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ORIGINAL PAPER

Tilting Torsion-Free Classes in the Category of Comodules

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Abstract
A well-known Assem–Smalϕ theorem on tilting modules plays an important role in tilting theory. In this paper, we prove that a version of Assem–Smalϕ theorem still holds in the category of comodules. Following this result, we characterize the tilting torsion-free classes in the category of comodules using the precover theory.

Keywords Precover · Tilting comodule · Cofinendo · Tilting torsion-free class

Mathematics Subject Classification 16T15 · 18G05

1 Introduction

Tilting theory is an important tool for the study of the representation theory of algebras. In particular, the finitely generated tilting modules play an important role in the representation theory of finite-dimensional algebras in [2,4,5]. It is well known that each finitely generated tilting module gives rise to a torsion theory ($\mathcal{T} = \mathrm{Gen}(T)$, $\mathcal{F} = \mathrm{Ker}(\mathrm{Hom}_{\mathrm{R}}(T, -))$). However, the converse is not necessarily true. It is natural then to ask when is a torsion theory generated by a finitely generated tilting module? For finite-dimensional algebras, Assem [3] and Smalϕ [28] proved that a given torsion class $\mathcal{T}$ which meet some conditions can be generated by a finitely generated tilting modules T such that $\mathcal{T} = \mathrm{Gen}(T)$. This result is also called Assem–Smalϕ theorem. Colby and Fuller generalized the notion of finitely generated tilting module over a finite-dimensional algebra, see [8]. Later, Colpi and Trlifaj generalized clas-

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sical tilting modules over any rings to the infinite-dimensional case. They proved that Assem–Smalϕ theorem still holds over any rings, see [9, Corollary 2.7]. Inspired by Assem and Smalϕ, Angeleri Hügel, Tonolo and Trlifaj proved that the torsion classes over any rings generated by tilting modules (i.e. tilting torsion classes) can be characterized as special preenvelope classes of the module category, see [1, Theorem 2.1]. Dually to [9, Corollary 2.7], they also gave the Assem–Smalϕ theorem on the torsion-free classes cogenerated by cotilting modules, see [1, Lemma 2.4].

Dually to the representation theory of algebras, the researches about the representation theory of coalgebras have been on the rise. In 1977, I-peng Lin [14] introduced the definition of semiperfect coalgebras, and studied their properties. If every finite-dimensional left C-comodule has a projective cover, then a coalgebra C is called left semiperfect. Doi [10], Simson [17–27], Chin [6,7] and other scholars did some researches on the representation theory of coalgebras, which make it possible for the development of representation-infinite algebras. For example, Simson [17,18,25] introduced the concepts of wildness, tameness, discrete comodule type, polynomial growth, and the pure semisimplicity for basic coalgebras over an algebraically closed field K, see also [13,15]. In addition, Simson [25,26] introduced the fc-tame comodule type and the fc-wild comodule type of coalgebras. In 2000, Wang [31] investigated Morita–Takeuchi contexts coalgebras acting on graded coalgebras. In 2011, Keller and Yang [12] obtained some nice results about pseudocompact algebras and pseudocompact modules. According to their results , a better framework for cotilting of coalgebras are derived categories of comodule categories or pseudocompact module categories. In 2014, Zhang and Yao [32] characterized f-cotilting comodules, finitely copresented comodules, and the localization of classical tilting comodules. In 2016, Fu and Yao [11] proved the Auslander–Reiten formula for comodule categories and gave the applications to partial tilting comodules and tilting global dimension.

One of the open problems which Simson provided in [17] is to develop a (co)tilting theory for comodule categories. Wang [29,30] introduced the notion of tilting comodules and proved that each tilting comodule induces a torsion theory over semiperfect coalgebras. Different from the Wang's work, Simson [24] introduced the concepts of cotilting comodules and f-cotilting comodules. Similarly, Simson proved that each cotilting comodule can cogenerate a torsion theory over basic coalgebras. Therefore, it is necessary to consider whether Assem–Smalϕ theorem can be established in the category of comodules.

In this article, we give a positive answer to this basic question. Motivated by the work of Angeleri Hügel, Tonolo and Trlifaj, we further use the precover theory to give a characterization of the torsion-free classes generated by the tilting comodules.

The present article is organized as follows. In Sect. 2, we give some necessary preliminaries for the present paper. In Sect. 3, we introduce the definition of the pre(cover) of a comodule. In addition, some properties of the pre(cover) classes are discussed. In Sect. 4, we introduce the concept of cofinendo comodules. Furthermore, we prove that a precover class, as a pretorsion-free class, coincides with a comodule class which is cogenerated by a cofinendo comodule. In Sect. 5, we introduce the concept of $\mathcal{D}$-injective comodules, where $\mathcal{D}$ is a class of comodules in C-Comod. Moreover, we give the version of Assem–Smalϕ theorem and characterize the tilting torsion-free classes in the category of comodules.

2 Preliminaries

The reader is referred to [16] and [27] for terminology and notation in the study of comodule categories C-Comod.

