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ORIGINAL PAPER

A New Flow on Open Manifolds: Short Time Existence and Uniqueness

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Abstract
A new flow on an open manifold with bounded geometry has been considered. We prove short time existence and uniqueness for the flow. The approach is based on the study of the geometry of the manifold of Riemannian metrics on open manifolds. Moreover, some properties of this flow are investigated.

Keywords Ricci flow · Space of Riemannian metrics · Bounded geometry

Mathematics Subject Classification 53C44 · 58D17 · 58B25

1 Introduction

Hamilton introduced the Ricci flow on a compact manifold as an evolution equation for a family of Riemannian metrics [13]. Solutions of the following evolution on an n-dimensional Riemannian manifold M have been called a Ricci flow:

$$\begin{aligned} &\tfrac{\partial}{\partial t} g(t) = -2Ric(g(t)), \\ &g(0) = g_0 \end{aligned} \tag{1.1}$$

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where $Ric(g(t))$ denotes the Ricci curvature of the metric $g(t)$.

Short time existence and uniqueness of the Ricci flow on a compact and complete noncompact manifolds have been proved by different mathematicians (See[4,9,12, 13,15,17]). All proofs are based on considering Ricci flow as a partial differential equation. This approach is elaborated, but it is too long. In [10], the authors presented for the compact case an easy proof of short time existence and uniqueness of Ricci flow by considering the geometry of the manifold of Riemannian metrics $\mathfrak{M}$ on a compact manifold. Ebin studied the space of Riemannian metrics on a compact manifold from a geometric point of view [5]. The space of Riemannian metrics on a compact, oriented, smooth n-manifold M is an infinite-dimensional manifold. It is an open subset of a Fréchet space, and is a Fréchet manifold. After Ebin, the geometry of $\mathfrak{M}$ where M is a compact manifold, has been investigated in [8]. In 1995, Eichhorn studied the geometry of $\mathfrak{M}$ on an open manifold, i.e., a noncompact manifold without boundary [6]. Where he restricted himself to metrics with bounded geometry. The reason is the fact that the Sobolev analysis is then available, e.g., embedding theorems, module structure theorems, and many invariance properties. We recall that:

A noncompact Riemannian manifold (M, g) has bounded geometry up to order k if it satisfies the following conditions (I) and (B_k):

$$\begin{array}{ll} (I) & r_{inj}(M, g) > 0, \\ (B_k) & |\nabla^i R| \le C_i, \qquad 0 \le i \le k, \end{array}$$

where $r_{inj}(M, g) = inf_x r_{inj}(x)$ is the injectivity radius of (M, g) [6]. The condition $(B_\infty(M, g))$ means $|\nabla^i R| \le C_i, \quad i = 1, 2, \ldots$. Eichhorn introduced a suitable uniform structure on $\mathfrak{M}$ and obtained completed spaces ${}^{b,m}\mathfrak{M}$ or $\mathfrak{M}^r(I, B_k)$, and calculated for each component of $\mathfrak{M}^r(I, B_k)$ the infinite-dimensional geometry.

Theorem 1.1 [6] *Let M be an open manifold, $k \ge r > n/p + 1$. Then, $\mathfrak{M}^{p,r}(I, B_k)$ has a representation as a topological sum:*

$$\mathfrak{M}^{p,r}(I, B_k) = \sum_{i \in I} U^{p,r}(g_i).$$

For $g \in \mathfrak{M}^{p,r}(I, B_k)$:

$$U^{p,r}(g) = \{g' \in \mathfrak{M}^{p,r}(I, B_k) |{}^b|g - g'|_g < \infty, {}^b|g - g'|_{g'} < \infty, |g - g'|_{g,p,r} < \infty\}.$$

Also, he showed that

Theorem 1.2 [6] *Assume $k \ge r > n/p + 1$. Then, each component of the space $\mathfrak{M}^{p,r}(I, B_k)$ is a Banach manifold, and for $p = 2$, it is a Hilbert manifold.*

