Bulletin of the Iranian Mathematical Society (2021) 47:1325–1335
https://doi.org/10.1007/s41980-020-00444-z

ORIGINAL PAPER

Nonlinear Maps Preserving Mixed Product on Factors

Yuanyuan Zhao[1] · **Changjing Li**[1] · **Quanyuan Chen**[2]

Received: 2 May 2020 / Revised: 11 July 2020 / Accepted: 20 July 2020 / Published online: 25 August 2020

Abstract
Let $\mathcal{A}$ and $\mathcal{B}$ be two factors with $\dim\mathcal{A} > 4$. In this article, it is proved that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([A \bullet B, C]) = [\Phi(A) \bullet \Phi(B), \Phi(C)]$ for all $A, B, C \in \mathcal{A}$ if and only if Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.

Keywords Jordan $*$-product · Isomorphism · Factors

Mathematics Subject Classification 47B48 · 46L10

1 Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. Recall that a map $\Phi : \mathcal{A} \to \mathcal{B}$ preserves product or is multiplicative if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{A}$. The question of when a multiplicative map is additive was discussed in [16]. Motivated by this, many authors pay more attention to the maps on algebras preserving Lie product $[A, B] = AB - BA$ (for example, see [13–15,17,18]), or the skew Lie product $[A, B]_* = AB - BA^*$ (for example, see [1,3,7,11]), or the Jordan $*$-product $A \bullet B = AB + BA^*$ (for example,

Communicated by Shirin Hejazian.

✉ Changjing Li
lcjbxh@163.com

Yuanyuan Zhao
2934377003@qq.com

Quanyuan Chen
cqy0798@163.com

1 School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, People's Republic of China

2 College of Information, Jingdezhen Ceramic Institute, Jingdezhen 333403, People's Republic of China

see [4,8,12,22]). These results show that the (skew) Lie product or Jordan $*$-product structure is enough to determine the algebraic structure.

Recently, maps preserving the products of the mixture of (skew) Lie product and Jordan $*$-product have received a fair amount of attention. For example, Yang and Zhang [19] studied the nonlinear maps preserving the mixed skew Lie triple product $[[A, B]_*, C]$ on factors. Li et al. studied the nonlinear maps preserving the skew Lie triple product $[[A, B]_*, C]_*$ (for example, see [6,10]) and the Jordan triple $*$-product $A \bullet B \bullet C$ (for example, see [9,21]) on von Neumann algebras. In the present paper, we will establish the structure of the nonlinear maps preserving the mixed product $[A \bullet B, C]$ on factors.

Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the real field and complex field. A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. $\mathcal{A}$ is a factor means that its center is $\mathbb{C}I$. It is well known that the factor $\mathcal{A}$ is prime, in the sense that $A\mathcal{A}B = 0$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$.

2 Additivity

The main result in this section is the following.

Theorem 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two factors. Suppose that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([A \bullet B, C]) = [\Phi(A) \bullet \Phi(B), \Phi(C)]$ for all $A, B, C \in \mathcal{A}$. Then Φ is additive.*

Proof We will complete the proof by proving several claims. □

Claim 1 $\Phi(0) = 0$.

For every $A \in \mathcal{A}$, we have

$$\Phi(0) = \Phi([0 \bullet 0, A]) = [\Phi(0) \bullet \Phi(0), \Phi(A)].$$

Since Φ is surjective, there exists $A \in \mathcal{A}$ such that $\Phi(A) = 0$. So $\Phi(0) = 0$.

Choose an arbitrary nontrivial projection $P_1 \in \mathcal{A}$, write $P_2 = I - P_1$. Denote $\mathcal{A}_{ij} = P_i \mathcal{A} P_j, i, j = 1, 2$. Then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$, we may write $A = \sum_{i,j=1}^{2} A_{ij}$. In all that follows, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$.

Claim 2 For every $A_{12} \in \mathcal{A}_{12}$, $B_{21} \in \mathcal{A}_{21}$, we have

$$\Phi(A_{12} + B_{21}) = \Phi(A_{12}) + \Phi(B_{21}).$$

Choose $T = \sum_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{12} + B_{21}) - \Phi(A_{12}) - \Phi(B_{21}).$$

