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ORIGINAL PAPER

Blow-up solutions in a Cauchy problem of parabolic equations with spatial coefficients

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Abstract
This paper deals with a Cauchy problem of the parabolic equations

$$u_t = \Delta u + a_1(x)u^{p_1} + b_1(x)v^{q_1},\ v_t = \Delta v + a_2(x)u^{p_2} + b_2(x)v^{q_2},$$

where the exponents p_i, q_i $(i = 1, 2)$ are positive constants; the coefficients $a_i(x) \sim |x|^{\alpha_i}$ and $b_i(x) \sim |x|^{\beta_i}$ as $|x| \to +\infty$ with the parameters $\alpha_i,\ \beta_i \in R$. For α_i, $\beta_i \geq 0$, we determine the exponent regions where all of the solutions blow up for any nonnegative nontrivial initial data. For at least one negative parameter, we find different conditions on global existence of solutions according to different classifications of the parameters.

Keywords Cauchy problem · Reaction–diffusion equations · Fujita exponent

Mathematics Subject Classification 35K57 · 35K15 · 35K45 · 35B33

1 Introduction

In this paper, we study a Cauchy problem of the parabolic equations with different spatially dependent coefficients

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$$\begin{cases} u_t = \Delta u + a_1(x)u^{p_1} + b_1(x)v^{q_1}, & (x,t) \in R^N \times [0,T), \\ v_t = \Delta v + a_2(x)u^{p_2} + b_2(x)v^{q_2}, & (x,t) \in R^N \times [0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in R^N, \end{cases} \tag{1.1}$$

where the exponents $p_i,\ q_i$ $(i = 1, 2)$ are positive constants; the coefficients $a_i(x),\ b_i(x) \gneqq 0$ are locally Hölder-continuous satisfying that $a_i(x) \sim |x|^{\alpha_i}$ and $b_i(x) \sim |x|^{\beta_i}$ as $|x| \to +\infty$ for $\alpha_i,\ \beta_i \in R$ $(i = 1, 2)$; Initial data $u_0,\ v_0 \gneqq 0$ are nonnegative bounded continuous functions. The uniqueness and local existence of classical solutions can be obtained by the standard procedure in [14]. Nonlinear parabolic equations coupled via nonlinear sources just as (1.1) are widely used in chemical reactions, population dynamics, and heat transfer process, where the components of the solutions represent the thickness of two kinds of chemical reactants, the densities of two biological populations during a migration, and the temperature of two different materials during a propagation (see, for example, [4,5,12]).

It is well known that the Fujita blow-up exponent $p_c = 1+2/N$ is introduced for the Cauchy problem of $u_t = \Delta u + u^p$. If $1 < p \le p_c$, any nonnegative nontrivial solutions blow up in finite time. Pinsky studied the weighted equation $u_t = \Delta u + a(x)u^p$ with $a(x) \sim |x|^\alpha$ as $|x| \to +\infty$ and obtained the Fujita exponent $p_c = 1 + (2+\alpha)/N$ in [16]. Escobedo and Herrero studied the Fujita exponents of the problem (1.1) with $a_1 = b_2 = 0$ and $a_2 = b_1 = 1$ in [4].

Li, Sun and Zhang in [8] considered the Fujita exponent to the Cauchy problem of the following reaction–diffusion equations:

$$u_t = \Delta u + v^p, \quad v_t = \Delta v + a(x)u^q, \quad (x,t) \in R^N \times [0,T), \tag{1.2}$$

where $a(x) \sim |x|^m$ as $|x| \to +\infty$ and $m \in R$. They proved that (i) if $0 < pq \le 1$ and $m \ge 0$, all of the solutions are global; (ii) if $pq > 1$ and $m \ge 0$, there are no global solutions provided that $\max\left\{\frac{\frac{m+2}{2}p+1}{pq-1}, \frac{\frac{m+2}{2}+q}{pq-1}\right\} \ge N/2$; (iii) if $\max\left\{\frac{\frac{m+2}{2}p+1}{pq-1}, \frac{\frac{m+2}{2}+q}{pq-1}\right\} < N/2$ and $m \ge 0$, the problem (1.2) possesses both global and blow-up solutions. For $m < 0$, some results on the global existence of solutions are proved under some additional conditions.

Souplet and Tayachi in [15] discussed the following Cauchy problem:

$$u_t = \Delta u + u^{p_1} + v^{q_2}, \quad v_t = \Delta v + v^{p_2} + u^{q_1}, \quad (x,t) \in R^N \times (0,T), \tag{1.3}$$

with constants $p_i,\ q_i > 1$ $(i = 1, 2)$. If $p_1 > q_1 + 1$ or $p_2 > q_2 + 1$, then there exist initial data, such that non-simultaneous blow-up happens; If $p_1 < q_1 + 1$ and $p_2 < q_2 + 1$, then simultaneous blow-up occurs for every initial data. Rossi and Souplet in [13] studied the parabolic equations (1.3) in a bounded domain, subject to homogeneous Dirichlet boundary conditions. The coexistence of non-simultaneous and simultaneous blow-up was first observed in the exponent region $\{p_1 > q_1 + 1,\ p_2 > q_2 + 1\}$.

Liu and Lin discussed the Cauchy problem of the following parabolic equations in [10]:

$$u_t = \Delta u + b(x)u^\alpha v^p, \quad v_t = \Delta v + a(x)u^q v^\beta, \quad (x,t) \in R^N \times [0,T), \qquad (1.4)$$

where $a(x) \sim |x|^m$ and $b(x) \sim |x|^n$ as $|x| \to +\infty$ for $m, n \geq 0$. Problem (1.4) does not possess global solutions for nonnegative nontrivial initial data provided that $(1-\alpha)(1-\beta) < pq \leq (pq)_c$ or $1 < \alpha \leq \alpha_c := 1 + (2+n)/N$ or $1 < \beta \leq \beta_c := 1 + (2+m)/N$, where

$$\begin{aligned}(pq)_c : = (1-\alpha)(1-\beta) + \frac{2}{N}\max\Big\{p\left(\frac{m}{2}+1\right)\\ +(1-\beta)\left(\frac{n}{2}+1\right),\ q\left(\frac{n}{2}+1\right) + (1-\alpha)\left(\frac{m}{2}+1\right)\Big\}.\end{aligned}$$

If $pq > (pq)_c$, $\alpha > \alpha_c$, and $\beta > \beta_c$, both global solutions and blow-up solutions exist according to the choice of the initial data. Li and Sun in [7] studied a time-weighted parabolic system, subject to null Dirichlet boundary conditions. The critical Fujita exponents are prescribed by the weighted functions and the first eigenvalue of Laplacian operator with zero Dirichlet boundary. Some related results can also be found in the works [3,9,17–20] and [1,2,11] also.

It could be checked that the Fujita exponents in [10] are compatible with the ones in [4,8,16] when the exponents and the parameters in (1.4) were taken of the special values. Because of the different coupled relationship between (1.1) and (1.4), the blow-up and global criteria on solutions in [10] are not applicable to the ones of (1.1). The coupled parabolic equations in (1.1) are much more complicated than the ones in (1.2). There are also many classifications for the parameters of coefficients. It could be imagined that the singular phenomena of solutions are much more complicated than the ones in [8]. Inspired by the works [7,8,10], we want to determine the exponent regions where any nonnegative solutions blow up in finite time for any nontrivial nonnegative initial data. Moreover, we want to discuss the influence of the four parameters in the coefficients and show the quantitative conditions on the global existence of solution.

This paper is arranged as follows. In the next section, the main results are given with respect to different cases (Theorems 2.1–2.6). The proof of the theorems can be found in Sects. 3, 4, and 5, respectively. At the last section, we show the conclusion.

2 Main Results

If p_1, q_2, $p_2q_1 \leq 1$, the problem (1.1) turns into the subcritical one, where all of the nonnegative solutions are global for any nonnegative initial data. If $\max\{p_1, q_2, p_2q_1\} > 1$, there exist blow-up solutions for large initial data (see, e.g., [6]). For convenience, we give three notations

$$(p_1)_c : = 1 + \max\left\{\frac{2+\alpha_1}{N}, 0\right\}, \quad (q_2)_c := 1 + \max\left\{\frac{2+\beta_2}{N}, 0\right\},$$

$$(p_2q_1)_c := 1 + \frac{2}{N} \max \left\{ p_2 \left(\frac{\beta_1}{2} + 1 \right) + \frac{\alpha_2}{2} + 1,\ q_1 \left(\frac{\alpha_2}{2} + 1 \right) + \frac{\beta_1}{2} + 1,\ 0 \right\}.$$

Let $\tilde{c}_1, \tilde{c}_2 > 0$ be two constants satisfying that

$$\tilde{c}_1 |x|^{\alpha_i} \leq a_i(x) \leq \tilde{c}_2 |x|^{\alpha_i}, \quad \tilde{c}_1 |x|^{\beta_i} \leq b_i(x) \leq \tilde{c}_2 |x|^{\beta_i} \quad \text{for large } |x|,\ i = 1, 2. \tag{2.1}$$

Theorem 2.1 *Let α_1, α_2, β_1, and β_2 be positive. There are no global solutions of* (1.1) *for any nonnegative nontrivial initial data provided that*

$$1 < p_2q_1 \leq (p_2q_1)_c, \quad or \quad 1 < p_1 \leq (p_1)_c, \quad or \quad 1 < q_2 \leq (q_2)_c.$$

The blow-up criteria in Theorem 2.1 are compatible with the ones in [4,8,16]. The results of Theorem 2.1 and the ones in [8,10] show that the coefficients of the sources play an important role in distinguishing global solutions from blow-up solutions. Positive parameters of the coefficients are helpful for the existence of blow-up solutions when $|x|$ is large enough, while negative parameters are good for global existence of solutions.

If the exponents satisfy that

$$p_2q_1 > (p_2q_1)_c, \quad p_1 > (p_1)_c, \quad \text{and} \quad q_2 > (q_2)_c, \tag{2.2}$$

there are blow-up solutions for large initial data (see [6]). We care about the different quantitative conditions on the global existence of solutions according to different classifications of $\alpha_1, \alpha_2, \beta_1$, and β_2. The assumption (2.2) is necessary in the following theorems.