Now let K be a field and C be a vector space over K. We call the system(C, Δ, ε) or simply, C a K-coalgebra, if $(I \otimes \Delta)\Delta = (\Delta \otimes I)\Delta$, and $(I \otimes \varepsilon)\Delta = (\varepsilon \otimes I)\Delta$, where $\Delta : C \to C \otimes C$ and $\varepsilon : C \to K$ are K-linear maps. Let C be a coalgebra. A pair (M, ρ_M^-) is said to be a left C-comodule if M is a K-linear space, and $\rho_M^- : M \to C \otimes M$ is a K-linear map such that $(I \otimes \rho_M^-)\rho_M^- = (\Delta \otimes I)\rho_M^-$ and $(\varepsilon \otimes I)\rho_M^- = I$.

We assume, unless otherwise stated, that all comodules are left comodules in this paper. We use C-Comod to denote the category of left C-comodules. If M and N are two left C-comodules, we denote by $\mathrm{Hom}_C(M, N)$ the K-space of all the homomorphisms of comodules from M to N. Let C be a coalgebra. We denote by $\mathrm{Cogen}(T)$ the class of all subcomodules of products of copies of M, that is, for all $M \in \mathrm{Cogen}(T)$, there exists a monomorphism $\varphi : M \to T^I$, where I is an index set. In addition, $\mathrm{Copres}(T)$, the subclass of $\mathrm{Cogen}(T)$, is composed of the C-comodules which can be copresented by T. Thus, for every $M \in \mathrm{Copres}(T)$, there is an exact sequence $0 \to M \overset{\phi}{\to} T^I \overset{\theta}{\to} T^J$, where I, J are index sets.

Given a comodule class $\mathcal{M} \subseteq C$-Comod, we denote by $\mathrm{Add}(\mathcal{M})(\mathrm{add}(\mathcal{M}))$ the class consisting of all summands of (finite) direct sums of comodules in $\mathcal{M}$. Similarly, we denote by $\mathrm{Prod}(\mathcal{M})$ the class of all summands of any products of comodules in $\mathcal{M}$. Moreover, let

$$\mathcal{M}^{\perp} = \{X \in C\text{-Comod} \mid \mathrm{Ext}_C^1(M, X) = 0 \text{ for all } M \in \mathcal{M}\},$$
$$^{\perp}\mathcal{M} = \{X \in C\text{-Comod} \mid \mathrm{Ext}_C^1(X, M) = 0 \text{ for all } M \in \mathcal{M}\}.$$

If $\mathcal{M} = \{M\}$, then we just write $\mathrm{Add}M$, $\mathrm{add}M$, $M^{\perp}$ and $^{\perp}M$, respectively.

Let $\mathcal{D}$ be a class of comodules in C-Comod. Then $\mathcal{D}$ is a pretorsion-free class provided that $\mathcal{D}$ is closed under products and subcomodules.

3 Precovers and Covers

At first, we will introduce the concept of (pre)covers for comodules. Furthermore, we will discuss some properties of them.

Definition 3.1 Let $\mathcal{F}$ be a class of comodules in C-Comod and $M \in C$-Comod, then $\phi \in \mathrm{Hom}_C(X, M)$ with $X \in \mathcal{F}$ is an $\mathcal{F}$-precover of M if it satisfies that $\mathrm{Hom}_C(F, \phi) : \mathrm{Hom}_C(F, X) \to \mathrm{Hom}_C(F, M)$ is surjective for each $F \in \mathcal{F}$.

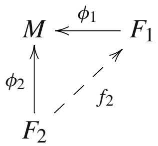
***Remark* 3.2** Let $\phi \in \mathrm{Hom}_C(X, M)$ be an $\mathcal{F}$-precover of M.

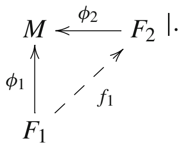
(i) ϕ is said to be an $\mathcal{F}$-cover of M, if $\phi g = \phi$ and $g \in \mathrm{End}_C(X)$ imply that g is an automorphism of X.
(ii) ϕ is called special if $\phi \in \mathrm{Hom}_C(X, M)$ is surjective and $\mathrm{Ker}\phi \in \mathcal{F}^{\perp}$.

We call $\mathcal{F} \subseteq C$-Comod a precover (cover) class if each comodule has an $\mathcal{F}$-precover ($\mathcal{F}$-cover).

Proposition 3.3 *Assume that $\phi_1 : F_1 \to M$ and $\phi_2 : F_2 \to M$ are two different $\mathcal{F}$-covers of M, then $F_1 \cong F_2$.*

Proof Take two different $\mathcal{F}$-covers of M, $\phi_1 : F_1 \to M$ and $\phi_2 : F_2 \to M$, then we have two comodule homomorphisms $f_1 : F_1 \to F_2$ and $f_2 : F_2 \to F_1$ satisfying the following commutative diagrams:

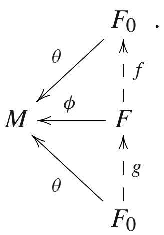




Thus, we have $\phi_2 = \phi_1 f_2$ and $\phi_1 = \phi_2 f_1$. So we obtain $\phi_2 = \phi_2 f_1 f_2$ and $\phi_1 = \phi_1 f_2 f_1$. By the premise, it follows that $f_2 f_1$ and $f_1 f_2$ are automorphisms. We infer that f_1, f_2 are both injective and surjective, that is, they are isomorphisms. Hence, $F_1 \cong F_2$. □