Since $[A_{12} \bullet P_1, P_1] = [B_{21} \bullet P_2, P_2] = 0$, it follows from Claim 1 that

$$[\Phi(A_{12} + B_{21}) \bullet \Phi(P_1), \Phi(P_1)] = \Phi([(A_{12} + B_{21}) \bullet P_1, P_1])$$

$$
\begin{aligned}
&= \Phi([B_{21} \bullet P_1, P_1]) \\
&= \Phi([A_{12} \bullet P_1, P_1]) + \Phi([B_{21} \bullet P_1, P_1]) \\
&= [(\Phi(A_{12}) + \Phi(B_{21})) \bullet \Phi(P_1), \Phi(P_1)]
\end{aligned}
$$

and

$$
\begin{aligned}
[\Phi(A_{12} + B_{21}) \bullet \Phi(P_2), \Phi(P_2)] &= \Phi([(A_{12} + B_{21}) \bullet P_2, P_2]) \\
&= \Phi([A_{12} \bullet P_2, P_2]) \\
&= \Phi([A_{12} \bullet P_2, P_2]) + \Phi([B_{21} \bullet P_2, P_2]) \\
&= [(\Phi(A_{12}) + \Phi(B_{21})) \bullet \Phi(P_2), \Phi(P_2)].
\end{aligned}
$$

Thus $\Phi([T \bullet P_1, P_1]) = [\Phi(T) \bullet \Phi(P_1), \Phi(P_1)] = 0$ and $\Phi([T \bullet P_2, P_2]) = [\Phi(T) \bullet \Phi(P_2), \Phi(P_2)] = 0$. Then $[T \bullet P_1, P_1] = [T \bullet P_2, P_2] = 0$, and so $T_{12} = T_{21} = 0$.

For every $C_{kl} \in \mathcal{A}_{kl}$, $1 \leq k \neq l \leq 2$, it follows from $[C_{kl} \bullet A_{12}, P_k] = [C_{kl} \bullet B_{21}, P_k] = 0$ that

$$
\begin{aligned}
[\Phi(C_{kl}) \bullet \Phi(A_{12} + B_{21}), \Phi(P_k)] &= \Phi([C_{kl} \bullet (A_{12} + B_{21}), P_k]) \\
&= \Phi([C_{kl} \bullet A_{12}, P_k]) + \Phi([C_{kl} \bullet B_{21}, P_k]) \\
&= [\Phi(C_{kl}) \bullet (\Phi(A_{12}) + \Phi(B_{21})), \Phi(P_k)].
\end{aligned}
$$

Thus $\Phi([C_{kl} \bullet T, P_k]) = [\Phi(C_{kl}) \bullet \Phi(T), \Phi(P_k)] = 0$, and so $[C_{kl} \bullet T, P_k] = 0$, which implies that $C_{kl} T_{ll} = 0$ for every $C_{kl} \in \mathcal{A}_{kl}$. Note that $\mathcal{A}$ is prime, it follows that $T_{ll} = 0, l = 1, 2$. Hence $T = 0$. Now we have proved that $\Phi(A_{12} + B_{21}) = \Phi(A_{12}) + \Phi(B_{21})$.

Claim 3 For every $A_{11} \in \mathcal{A}_{11}$, $B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$, $D_{22} \in \mathcal{A}_{22}$, we have

$$
\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).
$$

Let $T = \sum_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ be such that

$$
\Phi(T) = \Phi(A_{11} + B_{12} + C_{21} + D_{22}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{21}) - \Phi(D_{22}).
$$

It follows from Claim 2 that

$$
\begin{aligned}
&[\Phi(P_1) \bullet \Phi(A_{11} + B_{12} + C_{21} + D_{22}), \Phi(P_2)] \\
&= \Phi([P_1 \bullet (A_{11} + B_{12} + C_{21} + D_{22}), P_2]) \\
&= \Phi([P_1 \bullet (B_{12} + C_{21}), P_2]) \\
&= \Phi([P_1 \bullet (B_{12} + C_{21}), P_2]) + \Phi([P_1 \bullet A_{11}, P_2]) + \Phi([P_1 \bullet D_{22}, P_2]) \\
&= [\Phi(P_1) \bullet (\Phi(B_{12}) + \Phi(C_{21})), \Phi(P_2)] \\
&\quad + [\Phi(P_1) \bullet \Phi(A_{11}), \Phi(P_2)] + [\Phi(P_1) \bullet \Phi(D_{22}), \Phi(P_2)] \\
&= [\Phi(P_1) \bullet (\Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})), \Phi(P_2)].
\end{aligned}
$$

