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ORIGINAL PAPER

Mild Solution and Approximate Controllability of Second-Order Retarded Systems with Control Delays and Nonlocal Conditions

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Abstract

This work studies the approximate controllability of a class of second-order retarded semilinear differential equations with nonlocal conditions and with delays in control. First, we deduce the existence of mild solutions using cosine family and fixed point approach. For this, the nonlinear function is supposed to be locally Lipschitz. Controllability of the system is shown using an approximate and iterative technique. The results are illustrated using an example.

Keywords Approximate controllability · Cosine family · Fixed point · Mild solution

Mathematics Subject Classification 34H05 · 93B05 · 46N10 · 46N20

1 Introduction

Let $Z = L_p([0, c]; V)$ be a function space, where $p > 1$ and V is a Banach space. Let $\mathfrak{C}_t = C([-a, t]; V)$ denotes the set of all V-valued continuous functions defined on $[-a, t]$ with the norm $\|z\|_{\mathfrak{C}_t} = \sup_{-a\leq\varrho\leq t} \|z(\varrho)\|$. Consider the semilinear system:

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$$\left.\begin{aligned} &\ddot{z}(t) = Az(t) + \sum_{i=0}^{m} B_i u(t - a_i) \\ &\qquad + F(t, z_{\nu(t)}, u(t), u(t - \widehat{a}_1), \ldots, u(t - \widehat{a}_{\widehat{m}})), \quad t \in (0, c]; \\ &\dot{z}(0) = \mu_1; \\ &\psi(z) = h, \quad u(t) = 0, \quad t \in [-a, 0]; \end{aligned}\right\} \tag{1.1}$$

where the state $z(t) \in V$ and the control $u(\cdot) \in U = L_p([0, c]; V')$, V' is another Banach space; a_i and $\widehat{a}_j$, $j = 1, 2, \ldots, \widehat{m}$ are fixed delays, such that $0 = a_0 < a_1 < a_2 < \cdots < a_m < c$, $0 < \widehat{a}_1 < \widehat{a}_2 < \cdots < \widehat{a}_{\widehat{m}} < c$ and $a = \max\{a_m, \widehat{a}_{\widehat{m}}\}$. A generates a strongly continuous cosine family $\{\mathcal{C}(t) : t \in \mathbb{R}\}$ on V; $B_0, B_1, \ldots, B_m$ are continuous linear maps from V' to V; $\nu : [0, c] \to [0, c]$ is non-expansive and nondecreasing map satisfying $\nu(t) \leq t$; the function $z_{\nu(t)} \in \mathfrak{C}_0$ is defined by $z_{\nu(t)}(\varsigma) = z(\nu(t) + \varsigma), \quad \varsigma \in [-a, 0]$; and $F : [0, c] \times \mathfrak{C}_0 \times \underbrace{V' \times V' \times \cdots \times V'}_{(\widehat{m}+1) \text{ times}} \to V$ is nonlinear. ψ and h together represent the nonlocal delay condition.

Controllability is a fundamental property of dynamical systems, which was introduced by Kalman [1] in 1960. He discussed the controllability of deterministic linear systems. Roughly speaking, a dynamical system is called controllable during some bounded time interval, over a space V, if we can steer that system from any initial state to any final state in V during that time interval, using a set of control functions. There are several types of controllability, namely; total state controllability, complete state controllability, approximate controllability, trajectory controllability, null controllability, interior controllability, exact controllability, constrained controllability, partial approximate controllability, etc. Some results on various types of controllability can be seen in [2–10]. According to mathematical viewpoint, it is very important to recognize the difference between approximate and exact controllability. Approximate controllability empowers to steer the system to any given neighborhood of any final state, but by exact controllability, we mean that the system can be steered to any given final state. Obviously, the condition of exact controllability is necessarily stronger than approximate controllability. Approximate controllability makes free to steer a system to states belonging to a dense subset of the state space. Therefore, it is reasonable to investigate the approximate controllability of a system.

There are many natural incidents which involve a noteworthy memory effect. Such real-life problems can be modeled by retarded differential equations. For example, problems of economics, physical sciences, chemical sciences, biosciences, and medicine are affected by their history results. Therefore, it is very important to discuss the controllability of a retarded system. For any real world phenomena, it is always better to represent an abstract model with delays appearing in more than one step. For example, in patient's treatment, there may be delays in the availability of doctors and in the diagnosis. In the abstract system (1.1), the delay term $z_{\nu(t)} \in \mathfrak{C}_0$ represents the notion of generalized delay, where delays occur in two steps. For some results on the controllability of retarded systems of fractional and integer order, we refer [11–14] and some references therein.

Existence of solutions for nonlocal abstract Cauchy problems has been proven by Byszewski and Lakshmikantham [15]. Results for the existence of solutions of nonlocal semilinear systems with and without impulses can be seen in [16–18]. The controllability results for nonlocal semilinear systems of fractional and integer order have been proved in many articles (see [13,14,19–22] and references therein). Controllability of linear systems with control delays was proved by Klamka [23]. Constrained controllability results for semilinear systems with multiple delays in control can be seen in [24–26]. Klamka [27] proved the stochastic controllability of system with multiple delays in control. Shen [28] proved the stochastic controllability results for impulsive systems with multiple delays in control. However, to the best of our knowledge, there is no result on approximate controllability of nonlocal semilinear retarded systems of second order with multiple delays in control and locally Lipschitz non-linearity.

The article is structured like this: Sect. 2 contains the preliminaries. Existence and controllability results are derived in Sects. 3 and 4, respectively. An example is provided to illustrate the theory in Sect. 5.

2 Preliminaries

We present some definitions and preliminary facts which are to be used in forthcoming sections. First, we define sine and cosine family, because mild solutions for the systems of second order are defined in terms of these families (for details, see [29,30]). For this, let $\mathcal{B}(V)$ denotes the set of bounded linear maps from V to itself.

Definition 2.1 A family of operators $\{\mathcal{C}(t) : t \in \mathbb{R}\} \subset \mathcal{B}(V)$ is called strongly continuous cosine family if:

(i) $\mathcal{C}(0)z = z$ for all $z \in V$;
(ii) $2\mathcal{C}(s)\mathcal{C}(t) = \mathcal{C}(s-t) + \mathcal{C}(s+t)$ for all $s, t \in \mathbb{R}$;
(iii) $\mathcal{C}(t)$ is strongly continuous in t.