Proposition 3.4 *If M has an $\mathcal{F}$-cover and $\phi : F \to M$ is an $\mathcal{F}$-precover. Then there exist subcomodules F' and K of F such that $F = F' \oplus K$ and the restriction of ϕ over F' is an $\mathcal{F}$-cover of M, where $K \subset \mathrm{Ker}\phi$.*

Proof Take an $\mathcal{F}$-cover $\theta : F_0 \to M$ of M. Then we have the following commutative diagram:



Therefore, $\phi = \theta f$ and $\theta = \phi g$. So $\theta = \theta f g$. By the assumption, fg is an automorphism of F_0. Furthermore, we have $F = \mathrm{Img} \oplus \mathrm{Ker} f$. Hence, $K = \mathrm{Ker} f \subset \mathrm{Ker}\phi$, $F' = \mathrm{Img} \cong F_0$ and $F' \to M$ is an $\mathcal{F}$-cover of M. □

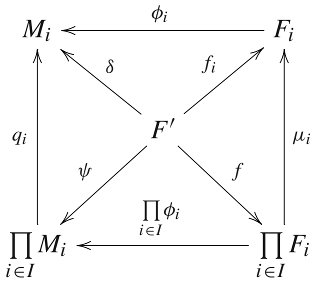
Corollary 3.5 *Assume that M has an $\mathcal{F}$-cover. Let $\phi : F \to M$ be an $\mathcal{F}$-precover. Then ϕ is a cover if and only if there does not exist direct sum decomposition $F = F' \oplus K$, and $K \neq 0$, $K \subset \mathrm{Ker}\phi$.*

Proof Necessity. Assume that $\phi : F \to M$ is an $\mathcal{F}$-cover. Take a decomposition $F = F' \oplus K$ and $K \neq 0$, $K \subset \mathrm{Ker}\phi$. Let $f : F \to F$ be the comodule homomorphism defined by $f(x + y) = x$, where $x \in F'$ and $y \in K$. It is easy to verify that $\phi = \phi f$ holds. Since $\phi : F \to M$ is an $\mathcal{F}$-cover, f is an automorphism. Hence, we get $K = 0$, which is a contradiction.

Conversely, it is immediate to conclude the sufficiency from Propositions 3.3 and 3.4. □

Lemma 3.6 *Assume that the comodule class $\mathcal{F}$ is closed under direct products. Let $\phi_i : F_i \to M_i$ be an $\mathcal{F}$-precover for every $i \in I$, where I is an index set. Then $\prod_{i\in I}\phi_i : \prod_{i\in I} F_i \to \prod_{i\in I} M_i$ is an $\mathcal{F}$-precover.*

Proof Since $\phi_i : F_i \to M_i$ is an $\mathcal{F}$-precover for every $i \in I$, where I is an index set, $\mathrm{Hom}_C(F', \phi_i)$: $\mathrm{Hom}_C(F', F_i) \to \mathrm{Hom}_C(F', M_i)$ is surjective for any $F' \in \mathcal{F}$. Take any homomorphism $\psi : F' \to \prod_{i\in I} M_i$. Let $q_i : \prod_{i\in I} M_i \to M_i$ be the canonical projection, then we obtain the following commutative diagram:



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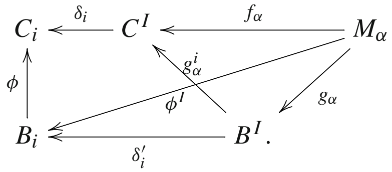
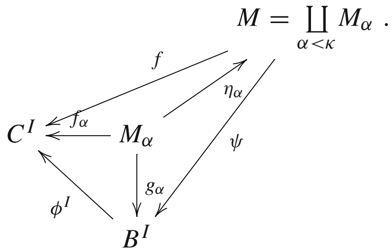
where μ_i is the canonical projection. Since $\phi_i : F_i \to M_i$ is an $\mathcal{F}$-precover, we have $q_i\psi = \phi_i f_i$. By the property of products, there is a unique homomorphism $f : F' \to \prod_{i\in I} F_i$ such that $\mu_i f = f_i$, i.e. $f|_{F_i} = f_i$. So we have $\psi = (\prod_{i\in I}\phi_i) f$. Thus, $\mathrm{Hom}_C(F', \prod_{i\in I}\phi_i)$: $\mathrm{Hom}_C(F', \prod_{i\in I} F_i) \to \mathrm{Hom}_C(F', \prod_{i\in I} M_i)$ is surjective. Hence, $\prod_{i\in I}\phi_i$ is an $\mathcal{F}$-precover. □

Lemma 3.7 *Assume that $\phi : B \to C$ is an $\mathcal{M}$-precover of C for a comodule class $\mathcal{M} \subseteq C$*-Comod. *Then* $\mathrm{Hom}_C(B, C)$ *is a cyclic* $\mathrm{End}_C(B)$*-module and* $\mathrm{Add}(\mathcal{M}) \subseteq \mathrm{Cogen}(B)$.