This implies that $[P_1 \bullet T, P_2] = 0$. Thus $T_{12} = T_{21} = 0$. For every $E_{ij} \in \mathcal{A}_{ij} (i \neq j)$, we have

$$\begin{aligned}
&[\Phi(E_{12}) \bullet \Phi(A_{11} + B_{12} + C_{21} + D_{22}), \Phi(P_1)] \\
&= \Phi([E_{12} \bullet (A_{11} + B_{12} + C_{21} + D_{22}), P_1]) \\
&= \Phi([E_{12} \bullet D_{22}, P_1]) \\
&= \Phi([E_{12} \bullet D_{22}, P_1]) + \Phi([E_{12} \bullet A_{11}, P_1]) \\
&\quad + \Phi([E_{12} \bullet B_{12}, P_1]) + \Phi([E_{12} \bullet C_{21}, P_1]) \\
&= [\Phi(E_{12}) \bullet (\Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})), \Phi(P_1)]
\end{aligned}$$

and

$$\begin{aligned}
&[\Phi(E_{21}) \bullet \Phi(A_{11} + B_{12} + C_{21} + D_{22}), \Phi(P_2)] \\
&= \Phi([E_{21} \bullet (A_{11} + B_{12} + C_{21} + D_{22}), P_2]) \\
&= \Phi([E_{21} \bullet A_{11}, P_2]) \\
&= \Phi([E_{21} \bullet D_{22}, P_2]) + \Phi([E_{21} \bullet A_{11}, P_2]) \\
&\quad + \Phi([E_{21} \bullet B_{12}, P_2]) + \Phi([E_{21} \bullet C_{21}, P_2]) \\
&= [\Phi(E_{21}) \bullet (\Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})), \Phi(P_2)].
\end{aligned}$$

Then $[E_{ij} \bullet T, P_i] = 0$, and so $T_{11} = T_{22} = 0$. Hence $T = 0$. It follows that $\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})$.

Claim 4 For every $A_{ij} \in \mathcal{A}_{ij}$, $B_{ij} \in \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$, we have

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

It follows from $A_{ij} + B_{ij} = [\frac{I}{2} \bullet (P_i + A_{ij}), P_j + B_{ij}]$ and Claim 3 that

$$\begin{aligned}
&\Phi(A_{ij} + B_{ij}) = \Phi([\frac{I}{2} \bullet (P_i + A_{ij}), P_j + B_{ij}]) \\
&= [\Phi(\frac{I}{2}) \bullet \Phi(P_i + A_{ij}), \Phi(P_j + B_{ij})] \\
&= [\Phi(\frac{I}{2}) \bullet (\Phi(P_i) + \Phi(A_{ij})), \Phi(P_j) + \Phi(B_{ij})] \\
&= \Phi([\frac{I}{2} \bullet P_i, P_j]) + \Phi([\frac{I}{2} \bullet P_i, B_{ij}]) + \Phi([\frac{I}{2} \bullet A_{ij}, P_j]) + \Phi([\frac{I}{2} \bullet A_{ij}, B_{ij}]) \\
&= \Phi(A_{ij}) + \Phi(B_{ij}).
\end{aligned}$$

Claim 5 For every $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$, $i = 1, 2$, we have

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

Let

$$\Phi(T) = \Phi(A_{ii} + B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii}).$$

It is clear that

$$\begin{aligned}[\Phi(P_i) \bullet \phi(A_{ii} + B_{ii}), \phi(P_i)] &= \Phi([P_i \bullet (A_{ii} + B_{ii}), P_i]) \\ &= \Phi([P_i \bullet A_{ii}, P_i]) + \Phi([P_i \bullet B_{ii}, P_i]) \\ &= [\Phi(P_i) \bullet (\Phi(A_{ii}) + \Phi(B_{ii})), \Phi(P_i)].\end{aligned}$$

Thus $[P_i \bullet T, P_i] = 0$, which implies that $T_{12} = T_{21} = 0$.

For every $C_{ji} \in \mathcal{A}_{ij}$, $j \neq i$, it follows from Claims 3 and 4 that

$$\begin{aligned}[\Phi(C_{ji}) \bullet \Phi(A_{ii} + B_{ii}), \Phi(P_i)] &= \Phi([C_{ji} \bullet (A_{ii} + B_{ii}), P_i]) \\ &= \Phi(C_{ji}A_{ii}) + \Phi(C_{ji}B_{ii}) - \Phi(A_{ii}C_{ji}^*) - \Phi(B_{ii}C_{ji}^*) \\ &= \Phi([C_{ji} \bullet A_{ii}, P_i]) + \Phi([C_{ji} \bullet B_{ii}, P_i]) \\ &= [\Phi(C_{ji}) \bullet \Phi(A_{ii}), \Phi(P_i)] + [\Phi(C_{ji}) \bullet \Phi(B_{ii}), \Phi(P_i)] \\ &= [\Phi(C_{ji}) \bullet (\Phi(A_{ii}) + \Phi(B_{ii})), \Phi(P_i)]\end{aligned}$$