The sine family $\{\mathcal{S}(t) : t \in \mathbb{R}\}$ associated with $\{\mathcal{C}(t) : t \in \mathbb{R}\}$ is defined as:

$$\mathcal{S}(t)z = \int_0^t \mathcal{C}(s)z\mathrm{d}s, \quad z \in V, \quad t \in \mathbb{R}.$$

Throughout this article, we suppose that $\|\mathcal{C}(t)\| \leq \eta_1$ and $\|\mathcal{S}(t)\| \leq \eta_2$, $0 \leq t \leq c$, where η_1 and η_2 are constants. We also utilize the set:

$$V_1 = \{z \in V : \mathcal{C}(t)z \quad \text{is continuously differentiable}\}.$$

Lemma 2.2 [29] *If A generates a strongly continuous cosine family $\{\mathcal{C}(t) : t \in \mathbb{R}\}$, then:*

(i) $\mathcal{S}(t) = -\mathcal{S}(-t)$ *and* $\mathcal{C}(t) = \mathcal{C}(-t)$ *for all* $t \in \mathbb{R}$;
(ii) $\mathcal{S}(s), \mathcal{S}(t), \mathcal{C}(s)$ *and* $\mathcal{C}(t)$ *commute for all* $s, t \in \mathbb{R}$;
(iii) $2\mathcal{S}(s)\mathcal{C}(t) = \mathcal{S}(s-t) + \mathcal{S}(s+t)$ *for all* $s, t \in \mathbb{R}$;
(iv) $\mathcal{C}(s)\mathcal{S}(t) + \mathcal{C}(t)\mathcal{S}(s) = \mathcal{S}(s+t)$ *for all* $s, t \in \mathbb{R}$;
(v) $2A\mathcal{S}(s)\mathcal{S}(t) = \mathcal{C}(s+t) - \mathcal{C}(s-t)$ *for all* $s, t \in \mathbb{R}$;
(vi) *if* $z \in V_1$, *then* $\mathcal{S}(t)z \in D(A)$ *and* $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(t)z = A\mathcal{S}(t)z$;

(vii) *if* $z \in V_1$, *then* $\mathcal{S}(t)z \in D(A)$ *and* $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{S}(t)z = A\mathcal{S}(t)z$.

Definition 2.3 Suppose $\wp \in \mathfrak{C}_0$ satisfies $\psi(\wp) = h$. A function $z(\cdot) \in \mathfrak{C}_c$ is called the mild solution of (1.1) if:

$$z(t) = \begin{cases} \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \int_0^t \mathcal{S}(t-s)\left(\sum_{i=0}^{m} B_i u(s-a_i)\right) \mathrm{d}s \\ \qquad + \int_0^t \mathcal{S}(t-s)F(s, z_{\nu(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}}))\, \mathrm{d}s, & t \in (0, c]; \\ \wp(t), & t \in [-a, 0]. \end{cases} \tag{2.1}$$

Moreover, if $\wp(0) \in V_1$, then $\dot{z}(t)$ is continuous on $[0, c]$, and it is given by:

$$\dot{z}(t) = A\mathcal{S}(t)\wp(0) + \mathcal{C}(t)\mu_1 + \int_0^t \mathcal{C}(t-s)\left(\sum_{i=0}^{m} B_i u(s-a_i)\right) \mathrm{d}s$$
$$+ \int_0^t \mathcal{C}(t-s)F(s, z_{\nu(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}}))\, \mathrm{d}s.$$

The systems

$$\left.\begin{aligned} &\ddot{z}(t) = Az(t) + \sum_{i=0}^{m} B_i u(t-a_i), \quad t \in (0, c]; \\ &\dot{z}(0) = \mu_1, \quad z(0) = \wp(0); \\ &u(t) = 0, \quad t \in [-a, 0] \end{aligned}\right\} \tag{2.2}$$

and

$$\left.\begin{aligned} &\ddot{z}(t) = Az(t) + B_0 u(t), \quad t \in (0, c]; \\ &\dot{z}(0) = \mu_1, \quad z(0) = \wp(0) \end{aligned}\right\} \tag{2.3}$$

are corresponding linear systems with delays and without delay, respectively, if $\psi(\wp) = h$.

Definition 2.4 The system given by (1.1) is said to be approximately controllable on $[0, c]$, if, for every given $\epsilon > 0$ and a final state $z_c \in V$, one can find a control $u \in U$, such that the mild solution $z(t)$ corresponding to the control u satisfies $\|z(c) - z_c\| \leq \epsilon$.

3 Existence of Mild Solution

To discuss the existence result, we suppose the following:

H_1 $\psi : C([-a, 0]; V_1) \to C([-a, 0]; V_1)$ and there exists a unique function $\wp \in C([-a, 0]; V_1)$ satisfying $\psi(\wp) = h$.

H_2 F is continuous in t and locally Lipschitz in z that is there exists a constant $\lambda_r > 0$ satisfying:

$$\|F(t, z_1, u_0, u_1, \dots, u_{\widehat{m}}) - F(t, z_2, u_0, u_1, \dots, u_{\widehat{m}})\| \leq \lambda_r \|z_1 - z_2\|_{\mathfrak{C}_0}$$

for all $t \in [0, c]$; $z_\ell \in \mathfrak{C}_0$ with $\|z_\ell\|_{\mathfrak{C}_0} \leq r$, $\ell = 1, 2$ and $u_j \in V'$, $j = 0, 1, 2, \dots \widehat{m}$.

H_3 There exists a $\kappa > 0$ satisfying:

$$\|F(t, z, u_0, u_1, \dots, u_{\widehat{m}})\| \leq \kappa(1 + \|z\|_{\mathfrak{C}_0} + \|u_0\| + \|u_1\| + \cdots + \|u_{\widehat{m}}\|)$$

for all $t \in [0, c]$; $z \in \mathfrak{C}_0$, and $u_j \in V'$, $j = 0, 1, 2, \dots, \widehat{m}$.

First, we deduce the following lemma:

Lemma 3.1 *Let $z(t)$ be continuous on $[-a, c)$. If k_1 and k_2 be two positive constants, such that:*

$$\|z(t)\| \leq k_1 + k_2 \int_0^t \|z_{\nu(s)}\|_{\mathfrak{C}_0} \, \mathrm{d}s \quad \forall t \in [0, c).$$

Then,

$$\|z(t)\| \leq (M_\wp + k_1)e^{k_2 c} \quad \forall t \in [0, c),$$

where $z(t) = \wp(t)$ for $t \in [-a, 0]$ and $M_\wp = \sup_{t \in [-a,0]} \|\wp(t)\|$.