Ricci flow on open manifolds with some conditions has been investigated by some authors (you can see [2,3,7,14]). The investigations described here are motivated by the paper [10] where we considered the Ricci flow as a curve on the manifold of Riemannian metrics on a compact manifold and by [6] where the geometry of the space of Riemannian metrics on the open manifold has been presented and some results have been given. The approach used in [10] is not suitable for the Ricci flow on an open manifold. The major problem is that the Sobolev norm of $Ric(g)$ itself is mostly infinite, since any manifold of bounded geometry has infinite volume, but the Sobolev norm of $Ric(g) - Ric(g_0)$ is finite. This point encouraged us to define a new flow on an open manifold with bounded geometry, which is similar to Ricci flow. We consider the following evolution differential equation on an open manifold with bounded geometry:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2(Ric(g(t)) - Ric(g_0)) \\ g(0) = \tilde{g}, \end{cases} \tag{1.2}$$

where $Ric(g(t))$ and $Ric(g_0)$ denote the Ricci curvature of the metric $g(t)$ and g_0, respectively, and $\tilde{g}$ belongs to $comp(g_0) = U^{2,r}(g_0)$. A new and easy proof for the short time existence and uniqueness of the flow (1.2) on an open manifold will be presented. The proof is based on the geometry of $\mathfrak{M}$ and considering solutions of the flow as a curve in this space. Moreover, some properties of the flow on open manifolds with bounded geometry have been obtained.

2 New Flow

The geometry of the space of Riemannian metrics on an open manifolds M with bounded geometry has been studied by Eichhorn [6]. We consider the flow (1.2) on an open manifold M.

If the initial metric $g(0) = g_0$ that is $\tilde{g} = g_0$, then $\frac{\partial}{\partial t} g(t) = 0$, and thus, Eq. (1.2) has the trivial solution $g(t) \equiv g_0$. Now, we assume that $\tilde{g} \in comp(g_0)$ and investigate some properties of this flow.

2.1 Exact Solutions

Suppose the initial metric is Einstein, that is $Ric(\tilde{g}) = \lambda \tilde{g}$ for some $\lambda \in \mathbb{R}$. Solutions of the flow (1.2) can be Einstein, i.e., $Ric(g(t)) = \lambda g(t)$. Indeed, we make the answer for (1.2) as follows:

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2(Ric(g(t)) - Ric(g_0)) \\ &= -2(\lambda g(t) - Ric(g_0)). \end{aligned}$$

Now:

$$\frac{\partial}{\partial t} g_{ij}(t) + 2\lambda g_{ij}(t) = 2Ric(g_0)_{ij}$$

is a linear ordinary differential equation. We have:

$$
\begin{aligned}
g_{ij}(t) &= \exp^{-\int 2\lambda \mathrm{d}t}[\int \exp^{\int 2\lambda \mathrm{d}t} \times 2Ric(g_0)_{ij}\mathrm{d}t + C_{ij}] \\
&= \frac{1}{\lambda}Ric(g_0)_{ij} + C_{ij}\exp^{-2\lambda t}.
\end{aligned}
$$

Then, with initial condition $g(0) = \tilde{g}$, solution to (1.2) is in the following form:

$$
g_{ij}(t) = \frac{1}{\lambda}Ric(g_0)_{ij}(1 - e^{-2\lambda t}) + \tilde{g}_{ij}e^{-2\lambda t},
$$

which is a Riemannian metric if $\lambda Ric(g_0)$ is positive definite.

Remark 2.1 Suppose equation (1.2) has a solution which is of the form:

$$
g(t) = f(t)\tilde{g},
$$

where $f(t)$ is a positive function. We compute:

$$
\frac{\partial}{\partial t}g(t) = f'(t)\tilde{g}.
$$

Also, we require:

$$
f'(t)\tilde{g} = -2(Ric(g(t)) - Ric(g_0)).
$$

Therefore:

$$
\begin{aligned}
f'(t)\tilde{g} &= -2(\lambda\tilde{g} - Ric(g_0)) \\
f(t)\tilde{g} &= -2(\lambda\tilde{g} - Ric(g_0))t + C, \quad C = \tilde{g} \\
f(t)I &= -2(\lambda I - \tilde{g}^{-1}Ric(g_0))t + I,
\end{aligned}
$$

which says that $\tilde{g}^{-1}Ric(g_0))$ must be diagonal.

2.2 Soliton Associated with Flow (1.2)

In this subsection, we can define soliton associated with the flow (1.2) similar to Ricci soliton in the compact case. Self-similar solutions for the flow (1.2) can be described as follows: Assume that:

$$
g(t) = \sigma(t)\phi_t^*(\tilde{g}) \tag{2.1}
$$

is a solution of (1.2), where $\sigma(t)$ and ϕ_t are scalar and isometries of g_0, respectively, such that $\sigma(0) = 1$ and $\phi(0) = id_M$. On the other hand, suppose there is a fixed Riemannian metric in such a way that the identity:

$$
-2(Ric(\tilde{g}) - Ric(g_0)) = 2\lambda\tilde{g} + \mathfrak{L}_V(\tilde{g}) \tag{2.2}
$$

holds for some constant λ and some complete Killing vector field V with respect to g_0. In this case, we say that g_0 is a Ricci soliton associated with the flow (1.2). The same is for a Ricci soliton in the compact case we can prove the following observation about Ricci soliton associated with the flow (1.2).