and

$$\begin{aligned}[\Phi(C_{ij}) \bullet \Phi(A_{ii} + B_{ii}), \Phi(P_i)] &= \Phi([C_{ij} \bullet (A_{ii} + B_{ii}), P_i]) \\ &= \Phi([C_{ij} \bullet A_{ii}, P_i]) + \Phi([C_{ij} \bullet B_{ii}, P_i]) \\ &= [\Phi(C_{ij}) \bullet \Phi(A_{ii}), \Phi(P_i)] + [\Phi(C_{ij}) \bullet \Phi(B_{ii}), \Phi(P_i)] \\ &= [\Phi(C_{ij}) \bullet (\Phi(A_{ii}) + \Phi(B_{ii})), \Phi(P_i)].\end{aligned}$$

Then $[C_{ji} \bullet T, P_i] = [C_{ij} \bullet T, P_i] = 0$, and so $T_{11} = T_{22} = 0$. Hence $T = 0$. It follows that $\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii})$.

Claim 6 Φ is additive.

Let $A = \sum_{i,j=1}^{2} A_{ij}$, $B = \sum_{i,j=1}^{2} B_{ij} \in \mathcal{A}$. By Claims 3, 4 and 5, we have

$$\begin{aligned}\Phi(A + B) &= \Phi\left(\sum_{i,j=1}^{2} A_{ij} + \sum_{i,j=1}^{2} B_{ij}\right) = \Phi\left(\sum_{i,j=1}^{2} (A_{ij} + B_{ij})\right) \\ &= \sum_{i,j=1}^{2} \Phi(A_{ij} + B_{ij}) = \sum_{i,j=1}^{2} \Phi(A_{ij}) + \sum_{i,j=1}^{2} \Phi(B_{ij}) \\ &= \Phi\left(\sum_{i,j=1}^{2} A_{ij}) + \Phi(\sum_{i,j=1}^{2} B_{ij}\right) = \Phi(A) + \Phi(B).\end{aligned}$$

3 Main Result

Our main result in this paper reads as follows.

Theorem 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two factors with dim$\mathcal{A} > 4$. Then a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([A \bullet B, C]) = [\Phi(A) \bullet \Phi(B), \Phi(C)]$ for all $A, B, C \in \mathcal{A}$ if and only if Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.*

Proof Clearly, we only need prove the necessity. By Theorem 2.1, we obtain the additivity of Φ. Now we will complete the proof of main theorem by proving several steps.

Step 1. $\Phi(\mathbb{C}I) = \mathbb{C}I$.

Let $B \in \mathcal{A}$ such that $\Phi(B) = I$. Then

$$0 = \Phi([B \bullet C, \lambda I]) = [\Phi(B) \bullet \Phi(C), \Phi(\lambda I)] = 2[\Phi(C), \Phi(\lambda I)]$$

for all $C \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. It follows from the surjectivity of Φ that $\Phi(\lambda I) \in \mathbb{C}I$, and then $\Phi(\mathbb{C}I) \subseteq \mathbb{C}I$. By considering Φ^{-1}, we can obtain that $\Phi(\mathbb{C}I) = \mathbb{C}I$. □

Step 2. For all $A, B \in \mathcal{A}$, $[\Phi(A), \Phi(B)] = 0$ if and only if $[A, B] = 0$.

It follows from $\Phi(I) \in \mathbb{C}I$ and the additivity of Φ that

$$\begin{aligned} 2\Phi([A, B]) = \Phi(2[A, B]) &= \Phi([I \bullet A, B]) \\ &= [\Phi(I) \bullet \Phi(A), \Phi(B)] \\ &= (\Phi(I) + \Phi(I)^*)[\Phi(A), \Phi(B)] \end{aligned}$$

for all $A, B \in \mathcal{A}$. If $\Phi(I)+\Phi(I)^* = 0$, then $\Phi([A, B]) = 0$, and so $[A, B] = 0$ for all $A, B \in \mathcal{A}$. This contradiction implies that $\Phi(I)+\Phi(I)^* \neq 0$. Hence $[\Phi(A), \Phi(B)] = 0$ if and only if $[A, B] = 0$. □

It follows from Step 2 that Φ is an additive bijection that preserves commutativity in both directions. Hence by [2, Corollary 3.8]

$$\Phi(A) = a\theta(A) + \xi(A)$$

for all $A \in \mathcal{A}$, where $a \in \mathbb{C}$ is a nonzero scalar, $\theta : \mathcal{A} \to \mathcal{B}$ is an additive Jordan isomorphism, and $\xi : \mathcal{A} \to \mathbb{C}I$ is an additive map. It is easy to check that $\theta(iI) = \pm iI$.