Proof Let $\widehat{t} \in [0, c)$ be arbitrary. Then,

one can find a $t^* \in [-a, \widehat{t}]$ satisfying:

$$\sup_{\varsigma \in [-a,0]} \|z(\nu(\widehat{t}) + \varsigma)\| = \|z(t^*)\|.$$

Now if $t^* \in [-a, 0]$, then:

$$\begin{aligned} \sup_{\varsigma \in [-a,0]} \|z(\nu(\widehat{t}) + \varsigma)\| &= \|z(t^*)\| \\ &\leq M_\wp \\ &< M_\wp + k_1 + k_2 \int_0^{\widehat{t}} \|z_{\nu(s)}\|_{\mathfrak{C}_0} \, \mathrm{d}s. \end{aligned}$$

If $t^* \in (0, \widehat{t}]$, then:

$$\begin{aligned} \sup_{\varsigma \in [-a,0]} \|z(\nu(\widehat{t}) + \varsigma)\| &= \|z(t^*)\| \\ &\leq k_1 + k_2 \int_0^{t^*} \|z_{\nu(s)}\|_{\mathfrak{C}_0} \, \mathrm{d}s \end{aligned}$$

$$\leq M_{\wp} + k_1 + k_2 \int_0^{\widehat{t}} \|z_{\nu(s)}\|_{\mathfrak{C}_0} \, \mathrm{d}s.$$

Thus:

$$\|z(\widehat{t})\| \leq \sup_{\varsigma \in [-a,0]} \|z(\nu(\widehat{t}) + \varsigma)\| \leq M_{\wp} + k_1 + k_2 \int_0^{\widehat{t}} \|z_{\nu(s)}\|_{\mathfrak{C}_0} \, \mathrm{d}s$$
$$\Rightarrow \|z(t)\| \leq M_{\wp} + k_1 + k_2 \int_0^{t} \|z_{\nu(s)}\|_{\mathfrak{C}_0} \, \mathrm{d}s \quad \forall t \in [0, c).$$

In view of Gronwall's inequality, we obtain:

$$\|z(t)\| \leq (M_{\wp} + k_1) e^{k_2 c} \quad \forall t \in [0, c).$$

□

Theorem 3.2 *Under hypotheses* $[H_1]$–$[H_3]$, *the system* (1.1) *has a unique mild solution for each* $u(\cdot) \in U$ *and* $\mu_1 \in V$.

Proof Let $0 < c_1 < c$ and $\max\left\{\|B_0\|, \|B_1\|, \ldots, \|B_m\|\right\} \leq M_B$. Define a mapping $\Phi : \mathfrak{C}_{c_1} \to \mathfrak{C}_{c_1}$ by:

$$(\Phi z)(t) = \begin{cases} \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \displaystyle\int_0^t \mathcal{S}(t-s) \left(\sum_{i=0}^m B_i u(s - a_i) \right) \mathrm{d}s \\ \quad + \displaystyle\int_0^t \mathcal{S}(t-s) F(s, z_{\nu(s)}, u(s), u(s - \widehat{a}_1), \ldots, u(s - \widehat{a}_{\widehat{m}})) \, \mathrm{d}s, & t \in (0, c_1]; \\ \wp(t), & t \in [-a, 0]; \end{cases} \tag{3.1}$$

and consider the ball:

$$B_{r_0} = \left\{ z(\cdot) \in \mathfrak{C}_{c_1} : \|z\|_{\mathfrak{C}_{c_1}} \leq r_0, z(0) = \wp(0) \text{ and } \dot{z}(0) = \mu_1 \right\}.$$

Then, for any $z(\cdot) \in B_{r_0}$ and $0 \leq s \leq c_1$:

$$\|z_{\nu(s)}\|_{\mathfrak{C}_0} = \sup_{\varsigma \in [-a,0]} \|z(\nu(s) + \varsigma)\| \leq \sup_{\varrho \in [-a, c_1]} \|z(\varrho)\| \leq r_0.$$

Thus:

$$\|(\Phi z)(t)\| \leq \eta_1 \|\wp(0)\| + \eta_2 \|\mu_1\| + \eta_2 M_B \left(\int_0^t \sum_{i=0}^m \|u(s - a_i)\| \, \mathrm{d}s \right)$$
$$+ \eta_2 \int_0^t \|F(t, z_{\nu(s)}, u(s), u(s - \widehat{a}_1), \ldots, u(s - \widehat{a}_{\widehat{m}}))$$

$$- F(t, 0, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}}))\| \,\mathrm{d}s$$
$$+ \eta_2 \int_0^t \|F(t, 0, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}}))\| \,\mathrm{d}s.$$

Using $[H_2]$ and $[H_3]$, we obtain:

$$\begin{aligned}
\|(\Phi z)(t)\| &\le \eta_1 \|\wp(0)\| + \eta_2\|\mu_1\| + (m+1)\eta_2 M_B c^{1-\frac{1}{p}}\|u\|_U + \eta_2\lambda_{r_0}\int_0^t \|z_{\nu(s)}\|_{\mathfrak{C}_0}\,\mathrm{d}s \\
&\quad + \eta_2\kappa\int_0^t [1 + \|u(s)\| + \|u(s-\widehat{a}_1)\| + \cdots + \|u(s-\widehat{a}_{\widehat{m}})\|]\,\mathrm{d}s \\
&\le \eta_1 \|\wp(0)\| + \eta_2\|\mu_1\| + (m+1)\eta_2 M_B c^{1-\frac{1}{p}}\|u\|_U \\
&\quad + \eta_2\lambda_{r_0} r_0 c_1 + \eta_2\kappa\left[c_1 + (\widehat{m}+1){c_1}^{1-\frac{1}{p}}\|u\|_U\right] \\
&= \eta_1 \|\wp(0)\| + \eta_2\|\mu_1\| + (m+1)\eta_2 M_B c^{1-\frac{1}{p}}\|u\|_U \\
&\quad + \eta_2\left[\lambda_{r_0} r_0 c_1 + \kappa\left(c_1 + (\widehat{m}+1){c_1}^{1-\frac{1}{p}}\|u\|_U\right)\right].
\end{aligned}$$

Now, choosing $r_0 = 2\left[\eta_1\|\wp(0)\| + \eta_2\|\mu_1\| + (m+1)\eta_2 M_B c^{1-\frac{1}{p}}\|u\|_U\right] + 1$ and c_1 small enough, such that:

$$\eta_2\left[\lambda_{r_0} r_0 c_1 + \kappa\left(c_1 + (\widehat{m}+1){c_1}^{1-\frac{1}{p}}\|u\|_U\right)\right] \le \eta_1\|\wp(0)\|$$
$$+\eta_2\|\mu_1\| + (m+1)\eta_2 M_B c^{1-\frac{1}{p}}\|u\|_U + 1.$$

Then:

$$\begin{aligned}
\|(\Phi z)(t)\| &\le 2\left[\eta_1\|\wp(0)\| + \eta_2\|\mu_1\| + (m+1)\eta_2 M_B c^{1-\frac{1}{p}}\|u\|_U\right] + 1 \\
&= r_0.
\end{aligned}$$

Therefore, Φ maps B_{r_0} into itself.