Proof Since $\phi : B \to C$ is an $\mathcal{M}$-precover of C, we have an epimorphism $\mathrm{Hom}_C(B, \phi) : \mathrm{End}_C(B) \to \mathrm{Hom}_C(B, C) \cong B$. The scalar product of $\mathrm{Hom}_C(B, C)$ to be a left $\mathrm{End}_C(B)$-module is given by $(\alpha f)(b) = f(\alpha(b))$, where $b \in B, \alpha \in \mathrm{End}_C(B)$ and $f \in \mathrm{Hom}_C(B, C)$. It is easy to know that $\mathrm{Hom}_C(B, C)$ is a cyclic $\mathrm{End}_C(B)$-module since $\mathrm{Hom}_C(B, \phi)$ is surjective. Let $M = \coprod_{\alpha<\kappa} M_\alpha$, where $\{M_\alpha | \alpha < \kappa\} \subseteq \mathcal{M}$. Then there is an injection map $\eta_\alpha : M_\alpha \hookrightarrow M = \coprod_{\alpha<\kappa} M_\alpha$. If we take a monomorphism $f : M = \coprod_{\alpha<\kappa} M_\alpha \to C^I$, then we have an $f_\alpha : M_\alpha \to C^I$ such that $f\eta_\alpha = f_\alpha$ by the property of coproducts. Take the ith projection $\delta_i : C^I \to C_i$, where $C_i = C$ and $\delta'_i : B^I \to B_i$, where $B_i = B$. Then we have $\delta_i\phi^I = \phi\delta'_i$, where $\phi^I : B^I \to C^I$ is induced by ϕ. Since $\phi : B \to C$ is an $\mathcal{M}$-precover, there is a $g^i_\alpha : M_\alpha \to B_i$ such that $\phi g^i_\alpha = \delta_i f_\alpha$. By the property of products, there exists a homomorphism $g_\alpha : M_\alpha \to B^I$ such that $\delta'_i g_\alpha = g^i_\alpha$. Thus, we have the following commutative diagram:



Furthermore, we have $\delta_i \phi^I g_\alpha = \delta_i f_\alpha$. By the property of products, we have the uniqueness of f_α, i.e. $f_\alpha = \phi^I g_\alpha$. It follows that there is a unique homomorphism $\psi : \coprod_{\alpha<\kappa} M_\alpha \to B^I$ such that $\psi\eta_\alpha = g_\alpha$ from the property of coproducts. Therefore, we obtain the following commutative diagram:



Thus, $\phi^I \psi \eta_\alpha = f_\alpha = f\eta_\alpha$. By the property of a coproduct, we have the uniqueness of f, that is, $\phi^I \psi = f$. Since f is a monomorphism, ψ is injective. Therefore, we have $M \in \mathrm{Cogen}(B)$, i.e. $\mathrm{Add}(\mathcal{M}) \subseteq \mathrm{Cogen}(B)$. □

4 Cofinendo Comodules

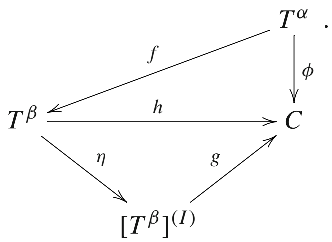
In this section, we will introduce the definition of cofinendo comodules, and research their properties.

Definition 4.1 Let C be a coalgebra. A C-comodule D is called cofinendo, if there exist a cardinal γ and a comodule homomorphism $f : D^\gamma \to C$ such that for every cardinal α, all comodule homomorphisms $D^\alpha \to C$ factorizes through f.

Lemma 4.2 *Let C be a coalgebra and T be a C-comodule, then the following are equivalent:*

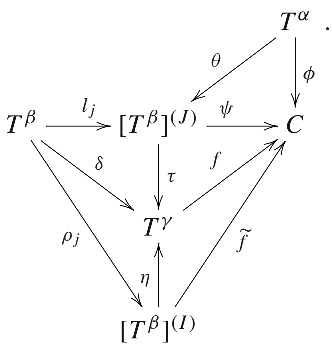
(1) *T is cofinendo;*
(2) *there exists a cardinal β such that for each cardinal α, all comodule homomorphisms $T^\alpha \to C$ factorize through some coproducts of copies of T^β.*

Proof (1) $\Rightarrow$ (2) Since T is cofinendo, there exists a β such that for each α, a comodule homomorphism $\phi : T^\alpha \to C$ factorizes through $h : T^\beta \to C$. Let $I = \mathrm{Hom}_C(T^\beta, C)$. Take $\eta : T^\beta \to [T^\beta]^{(I)}$ as the injection map of the hth copy of T^β and let $g : [T^\beta]^{(I)} \to C$ be the codiagonal map induced by all maps in I. Thus, we have the following commutative diagram:



Therefore, $\phi = g\eta f$, and we obtain (2).