Step 3. For every $A, B \in \mathcal{A}$, we have

(1) $\Phi(iA) - \theta(iI)\Phi(A) \in \mathbb{C}I$,
(2) $\Phi([A, B]) = \epsilon[\Phi(A), \Phi(B)]$, where $\epsilon \in \{1, -1\}$.

(1) Let $A \in \mathcal{A}$. Then

$$\Phi(iA) - \theta(iI)\Phi(A) = a\theta(iA) + \xi(iA) - \theta(iI)\Phi(A)$$

$$
\begin{aligned}
&= a\theta(iI)\theta(A) + \xi(iA) - \theta(iI)\Phi(A) \\
&= \theta(iI)(a\theta(A) + \xi(A)) + \xi(iA) - \theta(iI)\xi(A) - \theta(iI)\Phi(A) \\
&= \xi(iA) - \theta(iI)\xi(A) \in \mathbb{C}I
\end{aligned}
$$

(2) It follows from Step 1 that $\frac{1}{2}(\Phi(I) + \Phi(I)^*) = \epsilon I$ for some $\epsilon \in \mathbb{C}$. By Step 2, we have

$$\Phi([A, B]) = \frac{1}{2}(\Phi(I) + \Phi(I)^*)[\Phi(A), \Phi(B)] = \epsilon[\Phi(A), \Phi(B)]$$

for all $A, B \in \mathcal{A}$. For every $A \in \mathcal{A}$ with $A = -A^*$, we have

$$
\begin{aligned}
[\Phi(A) \bullet \Phi(B), \Phi(C)] &= \Phi([A \bullet B, C]) = \Phi([[A, B], C]) \\
&= \epsilon^2[[\Phi(A), \Phi(B)], \Phi(C)]
\end{aligned}
$$

for all $B, C \in \mathcal{A}$. Thus

$$(1 - \epsilon^2)\Phi(A)\Phi(B) + \Phi(B)(\epsilon^2\Phi(A) + \Phi(A)^*) \in \mathbb{C}I \tag{3.1}$$

for all $B \in \mathcal{A}$ and $A \in \mathcal{A}$ with $A = -A^*$. Let $P_1 \in \mathcal{B}$ be a nontrivial projection. Then there exists $D \in \mathcal{A}$ such that $\Phi(D) = P_1$. Taking $B = D$ in (3.1), we have

$$(1 - \epsilon^2)\Phi(A)P_1 + P_1(\epsilon^2\Phi(A) + \Phi(A)^*) \in \mathbb{C}I.$$

This yields

$$(1 - \epsilon^2)P_2\Phi(A)P_1 = 0 \tag{3.2}$$

for all $A \in \mathcal{A}$ with $A = -A^*$, where $P_2 = I - P_1$. Then by assertion (1) and (3.2),

$$(1 - \epsilon^2)P_2\Phi(iB)P_1 = 0 \tag{3.3}$$

for all $B \in \mathcal{A}$ with $B = -B^*$. It follows from (3.2) and (3.3) that

$$(1 - \epsilon^2)P_2\Phi(C)P_1 = 0$$

for all $C \in \mathcal{A}$. Hence $\epsilon \in \{1, -1\}$.

□

Remark 3.2 Let ϵ be as above, and let $\Psi = \epsilon\Phi$. It follows from Theorem 2.1 and Step 3(2) that $\Psi : \mathcal{A} \to \mathcal{B}$ is an additive bijection and satisfies

$$\Psi([A \bullet B, C]) = [\Psi(A) \bullet \Psi(B), \Psi(C)]$$

and

$$\Psi([A, B]) = [\Psi(A), \Psi(B)]$$

for all A, B, C in $\mathcal{A}$. Hence by [20, Theorem 2.1], there exists an additive map $f : \mathcal{A} \to \mathbb{C}I$ with $f([A, B]) = 0$ for all $A, B \in \mathcal{A}$ such that one of the following statements holds:

(1) $\Psi(A) = \varphi(A) + f(A)$ for all A in $\mathcal{A}$, where $\varphi : \mathcal{A} \to \mathcal{B}$ is an additive isomorphism;
(2) $\Psi(A) = -\varphi(A) + f(A)$ for all A in $\mathcal{A}$, where $\varphi : \mathcal{A} \to \mathcal{B}$ is an additive anti-isomorphism.