The first result is given for the positive parameters α_1, α_2, β_1, and β_2.

Theorem 2.2 *Let* (2.2) *be in force. Assume the initial data satisfy*

$$u_0(y) \leq \delta_1 G(k, 0, y), \quad v_0(y) \leq \delta_2 G(k, 0, y), \tag{2.3}$$

for any $k > 0$ and some constants $\delta_1, \delta_2 > 0$. There exist global solutions of (1.1) *provided that positive parameters β_2, α_1, β_1, and α_2 satisfy that*

$$\beta_2 > q_1 N, \tag{2.4}$$

$$\alpha_1 > p_2 N, \tag{2.5}$$

$$(q_2 - 1)E_v < \min\{1,\ N(q_2 - 1)/2 - 1 - \beta_2/2\}, \tag{2.6}$$

$$(p_1 - 1)E_u < \min\{1,\ N(p_1 - 1)/2 - 1 - \alpha_1/2\}, \tag{2.7}$$

where $E_v := \frac{N}{2} - \frac{(\frac{\beta_1}{2}+1)p_2 + \frac{\alpha_2}{2} + 1}{p_2q_1 - 1}$ *and* $E_u := \frac{N}{2} - \frac{(\frac{\alpha_2}{2}+1)q_1 + \frac{\beta_1}{2} + 1}{p_2q_1 - 1}$.

It can be checked that the conditions (2.4–2.7) in Theorem 2.2 are easy to meet. In fact, (2.2) deduces that $p_1 > 1$ and $q_2 > 1$; At first, we easily choose β_2 and α_1, such that (2.4–2.5) hold for any $N \geq 1$. Then, one could choose suitable β_1 and α_2, such that (2.6–2.7) hold.

The following four results of (1.1) are given, containing at least one negative parameter in the coefficients.

Theorem 2.3 *Let only* **one** *of the parameters* α_1, α_2, β_1, *and* β_2 *be negative and* (2.2) *be in force. Assume the initial data satisfy* (2.3).

(i) $\alpha_1 < 0$. *If the positive parameters* β_2, α_2, *and* β_1 *satisfy*

$$(p_1 - 1)E_u \leq \frac{2 + \alpha_1}{2} \quad \text{for } p_1 - 1 > p_2, \tag{2.8}$$

(2.4) *and* (2.6), *then system* (1.1) *has global solutions.*

(ii) $\beta_2 < 0$. *If the positive parameters* α_1, α_2, *and* β_1 *satisfy*

$$(q_2 - 1)E_v \leq \frac{2 + \beta_2}{2} \quad \text{for } q_2 - 1 > q_1, \tag{2.9}$$

(2.5) *and* (2.7), *then system* (1.1) *has global solutions.*

(iii) $\alpha_2 < 0$ *or* $\beta_1 < 0$. *If the positive parameters* β_2, α_1, *and* β_1, *or* β_2, α_1, *and* α_2, *respectively, satisfy* (2.4–2.7) *for* $q_2 - 1 > q_1$, *then system* (1.1) *has global solutions.*

Theorem 2.4 *Let* **two** *of the parameters* α_1, α_2, β_1, *and* β_2 *be negative and* (2.2) *be in force. Assume the initial data satisfy* (2.3).

(i) $\beta_1, \beta_2 < 0$ *or* $\alpha_2, \beta_2 < 0$. *If positive parameters* α_1 *and* α_2, *or* α_1 *and* β_1, *respectively, satisfy* (2.5), (2.7) *and* (2.9) *for* $q_2 - 1 > q_1$, *then system* (1.1) *has global solutions.*

(ii) $\alpha_1, \alpha_2 < 0$ *or* $\alpha_1, \beta_1 < 0$. *If positive parameters* β_2 *and* β_1, *or* β_2 *and* α_2, *respectively, satisfy* (2.4), (2.6) *and* (2.8) *for* $p_1 - 1 > p_2$, *then system* (1.1) *has global solutions.*

(iii) $\alpha_1, \beta_2 < 0$ *or* $\alpha_2, \beta_1 < 0$. *If* (2.4–2.7) *are true, then system* (1.1) *has global solutions.*

Theorem 2.5 *Let* **three** *of the parameters* α_1, α_2, β_1, *and* β_2 *be negative and* (2.2) *be in force. Assume the initial data satisfy* (2.3).

(i) $\alpha_1, \alpha_2, \beta_1 < 0$. *If* (2.4), (2.6) *and* (2.8) *are true and* $p_1 - 1 > p_2$, *then system* (1.1) *has global solutions.*

(ii) $\alpha_2, \beta_1, \beta_2 < 0$. *If* (2.5), (2.7) *and* (2.9) *are true and* $q_2 - 1 > q_1$, *then system* (1.1) *has global solutions.*

(iii) $\alpha_1, \alpha_2, \beta_2 < 0$ *or* $\alpha_1, \beta_1, \beta_2 < 0$. *If the parameter* β_1 *or* α_2, *respectively, satisfies* (2.8–2.9) *and* $q_2 - 1 > q_1$ *and* $p_1 - 1 > p_2$, *then system* (1.1) *has global solutions.*

Theorem 2.6 *Let* **all** *of the parameters* α_1, α_2, β_1 *and* β_2 *be negative and* (2.2) *be in force. Assume the initial data satisfy* (2.3). *If* (2.8–2.9) *are true and* $q_2 - 1 > q_1$ *and* $p_1 - 1 > p_2$, *then system* (1.1) *has global solutions.*

3 Proof of Theorem 2.1

Before the proof of Theorem 2.1, we give five lemmas. Denote $u(t) := u(x, t)$ and $v(t) := v(x, t)$ for simplicity. Let $G(t, x, y) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x-y|^2}{4t}}$ be the fundamental solution of the heat equation in R^N and

$$(S(t)w_0)(x) := \int_{R^N} G(t, x, y)w_0(y)\mathrm{d}y.$$

The first three lemmas are just [8, Lemmas 4.1–4.3], respectively. For completeness, we introduce the three results here.

Lemma 3.1 *For any* $m \geq 0$ *and* $t > 0$*, the function* $H(x) = \int_{R^N} G(t, x, y)(1 + |y|)^m \mathrm{d}y$ *attains its minimum at* $x = 0$. □

Lemma 3.2 *For any* $\beta > 1$, $m \geq 0$, *and* $k = 1, 2, ...$, *there is a constant* $C_1 > 0$ *independent of k, such that*

$$\left(\int_{R^N} G(t, 0, y)(1 + |y|)^{\frac{m}{\beta^k}} \mathrm{d}y\right)^{\beta^k} \geq C_1 t^{\frac{m}{2}} \quad \textit{for } t \geq 0.$$

Moreover, the same result holds for $m < 0$ *and* $t \geq 1$. □

Lemma 3.3 *For any* $k > 0$ *and* $m \geq 0$, *there is a constant* $C_2 > 0$ *independent of k, such that*

$$\int_0^1 r^k [r(1-r)]^{\frac{m}{2}} \mathrm{d}r \geq \frac{C_2}{(k + \frac{m}{2} + 1)^{\frac{m}{2}+1}}.$$

□

The following three estimates are inspired by the [8, Lemmas 4.4, 4.7, and 4.8], respectively. Since the coupled relationship in (1.4) is much complicated than the main system in [8], similarly to them, we show the proof of the following three lemmas for completeness.

Lemma 3.4 *Let* $(u(t), v(t))$ *be a global solution of* (1.1) *with* $p_2 \geq 1$, $q_1 \geq 1$, *and* $p_2 q_1 > 1$. *Then, there exists a positive constant* $C = C(p_2, q_1, \alpha_2, \beta_1)$, *such that for any* $t > 0$

$$t^{\frac{(\frac{\alpha_2}{2}+1)q_1 + \frac{\beta_1}{2}+1}{p_2 q_1 - 1}} \|S(t)u_0\|_\infty \leq C, \qquad t^{\frac{(\frac{\beta_1}{2}+1)p_2 + \frac{\alpha_2}{2}+1}{p_2 q_1 - 1}} \|S(t)v_0\|_\infty \leq C.$$

Proof Considering that

$$\begin{cases} u(t) = S(t)u_0 + \int_0^t S(t-s)a_1 u^{p_1}(s)\mathrm{d}s + \int_0^t S(t-s)b_1 v^{q_1}(s)\mathrm{d}s, \\ v(t) = S(t)v_0 + \int_0^t S(t-s)a_2 u^{p_2}(s)\mathrm{d}s + \int_0^t S(t-s)b_2 v^{q_2}(s)\mathrm{d}s, \end{cases} \tag{3.1}$$

and by (3.1), we have

$$u(t) \geq S(t)u_0, \quad v(t) \geq S(t)v_0, \tag{3.2}$$

$$u(t) \geq \int_0^t S(t-s)b_1 v^{q_1}(s)\mathrm{d}s, \quad v(t) \geq \int_0^t S(t-s)a_2 u^{p_2}(s)\mathrm{d}s. \tag{3.3}$$

Using (3.2) and (3.3), we obtain $u(x,t) \geq b_1(x)t(S(t)v_0(x))^{q_1}$ and

$$\begin{aligned} v(t) &\geq \int_0^t S(t-s)a_2 u^{p_2}(s)\mathrm{d}s \\ &\geq \int_0^t s^{p_2} \left(\int_{R^N}\int_{R^N} G\left(t-s,x,y\right) a_2^{\frac{1}{p_2 q_1}}(y) G(s,y,z) b_1^{\frac{1}{q_1}}(y) v_0(z)\mathrm{d}z\mathrm{d}y \right)^{p_2 q_1} \mathrm{d}s \\ &\geq \int_0^t s^{p_2} \left\{ \int_{R^N}\int_{R^N} [4\pi(t-s)]^{-\frac{N}{2}} (4\pi s)^{-\frac{N}{2}} e^{-\frac{|y-x|^2}{4(t-s)} - \frac{|z-y|^2}{4s}} a_2^{\frac{1}{p_2 q_1}} \right. \\ &\quad \left. (y) b_1^{\frac{1}{q_1}}(y) v_0(z)\mathrm{d}z\mathrm{d}y \right\}^{p_2 q_1} \mathrm{d}s. \end{aligned}$$