Lemma 2.2 *If $g(t)$ is a solution of the flow (1.2) having the special form (2.1), then there exists a Killing vector field X with respect to g_0, such that (g_0, X) solves (2.2). Conversely, given any solution of (2.2), there exist $1-$ parameter families of scalars $\sigma(t)$ and isometries $\phi(t)$ of M, such that $g(t)$ becomes a solution of the flow (1.2) when $g(t)$ is defined by (2.1).*

Proof First, suppose that there is a solution of the flow (1.2) having the form (2.1). We may assume without loss of generality that $\sigma(0) = 1$ and $\phi_0 = id$. Then, we have:

$$\frac{\partial g}{\partial t} = \sigma'(t)\phi_t^*(\tilde{g}) + \sigma(t)\frac{d}{dt}\phi_t^*(\tilde{g})$$
$$\frac{\partial}{\partial t} g(t)|_{t=0} = 2\lambda\tilde{g} + \mathfrak{L}_V(\tilde{g}),$$

where $V(t)$ is the family of vector fields generating the diffeomorphisms ϕ_t which is also the Killing vector field with respect to g_0. This implies that g_0 satisfies (2.2) with $\lambda = \frac{1}{2}\sigma'(0)$ and $X = V(0)$. Conversely, suppose that g_0 and $\tilde{g}$ satisfy (2.2). Define $\sigma(t) = 1 - 2\lambda t$ and $X(t) = \frac{1}{\sigma(t)}V$, let $\phi(t)$ denote the diffeomorphisms generated by the family X_t, where $\phi(0) = id_M$ and $g(t)$ is a smooth 1-parameter family of metrics on M defined by:

$$g(t) = \sigma(t)\phi_t^*(\tilde{g}).$$

It is a solution to Eq. (1.2) with $g(0) = \tilde{g}$ by the following computations:

$$\begin{aligned}\frac{\partial g}{\partial t} &= \sigma'(t)\phi_t^*(\tilde{g}) + \sigma(t)\phi_t^*(\mathfrak{L}_X\tilde{g})\\ &= \phi_t^*(-2\lambda\tilde{g} + \mathfrak{L}_V\tilde{g})\\ &= \phi_t^*(-2(Ric(\tilde{g}) - Ric(g_0)))\\ &= -2(Ric(\phi_t^*(\tilde{g}) - Ric(\phi_t^*(g_0)))\\ &= -2(Ric(\phi_t^*(\tilde{g}) - Ric(g_0))).\end{aligned}$$

□

Definition 2.3 Let g_0 be a Riemannian metric on an open manifold M. A complete Riemannian metric $\tilde{g}_{ij}$ on M is called a Ricci soliton associated with 1.2 if there exists a smooth Killing vector field $V = (V_i)$ with respect to g_0, such that the Ricci tensor $Ric(\tilde{g}_{ij})$ of the metric $\tilde{g}_{ij}$ satisfies the equation:

$$-2((Ric\tilde{g})_{ij} - (Ricg_0)_{ij}) = 2\lambda\tilde{g}_{ij} + \tilde{\nabla}_i V_j + \tilde{\nabla}_j V_i \tag{2.3}$$

for some constant λ. Moreover, if V is a gradient vector field, then we have a gradient Ricci soliton associated with 1.2, satisfying the equation:

$$(Ric\tilde{g})_{ij} - (Ricg_0)_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j f = \lambda\tilde{g}_{ij} \tag{2.4}$$

for some smooth function f on M. The function f is called a potential function of the soliton associated with (1.2). Extending to our new setting the soliton terminology, we say that the soliton associated with (1.2) is shrinking, steady, and expanding according as λ is positive, zero, and negative, respectively.

Lemma 2.4 *Let $\tilde{g}_{ij}$ be a complete gradient Ricci soliton associated with 1.2 with potential function f. Then, we have:*

$$\tilde{\nabla}_i\tilde{R} + 2Ric(\tilde{g}_{ij})\tilde{\nabla}_j f - 2\tilde{g}^{jk}(\tilde{\nabla}_i Ric(g_0)_{jk} - \tilde{\nabla}_j Ric(g_0)_{ik}) = 0 \tag{2.5}$$

for some constant C and Riemannian metric g_0. Here, $\tilde{R}$ denotes the scalar curvature of $\tilde{g}$.