Step 4. Statement (2) of Remark 3.2 does not occur.

Indeed, if $\Psi = -\varphi + f$, where $\varphi : \mathcal{A} \to \mathcal{B}$ is an additive anti-isomorphism and $f : \mathcal{A} \to \mathbb{C}I$ is an additive map with $f([A, B]) = 0$ for all $A, B \in \mathcal{A}$, then

$$\Psi([A \bullet B, C]) = -\varphi([A \bullet B, C]) = [\varphi(B)\varphi(A) + \varphi(A^*)\varphi(B), \varphi(C)]$$

for all A, B, C in $\mathcal{A}$. On the other hand,we have

$$\begin{aligned}\Psi([A \bullet B, C]) &= [\Psi(A) \bullet \Psi(B), \Psi(C)] \\ &= [(-\varphi(A) + f(A)) \bullet (-\varphi(B) + f(B)), (-\varphi(C) + f(C))] \\ &= [(\varphi(A) - f(A)) \bullet (-\varphi(B) + f(B)), \varphi(C)] \\ &= [\varphi(A) \bullet (-\varphi(B)) + \varphi(A) \bullet f(B) + f(A) \bullet \varphi(B), \varphi(C)].\end{aligned}$$

It follows from the surjectivity of φ that

$$\begin{aligned}&(\varphi(A^*) + \varphi(A))\varphi(B) + (\varphi(B) - f(B))(\varphi(A)^* \\ &\quad + \varphi(A)) + (f(A) + f(A)^*)\varphi(B) \in \mathbb{C}I\end{aligned} \tag{3.4}$$

for all $A, B \in \mathcal{A}$. Let $P \in \mathcal{A}$ be a nontrivial projection. Then $\varphi(P)$ is a nontrivial idempotent in $\mathcal{B}$. Taking $B = P$ in (3.4) and multiplying (3.4) on the right-hand side by $\varphi(P^\perp)$ and on the left-hand side by $\varphi(P)$, we get

$$(I - f(P))\varphi(P)(\varphi(A)^* + \varphi(A))\varphi(P^\perp) = 0 \tag{3.5}$$

for all $A \in \mathcal{A}$. Replacing $\varphi(A)$ by $i\varphi(A)$ in (3.5), we have

$$(I - f(P))\varphi(P)(\varphi(A)^* - \varphi(A))\varphi(P^\perp) = 0. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$(I - f(P))\varphi(P)\varphi(A)\varphi(P^\perp) = 0$$

for all $A \in \mathcal{A}$. Hence $f(P) = I$ for any nontrivial projection $P \in \mathcal{A}$. Taking $B = P$ in (3.4) and multiplying (3.4) on the right-hand side by $\varphi(P^{\perp})$, we have

$$\varphi(P^{\perp})(\varphi(A)^* + \varphi(A))\varphi(P^{\perp}) \in \mathbb{C}\varphi(P^{\perp}) \tag{3.7}$$

for all $A \in \mathcal{A}$ and any nontrivial projection $P \in \mathcal{A}$. Replacing $\varphi(A)$ by $i\varphi(A)$ in (3.7), we can obtain that

$$\varphi(P^{\perp}AP^{\perp}) = \varphi(P^{\perp})\varphi(A)\varphi(P^{\perp}) \in \mathbb{C}\varphi(P^{\perp}) = \varphi(\mathbb{C}P^{\perp})$$

for all $A \in \mathcal{A}$ and any nontrivial projection $P \in \mathcal{A}$. This implies that

$$P^{\perp}\mathcal{A}P^{\perp} = \mathbb{C}P^{\perp}, \; P\mathcal{A}P = \mathbb{C}P$$

for any nontrivial projection $P \in \mathcal{A}$. It follows that $\mathcal{A}$ is isomorphic to $M_2(\mathbb{C})$, the algebra of all 2×2 matrices over $\mathbb{C}$, which contradicts the assumption that $\dim\mathcal{A} > 4$.

□

Step 5. Ψ is an additive *-isomorphism .