(2) $\Rightarrow$ (1) For any cardinal α, by the assumption there exists a cardinal β such that each comodule homomorphism $\phi : T^\alpha \to C$ factorizes through $\psi : [T^\beta]^{(J)} \to C$, i.e. there is a $\theta : T^\alpha \to [T^\beta]^{(J)}$ such that $\phi = \psi\theta$. Take $I = \mathrm{Hom}_C(T^\beta, C)$, we denote by $\widetilde{f} : [T^\beta]^{(I)} \to C$ the codiagonal map induced by all comodule homomorphisms in I. Now, there are cardinal γ and comodule homomorphism $f : T^\gamma \to C$ such that there exists an embedding map $\eta : [T^\beta]^{(I)} \hookrightarrow T^\gamma$ and $f \mid_{[T^\beta]^{(I)}} = \widetilde{f}$. Let $l_j : T^\beta \hookrightarrow [T^\beta]^{(J)}$ be the embedding map of the jth copy, where $j \in J$, of T^β in $[T^\beta]^{(J)}$. Choose $\rho_j : T^\beta \to [T^\beta]^{(I)}$ as the injection map of the j]th copy, where $j \in I$, of T^β in $[T^\beta]^{(I)}$. Let $\delta = \eta\rho_j : T^\beta \to T^\gamma$ be the composition of η and ρ_j, then there is a unique $\tau : [T^\beta]^{(J)} \to T^\gamma$ by the property of coproducts such that $\delta = \tau l_j$. Therefore, we obtain the following commutative diagram:

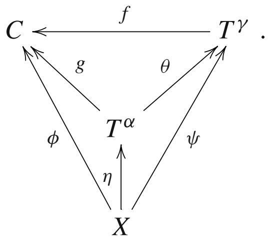


Thus, by computing we obtain $\psi l_j = \widetilde{f}\rho_j = f\eta\rho_j = f\tau l_j$. Hence, $\psi = f\tau$ by the property of a coproduct. Furthermore, we obtain $\phi = f\tau\theta$. Therefore, T is cofinendo. □

Proposition 4.3 *Let C be a coalgebra. Then the following three conditions are equivalent for a comodule T:*

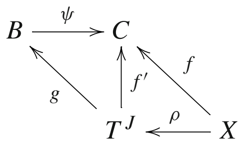
(1) *T is cofinendo;*
(2) *C has a* Prod*(T)-precover;*
(3) Cogen*(T) is a precover class.*

Proof (1) $\Rightarrow$ (2) For a morphism $\phi : X \to C$, where $X \in \text{Prod}(T)$, then there is a monomorphism $\eta : X \to T^\alpha$, where α is a cardinal. Furthermore, there is a $g : T^\alpha \to C$ such that $g\eta = \phi$ because C is injective. Since T is cofinendo, for any cardinal α and $g : T^\alpha \to C$, there exist a cardinal γ and a $f : T^\gamma \to C$ such that g factorizes through f. Thus, we have the following commutative diagram:

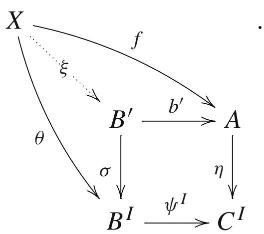


Hence, $f\theta\eta = \phi$. Therefore, we have $\psi = \theta\eta$ and hence we conclude that $f : T^\gamma \to C$ is a Prod(T)-precover.

(2) $\Rightarrow$ (3) Take a Prod(T)-precover $\psi : B \to C$. First, we show that ψ is also a Cogen(T)-precover. Assume that $f : X \to C$ is any comodule homomorphism, where $X \in \text{Cogen}(T)$. Then there is a monomorphism $\rho : X \to T^J$ for some cardinal J. By the injectivity of C in C-Comod there exists an $f' : T^J \to C$ such that $f = f'\rho$. Hence, there is a $g : T^J \to B$ such that $f' = \psi g$ because $\psi : B \to C$ is a Prod(T)-precover. Therefore, we have the following commutative diagram:



So $\psi g\rho = f'\rho = f$. Hence, ψ is also a Cogen(T)-precover. Let A be any C-comodule. There exists a monomorphism $\eta : A \to C^I$ since C is an cogenerator. Then we obtain the following pullback diagram:



By the monomorphism η, we infer that σ is a monomorphism. Furthermore, we know $B' \in \text{Cogen}(T)$. It follows from Lemma 3.6 that ψ^I is a Cogen(T)-precover. Therefore, for an $f : X \to A$, where $X \in \text{Cogen}(T)$, there exists a $\theta : X \to B^I$ such that $\eta f = \psi^I\theta$. By the property of pullbacks, there is a unique $\xi : X \to B'$ such that $f = b'\xi$. Thus, b' is a Cogen(T)-precover. Furthermore, Cogen(T) is a precover class.

(3) ⇒ (1) Let $\phi : X \to C$ be a Cogen(T)-precover of C, where $X \in$ Cogen(T). Then $\mathrm{Hom}_C(T^\alpha, \phi) : \mathrm{Hom}_C(T^\alpha, X) \to \mathrm{Hom}_C(T^\alpha, C)$ is surjective. Thus, for any $g : T^\alpha \to C$, there is an $h : T^\alpha \to X$ such that $\phi h = g$. Since $X \in$ Cogen(T), we have a monomorphism $\eta : X \to T^\gamma$, where γ is a cardinal. Since C is an injective cogenerator, there is an $f : T^\gamma \to C$ such that $f\eta = \phi$. Thus, $g = f\eta h$. Therefore, T is cofinendo. □

Corollary 4.4 *Let $\mathcal{F} \subseteq C$-Comod be a pretorsion-free class. Then the following are equivalent:*

(1) *$\mathcal{F}$ is a precover class;*
(2) *C has an $\mathcal{F}$-precover;*
(3) *$\mathcal{F} =$ Cogen(T) for a cofinendo comodule T.*

Proof (1) ⇒ (2) It is clear.