Now, take $z_1, z_2 \in B_{r_0}$, then:

$$\begin{aligned}
\|(\Phi z_1)(t) - (\Phi z_2)(t)\| &\le \eta_2\int_0^t \|F(s, (z_1)_{\nu(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}})) \\
&\quad - F(s, (z_2)_{\nu(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}}))\|\,\mathrm{d}s \\
&\le \eta_2\lambda_{r_0}\int_0^t \|(z_1)_{\nu(s)} - (z_2)_{\nu(s)}\|_{\mathfrak{C}_0}\,\mathrm{d}s \\
&\le \eta_2\lambda_{r_0} t\|z_1 - z_2\|_{\mathfrak{C}_{c_1}}.
\end{aligned}$$

Repeating the above process, one can obtain:

$$\|(\Phi^n z_1)(t) - (\Phi^n z_2)(t)\| \le \frac{(\eta_2\lambda_{r_0} t)^n}{n!}\|z_1 - z_2\|_{\mathfrak{C}_{c_1}}$$

$$\leq \frac{\left(\eta_2 \lambda_{r_0} c\right)^n}{n!} \|z_1 - z_2\|_{\mathfrak{C}_{c_1}},$$

which shows that Φ^n is a contraction map for sufficiently large value of n. By Banach fixed point theorem, Φ has a fixed point in B_{r_0}. Hence, (2.1) is a mild solution on $[-a, c_1]$. In similar way, the existence of mild solution on $[c_1, c_2]$, where $c_1 < c_2$, can be shown. Applying the above technique, one can deduce that (2.1) is a mild solution on the maximal existing interval $[-a, c^*)$, $c^* \leq c$. Next, we show the boundedness of solution. Clearly, $z(t)$ is bounded on $[-a, 0]$. Now, for $t \in [0, c^*)$:

$$\begin{aligned}
\|z(t)\| \leq\ & \eta_1 \|\wp(0)\| + \eta_2 \|\mu_1\| + (m+1)\eta_2 M_B \int_0^t \|u(s)\| \,\mathrm{d}s \\
& + \eta_2 \kappa \int_0^t \left(1 + \|z_{\nu(s)}\|_{\mathfrak{C}_0} + (\widehat{m}+1)\|u(s)\|\right) \mathrm{d}s \\
\leq\ & \eta_1 \|\wp(0)\| \\
& + \eta_2 \left[\|\mu_1\| + \left((m+1)M_B + (\widehat{m}+1)\kappa\right) c^{1-\frac{1}{p}} \|u\|_U + \kappa c \right] \\
& + \eta_2 \kappa \int_0^t \|z_{\nu(s)}\|_{\mathfrak{C}_0} \,\mathrm{d}s.
\end{aligned}$$

By Lemma 3.1, we have:

$$\begin{aligned}
\|z(t)\| \leq \Big[& \eta_1 \|\wp(0)\| + \eta_2 (\|\mu_1\| + \big((m+1)M_B \\
& + (\widehat{m}+1)\kappa\big) c^{1-\frac{1}{p}} \|u\|_U + \kappa c) + M_{\wp} \Big] e^{\eta_2 \kappa c},
\end{aligned}$$

which shows that $z(t)$ is bounded on $[-a, c^*)$, and hence, it is defined on $[-a, c]$. For uniqueness, suppose z_1 and z_2 be two solutions of (1.1) for the same control function u. Then, $z_1(t) = z_2(t) = \wp(t)$ for $t \in [-a, 0]$. Now, for $t \in [0, c]$, let:

$$a^* = \max \left\{ \|z_1\|_{\mathfrak{C}_c}, \|z_2\|_{\mathfrak{C}_c} \right\}.$$

Then:

$$\begin{aligned}
\|z_1(t) - z_2(t)\|_V \leq\ & \eta_2 \int_0^t \Big\| F\big(s, (z_1)_{\nu(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}})\big) \\
& - F\big(s, (z_2)_{\nu(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}})\big) \Big\| \,\mathrm{d}s \\
\leq\ & \eta_2 \lambda_{a^*} \int_0^t \left\| (z_1)_{\nu(s)} - (z_2)_{\nu(s)} \right\|_{\mathfrak{C}_0} \mathrm{d}s \\
\leq\ & \eta_2 \lambda_{a^*} \int_0^c \left\| (z_1)_{\nu(s)} - (z_2)_{\nu(s)} \right\|_{\mathfrak{C}_0} \mathrm{d}s.
\end{aligned}$$

Therefore:

$$\|(z_1)_{\nu(t)} - (z_2)_{\nu(t)}\|_{\mathfrak{C}_0} \le \eta_2 \lambda_{a^*} \int_0^c \left\|(z_1)_{\nu(s)} - (z_2)_{\nu(s)}\right\|_{\mathfrak{C}_0} \mathrm{d}s.$$

By Gronwall's inequality, we obtain $(z_1)_{\nu(t)} = (z_2)_{\nu(t)}$ for all $t \in [0, c]$, and consequently, $z_1 = z_2$. This completes the proof. □

Remark 3.3 It is notable that if the map ψ is not injective, then the system (1.1) may have more than one solution for a fixed control $u(\cdot) \in U$.

4 Approximate Controllability

The subsequent discussion needs the following hypotheses:

H_4 The system (2.3) is approximately controllable.
H_5 There exists a function $q(\cdot) \in L_1[0, c]$ satisfying:

$$\|F(t, z, u_0, u_1, \ldots, u_{\widehat{m}})\| \le q(t)$$

for all $(t, z, u_0, u_1, \ldots, u_{\widehat{m}}) \in [0, c] \times \mathfrak{C}_0 \times V' \times V' \times \cdots \times V'$.
First, we prove the controllability of the corresponding linear system using the technique similar to [32].