It can be checked that

$$-\frac{|y-x|^2}{4(t-s)} - \frac{|z-y|^2}{4s} = -\frac{t}{4s(t-s)} \left| y - \frac{sx+(t-s)z}{t} \right|^2 - \frac{|x-z|^2}{4t}.$$

Then, by (2.1) and Lemma 3.2, one can obtain that $a_2(x) \geq c_1(1+|x|)^{\alpha_2}$ and $b_1(x) \geq c_1(1+|x|)^{\beta_1}$ for large $|x|$, and hence

$$\begin{aligned} v(t) &\geq \int_0^t s^{p_2} \left\{ \int_{R^N}\int_{R^N} [4\pi(t-s)]^{-\frac{N}{2}} (4\pi s)^{-\frac{N}{2}} e^{-\frac{t|y|^2}{4s(t-s)} - \frac{|x-z|^2}{4t}} \right. \\ &\quad \left. c_1(1+|y|)^{\frac{\alpha_2+p_2\beta_1}{p_2 q_1}} v_0(z)\mathrm{d}z\mathrm{d}y \right\}^{p_2 q_1} \mathrm{d}s \\ &\geq \int_0^t s^{p_2} \left[\int_{R^N}\int_{R^N} G\left(\frac{(t-s)s}{t}, 0, y \right) c_1(1+|y|)^{\frac{\alpha_2+p_2\beta_1}{p_2 q_1}} \mathrm{d}y G(t,x,z) v_0(z)\mathrm{d}z \right]^{p_2 q_1} \mathrm{d}s \\ &\geq \int_0^t s^{p_2} c_1 C_1 \left[\frac{(t-s)s}{t} \right]^{\frac{\alpha_2+p_2\beta_1}{2}} \mathrm{d}s (S(t)v_0)^{p_2 q_1}, \end{aligned}$$

where C_1 is given in Lemma 3.2. Using Lemma 3.3, we let $s = rt$ and obtain

$$\int_0^t s^{p_2} \left[\frac{s(t-s)}{t} \right]^{\frac{\alpha_2+p_2\beta_1}{2}} \mathrm{d}s \geq \frac{C_2 t^{\frac{\alpha_2+p_2\beta_1}{2}+p_2+1}}{\left(\frac{\alpha_2+p_2\beta_1}{2} + p_2 + 1 \right)^{\frac{\alpha_2+p_2\beta_1}{2}+1}}.$$

In fact, in Lemma 3.3, $\int_0^1 r^k[r(1-r)]^{\frac{m}{2}}\mathrm{d}r$ is bounded if $k \le 0$ or $m < 0$. Here, we solve the case for $p_2 > 0$. Then, we have

$$v(t) \ge \frac{c_1C_1C_2t^{\left(\frac{\beta_1}{2}+1\right)p_2+\frac{\alpha_2}{2}+1}(S(t)v_0)^{p_2q_1}}{\left[\left(\frac{\beta_1}{2}+1\right)p_2+\frac{\alpha_2}{2}+1\right]^{\frac{\alpha_2+p_2\beta_1}{2}+1}},$$

$$\text{similarly, } u(t) \ge \frac{c_1C_1C_2t^{(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1}(S(t)u_0)^{p_2q_1}}{\left[\left(\frac{\alpha_2}{2}+1\right)q_1+\frac{\beta_1}{2}+1\right]^{\frac{\alpha_2+p_2\beta_1}{2}+1}}, \tag{3.4}$$

where C_1 and c_1 are two positive constants.

Substituting (3.4) into (3.1), we have

$$\begin{aligned} v(t) &\ge \int_0^t S(t-s)a_2u^{p_2}(s)\mathrm{d}s \\ &\ge \int_0^t S(t-s)a_2\left\{\frac{c_1C_1C_2s^{(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1}(S(s)u_0)^{p_2q_1}}{\left[\left(\frac{\alpha_2}{2}+1\right)q_1+\frac{\beta_1}{2}+1\right]^{\frac{\alpha_2+p_2\beta_1}{2}+1}}\right\}^{p_2}\mathrm{d}s \\ &= \frac{(c_1C_1C_2)^{p_2}}{\left[\left(\frac{\alpha_2}{2}+1\right)q_1+\frac{\beta_1}{2}+1\right]^{\left(\frac{\alpha_2+p_2\beta_1}{2}+1\right)p_2}}\int_0^t S(t-s)a_2s^{\left[(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1\right]p_2} \\ &\quad (S(s)u_0)^{p_2^2q_1}\mathrm{d}s. \end{aligned}$$

In fact, by Lemma 3.2, we have

$$\begin{aligned} &\int_0^t S(t-s)a_2(S(s)u_0)^{p_2^2q_1}\mathrm{d}s \\ &\ge \int_0^t\left(\int_{R^N}\int_{R^N} G(t-s,x,y)\,a_2^{\frac{1}{p_2^2q_1}}(y)G(s,y,z)u_0(z)\mathrm{d}z\mathrm{d}y\right)^{p_2^2q_1}\mathrm{d}s \\ &\ge \int_0^t\left\{\int_{R^N}\int_{R^N}[4\pi(t-s)]^{-\frac{N}{2}}(4\pi s)^{-\frac{N}{2}}e^{-\frac{t|y|^2}{4s(t-s)}-\frac{|x-z|^2}{4t}}a_2^{\frac{1}{p_2^2q_1}}(y)u_0(z)\mathrm{d}z\mathrm{d}y\right\}^{p_2^2q_1}\mathrm{d}s \\ &\ge \int_0^t\left[\int_{R^N} G\left(\frac{(t-s)s}{t},0,y\right)(1+|y|)^{\frac{\alpha_2}{p_2^2q_1}}S(t)u_0\right]^{p_2^2q_1}\mathrm{d}s \\ &\ge \int_0^t C_1\left[\frac{(t-s)s}{t}\right]^{\frac{\alpha_2}{2}}(S(t)u_0)^{p_2^2q_1}\mathrm{d}s. \end{aligned}$$

By Lemma 3.3, we have

$$\begin{aligned}
&\int_0^t S(t-s)a_2 s^{\left[(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1\right]p_2}(S(s)u_0)^{p_2^2q_1}\mathrm{d}s \\
&\geq \int_0^t S^{p_2^2q_1}(t-s)a_2 s^{\left[(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1\right]p_2}(S(s)u_0)^{p_2^2q_1}\mathrm{d}s \\
&\geq \int_0^t C_1\left[\frac{(t-s)s}{t}\right]^{\frac{\alpha_2}{2}}(S(s)u_0)^{p_2^2q_1}s^{\left[(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1\right]p_2}\mathrm{d}s \\
&\geq C_1 t^{\left[(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1\right]p_2+\frac{\alpha_2}{2}+1}\int_0^1 r^{\left[(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1\right]p_2}[r(1-r)]^{\frac{\alpha_2}{2}}\mathrm{d}r(S(t)u_0)^{p_2^2q_1} \\
&\geq \frac{C_1C_2t^{\left[(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1\right]p_2+\frac{\alpha_2}{2}+1}(S(t)u_0)^{p_2^2q_1}}{\left\{\left[\left(\frac{\alpha_2}{2}+1\right)q_1+\frac{\beta_1}{2}+1\right]p_2+\frac{\alpha_2}{2}+1\right\}^{\frac{\alpha_2}{2}+1}}.
\end{aligned}$$

For convenience, we denote $E := (\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1$, $F := (\frac{\beta_1}{2}+1)p_2+\frac{\alpha_2}{2}+1$. Therefore, we have

$$v(t) \geq \frac{(c_1C_1C_2)^{p_2+1}t^{Ep_2+\frac{\alpha_2}{2}+1}(S(t)u_0)^{p_2^2q_1}}{E^{\left(\frac{\alpha_2+p_2\beta_1}{2}+1\right)p_2}(Ep_2+\frac{\alpha_2}{2}+1)^{\frac{\alpha_2}{2}+1}},$$

$$\text{similarly,}\ \ u(t) \geq \frac{(c_1C_1C_2)^{q_1+1}t^{Fq_1+\frac{\beta_1}{2}+1}(S(t)v_0)^{p_2q_1^2}}{F^{\left(\frac{\beta_1+q_1\alpha_2}{2}+1\right)q_1}(Fq_1+\frac{\beta_1}{2}+1)^{\frac{\beta_1}{2}+1}}.$$

Using Lemmas 3.2 and 3.3, we obtain

$$\begin{aligned}
&\int_0^t S(t-s)a_2 s^{(Fq_1+\frac{\beta_1}{2}+1)p_2}\mathrm{d}s \\
&\geq c_1C_1\int_0^t s^{(Fq_1+\frac{\beta_1}{2}+1)p_2}\left[\frac{s(t-s)}{t}\right]^{\frac{\alpha_2}{2}}\mathrm{d}s \\
&= c_1C_1t^{(Fq_1+\frac{\beta_1}{2}+1)p_2+\frac{\alpha_2}{2}+1}\int_0^1 r^{(Fq_1+\frac{\beta_1}{2}+1)p_2}[r(1-r)]^{\frac{\alpha_2}{2}}\mathrm{d}r \\
&\geq \frac{c_1C_1C_2t^{(Fq_1+\frac{\beta_1}{2}+1)p_2+\frac{\alpha_2}{2}+1}}{[(Fq_1+\frac{\beta_1}{2}+1)p_2+\frac{\alpha_2}{2}+1]^{\frac{\alpha_2}{2}+1}} \geq \frac{c_1C_1C_2t^{Fp_2q_1+F}}{(Fp_2q_1+F)^{\frac{\alpha_2}{2}+1}}.
\end{aligned}$$