By Step 4, now we obtain that $\Psi = \varphi + f$, where $\varphi : \mathcal{A} \to \mathcal{B}$ is an additive isomorphism and $f : \mathcal{A} \to \mathbb{C}I$ is an additive map with $f([A, B]) = 0$ for all $A, B \in \mathcal{A}$. Thus

$$\Psi([A \bullet B, C]) = \varphi([A \bullet B, C]) = [\varphi(A)\varphi(B) + \varphi(B)\varphi(A^*), \varphi(C)]$$

for all $A, B, C \in \mathcal{A}$. On the other hand, we have

$$\begin{aligned}\Psi([A \bullet B, C]) &= [\Psi(A) \bullet \Psi(B), \Psi(C)] \\ &= [(\varphi(A) + f(A)) \bullet (\varphi(B) + f(B)), (\varphi(C) + f(C))] \\ &= [(\varphi(A) + f(A)) \bullet (\varphi(B) + f(B)), \varphi(C)] \\ &= [\varphi(A) \bullet \varphi(B) + f(A) \bullet \varphi(B) + \varphi(A) \bullet f(B), \varphi(C)].\end{aligned}$$

It follows from the surjectivity of φ that

$$\varphi(B)(\varphi(A)^* - \varphi(A^*)) + \varphi(B)(f(A)^* + f(A)) + f(B)(\varphi(A)^* + \varphi(A)) \in \mathbb{C}I \tag{3.8}$$

for all $A, B \in \mathcal{A}$. Let $\lambda \in \mathbb{C}$, and let $P \in \mathcal{A}$ be a nontrivial projection. Multiplying (3.8) on the left-hand side by $\varphi(P^{\perp})$ and on the right-hand side by $\varphi(P)$, and then taking $B = \lambda P$, we have

$$f(\lambda P)\varphi(P^{\perp})(\varphi(A)^* + \varphi(A))\varphi(P) = 0 \tag{3.9}$$

for all $A \in \mathcal{A}$. Similarly, we can obtain from (3.9) that

$$f(\lambda P)\varphi(P^{\perp})\varphi(A)\varphi(P) = 0$$

for all $A \in \mathcal{A}$. Then $f(\lambda P) = 0$ for all $\lambda \in \mathbb{C}$ and any nontrivial projection $P \in \mathcal{A}$. This yields that

$$f(\lambda I) = f(\lambda P) + f(\lambda P^{\perp}) = 0$$

for all $\lambda \in \mathbb{C}$. Since every A in $\mathcal{A}$ can be written as a finite linear combination of projections in $\mathcal{A}$ (see [5]), it follows that $f(A) = 0$ for all $A \in \mathcal{A}$. Now (3.8) becomes

$$\varphi(B)(\varphi(A)^* - \varphi(A^*)) \in \mathbb{C}I \tag{3.10}$$

for all $A, B \in \mathcal{A}$. In particular, $\varphi(A)^* - \varphi(A^*) \in \mathbb{C}I$ for all $A \in \mathcal{M}$. If $\varphi(A)^* - \varphi(A^*) \neq 0$ for some $A \in \mathcal{A}$, then by (3.10), $\varphi(B) \in \mathbb{C}I$ for all $B \in \mathcal{A}$. This contradiction implies that $\varphi(A)^* = \varphi(A^*)$ for all $A \in \mathcal{A}$. Hence $\Psi = \varphi$ is an additive $*$-isomorphism. □

Step 6. Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.

By Step 5, it is easy to check that $\Psi(iI) = \pm iI$ and $\Psi(qI) = qI$ for every rational number q. Let A be a positive element in $\mathcal{A}$. Then $A = B^2$ for some self-adjoint element $B \in \mathcal{A}$. It follows from Step 5 that $\Psi(A) = \Psi(B)^2$ and $\Psi(B)$ is self-adjoint. So $\Psi(A)$ is positive. This shows that Ψ preserves positive elements. Let $\lambda \in \mathbb{R}$ be any real number. Choose sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers such that $a_n \leq \lambda \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda$. It follows from

$$a_n I \leq \lambda I \leq b_n I$$

that

$$a_n I \leq \Psi(\lambda I) \leq b_n I.$$

Taking the limit, we get that $\Psi(\lambda I) = \lambda I$. Hence for all $A \in \mathcal{A}$,

$$\Psi(\lambda A) = \Psi((\lambda I)A) = \Psi(\lambda I)\Psi(A) = \lambda\Psi(A).$$

Hence Ψ is real linear. It follows from $\Psi(iI) = \pm iI$ that Ψ is linear or conjugate linear. Since $\Phi = \epsilon\Psi$, $\epsilon \in \{1, -1\}$, now we can obtain that Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism. □

From Theorem 3.1 and the fact that every ring isomorphism between type I factors is spatial, we have the following corollary.