Theorem 4.1 *Under hypotheses* $[H_1]$ *and* $[H_4]$, *the corresponding linear delay system* (2.2) *is approximately controllable.*

Proof Set $c = a_{m+1}$ and $r = \min\{a_1, a_2 - a_1, a_3 - a_2, \ldots, a_{m+1} - a_m\}$. Since $0 = a_0 < a_1 < a_2 < \cdots < a_m < a_{m+1}$. Therefore, for each a_{i+1}, one can find a positive integer n_i and a constant $\alpha_i \in [0, r)$ satisfying $a_{i+1} = a_i + n_i r + \alpha_i$, $i = 1, 2, \ldots, m$.

Case 1: If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are positive.

Let $\tilde{z}_0; \tilde{z}_{11}, \tilde{z}_{12} \ldots, \tilde{z}_{1n_1}, \tilde{z}_{1\overline{n_1+1}}; \tilde{z}_{21}, \tilde{z}_{22}, \ldots, \tilde{z}_{2n_2}, \tilde{z}_{2\overline{n_2+1}}; \cdots; \tilde{z}_{m1}, \tilde{z}_{m2}, \ldots, \tilde{z}_{mn_m}, z_c$ be given in V , where z_c is the final state. Consider the system:

$$\left.\begin{aligned} &\ddot{\xi}(t) = A\xi(t) + B_0 u(t), \quad t \in (0, a_1];\\ &\dot{\xi}(0) = \xi_1 = \mu_1;\\ &\xi(0) = \wp(0). \end{aligned}\right\} \tag{4.1}$$

Let $\epsilon > 0$ be given. Set $\tilde{\xi}_0 = \tilde{z}_0$. By $[H_4]$, one can find a control u_0, such that the mild solution $\xi(t)$ of (4.1) given by:

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \int_0^t \mathcal{S}(t-s)B_0 u_0(s)\,\mathrm{d}s, \quad 0 < t \le a_1,$$

satisfies $\|\xi(a_1) - \tilde{\xi}_0\| \le \epsilon$.

Let

$$w_0(t) = \begin{cases} 0, & t \in [-a, 0]; \\ u_0(t), & t \in [0, a_1] \end{cases}$$

and

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \int_0^t \mathcal{S}(t-s)\left(\sum_{i=0}^{m} B_i w_0(s-a_i)\right) \mathrm{d}s,$$
$$0 < t \le a_1.$$

Then:

$$\begin{aligned} \|z(a_1) - \tilde{z}_0\| &= \|\xi(a_1) - \tilde{\xi}_0\| \\ &\le \epsilon. \end{aligned}$$

Denote $\xi(a_1)$ by ξ_{a_1} and $\dot{\xi}(a_1)$ by $\dot{\xi}_{a_1}$ and consider the system:

$$\left.\begin{array}{l} \ddot{\xi}(t) = A\xi(t) + B_0 u(t), \quad t \in (a_1, a_1 + r]; \\ \dot{\xi}(a_1) = \dot{\xi}_{a_1}; \\ \xi(a_1) = \xi_{a_1}. \end{array}\right\} \tag{4.2}$$

Set $\tilde{\xi}_{11} = \tilde{z}_{11} - \chi_{a_1+r}$, where $\chi_{a_1+r} = \int_0^{a_1+r} \mathcal{S}(a_1+r-s)\big(\sum_{i=1}^{m} B_i w_0(s-a_i)\big)\,\mathrm{d}s = \int_0^{a_1+r} \mathcal{S}(a_1+r-s) B_1 w_0(s-a_1)\,\mathrm{d}s$ is known.

Again by $[H_4]$, one can find a control u_{11}, such that the mild solution $\xi(t)$ of (4.2) given by:

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \int_0^t \mathcal{S}(t-s) B_0 u_{11}(s)\,\mathrm{d}s, \quad a_1 < t \le a_1 + r,$$

satisfies $\|\xi(a_1+r) - \tilde{\xi}_{11}\| \le \epsilon$.

Let

$$w_{11}(t) = \begin{cases} w_0(t), & t \in [0, a_1]; \\ u_{11}(t), & t \in (a_1, a_1 + r] \end{cases}$$

and

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \int_0^t \mathcal{S}(t-s)\left(\sum_{i=0}^{m} B_i w_{11}(s-a_i)\right)\mathrm{d}s, \quad a_1 < t \le a_1 + r.$$

Then:

$$\|z(a_1+r) - \tilde{z}_{11}\| = \|\xi(a_1+r) + \chi_{a_1+r} - \tilde{z}_{11}\|$$

$$= \|\xi(a_1 + r) - \tilde{\xi}_{11}\|$$
$$\leq \epsilon.$$

Continuing in similar fashion, at the $(n_1 + 2)$th step, we get:

$$\left.\begin{array}{l} \ddot{\xi}(t) = A\xi(t) + B_0 u(t), \quad t \in (a_1 + n_1 r, a_2]; \\ \dot{\xi}(a_1 + n_1 r) = \dot{\xi}_{a_1+n_1 r}; \\ \xi(a_1 + n_1 r) = \xi_{a_1+n_1 r}. \end{array}\right\} \tag{4.3}$$

Set $\tilde{\xi}_{1\overline{n_1+1}} = \tilde{z}_{1\overline{n_1+1}} - \chi_{a_2}$, where $\chi_{a_2} = \int_0^{a_2} \mathcal{S}(a_2 - s)\big(\sum_{i=1}^m B_i w_{1n_1}(s - a_i)\big)\,\mathrm{d}s = \int_0^{a_2} \mathcal{S}(a_2 - s) B_1 w_{1n_1}(s - a_1)\,\mathrm{d}s$ is known. Then, one can find a control $u_{1\overline{n_1+1}}$, such that the mild solution $\xi(t)$ of (4.3) given by:

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \int_0^t \mathcal{S}(t-s) B_0 u_{1\overline{n_1+1}}(s)\,\mathrm{d}s, \quad a_1 + n_1 r < t \leq a_2,$$

satisfies $\|\xi(a_2) - \tilde{\xi}_{1\overline{n_1+1}}\| \leq \epsilon$.