(2) ⇒ (3) Let $\phi : T \to C$ be an $\mathcal{F}$-precover of C. Then we have $\mathcal{F} \subseteq \mathrm{Add}(\mathcal{F}) \subseteq$ Cogen(T) by Lemma 3.7. Since $\mathcal{F} \subseteq C$-Comod is a pretorsion-free class, we have Cogen$(T) \subseteq \mathcal{F}$. Thus, $\mathcal{F} =$ Cogen(T). Since C has an $\mathcal{F}$-precover, $\mathrm{Hom}_C(T^\alpha, \phi)$ is surjective for any cardinal α and $\phi : X \to C$, where $X \in$ Cogen(T). That is, for any $\psi : T^\alpha \to C$, there is a $\delta : T^\alpha \to X$ such that $\phi\delta = \psi$. From $T \in$ Cogen(T) it follows that $\eta : X \to T^\gamma$ is injective for some cardinal γ. Thus, there is a $\theta : T^\gamma \to C$ such that $\theta\eta = \phi$ by the injectivity of C in C-Comod. So for any cardinal α, and $\psi : T^\alpha \to C$ there exist a cardinal γ, and $\theta : T^\gamma \to C$ such that ψ factorizes through θ. Hence, T is a cofinendo comodule, and $\mathcal{F} =$ Cogen(T).

(3) ⇒ (1) It is easily obtained by the proof of (1) ⇒ (3) in Proposition 4.3. □

5 Main Results

In this section, we introduce the concept of a $\mathcal{D}$-injective comodule, where $\mathcal{D}$ is a class of comodules in C-Comod. Moreover, we obtain some results for an arbitrary tilting torsion-free class.

Let $\mathcal{D}$ be a class of comodules in C-Comod. For a comodule $M \in C$-Comod, we call M $\mathcal{D}$-injective if $\mathrm{Hom}_C(-, M)$ is exact on the short exact sequences of this form: $0 \to X \to U \to V \to 0$, where $X, U, V \in \mathcal{D}$.

Definition 5.1 [30] T is called a tilting comodule if T satisfies the following three conditions:

(1) inj.dim$(T) \leq 1$;
(2) $\mathrm{Ext}^1_C(T^X, T) = 0$ for any cardinal X;
(3) there exists an exact sequence $0 \to T_2 \to T_1 \to C \to 0$, where $T_i \in$ Prod T.

***Remark* 5.2** $\mathcal{D}$ is said to be a tilting torsion-free class provided that $\mathcal{D} =$ Cogen(M) for a tilting comodule M.

Definition 5.3 Assume that C is a coalgebra and C-Comod has a projective generator W, then M is said to be faithful, if a projective generator W is cogenerated by a comodule M.

Lemma 5.4 *Let T be a C-comodule. If* $\mathrm{Cogen}(T) = {}^{\perp}T$*, then* $\mathrm{Cogen}(T) = \mathrm{Copres}(T)$.

Proof Assume that $M \in \mathrm{Cogen}(T)$ and $X = \mathrm{Hom}_C(M, T)$. Let $\eta : M \to T^X$ be the diagonal morphism $\eta(m) = (x(m))_{x \in X}$. We know that η is injective because M is cogenerated by T. Let $D = \mathrm{Coker}(\eta)$, then there is the following exact sequence:

$$0 \to M \xrightarrow{\eta} T^X \to D = \mathrm{Coker}(\eta) \to 0. \tag{5.1}$$

Applying $\mathrm{Hom}_C(-, T)$ to (5.1), we obtain the following long exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}_C(D, T) \to \mathrm{Hom}_C(T^X, T) \xrightarrow{\eta^*} \mathrm{Hom}_C(M, T) \to \\ \mathrm{Ext}^1_C(D, T) \to \mathrm{Ext}^1_C(T^X, T) = 0. \end{aligned}$$

By construction, we know that η^* is surjective. Hence, $\mathrm{Ext}^1_C(D, T) = 0$, that is, $D \in \mathrm{Cogen}(T)$. □

Proposition 5.5 *Let C be a coalgebra,* ${}^{\perp}T = \mathrm{Cogen}(T)$*, and W be a projective generator in C*-Comod*. If M is an injective comodule, then there is an short exact sequence* $0 \to T_1 \to T_0 \to M \to 0$*, where* $T_0, T_1 \in \mathrm{Prod}(T)$.