Corollary 3.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be two type I factors acting on a complex Hilbert spaces $\mathcal{H}$ with $\dim\mathcal{H} > 2$. Then a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([A \bullet B, C]) = [\Phi(A) \bullet \Phi(B), \Phi(C)]$ for all $A, B, C \in \mathcal{A}$ if and only if there exists $\epsilon \in \{1, -1\}$ such that $\Phi(A) = \epsilon UAU^*$ for all $A \in \mathcal{A}$, where U is a unitary or conjugate unitary operator.*

Acknowledgements The authors would like to thank the referee for the very thorough reading of the paper and many helpful comments. The second author is supported by the Natural Science Foundation of Shandong Province, China (Grant No. ZR2018BA003) and the National Natural Science Foundation of China (Grant No. 11801333).

References

1. Bai, Z., Du, S.: Maps preserving products $XY - YX^*$ on von Neumann algebras. J. Math. Anal. Appl. **386**, 103–109 (2012)
2. Banning, R., mathieu, M.: Commutativity preserving mapping on semiprime rings. Commn. Algebra **25**, 247–265 (1997)
3. Cui, J., Li, C.K.: Maps preserving product $XY - YX^*$ on factor von Neumann algebras. Linear algebra Appl. **431**, 833–842 (2009)
4. Dai, L., Lu, F.: Nonlinear maps preserving Jordan $*$-products. J. Math. Anal. Appl. **409**, 180–188 (2014)
5. Fillmore, P., Topping, D.: Operator algebra generated by projections. Duke Math. J. **34**, 333–336 (1967)
6. Li, C., Chen, Q., Wang, T.: Nonlinear maps preserving the Jordan triple $*$-product on factors. Chin. Ann. Math. Ser. B **39**, 633–642 (2018)
7. Li, C., Chen, Q.: Strong skew commutativity preserving maps on rings with involution. Acta Math. Sinica (Engl. Ser.) **32**, 745–752 (2016)
8. Li, C., Lu, F., Fang, X.: Nonlinear mappings preserving new product $XY+YX^*$ on factor von Neumann algebras. Linear Algebra Appl. **438**, 2339–2345 (2013)
9. Li, C., Lu, F.: Nonlinear maps preserving the Jordan triple 1-$*$-product on von Neumann algebras. Complex Anal. Oper. Theory **11**, 109–117 (2017)
10. Li, C., Lu, F.: Nonlinear maps preserving the Jordan triple $*$-product on von Neumann algebras. Ann. Funct. Anal. **7**, 496–507 (2016)
11. Li, C., Zhao, F., Chen, Q.: Nonlinear skew Lie triple derivations between factors. Acta Math. Sinica (Engl. Ser.) **32**, 821–830 (2016)
12. Li, C., Zhao, F., Chen, Q.: Nonlinear maps preserving product $X^*Y + Y^*X$ on von Neumann algebras. Bull. Iran. Math. Soc. **44**, 729–738 (2018)
13. Li, C., Lu, F.: 2-local $*$-Lie isomorphisms of operator algebras. Aequ. Math. **90**, 905–916 (2016)
14. Li, C., Lu, F.: 2-local Lie isomorphisms of nest algebras. Oper. Matrices **10**, 425–434 (2016)
15. Marcoux, L.W.: Lie isomorphism of nest algebras. J. Funct. Anal. **164**, 163–180 (1999)
16. Martindale III, W.S.: When are multiplicative mappings additive? Proc. Am. Math. Soc. **21**, 695–698 (1969)
17. Mires, C.R.: Lie isomorphisms of operator algebras. Pacific J. Math. **38**, 717–735 (1971)
18. Mires, C.R.: Lie isomorphisms of factors. Trans. Am. Math. Soc. **147**, 55–63 (1970)
19. Yang, Z., Zhang, J.: Nonlinear maps preserving the mixed skew Lie triple product on factor von Neumann algebras. Ann. Funct. Anal. **10**, 325–336 (2019)
20. Zhang, J., Zhang, F.: Nolinear maps preserving Lie product on factor von Neumann algebras. Linear algebra Appl. **429**, 18–30 (2008)
21. Zhao, F., Li, C.: Nonlinear maps preserving the Jordan triple $*$-product between factors. Indag. Math. **29**, 619–627 (2018)
22. Zhao, F., Li, C.: Nonlinear $*$-Jordan triple derivations on von Neumann algebras. Math. Slovaca **68**, 163–170 (2018)