Let

$$w_{1\overline{n_1+1}}(t) = \begin{cases} w_{1n_1}(t), & t \in (a_1 + \overline{n_1 - 1}r, a_1 + n_1 r]; \\ u_{1\overline{n_1+1}}(t), & t \in (a_1 + n_1 r, a_2] \end{cases}$$

and

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \int_0^t \mathcal{S}(t-s)\left(\sum_{i=0}^m B_i w_{1\overline{n_1+1}}(s - a_i)\right)\mathrm{d}s,$$
$$a_1 + n_1 r < t \leq a_2.$$

Then:

$$\begin{aligned} \|z(a_2) - \tilde{z}_{1\overline{n_1+1}}\| &= \|\xi(a_2) + \chi_{a_2} - \tilde{z}_{1\overline{n_1+1}}\| \\ &= \|\xi(a_2) - \tilde{\xi}_{1\overline{n_1+1}}\| \\ &\leq \epsilon. \end{aligned}$$

Repeating the above process, at the last step that is $(n_1 + n_2 + \cdots + n_m + m + 1)$th step, we get:

$$\left.\begin{array}{l} \ddot{\xi}(t) = A\xi(t) + B_0 u(t), \quad t \in (a_m + n_m r, c]; \\ \dot{\xi}(a_m + n_m r) = \dot{\xi}_{a_m+n_m r}; \\ \xi(a_m + n_m r) = \xi_{a_m+n_m r}. \end{array}\right\}. \tag{4.4}$$

Set $\tilde{\xi}_c = \tilde{z}_c - \chi_c$, where $\chi_c = \int_0^c \mathcal{S}(c - s)\big(\sum_{i=1}^m B_i w_{mn_m}(s - a_i)\big)\,\mathrm{d}s$ is known from previous step. Then, one can find a control $u_{m\overline{n_m+1}}$, such that the mild solution $\xi(t)$

of (4.4) given by:

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \int_0^t \mathcal{S}(t-s)B_0 u_{m\overline{n_m+1}}(s)\,\mathrm{d}s, \quad a_m + n_m r < t \le c,$$

satisfies $\|\xi(c) - \tilde{\xi}_c\| \le \epsilon$.

Let

$$w_{m\overline{n_m+1}}(t) = \begin{cases} w_{mn_m}(t), & t \in (a_m + \overline{n_m - 1}r, a_m + n_m r]; \\ u_{m\overline{n_m+1}}(t), & t \in (a_m + n_m r, c] \end{cases}$$

and

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \int_0^t \mathcal{S}(t-s)\left(\sum_{i=0}^m B_i w_{m\overline{n_m+1}}(s - a_i)\right)\mathrm{d}s,$$
$$a_m + n_m r < t \le c.$$

Then:

$$\begin{aligned} \|z(c) - z_c\| &= \|\xi(c) + \chi_c - z_c\| \\ &= \|\xi(c) - \tilde{\xi}_c\| \\ &\le \epsilon. \end{aligned}$$

Now, define the control $w(\cdot)$ on $[-a, c]$ as:

$$w(t) = \begin{cases} w_0, & t \in [-a, a_1]; \\ w_i(t), & t \in (a_i, a_{i+1}], \quad i = 1, 2, \ldots, m; \end{cases}$$

where:

$$w_i(t) = \begin{cases} w_{ij}(t), & t \in (a_i + \overline{j-1}r, a_i + jr], \quad j = 1, 2, \ldots, n_i; \\ u_{i\overline{n_i+1}}(t), & t \in (a_i + n_i r, a_{i+1}]. \end{cases}$$

Then:

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)\mu_1 + \int_0^t \mathcal{S}(t-s)\left(\sum_{i=0}^m B_i w(s - a_i)\right)\mathrm{d}s, \quad 0 < t \le c$$

is the mild solution of (2.2) for the control function w, and it satisfies $\|z(c) - z_c\| \le \epsilon$. For other cases, the proof is similar. □

Theorem 4.2 *Under hypotheses* $[H_1]$–$[H_5]$, *the semilinear system* (1.1) *is approximately controllable.*

Proof Since $q(\cdot) \in L_1[0, c]$, one can find an increasing sequence $\langle c_n \rangle$ in $[0, c]$, such that $c_n \to c$ and:

$$\int_{c_n}^{c} q(t)\,\mathrm{d}t \to 0, \quad \text{as} \quad n \to \infty.$$

Now, by approximate controllability of (2.2), for any given $\epsilon > 0$ and $z_c \in V$, one can find a control $\tilde{u}_0 \in U$ satisfying:

$$\left\| z_c - \mathcal{C}(c)\wp(0) - \mathcal{S}(c)\mu_1 - \int_0^c \mathcal{S}(c-s)\left(\sum_{i=0}^{m} B_i \tilde{u}_0(s-a_i)\right)\mathrm{d}s \right\| \le \frac{\epsilon}{2}.$$

Denote $z_1 = z(c_1, \wp, \tilde{u}_0)$ and $\dot{z}_1 = \dot{z}(c_1, \wp, \tilde{u}_0)$, where $z(t, \wp, \tilde{u}_0)$ is the mild solution of (1.1) for the control $\tilde{u}_0$. Again by approximate controllability of (2.2), one can find a control $\tilde{u}_1 \in L_p([c_1, c]; V')$ satisfying:

$$\left\| z_c - \mathcal{C}(c-c_1)z_1 - \mathcal{S}(c-c_1)\dot{z}_1 - \int_{c_1}^c \mathcal{S}(c-s)\left(\sum_{i=0}^{m} B_i \tilde{u}_1(s-a_i)\right)\mathrm{d}s \right\| \le \frac{\epsilon}{2}.$$

Define:

$$\tilde{w}_1(t) = \begin{cases} \tilde{u}_0(t), & t \in [0, c_1); \\ \tilde{u}_1(t), & t \in [c_1, c]. \end{cases}$$

Clearly, $\tilde{w}_1(\cdot) \in U$. Continuing in this manner, one can obtain three sequences z_n, $\tilde{u}_n$ and $\tilde{w}_n$, such that $\tilde{u}_n(\cdot) \in L_p([c_n, c]; V')$, $\tilde{w}_n(\cdot) \in U$ given by:

$$\tilde{w}_n(t) = \begin{cases} \tilde{u}_{n-1}(t), & t \in [0, c_n); \\ \tilde{u}_n(t), & t \in [c_n, c] \end{cases}$$

and $z_n = z(c_n, \wp, \tilde{u}_{n-1})$, $\dot{z}_n = \dot{z}(c_n, \wp, \tilde{u}_{n-1})$ with:

$$\left\| z_c - \mathcal{C}(c-c_n)z_n - \mathcal{S}(c-c_1)\dot{z}_n - \int_{c_n}^c \mathcal{S}(c-s)\left(\sum_{i=0}^{m} B_i \tilde{u}_n(s-a_i)\right)\mathrm{d}s \right\| \le \frac{\epsilon}{2}.$$