Proof Take an epimorphism $\theta : W^{(\alpha)} \to M$, where α is a cardinal, and W is a projective generator. Since $W^{(\alpha)} \in {}^{\perp}T = \mathrm{Cogen}(T)$, there is a monomorphism $\phi : W^{(\alpha)} \to T^{\beta}$, where β is some cardinal. Thus, there is an $f : T^{\beta} \to M$ such that $f\phi = \theta$ since M is injective. Furthermore, f is an epimorphism. Let $K = \mathrm{Ker} f$. We obtain the short exact sequence $0 \to K \to T^{\beta} \xrightarrow{f} M \to 0$. By Lemma 5.4, there exists an exact sequence $0 \to K \to T^{\gamma} \to L \to 0$, where $L \in \mathrm{Cogen}(T)$. Then we have the pushout diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K & \longrightarrow & T^{\beta} & \xrightarrow{f} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & T^{\gamma} & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & L & = & L & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} .$$

Hence, we obtain the following short exact sequence:

$$0 \to T^{\gamma} \to E \to M \to 0. \tag{5.2}$$

Note that $E \in {}^{\perp}T = \mathrm{Cogen}(T)$ because L and T^{β} are in ${}^{\perp}T$. From Proposition 5.4 again, there exists an exact sequence

$$0 \to E \to T^{\delta} \to Y \to 0, \tag{5.3}$$

where $Y \in \mathrm{Cogen}(T) = {}^{\perp}T$. Applying $\mathrm{Hom}_C(Y, -)$ to (5.2), we obtain the following long exact sequence:

$$0 \to \mathrm{Hom}_C(Y, T^{\gamma}) \to \mathrm{Hom}_C(Y, E) \to \mathrm{Hom}_C(Y, M) \to$$
$$\mathrm{Ext}^1_C(Y, T^{\gamma}) \to \mathrm{Ext}^1_C(Y, E) \to \mathrm{Ext}^1_C(Y, M) \to \cdots .$$

Since $\mathrm{Ext}^1_C(Y, T^{\gamma}) \cong \mathrm{Ext}^1_C(Y, T)^{\gamma} = 0$ and $\mathrm{Ext}^1_C(Y, M) = 0$, we have $\mathrm{Ext}^1_C(Y, E) = 0$, that is, the exact sequence (5.3) is split. Hence, $E \in \mathrm{Prod}(T)$. □

Now, we prove that the following theorem which can be viewed as the version of Assem–Smalϕ theorem in the category of comodules.

Theorem 5.6 *Let C be a semiperfect coalgebra and $\mathcal{F} \subseteq C$-Comod be a class of comodules. Then $\mathcal{F}$ is a tilting torsion-free class if and only if $\mathcal{F} = \mathrm{Cogen}(T)$, where T is a faithful, cofinendo and $\mathcal{F}$-injective comodule.*

Proof Necessity. Since $\mathcal{F}$ is a tilting torsion-free class, we have $\mathcal{F} = \mathrm{Cogen}(T) = {}^{\perp}T$, where T is a tilting comodule. Since $W \in {}^{\perp}T = \mathrm{Cogen}(T)$, T is faithful. Take an exact sequence

$$0 \to X \to Y \to N \to 0, \tag{5.4}$$

where $X, Y, N \in \mathcal{F}$. Applying $\mathrm{Hom}_C(-, T)$ to (5.4), we obtain the following exact sequence:

$$0 \to \mathrm{Hom}_C(N, T) \to \mathrm{Hom}_C(Y, T) \to \mathrm{Hom}_C(X, T) \to \mathrm{Ext}^1_C(N, T),$$

where we have $\mathrm{Ext}^1_C(N, T) = 0$ because $N \in \mathcal{F} = {}^{\perp}T$. So T is $\mathcal{F}$-injective. As T is a tilting comodule, there is the following short exact sequence:

$$0 \to T_1 \to T_0 \xrightarrow{\delta} C \to 0, \tag{5.5}$$

where $T_0, T_1 \in \mathrm{Prod}(T)$. Applying $\mathrm{Hom}_C(L, -)$ to (5.5), we derive the following long exact sequence:

$$0 \to \mathrm{Hom}_C(L, T_1) \to \mathrm{Hom}_C(L, T_0) \xrightarrow{\delta^*} \mathrm{Hom}_C(L, C) \to \mathrm{Ext}^1_C(L, T_1).$$

We have $\mathrm{Ext}^1_C(L, T_1) = 0$ for all $L \in \mathrm{Prod}(T)$. Furthermore, δ^* is surjective. Thus, the comodule homomorphism $\delta : T_0 \to C$ is a Prod(T)-precover of C. By Proposition 4.3, T is cofinendo.

Sufficiency. Let $\mathrm{Cogen}(T) = \mathcal{F}$ with T faithful, cofinendo and $\mathcal{F}$-injective. From Proposition 4.3, we obtain an $\mathcal{F}$-precover $\theta : T^\gamma \to C$ of an injective cogenerator C. Since T is faithful, $\mathcal{F}$ contains all projective comodules. So there is a projective comodule X such that $g : X \to C$ is surjective. Furthermore, there exists an $f : X \to T^\gamma$ such that $\theta f = g$ by the property of precovers. So θ is surjective, and there is the following exact sequence:

$$0 \to K \to T^\gamma \xrightarrow{\theta} C \to 0. \tag{5.6}$$

Assume that $M \in \mathcal{F}$, then we take any exact sequence

$$0 \to F' \to F \to M \to 0 \tag{5.7}$$

in $\mathcal{F}$, where F is a projective comodule. Applying $\mathrm{Hom}_C(-, T)$ to (5.7), we derive the following long exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}_C(M, T) \to \mathrm{Hom}_C(F, T) \to \mathrm{Hom}_C(F', T) \to \\ \mathrm{Ext}^1_C(M, T) \to \mathrm{Ext}^1_C(F, T) \to \mathrm{Ext}^1_C(F', T) \to \cdots . \end{aligned}$$