Therefore, there is

$$v(t) \geq \int_0^t S(t-s)a_2\left[\frac{(c_1C_1C_2)^{q_1+1}s^{Fq_1+\frac{\beta_1}{2}+1}(S(s)v_0)^{p_2q_1^2}}{F^{\left(\frac{\beta_1+q_1\alpha_2}{2}+1\right)q_1}\left(Fq_1+\frac{\beta_1}{2}+1\right)^{\frac{\beta_1}{2}+1}}\right]^{p_2}\mathrm{d}s$$

$$= \frac{(c_1C_1C_2)^{(q_1+1)p_2}}{F^{\left(\frac{\beta_1+q_1\alpha_2}{2}+1\right)p_2q_1}\left(Fq_1+\frac{\beta_1}{2}+1\right)^{\left(\frac{\beta_1}{2}+1\right)p_2}}$$
$$\times\int_0^t S(t-s)a_2 s^{(Fq_1+\frac{\beta_1}{2}+1)p_2}(S(s)v_0)^{(p_2q_1)^2}\mathrm{d}s$$
$$\geq \frac{(c_1C_1C_2)^{(p_2q_1+1)}t^{Fp_2q_1+F}(S(t)v_0)^{(p_2q_1)^2}}{F^{\left(\frac{\beta_1+q_1\alpha_2}{2}+1\right)p_2q_1}\left(Fq_1+\frac{\beta_1}{2}+1\right)^{\left(\frac{\beta_1}{2}+1\right)p_2}(Fp_2q_1+F)^{\frac{\alpha_2}{2}+1}}.$$

Using the induction process, we have

$$v(t)\geq (c_1C_1C_2)^{\frac{(p_2q_1)^k-1}{p_2q_1}}A_kB_kt^{\frac{(p_2q_1)^k-1}{p_2q_1}}(S(t)v_0)^{(p_2q_1)^k}, \tag{3.5}$$

where constants

$$A_k := F^{-\left(\frac{\beta_1+q_1\alpha_2}{2}+1\right)(p_2q_1)^{k-1}}\prod_{i=1}^{k-1}\left[Fq_1\frac{(p_2q_1)^i-1}{p_2q_1-1}+\frac{\beta_1}{2}+1\right]^{-\left(\frac{\beta_1}{2}+1\right)(p_2q_1)^{k-i}p_2},$$
$$B_k := (Fp_2q_1+F)^{-(\frac{\alpha_2}{2}+1)(p_2q_1)^{k-2}}\prod_{i=1}^{k-1}\left[\frac{(p_2q_1)^i-1}{p_2q_1-1}\right]^{-(\frac{\alpha_2}{2}+1)(p_2q_1)^{k-i}}.$$

Then by (3.5), we have

$$t^{\left[\left(\frac{\beta_1}{2}+1\right)p_2+\frac{\alpha_2}{2}+1\right]\frac{(p_2q_1)^k-1}{p_2q_1-1}\frac{1}{(p_2q_1)^k}}(S(t)v_0)$$
$$\leq (c_1C_1C_2)^{-\frac{(p_2q_1)^k-1}{p_2q_1-1}\frac{1}{(p_2q_1)^k}}A_k^{-\frac{1}{(p_2q_1)^k}}B_k^{-\frac{1}{(p_2q_1)^k}}\|v(t)\|_\infty^{\frac{1}{(p_2q_1)^k}}.$$

It can be found out that $(A_kB_k)^{-\frac{1}{(p_2q_1)^k}}$ has a finite limit as $k\to\infty$. Thus, for some constant $C>0$ and any $t\in[0,T)$

$$t^{\frac{\left(\frac{\beta_1}{2}+1\right)p_2+\frac{\alpha_2}{2}+1}{p_2q_1-1}}\|S(t)v_0\|_\infty\leq C<+\infty.$$

Then, we treat $(u(t+\tau), v(t+\tau))$ for $t,\tau\geq 0$ as the solution of (1.1) with the initial value $(u(\tau), v(\tau))$. Replace (u_0, v_0) by $(u(\tau), v(\tau))$ and those estimates hold also. Setting $t=\tau$, one can obtain the conclusion. The proof for $u(t)$ is similar. □

Lemma 3.5 *Assume the global solution* (u, v) *of* (1.1) *satisfies that*

$$u(x,t)\geq c_0t^{l_1}e^{-\frac{|x|^2}{t}},\quad v(x,t)\geq c_0t^{l_2}e^{-\frac{|x|^2}{t}},\quad t\geq t_0>0,\ x\in R^N, \tag{3.6}$$

where t_0, $c_0 > 0$, *and* $l_1 \in [-N/2, \infty)$, $l_2 \in [-N/2, \infty)$. *Then, for* $a_i(x) \sim |x|^{\alpha_i}$, $b_i(x) \sim |x|^{\beta_i}$, $i = 1, 2$, *as* $|x| \to \infty$, *there exist positive constants* c, t_1, *such that for* $t \geq t_1$

$$\int_0^t S(t-s)b_1 v^{q_1}(s)\mathrm{d}s \geq \begin{cases} ct^{1+q_1 l_2+\frac{\beta_1}{2}} e^{-\frac{|x|^2}{t}}, & \text{if } 1+q_1 l_2+\frac{\beta_1}{2} > -\frac{N}{2}, \\ ct^{-\frac{N}{2}}\log(1+t)e^{-\frac{|x|^2}{t}}, & \text{if } 1+q_1 l_2+\frac{\beta_1}{2} = -\frac{N}{2}; \end{cases} \tag{3.7}$$

$$\int_0^t S(t-s)a_2 u^{p_2}(s)\mathrm{d}s \geq \begin{cases} ct^{1+p_2 l_1+\frac{\alpha_2}{2}} e^{-\frac{|x|^2}{t}}, & \text{if } 1+p_2 l_1+\frac{\alpha_2}{2} > -\frac{N}{2}, \\ ct^{-\frac{N}{2}}\log(1+t)e^{-\frac{|x|^2}{t}}, & \text{if } 1+p_2 l_1+\frac{\alpha_2}{2} = -\frac{N}{2}. \end{cases} \tag{3.8}$$

Proof It follows from (3.3) and (3.6) that:

$$u(t) \geq c_0^{q_1} \int_{t_0}^t \int_{R^N} [4\pi(t-s)]^{-\frac{N}{2}} e^{-\frac{|y-x|^2}{4(t-s)}} b_1(y) s^{q_1 l_2} e^{-\frac{q_1|y|^2}{s}} \mathrm{d}y\mathrm{d}s, \quad t \geq t_0.$$

Here

$$e^{-\frac{|y-x|^2}{4(t-s)}-\frac{q_1|y|^2}{s}} = e^{-\frac{|y-r(s,t)x|^2}{4r(s,t)(t-s)}} e^{-\frac{q_1 r(s,t)|x|^2}{s}},$$

and for $s \in [0, \frac{t}{2}]$

$$-\frac{q_1 r(s,t)}{s}|x|^2 - \frac{r(s,t)}{2(t-s)}|x|^2 > -\frac{|x|^2}{t},$$

where $r(s,t) = \frac{s}{s+4q_1(t-s)}$. Then

$$\begin{aligned} u(t) &= c_0^{q_1} \int_{t_0}^t \int_{R^N} G(r(s,t)(t-s), r(s,t)x, y) e^{-\frac{q_1 r(s,t)|x|^2}{s}} b_1(y) s^{q_1 l_2} r^{\frac{N}{2}}(s,t)\mathrm{d}y\mathrm{d}s \\ &\geq c_0^{q_1} \int_{t_0}^t \int_{R^N} 2^{-\frac{N}{2}} G\left(\frac{r(s,t)(t-s)}{2}, 0, y\right) e^{-\frac{q_1 r(s,t)}{s}|x|^2 - \frac{r(s,t)}{2(t-s)}|x|^2} \\ &\quad b_1(y) s^{q_1 l_2} r^{\frac{N}{2}}(s,t)\mathrm{d}y\mathrm{d}s \\ &\geq c_0^{q_1} 2^{-\frac{N}{2}} \int_{t_0}^t \int_{R^N} G\left(\frac{r(s,t)(t-s)}{2}, 0, y\right) e^{-\frac{|x|^2}{t}} b_1(y) s^{q_1 l_2} r^{\frac{N}{2}}(s,t)\mathrm{d}y\mathrm{d}s. \end{aligned}$$

For $t \geq 2t_0$, we obtain

$$\begin{aligned} &\int_0^t S(t-s)b_1 v^{q_1}(s)\mathrm{d}s \\ &\quad \geq c_0^{q_1} 2^{-\frac{N}{2}} e^{-\frac{|x|^2}{t}} \int_{t_0}^{\frac{t}{2}} \int_{R^N} G\left(\frac{r(s,t)(t-s)}{2}, 0, y\right) b_1(y) s^{q_1 l_2} r^{\frac{N}{2}}(s,t)\mathrm{d}y\mathrm{d}s. \end{aligned}$$

A simple calculation reveals, for sufficiently large $k > 0$, if $t \geq 2k$ and $k \leq s \leq t-k$, then $r(s,t)(t-s)/2 \geq 1$. Without loss of generality, we take $k \geq t_0$ and $r_1 = s/t$. By [8, Lemma 2.2], we have

$$\begin{aligned} u(t) &\geq c_1 c_0^{q_1} 2^{-\frac{N+\beta_1}{2}} e^{-\frac{|x|^2}{t}} \int_k^{\frac{t}{2}} (t-s)^{\frac{\beta_1}{2}} s^{q_1 l_2} r^{\frac{N+\beta_1}{2}} \mathrm{d}s \\ &\geq c_1 c_0^{q_1} 2^{-\frac{N+\beta_1}{2}} e^{-\frac{|x|^2}{t}} \int_{\frac{k}{t}}^{\frac{1}{2}} (t-tr_1)^{\frac{\beta_1}{2}} (tr_1)^{q_1 l_2} r^{\frac{N+\beta_1}{2}} t \mathrm{d}r_1 \\ &= c_1 c_0^{q_1} 2^{-\frac{N+\beta_1}{2}} e^{-\frac{|x|^2}{t}} \int_{\frac{k}{t}}^{\frac{1}{2}} (t-tr_1)^{\frac{\beta_1}{2}} (tr_1)^{q_1 l_2} \left[\frac{r_1 t}{r_1 t + 4q_1(t-tr_1)}\right]^{\frac{N+\beta_1}{2}} t \mathrm{d}r_1 \\ &= c_1 c_0^{q_1} 2^{-\frac{N+\beta_1}{2}} e^{-\frac{|x|^2}{t}} t^{1+q_1 l_2 + \frac{\beta_1}{2}} \int_{\frac{k}{t}}^{\frac{1}{2}} \frac{r_1^{q_1 l_2 + \frac{\beta_1+N}{2}} (1-r_1)^{\frac{\beta_1}{2}}}{[r_1 + 4q_1(1-r_1)]^{\frac{N+\beta_1}{2}}} \mathrm{d}r_1, \quad t \geq 2k. \end{aligned} \tag{3.9}$$

