow \mathrm{Hom}_R(\mathrm{Cone}(f), N) \longrightarrow 0.$$

So the long exact sequence in (1) holds.

2. By Lemma 4.6 we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{P}_N & \xrightarrow{\alpha} & \mathbf{P}_{N'} & \xrightarrow{\beta} & \mathbf{P}_{N''} & \longrightarrow & 0 \\
 & & \downarrow{s} & & \downarrow{s'} & & \downarrow{s''} & & \\
0 & \longrightarrow & \mathbf{G}_N & \xrightarrow{\gamma} & \mathbf{G}_{N'} & \xrightarrow{\delta} & \mathbf{G}_{N''} & \longrightarrow & 0 \\
 & & \downarrow{t} & & \downarrow{t'} & & \downarrow{t''} & & \\
0 & \longrightarrow & N & \xrightarrow{f} & N^{'} & \xrightarrow{g} & N^{''} & \longrightarrow & 0,
\end{array}$$

where $\mathbf{P}_N \xrightarrow{\pi} N \rightarrow 0$, $\mathbf{P}_{N'} \xrightarrow{\pi'} N^{'} \rightarrow 0$ and $\mathbf{P}_{N''} \xrightarrow{\pi''} N^{''} \rightarrow 0$ are projective resolutions of $N$, $N^{'}$ and $N^{''}$ resp., $\mathbf{G}_N \xrightarrow{t} N \rightarrow 0$, $\mathbf{G}_{N'} \xrightarrow{t'} N^{'} \rightarrow 0$ and $\mathbf{G}_{N''} \xrightarrow{t''} N^{''} \rightarrow 0$ are proper left $\mathcal{X}$-$\mathcal{GP}$-resolutions of $N$, $N^{'}$ and $N^{''}$, resp. Also the upper two rows are degreewise split-exact sequences of complexes. Since $\alpha$ and $\gamma$ are injective, so $\begin{pmatrix} \Sigma\alpha & 0 \\ 0 & \gamma \end{pmatrix} : \mathrm{Cone}(s) \rightarrow \mathrm{Cone}(s')$ is injective. Similarly, since $\beta$, $\delta$ are epic, then $\begin{pmatrix} \Sigma\beta & 0 \\ 0 & \delta \end{pmatrix} : \mathrm{Cone}(s') \rightarrow \mathrm{Cone}(s'')$ is epic. Hence, we have degreewise split exact sequence of complexes

$$0 \longrightarrow \mathrm{Cone}(s) \longrightarrow \mathrm{Cone}(s^{'}) \longrightarrow \mathrm{Cone}(s^{''}) \longrightarrow 0.$$

Applying $\mathrm{Hom}_R(-, M)$ to it, we get the exact sequence of complexes

$$0 \rightarrow \mathrm{Hom}_R(\mathrm{Cone}(s''), M) \rightarrow \mathrm{Hom}_R(\mathrm{Cone}(s'), M) \rightarrow \mathrm{Hom}_R(\mathrm{Cone}(s), M) \rightarrow 0.$$

Hence, the long exact sequence in (2) holds. □

Dually, we have

**Proposition 4.8** *Let* $\mathbf{M} = 0 \rightarrow M \xrightarrow{f} M^{'} \xrightarrow{g} M^{''} \rightarrow 0$ *be a* $\mathcal{Y}$-$\mathcal{GI}$*-acyclic sequence of* $R$*-modules,* $N$ *be an arbitrary* $R$*-module.*

1. *If* $N$ *has a proper right* $\mathcal{Y}$-$\mathcal{GI}$*-resolution, then there exists a natural morphism* $\widehat{\vartheta}^n_{\mathcal{YGI}}(\mathbf{M}, N)$*, such that the following sequence is exact.*

$$\cdots \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(M'', N) \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(g,N)} \widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(M', N) \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(f,N)} \widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(M, N)$$

$$\xrightarrow{\widehat{\vartheta}^n_{\mathcal{YGI}}(\mathbf{M},N)} \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{YGI}}(M'', N) \xrightarrow{\widehat{\mathrm{Ext}}^{n+1}_{\mathcal{YGI}}(g,N)} \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{YGI}}(M', N) \xrightarrow{\widehat{\mathrm{Ext}}^{n+1}_{\mathcal{YGI}}(f,N)} \cdots .$$