Let $z(t, \wp, \tilde{w}_n)$ be the mild solution of (1.1) associated with $\tilde{w}_n$. Denote:

$$G_{F,B}(s) = \sum_{i=0}^{m} B_i \tilde{w}_n(s-a_i) + F(s, z_{v(s)}, \tilde{w}_n(s), \tilde{w}_n(s-\widehat{a}_1), \ldots, \tilde{w}_n(s-\widehat{a}_{\widehat{m}})).$$

Then:

$$
\begin{aligned}
z(c,\wp,\tilde{w}_n) &= \mathcal{C}(c)\wp(0)+\mathcal{S}(c)\mu_1+\int_0^c \mathcal{S}(c-s)G_{F,B}(s)\,ds\\
&= \mathcal{C}(c-c_n+c_n)\wp(0)+\mathcal{S}(c-c_n+c_n)\mu_1+\int_0^{c_n}\mathcal{S}(c-c_n+c_n-s)G_{F,B}(s)\,ds\\
&\quad+\int_{c_n}^{c}\mathcal{S}(c-s)G_{F,B}(s)\,ds\\
&= \big(\mathcal{C}(c-c_n)\mathcal{C}(c_n)+A\mathcal{S}(c-c_n)\mathcal{S}(c_n)\big)\wp(0)+\big(\mathcal{S}(c-c_n)\mathcal{C}(c_n)+\mathcal{S}(c_n)\mathcal{C}(c-c_n)\big)\mu_1\\
&\quad+\int_0^{c_n}\big(\mathcal{S}(c-c_n)\mathcal{C}(c_n-s)+\mathcal{S}(c_n-s)\mathcal{C}(c-c_n)\big)G_{F,B}(s)\,ds\\
&\quad+\int_{c_n}^{c}\mathcal{S}(c-s)G_{F,B}(s)\,ds\\
&= \mathcal{C}(c-c_n)\left(\mathcal{C}(c_n)\wp(0)+\mathcal{S}(c_n)\mu_1+\int_0^{c_n}\mathcal{S}(c_n-s)G_{F,B}(s)\,ds\right)\\
&\quad+\mathcal{S}(c-c_n)\left(A\mathcal{S}(c_n)\wp(0)+\mathcal{C}(c_n)\mu_1+\int_0^{c_n}\mathcal{C}(c_n-s)G_{F,B}(s)\,ds\right)\\
&\quad+\int_{c_n}^{c}\mathcal{S}(c-s)G_{F,B}(s)\,ds\\
&= \mathcal{C}(c-c_n)\Bigg[\mathcal{C}(c_n)\wp(0)+\mathcal{S}(c_n)\mu_1+\int_0^{c_n}\mathcal{S}(c_n-s)\Bigg(\sum_{i=0}^{m}B_i\tilde{u}_{n-1}(s-a_i)\\
&\quad+F(s,z_{\nu(s)},\tilde{u}_{n-1}(s),\tilde{u}_{n-1}(s-\widehat{a}_1),\ldots,\tilde{u}_{n-1}(s-\widehat{a}_{\widehat{m}}))\Bigg)ds\Bigg]\\
&\quad+\mathcal{S}(c-c_n)\Bigg[A\mathcal{S}(c_n)\wp(0)+\mathcal{C}(c_n)\mu_1+\int_0^{c_n}\mathcal{C}(c_n-s)\Bigg(\sum_{i=0}^{m}B_i\tilde{u}_{n-1}(s-a_i)\\
&\quad+F(s,z_{\nu(s)},\tilde{u}_{n-1}(s),\tilde{u}_{n-1}(s-\widehat{a}_1),\ldots,\tilde{u}_{n-1}(s-\widehat{a}_{\widehat{m}}))\Bigg)ds\Bigg]\\
&\quad+\int_{c_n}^{c}\mathcal{S}(c-s)\Bigg(\sum_{i=0}^{m}B_i\tilde{u}_n(s-a_i)\\
&\quad+F(s,z_{\nu(s)},\tilde{u}_n(s),\tilde{u}_n(s-\widehat{a}_1),\ldots,\tilde{u}_n(s-\widehat{a}_{\widehat{m}}))\Bigg)ds\\
&= \mathcal{C}(c-c_n)z_n+\mathcal{S}(c-c_n)\dot{z}_n+\int_{c_n}^{c}\mathcal{S}(c-s)\left(\sum_{i=0}^{m}B_i\tilde{u}_n(s-a_i)\right)ds\\
&\quad+\int_{c_n}^{c}\mathcal{S}(c-s)F(s,z_{\nu(s)},\tilde{u}_n(s),\tilde{u}_n(s-\widehat{a}_1),\ldots,\tilde{u}_n(s-\widehat{a}_{\widehat{m}}))\,ds.
\end{aligned}
$$

Now:

$$
\begin{aligned}
\|z(c,\wp,\tilde{w}_n)-z_c\| \leq & \left\| z_c - \mathcal{C}(c-c_n)z_n - \mathcal{S}(c-c_n)\dot{z}_n \right. \\
& \left. - \int_{c_n}^{c} \mathcal{S}(c-s)\left(\sum_{i=0}^{m} B_i \tilde{u}_n(s-a_i)\right) \mathrm{d}s \right\| \\
& + \left\| \int_{c_n}^{c} \mathcal{S}(c-s) F(s, z_{\nu(s)}, \tilde{u}_n(s), \tilde{u}_n(s-\widehat{a}_1), \ldots, \tilde{u}_n(s-\widehat{a}_{\widehat{m}}))\, \mathrm{d}s \right\| \\
\leq & \frac{\epsilon}{2} + \eta_2 \int_{c_n}^{c} q(s)\, \mathrm{d}s \\
\leq & \frac{\epsilon}{2} + \eta_2 \frac{\epsilon}{2\eta_2} \quad \text{(if } n \text{ is large enough)} \\
= & \epsilon.
\end{aligned}
$$