Proof Let $\tilde{g}_{ij}$ be a complete gradient Ricci soliton associated with flow (1.2) on a manifold M, so that there exists a potential function f, such that the soliton equation (2.4) holds. Taking the covariant derivatives and using the commutating formula for covariant derivatives, we obtain:

$$\tilde{\nabla}_i R\tilde{i}c_{jk} - \tilde{\nabla}_j R\tilde{i}c_{ik} + \tilde{R}_{ijkl}\tilde{\nabla}_l f - (\tilde{\nabla}_i R\dot{i}c_{jk} - \tilde{\nabla}_j R\dot{i}c_{ik}) = 0.$$

Taking the trace on j and k and using the contracted second Bianchi identity:

$$\tilde{\nabla}_j Ric_{ik} = \frac{1}{2}\tilde{\nabla}_i\tilde{R},$$

we get

$$\tilde{\nabla}_i\tilde{R} - \frac{1}{2}\tilde{\nabla}_i R + Ric(\tilde{g}_{ij})\tilde{\nabla}_j f - \tilde{g}^{jk}(\tilde{\nabla}_i Ric(g_0)_{jk} - \tilde{\nabla}_j Ric(g_0)_{ik}) = 0. \tag{2.6}$$

Therefore:

$$\tilde{\nabla}_i\tilde{R} + 2Ric(\tilde{g}_{ij})\tilde{\nabla}_j f - 2\tilde{g}^{jk}(\tilde{\nabla}_i Ric(g_0)_{jk} - \tilde{\nabla}_j Ric(g_0)_{ik}) = 0. \tag{2.7}$$

□

2.3 Short Time Existence and Uniqueness

The first question in the study of every evolution differential equation is the short time existence of the solutions. In this subsection, we answer this question by considering the geometry of the space of Riemannian metrics.

Theorem 2.5 *Let M be an open manifold and $g_0 \in \mathfrak{M}^{2,r}(I, B_k)$ such that $k \geq r > n/2 + 1$. There exist a constant T, such that evolution equation (1.2) has a unique solution $g(t)$ for short time $[0, T)$ with initial metrics $g(0) = \tilde{g} \in comp(g_0) = U^{2,r}(g_0)$.*

Proof To prove the theorem, we define a vector field on $\mathfrak{M}^{2,r}(I, B_k)$, such that solution of flow (1.2) is its integral curve. We claim that $X = -2(Ric(g) - Ric(g_0))$ is the desired vector field in the tangent space of Riemannian metrics. We have to show that $-2(Ric(g) - Ric(g_0))$ remains in $T_g comp(g) = \Omega^{2,r}(S^2T^*, g)$. One can proceed as follows:

Assume $r = \infty$, consider $comp^{2,\infty}(g_0) \subset \mathfrak{M}^{2,\infty}(I, B_\infty) = \cap_r \mathfrak{M}^{2,r}(2, B_\infty)$. Since $g \in comp^{2,\infty}(g_0)$, we have:

$$\begin{aligned} |g - g_0|_{2,r} &< \infty && \forall r \\ Ric(g) - Ric(g_0) &\in \Omega^{2,r}(S^2T^*M) && \forall r; \end{aligned}$$

thus, $Ric(g) - Ric(g_0) \in Tcomp^{2,\infty}(g_0)$ A curve in $comp^{2,\infty}(g_0)$ ia a curve in any $comp^{2,r}(g_0), \quad r \geq 0$.

According to [6, Theorem 2.37], every component of $\mathfrak{M}^{2,r}(I, B_k)$ is a Hilbert manifold; consequently, the vector field $X = -2(Ric(g) - Ric(g_0))$ in any $comp^{2,r}(g_0)$ has a unique integral curve on the short time $[0, \epsilon)$. Now, it is clear that the solution of evolution equation (1.2) is an integral curve of X. □

Example 2.6 Euclidean space with the standard metric ($g_0 = \delta_i^j$) trivially has bounded geometry. The flow (1.2) in component $comp(g_0)$ with initial metric $\tilde{g} = g_0$ has trivial solution g_0.