By (3.9), one could find positive constants c_2 and t_1, such that

$$\int_0^t S(t-s) b_1 v^{q_1}(s) \mathrm{d}s \geq \; c_2 t^{1+q_1 l_2 + \frac{\beta_1}{2}} e^{-\frac{|x|^2}{t}}, \quad t \geq t_1.$$

This shows the first inequality in (3.7). If $1+q_1 l_2 + \frac{\beta_1}{2} = -\frac{N}{2}$, then $q_1 l_2 + \frac{\beta_1}{2} + \frac{N}{2} = -1$, and the integral of (3.7) is on $\log t$ for large t. Thus

$$\int_0^t S(t-s) b_1 v^{q_1}(s) \mathrm{d}s \geq \; c t^{-\frac{N}{2}} \log(1+t) e^{-\frac{|x|^2}{t}}, \quad t \geq t_1.$$

This finishes the proof of (3.7). Similarly, we have (3.8). □

Lemma 3.6 *Assume that $p_2 q_1 > 1$ and $(u(t), v(t))$ be a global solution of (1.1).*

(i) *If $\frac{(\frac{\alpha_2}{2}+1)q_1 + \frac{\beta_1}{2}+1}{p_2 q_1 - 1} > \frac{N}{2}$, then there exist positive constants r_1, t_{r_1}, and C, such that $u(x,t) \geq C t^{r_1} e^{-\frac{|x|^2}{t}}$ for $t \geq t_{r_1}$, $x \in R^N$;*
If $\frac{(\frac{\alpha_2}{2}+1)q_1 + \frac{\beta_1}{2}+1}{p_2 q_1 - 1} = \frac{N}{2}$, then there exist positive constants t_1, C, such that $u(x,t) \geq C t^{-\frac{N}{2}} \log(1+t) e^{-\frac{|x|^2}{t}}$ for $t \geq t_1$, $x \in R^N$.

(ii) *If $\frac{(\frac{\beta_1}{2}+1)p_2 + \frac{\alpha_2}{2}+1}{p_2 q_1 - 1} > \frac{N}{2}$, then there exist positive constants r_2, t_{r_2}, and C, such that $v(x,t) \geq C t^{r_2} e^{-\frac{|x|^2}{t}}$ for $t \geq t_{r_2}$, $x \in R^N$;*
If $\frac{(\frac{\beta_1}{2}+1)p_2 + \frac{\alpha_2}{2}+1}{p_2 q_1 - 1} = \frac{N}{2}$, then there exist positive constants t_2, C, such that $v(x,t) \geq C t^{-\frac{N}{2}} \log(1+t) e^{-\frac{|x|^2}{t}}$ for $t \geq t_2$, $x \in R^N$.

Proof By (3.1), we have

$$u(t) \geq S(t)u_0 + \int_0^t S(t-s)b_1 v^{q_1}(s)\mathrm{d}s, \tag{3.10}$$

$$v(t) \geq S(t)v_0 + \int_0^t S(t-s)a_2 u^{p_2}(s)\mathrm{d}s. \tag{3.11}$$

Using (3.11) and (3.10), we obtain

$$u(t) \geq S(t)u_0 + \int_0^t S(t-s)b_1 \left[\int_0^{s_1} S(s_1-s_2)a_2 u^{p_2}(s_2)\mathrm{d}s_2\right]^{q_1} \mathrm{d}s_1.$$

We have $u(t) \geq u_0(x,t) + u_1(x,t)$. Define

$$u_1(t) := \int_0^t S(t-s_1)b_1 \left[\int_0^{s_1} S(s_1-s_2)a_2 u^{p_2}(s_2)\mathrm{d}s_2\right]^{q_1} \mathrm{d}s_1. \tag{3.12}$$

Let $l_1 = l_2 = -\frac{N}{2}$. By [8, Lemma 2.2], substituting (3.8) into (3.12), we obtain

$$\begin{aligned} u_1(t) &\geq \int_0^t \int_{R^N} G(t-s_1,x,y)b_1(y)(cs_2^{1-\frac{N}{2}p_2+\frac{\alpha_2}{2}}e^{-\frac{|x|^2}{s_2}})^{q_1}\mathrm{d}y\mathrm{d}s_1 \\ &\geq C_1 t^{(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1-\frac{N}{2}p_2q_1}e^{-\frac{|x|^2}{t}}, \\ &\quad \text{similarly, } v_1(t) = C_1 t^{\left(\frac{\beta_1}{2}+1\right)p_2+\frac{\alpha_2}{2}+1-\frac{N}{2}p_2q_1}e^{-\frac{|x|^2}{t}}, \end{aligned}$$

where constant $C_1 > 0$. Let

$$u_2(t) := \int_0^t S(t-s_1)b_1 \left[\int_0^{s_1} S(s_1-s_2)a_2 u_1^{p_2}(s_2)\mathrm{d}s_2\right]^{q_1} \mathrm{d}s_1.$$

We have

$$\begin{aligned} u_2(t) &\geq \int_0^t \int_{R^N} G(t-s_1,x,y)b_1(y)\left\{cs_2^{\left[(1+\frac{\alpha_2}{2})q_1+1+\frac{\beta_1}{2}-\frac{N}{2}p_2q_1\right]p_2+1+\frac{\alpha_2}{2}}e^{-\frac{|x|^2}{s_2}}\right\}^{q_1}\mathrm{d}y\mathrm{d}s_1 \\ &\geq C_2 t^{\left\{\left[(1+\frac{\alpha_2}{2})q_1+1+\frac{\beta_1}{2}-\frac{N}{2}p_2q_1\right]p_2+1+\frac{\alpha_2}{2}\right\}q_1+1+\frac{\beta_1}{2}}e^{-\frac{|x|^2}{t}} \\ &= C_2 t^{\left[(1+\frac{\alpha_2}{2})q_1+1+\frac{\beta_1}{2}\right](p_2q_1+1)-\frac{N}{2}(p_2q_1)^2}e^{-\frac{|x|^2}{t}}. \end{aligned}$$

By induction, we have

$$\begin{aligned} u_k(t) &= \int_0^t S(t-s_1)b_1 \left(\int_0^{s_1} S(s_1-s_2)a_2 u_{k-1}^{p_2}(s_2)\mathrm{d}s_2\right)^{q_1}\mathrm{d}s_1 \\ &= C_k t^{\left[(1+\frac{\alpha_2}{2})q_1+1+\frac{\beta_1}{2}\right]\frac{\left[(p_2q_1)^k-1\right]}{p_2q_1-1}-\frac{N}{2}(p_2q_1)^k}e^{-\frac{|x|^2}{t}}. \end{aligned}$$

Then, there exist positive constants C_k, t_k, such that

$$u(x,t) \geq C_k u_k(x,t) \geq c_k C_k t^{\left[(1+\frac{\alpha_2}{2})q_1+1+\frac{\beta_1}{2}\right]\frac{\left[(p_2q_1)^k-1\right]}{p_2q_1-1}-\frac{N}{2}(p_2q_1)^k} e^{-\frac{|x|^2}{t}},$$
$$t \geq t_k,\ x \in R^N.$$

Since $p_2q_1 > 1$, we obtain that, if $\frac{(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1}{p_2q_1-1} > \frac{N}{2}$, there exists a constant $K > 0$, such that

$$r_1 = \frac{\left[\left(\frac{\alpha_2}{2}+1\right)q_1+\frac{\beta_1}{2}+1\right]\left[(p_2q_1)^K-1\right]}{p_2q_1-1} - \frac{N}{2}(p_2q_1)^K > 0,$$

and $u(x,t) \geq Ct^{r_1}e^{-\frac{|x|^2}{t}}$. If $\frac{(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1}{p_2q_1-1} = \frac{N}{2}$, by Lemma 3.5, we have $u(x,t) \geq Ct^{-\frac{N}{2}}\log(1+t)e^{-\frac{|x|^2}{t}}$. Similarly, the case (ii) of Lemma 3.6 is proved. □

Inspired by the proof of [10, Theorem 1.1(i)], we show the proof of Theorem 2.1 of the present paper.

Proof of Theorem 2.1 We only prove $1 < p_2q_1 \leq (p_2q_1)_c$. Assume for some $(u_0, v_0) \neq (0,0)$, system (1.1) has a solution which is bounded in any $S_T = [0,T) \times R^N$. By Lemma 3.6, if $\frac{(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1}{p_2q_1-1} = \frac{N}{2}$ and $p_2q_1 > 1$, then there exist constants $t_1, C > 0$ satisfying that

$$\begin{aligned} S(t)u(t) &\geq \int_{R^N} G(t,x,y)Ct^{-\frac{N}{2}}\log(1+t)e^{-\frac{|y|^2}{t}}\mathrm{d}y \\ &\geq Ct^{-\frac{N}{2}}\log(1+t)e^{-\frac{|x|^2}{t}}, \quad t \geq t_1,\ x \in R^N. \end{aligned}$$

Thus, for some $\tau > 0$

$$t^{\frac{\left(\frac{\alpha_2}{2}+1\right)q_1+\frac{\beta_1}{2}+1}{p_2q_1-1}} S(t)u(t) \geq C\log(1+t)e^{-\frac{|x|^2}{t}}, \quad t \geq \tau,\ x \in R^N,$$

and these imply the left-hand side comes to infinite at point $x = 0$ as $t \to +\infty$, a contradiction to Lemma 3.4. If $\frac{(\frac{\alpha_2}{2}+1)q_1+\frac{\beta_1}{2}+1}{p_2q_1-1} > \frac{N}{2}$ and $p_2q_1 > 1$, there exist constants $t_1, C > 0$, such that

$$S(t)u(t) \geq \int_{R^N} G(t,x,y)Ct^{r_1}e^{-\frac{|y|^2}{t}}\mathrm{d}y \geq Ct^{r_1}e^{-\frac{|x|^2}{t}}, \quad t \geq t_{r_1},\ x \in R^N.$$

Similarly, one could obtain a contradiction also at point $x = 0$ as $t \to +\infty$. The proof for $v(t)$ is similar. □

4 Proof of Theorem 2.2

The proof is inspired and similar to the part (I) in the proof of [8, Theorem 1.2(b) and Theorem 1.3].