2. *If $M$, $M'$ and $M''$ have proper right $\mathcal{Y}$-$\mathcal{GI}$-resolutions, then there exists a natural morphism $\widehat{\vartheta}^n_{\mathcal{YGI}}(N, \mathbf{M})$, such that the following sequence is exact.*

$$\cdots \longrightarrow \widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(N, M) \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(N,f)} \widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(N, M') \xrightarrow{\widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(N,g)} \widehat{\mathrm{Ext}}^n_{\mathcal{YGI}}(N, M'')$$

$$\xrightarrow{\widehat{\vartheta}^n_{\mathcal{YGI}}(N,\mathbf{M})} \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{YGI}}(N, M) \xrightarrow{\widehat{\mathrm{Ext}}^{n+1}_{\mathcal{YGI}}(N,f)} \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{YGI}}(N, M') \xrightarrow{\widehat{\mathrm{Ext}}^{n+1}_{\mathcal{YGI}}(N,g)} \cdots .$$

## 5 Applications

***Example 5.1*** Let $R$ be a Noetherian ring. By [5, Lemma 2.1], G-gldim$(R) \leq n < \infty$ if and only if $R$ is a $n$-Gorenstein ring. Hence, Theorem 3.6 is the conclusion in [13]: when $R$ is a Gorenstein ring, $\mathcal{D}^*_{\mathcal{GP}}(R)$ and $\mathcal{D}^*_{\mathcal{GI}}(R)$ are triangle-equivalent, which are called Gorenstein derived categories.

***Example 5.2*** Let $R$ be a commutative coherent ring. By [4] and [21, Theorem 3.1], G-wdim$(R) \leq$ G-gldim$(R) \leq n < \infty$ if and only if $R$ is a $n$-FC ring. Hence, Theorem 3.6 is the conclusion in [24]: when $R$ is a $n$-FC ring, $\mathcal{D}^*_{\mathcal{DP}}(R)$ are $\mathcal{D}^*_{\mathcal{DI}}(R)$ are triangle-equivalent, which are called Ding derived categories.

***Example 5.3*** When ring $R$ has finite global dimension, then Theorem 3.6 is the classical triangle-equivalence between injective derived categories and injective derived categories, that is the classical derived category $\mathcal{D}(R)$, and $\mathcal{GP} = \mathcal{X}$-$\mathcal{GP} = \mathcal{P}$. Hence, Gorenstein projective derived category and $\mathcal{D}^*_{\mathcal{GP}}(R)$, $\mathcal{X}$-Gorenstein projective derived category $\mathcal{D}^*_{\mathcal{XGP}}(R)$ and derived category $\mathcal{D}^*(R)$ coincide.

***Example 5.4*** Let $R$ be a ring with finite global Gorenstein AC-injective dimension and global Gorenstein AC-projective dimension, also each absolutely clean $R$-module has finite level dimension or each level $R$-module has finite absolutely clean dimension, then for $* \in \{blank, -, +, b\}$, $\mathcal{D}^*_{\widetilde{\mathcal{C}}}(R)$ and $\mathcal{D}^*_{\widetilde{\mathcal{F}}}(R)$ are are triangle-equivalent, where $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{F}}$ denote the classes of Gorenstein AC-projective modules and injective modules, resp. Other corresponding conclusions on morphisms and Generalized Tate cohomology can be easily obtained. In fact, a ring $R$ is called AC-Gorenstein if each absolutely clean $R$-module has finite level dimension, equivalently, each level $R$-module has finite absolutely clean dimension. (See [16, Definition 4.1]). By [16, Theorems 6.2 and 7.1], ${}^{\perp}(\widetilde{\mathcal{F}}) = (\widetilde{\mathcal{C}})^{\perp} = \omega$, where $\omega$ denotes the class of modules of finite level dimension, equivalently, of finite absolutely clean dimension.

***Remark 5.5*** Recently, the authors in [3] generalize the all existing version of Gorenstein modules. The discussions in current paper can be considered in this general setting.

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