Example 2.7 Examples of open manifolds with bounded geometry are homogeneous spaces or Riemannian coverings of closed manifolds. Given a left-invariant metric g_0 on $\widetilde{E(2)}$, the universal cover of group of isometries of the plane, with associated Milnor frame $\{\theta^1, \theta^2, \theta^3\}$ the metric and the Ricci tensors are diagonalized [16]. Writing

$$g_0 = A_0(\theta^1)^2 + B_0(\theta^2)^2 + C_0(\theta^3)^2$$

with $A_0 B_0 C_0 = 4$, the non-zero components of the Ricci tensor $\widetilde{E(2)}$ are:

$$\begin{aligned} Ric_{11} &= \frac{1}{2}A(A^2 - B^2) \\ Ric_{22} &= \frac{1}{2}B(B^2 - A^2) \\ Ric_{33} &= -\frac{1}{2}C(A - B)^2. \end{aligned}$$

Let

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2,$$

the new flow (1.2) in component $comp(g_0)$ with initial metric:

$$\tilde{g} = \tilde{A}(\theta^1)^2 + \tilde{B}(\theta^2)^2 + \tilde{C}(\theta^3)^2, \tag{2.8}$$

belongs to component $comp(g_0)$, becomes an ODE system in (A, B, C) as follows:

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= -(A(A^2 - B^2) - A_0(A_0^2 - B_0^2)) \\
\frac{\mathrm{d}B}{\mathrm{d}t} &= -(B(B^2 - A^2) - B_0(B_0^2 - A_0^2)) \\
\frac{\mathrm{d}C}{\mathrm{d}t} &= C(A - B)^2 - C_0(A_0 - B_0)^2,
\end{aligned}$$

which has unique solution with the initial condition (2.8).

Applying the approach used in [11], we obtain the following result:

Theorem 2.8 *The solution of evolution equation* (1.2) *as a curve starting from a* $g_0 \in \mathfrak{M}^{2,\infty}(I, B_\infty)$ *is not a geodesic.*

Proof Since a curve in $comp^{2,\infty}(g_0)$ is a curve in any $comp^{2,r}(g_0)$, $r \geq 0$, the geodesic on $comp^{2,\infty}(g_0)$ is a geodesic in any $comp^{2,r}(g_0)$, $r \geq 0$. The geodesics on $\mathfrak{M}^{2,r}(I, B_k)$ are solutions to the following second-order equation:

$$\ddot{g} = \dot{g}g^{-1}\dot{g} + \frac{1}{4}tr(g^{-1}\dot{g}g^{-1}\dot{g})g - \frac{1}{2}tr(g^{-1}\dot{g})\dot{g},$$

where $g = g(t)$, $\dot{g} = \frac{dg}{\mathrm{d}t}$ [6].

Let the solutions of the flow (1.2) satisfy the above equation; that is:

$$\frac{\partial}{\partial t}[-2(Ric(g) - Ric(g_0))] = \dot{g}g^{-1}\dot{g} + \frac{1}{4}tr(g^{-1}\dot{g}g^{-1}\dot{g})g - \frac{1}{2}tr(g^{-1}\dot{g})\dot{g};$$

then:

$$\begin{aligned}
\frac{\partial}{\partial t}Ric(g)_{ik} = {} & 4([Ric(g) - Ric(g_0)]g^{-1}[Ric(g) - Ric(g_0)])_{ik} \\
& + tr(g^{-1}[Ric(g) - Ric(g_0)]g^{-1}[Ric(g) - Ric(g_0)])g_{ik} \\
& - 2tr(g^{-1}[Ric(g) - Ric(g_0)])[Ric(g) - Ric(g_0)]_{ik}
\end{aligned}$$

On the other hand, one can easily conclude that the Ricci tensor under the flow (1.2) evolves by:

$$\begin{aligned}
\frac{\partial}{\partial t}R_{ik} = {} & g^{pq}(-\nabla^2_{q,k}R_{ip} + \nabla^2_{i,k}R_{pq} - \nabla^2_{q,i}R_{kp} + \nabla^2_{q,p}R_{ik}) \\
& + g^{pq}(-\nabla^2_{q,k}R^0_{ip} + \nabla^2_{i,k}R^0_{pq} - \nabla^2_{q,i}R^0_{kp} + \nabla^2_{q,p}R^0_{ik}),
\end{aligned}$$

where $R_{ik} = [Ric(g(t))]_{ik}$ and $R^0_{ik} = [Ric(g_0)]_{ik}$. (We obtained it by applying evolution equation for the Ricci tensor mentioned in [1])

Above equations show that the flow (1.2) cannot be a geodesic on the manifold of Riemannian metrics. □

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