Proof of Theorem 2.2 Define $u_0(x,t) = S(t)u_0$, $v_0(x,t) = S(t)v_0$, and

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t S(t-s)a_1 u_n^{p_1}(s)\mathrm{d}s + \int_0^t S(t-s)b_1 v_n^{q_1}(s)\mathrm{d}s, \quad (4.1)$$

$$v_{n+1}(x,t) = v_0(x,t) + \int_0^t S(t-s)a_2 u_n^{p_2}(s)\mathrm{d}s + \int_0^t S(t-s)b_2 v_n^{q_2}(s)\mathrm{d}s. \quad (4.2)$$

By induction, $u_{n+1}(x,t) \geq u_n(x,t)$ and $v_{n+1}(x,t) \geq v_n(x,t)$. If $u(x,t) = \lim_{n\to\infty} u_n(x,t) < \infty$ and $v(x,t) = \lim_{n\to\infty} v_n(x,t) < \infty$ for $x \in R^N$, $t \in [0,\infty)$, then by(4.1–4.2), (u,v) satisfies (3.1). Hence, (u,v) is global. Thus, it suffices to prove that if

$$u_0(y) \leq \delta_1 G(k,0,y), \quad v_0(y) \leq \delta_2 G(k,0,y),$$

for any $k > 0$ and some $\delta_1, \delta_2 > 0$, then

$$\sup_n u_n(x,t) < \infty \quad \sup_n v_n(x,t) < \infty \quad \text{for } x \in R_N,\ t \geq 0.$$

Consider

$$\begin{cases} u_n(x,t) \leq c_1(k+t)^{E_u} G(m_n(t+k),0,x) \text{ for } x \in R_N,\ t \geq 0, \\ v_n(x,t) \leq c_2(k+t)^{E_v} G(l_n(t+k),0,x) \quad \text{for } x \in R_N,\ t \geq 0, \end{cases} \quad (4.3)$$

where $c_1, c_2 > 0$ and

$$w(\varepsilon,N) = \begin{cases} \frac{\varepsilon}{2}, & \mathit{if}\,(\varepsilon,N) \in \{(-1,+\infty)\times[1,+\infty)\} \cup \{(-2,-1]\times[2,+\infty)\}, \\ -\frac{1}{2}, & \mathit{if}\,(\varepsilon,N) \in [-2,-1)\times\{1\}, \\ \frac{\varepsilon}{2}+\delta, & \mathit{if}\,(\varepsilon,N) = \{\{-1\}\}\times\{1\}\} \cup \{\{-2\}\times[2,+\infty)\}, \end{cases}$$

with $\varepsilon \in \{\alpha_i, \beta_i,\ i = 1,2\}$ and small $\delta > 0$. For $n = 0, 1, 2, ...$

$$m_n = 1,\ \frac{1+p_2}{p_2},\ \frac{1+q_1+p_2q_1}{p_2q_1},\ \frac{1+p_2+p_2q_1+p_2^2q_1}{p_2^2q_1}, ...,$$

$$l_n = 1,\ \frac{1+q_1}{q_1},\ \frac{1+p_2+p_2q_1}{p_2q_1},\ \frac{1+q_1+p_2q_1+p_2q_1^2}{p_2q_1^2}, ...,$$

that is, for $k = 0, 1, 2, \ldots$

$$m_n = \begin{cases} \dfrac{1 + q_1 - (p_2 q_1)^{k+1} - q_1 (p_2 q_1)^k}{(1 - p_2 q_1)(p_2 q_1)^k}, & n = 2k, \\ \dfrac{(1 + p_2)\left[1 - (p_2 q_1)^{k+1}\right]}{(1 - p_2 q_1) p_2^{k+1} q_1^k}, & n = 2k + 1; \end{cases}$$

$$l_n = \begin{cases} \dfrac{1 + p_2 - (p_2 q_1)^{k+1} - p_2 (p_2 q_1)^k}{(1 - p_2 q_1)(p_2 q_1)^k}, & n = 2k, \\ \dfrac{(1 + q_1)\left[1 - (p_2 q_1)^{k+1}\right]}{(1 - p_2 q_1) q_1^{k+1} p_2^k}, & n = 2k + 1. \end{cases}$$

One could see $\frac{m_n + q_1}{q_1} = l_{n+1}$, $\frac{l_n + p_2}{p_2} = m_{n+1}$, and $\{m_n\}$, $\{l_n\}$ are increasing with

$$\lim_{n \to \infty} m_n = \frac{p_2 q_1 + q_1}{p_2 q_1 - 1}, \quad \lim_{n \to \infty} l_n = \frac{p_2 q_1 + p_2}{p_2 q_1 - 1}.$$

Here, we only prove (4.3). First

$$u_0(x, t) = S(t) u_0 \le \delta_1 \int_{R^N} G(t, x, y) G(k, 0, y) \mathrm{d}y \le \delta_1 (k + t)^{E_u} G(m_n(t + k), 0, x),$$
$$v_0(x, t) = S(t) v_0 \le \delta_2 \int_{R^N} G(t, x, y) G(k, 0, y) \mathrm{d}y \le \delta_2 (k + t)^{E_v} G(l_n(t + k), 0, x).$$

Assume that

$$u_n(x, t) \le c_1 (k + t)^{E_u} G(m_n(t + k), 0, x) \quad \text{for } x \in R^N, \ t \ge 0, \tag{4.4}$$
$$v_n(x, t) \le c_2 (k + t)^{E_v} G(l_n(t + k), 0, x) \quad \text{for } x \in R^N, \ t \ge 0. \tag{4.5}$$

By (4.4) and (4.2), we see by [8, (2.1)] that

$$\begin{aligned} v_{n+1}(x, t) \le\ & v_0(x, t) + \int_0^t S(t - s) a_2 c_1^{p_2} (k + s)^{p_2 E_u} G^{p_2}(m_n(s + k), 0, x) \mathrm{d}s \\ & + \int_0^t S(t - s) b_2 c_2^{q_2} (k + s)^{q_2 E_v} G^{q_2}(l_n(s + k), 0, x) \mathrm{d}s \\ \le\ & v_0(x, t) + C_1 \int_0^t c_1^{p_2} (k + s)^{p_2 E_u + \frac{N}{2}(1 - p_2)} \\ & G\left(\frac{m_n(s + k) + p_2(t - s)}{p_2}, 0, x\right) \\ & \times \int_{R^N} a_2(y) G\left(\frac{(t - s) m_n(s + k)}{m_n(s + k) + p_2(t - s)}, y, \frac{m_n(s + k)}{m_n(s + k) + p_2(t - s)} x\right) \mathrm{d}y \mathrm{d}s \\ & + C_2 \int_0^t c_2^{q_2} (k + s)^{q_2 E_v + \frac{N}{2}(1 - q_2)} G\left(\frac{l_n(s + k) + q_2(t - s)}{q_2}, 0, x\right) \\ & \times \int_{R^N} b_2(y) G\left(\frac{(t - s) l_n(s + k)}{l_n(s + k) + q_2(t - s)}, y, \frac{l_n(s + k)}{l_n(s + k) + q_2(t - s)} x\right) \mathrm{d}y \mathrm{d}s. \end{aligned} \tag{4.6}$$

We study the case $\alpha_2, \beta_2 > 0$. By [8, Lemma 5.1], we have

$$\int_{R^N} a_2(y) G\left(\frac{(t-s)m_n(s+k)}{m_n(s+k)+p_2(t-s)}, y, \frac{m_n(s+k)}{m_n(s+k)+p_2(t-s)}x\right) \mathrm{d}y$$
$$\leq C' \left\{1 + \left[\frac{(t-s)m_n(s+k)}{m_n(s+k)+p_2(t-s)}\right]^{\frac{\alpha_2}{2}} + \left[\frac{m_n(s+k)}{m_n(s+k)+p_2(t-s)}\right]^{\alpha_2} |x|^{\alpha_2}\right\},$$
$$\int_{R^N} b_2(y) G\left(\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}, y, \frac{l_n(s+k)}{l_n(s+k)+q_2(t-s)}x\right) \mathrm{d}y$$
$$\leq C'' \left\{1 + \left[\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right]^{\frac{\beta_2}{2}} + \left[\frac{l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right]^{\beta_2} |x|^{\beta_2}\right\}.$$

Then

$$v_{n+1}(x,t) = v_0(x,t) + C_1 \int_0^t c_1^{p_2}(k+s)^{p_2 E_u + \frac{N}{2}(1-p_2)} G\left(\frac{m_n(s+k)+p_2(t-s)}{p_2}, 0, x\right)$$
$$\times \left\{1 + \left[\frac{(t-s)m_n(s+k)}{m_n(s+k)+p_2(t-s)}\right]^{\frac{\alpha_2}{2}} + \left[\frac{m_n(s+k)}{m_n(s+k)+p_2(t-s)}\right]^{\alpha_2} |x|^{\alpha_2}\right\} \mathrm{d}s$$
$$+ C_2 \int_0^t c_2^{q_2}(k+s)^{q_2 E_v + \frac{N}{2}(1-q_2)} G\left(\frac{l_n(s+k)+q_2(t-s)}{q_2}, 0, x\right)$$
$$\times \left\{1 + \left[\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right]^{\frac{\beta_2}{2}} + \left[\frac{l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right]^{\beta_2} |x|^{\beta_2}\right\} \mathrm{d}s.$$