Hence, the system (1.1) is approximately controllable. □

5 Example

Consider the semilinear wave equation for $0 \leq x \leq 1$ on $[0, c]$:

$$
\left.
\begin{aligned}
&\frac{\partial^2 \tilde{z}(t,x)}{\partial t^2} = \frac{\partial^2 \tilde{z}(t,x)}{\partial x^2} + \sum_{i=0}^{m} u(t-a_i, x) \\
&\qquad + F(t, \tilde{z}(\nu(t)+\varsigma, x), u(t,x), u(t-\widehat{a}_1, x), \ldots, u(t-\widehat{a}_{\widehat{m}})), \quad t \in [0, c]; \\
&\frac{\partial \tilde{z}}{\partial x}(t,0) = \frac{\partial \tilde{z}}{\partial x}(t,1) = 0, \quad t \in [0, c]; \\
&\frac{\partial \tilde{z}}{\partial t}(0,x) = \tilde{\mu}_1(x); \\
&\sum_{j=1}^{n} \beta_j \tilde{z}(t_j, x) = \tilde{z}_0(x);
\end{aligned}
\right\} \tag{5.1}
$$

where $-a \leq t_1 < t_2 < \cdots < t_n \leq 0$. The above equation can be converted in the abstract form (1.1), if we make the following setting:
(i) Define $A : D(A) \subseteq V \to V$ by $Ay = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, where $V = L_2[0, 1]$ and:

$$
\begin{aligned}
D(A) = \{ y \in V : y,\ y_x \quad & \text{are absolutely continuous}, \\
& y_{xx} \in V \quad \text{and} \quad y_x(0) = 0 = y_x(1) \}.
\end{aligned}
$$

For $\ell = 1, 2, \ldots$; take $f_\ell(x) = 2^{1/2} \cos l\pi x$ and $\sigma_\ell = (\ell\pi)^2$. Clearly, $0, \sigma_1, \sigma_2, \ldots$ are eigenvalues of A with eigenfunctions $1, f_1, f_2, \ldots$; respectively. Also, the set

$\{1, f_1, f_2, \ldots\}$ is an orthonormal basis for V. Then:

$$Ay = -\sum_{\ell=1}^{\infty} (\ell\pi)^2 \langle y, f_\ell \rangle f_\ell, \quad y \in V,$$

and A generates the cosine family $\{\mathcal{C}(t) : t \in \mathbb{R}\}$ given by:

$$\mathcal{C}(t)y = \sum_{\ell=1}^{\infty} \cos(\ell\pi t) \langle y, f_\ell \rangle f_\ell, \quad y \in V,$$

with corresponding sine family:

$$\mathcal{S}(t)y = \sum_{\ell=1}^{\infty} \frac{1}{\ell\pi} \sin(\ell\pi t) \langle y, f_\ell \rangle f_\ell, \quad y \in V.$$

Now define:

$$V' = \left\{ u(t) \in L_2[0,1] \,\middle|\, u(t) = \sum_{\ell=2}^{\infty} \delta_\ell f_\ell \quad \text{with} \quad \sum_{\ell=2}^{\infty} \delta_\ell^2 < \infty \right\}$$

and $B_i u(t) = u(t)$.

(ii) $\nu(t) = \frac{t^3}{1+c^3}$, $t \in [0, c]$, which satisfies $\nu(t) \le t$ and $\tilde{z}_{\nu(t)}(\varsigma, x) = \tilde{z}\left(\frac{t^3}{1+c^3} + \varsigma, x\right)$.

(iii) $\psi(z)(t) = \mathcal{P}(z)$ for $z \in \mathfrak{C}_0$, $t \in [-a, 0]$; $h(t) = z_0 = \tilde{z}_0(x)$, where $\mathcal{P} : \mathfrak{C}_0 \to V$ is such that $\mathcal{P}(z) = \sum_{j=1}^{n} \beta_j z(t_i)$ and $z(t_j) = \tilde{z}(t_j, x)$. We take $\wp(t) = \sum_{j=1}^{n} \wp(t_j)$, where $\wp(t_j) = \frac{1}{\beta_j}\frac{1}{n} z_0$, and then, for each $t \in [-a, 0]$:

$$\psi(\wp)(t) = \mathcal{P}(\wp) = \sum_{j=1}^{n} \beta_j \wp(t_j) = \sum_{j=1}^{n} \beta_j \frac{1}{\beta_j}\frac{1}{n} z_0 = z_0 = h(t).$$

If we take

$$F(t, z_{\nu(t)}, u(t), u(t-\widehat{a}_1), \ldots, u(t-\widehat{a}_{\widehat{m}})) = \left(\frac{t \left\| z_{\nu(t)} \right\|_{\mathfrak{C}_0}}{1 + \left\| z_{\nu(t)} \right\|_{\mathfrak{C}_0}} f_3(x) + \frac{t^2 (\|u(t)\| + \|u(t-\widehat{a}_1)\| + \cdots + \|u(t-\widehat{a}_{\widehat{m}})\|)}{1 + \|u(t)\| + \|u(t-\widehat{a}_1)\| + \cdots + \|u(t-\widehat{a}_{\widehat{m}})\|} f_4(x) \right),$$

then:

$$\begin{aligned} &\|F(t, z_{\nu(t)}, u(t), u(t-\widehat{a}_1), \ldots, u(t-\widehat{a}_{\widehat{m}}))\| \\ &\quad \le 2^{1/2}(t + t^2) \\ &\quad \le 2^{1/2}(c + c^2) \left(1 + \|z_{\nu(t)}\|_{\mathfrak{C}_0} + \|u(t)\| + \|u(t-\widehat{a}_1)\| + \cdots + \|u(t-\widehat{a}_{\widehat{m}})\|\right). \end{aligned}$$

Hence, $[H_3]$ and $[H_5]$ are satisfied.

Also:

$$\begin{aligned}&\|F(t,(z_1)_{\nu(t)},u(t),u(t-\widehat{a}_1),\ldots,u(t-\widehat{a}_{\widehat{m}}))\\&\quad-F(t,(z_2)_{\nu(t)},u(t),u(t-\widehat{a}_1),\ldots,u(t-\widehat{a}_{\widehat{m}}))\|\\&\le c\left\|(z_1)_{\nu(t)}-(z_2)_{\nu(t)}\right\|_{\mathfrak{C}_0}\end{aligned}$$

for any $(z_1)_{\nu(t)}, (z_2)_{\nu(t)} \in \mathfrak{C}_0$ and $u(t) \in V'$.
Hence, $[H_2]$ is satisfied. The linear part of (5.1) is approximately controllable (in fact, it is exactly controllable) [31]. Thus, by previous theorem, the system (5.1) is controllable.

6 Conclusion

In this article, approximate controllability for retarded systems of second order with control delays and nonlocal conditions has been discussed by assuming that the nonlinear term is locally Lipschitz which is a weaker condition than Lipschitz continuity. Using fixed point approach, the existence and uniqueness results have been derived. The system with delay effect appearing in two steps is considered. Here, the controllability results have been derived without assuming the inclusion relation among the range sets of the operators. However, conditions $[H_3]$ and $[H_5]$ are very strong, and may not be easily satisfied in many practical problems. For this reason, an study on approximate controllability of the same system without assuming the conditions $[H_3]$ and $[H_4]$ is a matter of next investigation.

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