Let

$$J_1 = \int_0^t (k+s)^{p_2 E_u + \frac{N}{2}(1-p_2)} G\left(\frac{m_n(s+k)+p_2(t-s)}{p_2}, 0, x\right) \mathrm{d}s,$$
$$J_2 = \int_0^t (k+s)^{p_2 E_u + \frac{N}{2}(1-p_2)} G\left(\frac{m_n(s+k)+p_2(t-s)}{p_2}, 0, x\right)$$
$$\left[\frac{(t-s)m_n(s+k)}{m_n(s+k)+p_2(t-s)}\right]^{\frac{\alpha_2}{2}} \mathrm{d}s,$$
$$J_3 = \int_0^t (k+s)^{p_2 E_u + \frac{N}{2}(1-p_2)} G\left(\frac{m_n(s+k)+p_2(t-s)}{p_2}, 0, x\right)$$
$$\left[\frac{m_n(s+k)}{m_n(s+k)+p_2(t-s)}\right]^{\alpha_2} |x|^{\alpha_2} \mathrm{d}s,$$
$$J_4 = \int_0^t (k+s)^{q_2 E_v + \frac{N}{2}(1-q_2)} G\left(\frac{l_n(s+k)+q_2(t-s)}{q_2}, 0, x\right) \mathrm{d}s,$$
$$J_5 = \int_0^t (k+s)^{q_2 E_v + \frac{N}{2}(1-q_2)} G\left(\frac{l_n(s+k)+q_2(t-s)}{q_2}, 0, x\right)$$

$$\left[\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right]^{\frac{\beta_2}{2}} \mathrm{d}s,$$

$$J_6 = \int_0^t (k+s)^{q_2 E_v + \frac{N}{2}(1-q_2)} G\left(\frac{l_n(s+k)+q_2(t-s)}{q_2}, 0, x\right)$$
$$\left[\frac{l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right]^{\beta_2} |x|^{\beta_2}\mathrm{d}s.$$

Then

$$v_{n+1}(x,t) \le v_0(x,t) + C(J_1+J_2+J_3+J_4+J_5+J_6)$$

with $C := \max\{c_1^{p_2}C_1, c_2^{q_2}C_2\}$. Since $\frac{m_n(s+k)+p_2(t-s)}{p_2} \le \frac{m_n+p_2}{p_2}(t+k) = l_{n+1}(t+k)$, one has

$$G\left(\frac{m_n(s+k)+p_2(t-s)}{p_2}, 0, x\right) \le \left[4\pi\frac{m_n(s+k)+p_2(t-s)}{p_2}\right]^{-\frac{N}{2}} e^{-\frac{|x|^2}{4\frac{(m_n+p_2)(t+s)}{p_2}}}$$
$$\le C\left[\frac{t+k}{m_n(s+k)+p_2(t-s)}\right]^{\frac{N}{2}} G(l_{n+1}(t+k), 0, x).$$

As for J_1, we have

$$J_1 \le C\int_0^t (k+s)^{p_2 E_u + \frac{N}{2}(1-p_2)} \left[\frac{t+k}{m_n(s+k)+p_2(t-s)}\right]^{\frac{N}{2}} G(l_{n+1}(t+k),0,x)\mathrm{d}s$$
$$= CG(l_{n+1}(t+k),0,x)\int_0^t (k+s)^{E_v - \frac{\alpha_2}{2} - 1}\left[\frac{t+k}{m_n(s+k)+p_2(t-s)}\right]^{\frac{N}{2}} \mathrm{d}s.$$

In view of $E_v > 0$, we obtain

$$J_1 = C(k+t)^{E_v} G(l_{n+1}(t+k),0,x)\int_0^t \frac{1}{(k+s)^{\frac{\alpha_2}{2}+1}}\left[\frac{t+k}{m_n(s+k)+p_2(t-s)}\right]^{\frac{N}{2}} \mathrm{d}s.$$

Let $r = s/t$. We have

$$\int_0^t \frac{1}{(k+s)^{\frac{\alpha_2}{2}+1}}\left[\frac{t+k}{m_n(s+k)+p_2(t-s)}\right]^{\frac{N}{2}} \mathrm{d}s$$
$$= \int_0^1 \frac{1}{(k+rt)^{\frac{\alpha_2}{2}+1}}\left[\frac{t+k}{m_n(rt+k)+p_2(t-rt)}\right]^{\frac{N}{2}} t\mathrm{d}r.$$

Moreover

$$\frac{t+k}{m_n(rt+k)+p_2(t-rt)} \to \frac{1}{m_n} \quad \text{as } t \to 0,$$

$$\frac{t+k}{m_n(rt+k)+p_2(t-rt)} \to \frac{1}{p_2-(p_2-m_n)r} \quad \text{as } t\to\infty.$$

Since $1\le m_n\le \frac{p_2q_1+q_1}{p_2q_1-1}$ and $r=\frac{p_2}{p_2-m_n}>1$, $\frac{t+k}{m_n(rt+k)+p_2(t-rt)}$ is bounded. By $\frac{\alpha_2}{2}>0$, we have $\frac{1}{(k+rt)^{\frac{\alpha_2}{2}+1}}t$ is integrable in $[0, 1]$ and $\int_0^1 \frac{1}{(k+rt)^{\frac{\alpha_2}{2}+1}}t\mathrm{d}r$ is bounded in $[0,\infty)$. Hence, $J_1\le C(k+t)^{E_v}G(l_{n+1}(t+k),0,x)$.

The estimate for J_2 is similar to the one for J_1 as $J_2\le C(k+t)^{E_v}G(l_{n+1}(t+k),0,x)$. We omit the detail here.

As for J_3, we have

$$J_3\le CG(l_{n+1}(t+k),0,x)\int_0^t (k+s)^{p_2E_u+\frac{N}{2}(1-p_2)}\left[\frac{m_n(s+k)}{m_n(s+k)+p_2(t-s)}\right]^{\alpha_2}$$
$$\times\left[\frac{t+k}{m_n(s+k)+p_2(t-s)}\right]^{\frac{N}{2}} e^{-\frac{p_2|x|^2}{4}R}|x|^{\alpha_2}\mathrm{d}s.$$

Here

$$R=\frac{m_nt+p_2k-(p_2-m_n)s}{[m_n(s+k)+p_2(t-s)][(m_n+p_2)(t+s)]}.$$

As for $e^{-\frac{p_2|x|^2}{4}R}|x|^{\alpha_2}$, let $z=|x|^2$ and

$$f(z)=e^{-\frac{p_2|x|^2}{4}R}|x|^{\alpha_2}=e^{-\frac{p_2Rz}{4}}|z|^{\frac{\alpha_2}{2}}.$$

$f(z)$ shows its maximum at $z=\frac{2\alpha_2}{p_2R}$ and its maximum is $e^{-\frac{\alpha_2}{2}}(\frac{2\alpha_2}{p_2R})^{\frac{\alpha_2}{2}}$. Then

$$J_3\le CG(l_{n+1}(t+k),0,x)\int_0^t (k+s)^{E_v-1}$$
$$\times\left[\frac{s+k}{m_n(s+k)+p_2(t-s)}\right]^{\frac{\alpha_2}{2}}\left[\frac{t+k}{m_n(s+k)+p_2(t-s)}\right]^{\frac{N}{2}+\frac{\alpha_2}{2}}\mathrm{d}s.$$

Similarly to the proof of J_1 and J_2, one could find a constant $C>0$, such that

$$J_3\le C(k+t)^{E_v}G(l_{n+1}(t+k),0,x).$$

Since the proofs for J_4, J_5, and J_6 are all very similar, we only prove J_5 here. In the case of $q_2>(q_2)_c$, if $\beta_2>q_1N$ and $E_v(q_2-1)<\min\left\{1,\ \frac{N}{2}(q_2-1)-1-\frac{\beta_2}{2}\right\}$, we obtain $q_2E_v+\frac{N}{2}(1-q_2)+\theta\le E_v$ and $\theta>\frac{\beta_2}{2}+1$, where θ is a constant. Therefore, we have

$$J_5\le C(k+t)^{E_v}G(l_{n+1}(t+k),0,x)\int_0^t \frac{1}{(k+s)^{\theta-\frac{\beta_2}{2}}}[4\pi l_{n+1}(t+k)]^{\frac{N}{2}}$$

$$\times \left[4\pi \frac{l_n(s+k)+q_2(t-s)}{q_2}\right]^{-\frac{N}{2}} e^{-\frac{|x|^2}{4}\left[\frac{q_2}{l_n(s+k)+q_2(t-s)}-\frac{1}{l_{n+1}(t+k)}\right]}$$
$$\left[\frac{t-s}{l_n(s+k)+q_2(t-s)}\right]^{\frac{\beta_2}{2}} \mathrm{d}s.$$

Since $\frac{l_n(s+k)+q_2(t-s)}{q_2} \le \frac{l_n+q_2}{q_2}(t+k)$, we have $e^{-\frac{|x|^2 q_2}{4[l_n(s+k)+q_2(t-s)]}} \le e^{-\frac{|x|^2}{4(l_n+q_2)(t+s)}}$. Let

$$R = \frac{q_2}{(l_n+q_2)(t+s)} - \frac{1}{l_{n+1}(t+k)}.$$

In virtue of $q_2 > (q_2)_c$ and $\frac{2+\beta_2}{N} > q_1$, one could find that $l_n > l_2 = 1+\frac{1}{q_1} > 1+\frac{1}{q_2-1}$. Hence, $R > 0$ and

$$J_5 = C(k+t)^{E_v} G(l_{n+1}(t+k), 0, x) \int_0^t \frac{1}{(k+s)^{\theta-\frac{\beta_2}{2}}}$$
$$\times \left[\frac{l_{n+1}(t+k)q_2}{l_n(s+k)+q_2(t-s)}\right]^{\frac{N}{2}} \left[\frac{t-s}{l_n(s+k)+q_2(t-s)}\right]^{\frac{\beta_2}{2}} \mathrm{d}s.$$

Let $r = s/t$. We have

$$\int_0^t \frac{1}{(k+s)^{\theta-\frac{\beta_2}{2}}} \left[\frac{l_{n+1}(t+k)q_2}{l_n(s+k)+q_2(t-s)}\right]^{\frac{N}{2}} \left[\frac{t-s}{l_n(s+k)+q_2(t-s)}\right]^{\frac{\beta_2}{2}} \mathrm{d}s$$
$$= \int_0^1 \frac{1}{(k+rt)^{\theta-\frac{\beta_2}{2}}} \left[\frac{l_{n+1}(t+k)q_2}{l_n(rt+k)+q_2(t-rt)}\right]^{\frac{N}{2}} \left[\frac{t-rt}{l_n(rt+k)+q_2(t-rt)}\right]^{\frac{\beta_2}{2}} t\mathrm{d}r.$$

Moreover

$$\frac{l_{n+1}(t+k)q_2}{l_n(rt+k)+q_2(t-rt)} \to \frac{l_{n+1}q_2}{l_n} \quad \text{as } t \to 0,$$
$$\frac{l_{n+1}(t+k)q_2}{l_n(rt+k)+q_2(t-rt)} \to \frac{l_{n+1}q_2}{q_2-(q_2-l_n)r} \quad \text{as } t \to \infty.$$

Since $1 \le l_n \le \frac{p_2q_1+p_2}{p_2q_1-1}$ and $r = \frac{q_2}{q_2-l_n} > 1$, there is a contradiction here with premise $r \in (0,1)$, and hence, the function $\frac{l_{n+1}(t+k)q_2}{l_n(rt+k)+q_2(t-rt)}$ is bounded. Similarly, $\frac{t-tr}{m_n(rt+k)+p_2(t-rt)}$ is also bounded. In virtue of $\theta - \frac{\beta_2}{2} > 1$, we know that $\frac{1}{(k+rt)^{\theta-\frac{\beta_2}{2}}}t$ is integrable in $[0,1]$ and $\int_0^1 \frac{1}{(k+rt)^{\theta-\frac{\beta_2}{2}}} t\mathrm{d}r$ is bounded in $[0,\infty)$. Then, there exists some positive constant C, such that $J_5 \le C(k+t)^{E_v} G(l_{n+1}(t+k), 0, x)$.

Consequently, we obtain

$$\begin{aligned} v_{n+1}(x,t) &\le v_0(x,t) + \tilde{C}(J_1 + J_2 + J_3 + J_4 + J_5 + J_6) \\ &\le (\delta_1 + \tilde{C})(k+t)^{E_v} G(l_{n+1}(t+k), 0, x), \end{aligned}$$

where $\tilde{C} = \max\{c_1^{p_1} C_1,\ c_2^{q_1} C_2\}$. The calculation steps for u_{n+1} and v_{n+1} are very similar and the result is

$$u_{n+1}(x,t) \le (\delta_2 + \hat{C})(k+t)^{E_u} G(m_{n+1}(t+k), 0, x),$$

where $\hat{C} = \max\{c_1^{q_2} C_1,\ c_2^{p_2} C_2\}$. We can do former procedure again and obtain that

$$\begin{cases} u_{n+2}(x,t) \le u_0(x,t) + C(k+t)^{E_u} G(m_{n+1}(t+k), 0, x), \\ v_{n+2}(x,t) \le v_0(x,t) + C(k+t)^{E_v} G(l_{n+1}(t+k), 0, x). \end{cases}$$

That is, if (4.4–4.5) hold for n, the estimates hold for $n+2$. This completes the proof. □

5 Proof of Theorems 2.3–2.6

In this section, we prove Theorems 2.3–2.6. We only give the proof of Theorem 2.4 (iii). The other cases can be proved similarly. The proof is inspired and similar to the part (II) in the proof of [8, Theorem 1.2(b) and Theorem 1.3].

Proof of Theorem 2.4(iii). We study the case for $\alpha_1, \beta_2 < 0$ and the other cases can be obtained similarly. There exists some positive constant C, such that $a_2(x) \le C(1+|x|)^{\alpha_2}$, $b_2(x) \le C(1+|x|)^{\beta_2}$. By [8, Lemma 5.2], we obtain from (4.6) that

$$\begin{aligned} &\int_{R^N} b_2(y) G\left(\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}, y, \frac{l_n(s+k)}{l_n(s+k)+q_2(t-s)} x\right) \mathrm{d}y \\ &\le \int_{R^N} b_2(y) G\left(\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}, 0, y\right) \mathrm{d}y. \end{aligned}$$

For $t \in [0,1]$ and $\beta_2 < 0$, $\int_{R^N} G(t,0,y)(1+|y|)^{\beta_2} \mathrm{d}y \le 1$. Using [8, Lemma 2.2]

$$\int_{R^N} G(t,0,y)(1+|y|)^{\beta_2} \mathrm{d}y \le C f(t), \quad t \ge 0,\ \beta_2 \in [-2, 0],$$

where

$$f(t) = \begin{cases} 1, & \text{if } t \le 1, \\ t^{w(\beta_2, N)}, & \text{if } t > 1. \end{cases}$$

By (4.6), we see

$$v_{n+1}(x,t) \le v_0(x,t) + C_1 \int_0^t c_1^{p_2}(k+s)^{p_2 E_u + \frac{N}{2}(1-p_2)} G\left(\frac{m_n(s+k)+p_2(t-s)}{p_2}, 0, x\right)$$
$$\times \left\{1 + \left[\frac{(t-s)m_n(s+k)}{m_n(s+k)+p_2(t-s)}\right]^{\frac{\alpha_2}{2}} + \left[\frac{m_n(s+k)}{m_n(s+k)+p_2(t-s)}\right]^{\alpha_2} |x|^{\alpha_2}\right\} \mathrm{d}s$$
$$+ C_2 \int_0^t c_2^{q_2}(k+s)^{q_2 E_v + \frac{N}{2}(1-q_2)} G\left(\frac{l_n(s+k)+q_2(t-s)}{q_2}, 0, x\right)$$
$$\times f\left(\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right) \mathrm{d}s.$$

Define

$$K_1 := \int_0^t (k+s)^{q_2 E_v + \frac{N}{2}(1-q_2)} G\left(\frac{l_n(s+k)+q_2(t-s)}{q_2}, 0, x\right)$$
$$f\left(\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right) \mathrm{d}s.$$

Therefore, $v_{n+1}(x,t) \le v_0(x,t) + C(J_1+J_2+J_3+K_1)$ with $C = \max\{c_1^{p_2}C_1, c_2^{q_2}C_2\}$. The discussion on J_1, J_2, and J_3 could be obtained by the former ones. We only deal with K_1. If $p_1 - 1 > p_2$,

$$K_1 \le G(l_{n+1}(t+k), 0, x) \int_0^t (k+s)^{q_2 E_v + \frac{N}{2}(1-q_2)}$$
$$\times \left[\frac{l_{n+1}(t+k)q_2}{l_n(s+k)+q_2(t-s)}\right]^{\frac{N}{2}} f\left(\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right) \mathrm{d}s.$$

From the analysis on J_5, we know that $\frac{l_{n+1}(t+k)q_2}{l_n(s+k)+q_2(t-s)}$ is bounded. Then

$$K_1 \le G(l_{n+1}(t+k), 0, x) \int_0^t (k+s)^{q_2 E_v + \frac{N}{2}(1-q_2)} f\left(\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right) \mathrm{d}s.$$

Let $r = s/t$. We obtain that

$$\int_0^t (k+s)^{q_2 E_v + \frac{N}{2}(1-q_2)} f\left(\frac{(t-s)l_n(s+k)}{l_n(s+k)+q_2(t-s)}\right) \mathrm{d}s$$
$$= \int_0^1 (k+rt)^{q_2 E_v + \frac{N}{2}(1-q_2)} f\left(\frac{(t-rt)l_n(rt+k)}{l_n(rt+k)+q_2(t-rt)}\right) t\mathrm{d}r.$$

If $\frac{(t-rt)l_n(rt+k)}{l_n(rt+k)+q_2(t-rt)} \le 1$, one can see that t is bounded. Then, if $(q_2-1)E_v \le \frac{2+\beta_2}{2}$, it is obvious that there exists a constant $C > 0$, such that

$$\int_0^1 (k+rt)^{q_2 E_v + \frac{N}{2}(1-q_2)} f\left(\frac{(t-rt)l_n(rt+k)}{l_n(rt+k)+q_2(t-rt)}\right) t\mathrm{d}r \le C(k+t)^{E_v}.$$

If $\frac{(t-rt)l_n(rt+k)}{l_n(rt+k)+q_2(t-rt)} > 1$, there exists a constant $C > 0$, such that $\frac{t-rt}{l_n(rt+k)+q_2(t-rt)} > C$. Then

$$\int_0^1 (k+rt)^{q_2 E_v + \frac{N}{2}(1-q_2)} f\left(\frac{(t-rt)l_n(rt+k)}{l_n(rt+k)+q_2(t-rt)}\right) t\mathrm{d}r$$
$$\leq \int_0^1 (k+rt)^{q_2 E_v + \frac{N}{2}(1-q_2)+w(\beta_2,N)} t\mathrm{d}r \leq C(k+t)^{E_v}.$$

Hence

$$v_{n+1}(x,t) \leq v_0(x,t) + c_i^p C_i(J_1+J_2+J_3+K_1)$$
$$\leq (\delta_2 + c^p C)(k+t)^{E_v} G(l_{n+1}(t+k),0,x).$$

Similarly, $u_{n+1}(x,t) \leq c(k+t)^{E_u} G(m_{n+1}(t+k),0,x)$. □

6 Conclusion

As noted above, for $\alpha_i, \beta_i > 0$, $i = 1, 2$, we prove that there are no global solutions of (1.1) for any nonnegative nontrivial initial data provided that

$$1 < p_2q_1 \leq (p_2q_1)_c, \quad or \quad 1 < p_1 \leq (p_1)_c, \quad or \quad 1 < q_2 \leq (q_2)_c.$$

Besides the case for $\alpha_i, \beta_i > 0$, $i = 1, 2$, at least one out of α_i, β_i, $i = 1, 2$ is negative, we show the global existence of solution provided that

$$p_2q_1 > (p_2q_1)_c, \quad p_1 > (p_1)_c, \quad \text{and} \quad q_2 > (q_2)_c,$$

where some other conditions on $p_i, q_i, \alpha_i, \beta_i$, $i = 1, 2$, are needed. Therefore, it was a pity that we have not obtain the precise Fujita exponents of (1.4). We thought about that the exponents $(p_2q_1)_c$, $(p_1)_c$, $(q_2)_c$ were correct, because they are compatible with the ones in [4,8,16]. In fact, the more complicated coupled relation and the unbounded variable coefficients bring much more difficulty in the discussion of the global existence of solutions. And the semigroup method used in [10] would not be used anymore. We need to overcome more difficulty in dealing with the interactive terms, such as J_i, $i = 1, 2, \cdots, 6$, in the proof of Theorems 2.2–2.6, respectively.

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