As F is projective, we have $\mathrm{Ext}^1_C(F, T) = 0$. Since T is $\mathcal{F}$-injective, we get $M \in {}^\perp T$. Hence, $\mathrm{Cogen}(K) \subseteq \mathcal{F} \subseteq {}^\perp T$. Now, we prove $\mathcal{F} = {}^\perp K$. Let M be a C-comodule. Applying $\mathrm{Hom}_C(M, -)$ to (5.6), we get the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_C(M, K) \to \mathrm{Hom}_C(M, T^\gamma) \xrightarrow{\theta^*} \mathrm{Hom}_C(M, C) \to \\ \mathrm{Ext}^1_C(M, K) \to \mathrm{Ext}^1_C(M, T^\gamma) \to \mathrm{Ext}^1_C(M, C) \to \cdots . \end{aligned} \tag{5.8}$$

If $M \in \mathcal{F}$, we know that θ^* is surjective by the property of precovers. Since $\mathcal{F} \subseteq {}^\perp T$, we obtain $\mathrm{Ext}^1_C(M, T^\gamma) = 0$. Furthermore, $\mathrm{Ext}^1_C(M, K) = 0$. Thus, $M \in {}^\perp K$. Conversely, if $M \in {}^\perp K$, then θ^* is surjective. Since C is an injective cogenerator, we have $\bigcap\{\mathrm{Ker}\phi \mid \phi \in \mathrm{Hom}_C(M, C)\} = 0$. Thus, we obtain the following commutative diagram:

$$\begin{array}{ccc} & & M \\ & \swarrow^{\psi_\phi} & \downarrow{\scriptstyle\phi} \\ T^\gamma & \xrightarrow{\theta} & C \end{array} .$$

So $\mathrm{Ker}\psi_\phi \subseteq \mathrm{Ker}\phi$. Furthermore, $\underset{\phi}{\cap}\mathrm{Ker}\psi_\phi \subseteq \underset{\phi}{\cap}\mathrm{Ker}\phi = 0$. Since $\bigcap\{\mathrm{Ker}\psi \mid \psi \in \mathrm{Hom}_C(M, T^\gamma\}) \subseteq \bigcap_\phi\{\mathrm{Ker}\psi_\phi \mid \psi_\phi \in \mathrm{Hom}_C(M, T^\gamma\}) = 0$, we have $M \in \mathcal{F}$. So

$\mathcal{F} = {}^{\perp}K$, and

$$\mathrm{Cogen}(T \oplus K) = \mathrm{Cogen}(T) = \mathcal{F} = {}^{\perp}K = {}^{\perp}(T \oplus K).$$

Hence, $T \oplus K$ is a tilting comodule cogenerating $\mathcal{F}$, and $\mathcal{F}$ is a tilting torsion-free class. □

Theorem 5.7 *Let C be a semiperfect coalgebra, and $\mathcal{T} \subseteq C$-*Comod *be a pretorsion-free class. Then the following are equivalent:*

(1) *$\mathcal{T}$ is a tilting torsion-free class;*
(2) *there is a special $\mathcal{T}$-precover of C;*
(3) *there is a $\mathcal{T}$-precover of C, $\psi : Q \to C$ such that Q is faithful, $\mathcal{T}$-injective, and ψ is surjective.*

Proof (1) $\Rightarrow$ (2) Take a tilting torsion-free class $\mathcal{T}$, then $\mathcal{T} = \mathrm{Cogen}(T) = {}^{\perp}T$ for a tilting comodule T. For a tilting comodule T, there exists the following exact sequence:

$$0 \to T_2 \to T_1 \xrightarrow{\tau} C \to 0, \tag{5.9}$$

where $T_i \in \mathrm{Prod}(T)$. Applying $\mathrm{Hom}_C(N, -)$ to (5.9), we get the exact sequence

$$0 \to \mathrm{Hom}_C(N, T_2) \to \mathrm{Hom}_C(N, T_1) \xrightarrow{\tau^*} \mathrm{Hom}_C(N, C) \to \mathrm{Ext}^1_C(N, T_2).$$

We obtain that $\tau^* : \mathrm{Hom}_C(N, T_1) \to \mathrm{Hom}_C(N, C)$ is surjective because Ext^1_C $(N, T_2) = 0$ for every $N \in \mathcal{T}$. Moreover, $\mathrm{Ext}^1_C(\mathcal{T}, T_2) = 0$, that is, $T_2 \in \mathcal{T}^{\perp}$. Hence, $\tau : T_1 \to C$ is a special $\mathcal{T}$-precover of C.

(2) $\Rightarrow$ (3) Take a special $\mathcal{T}$-precover $\psi : Q \to C$ of C, then ψ is surjective. For a projective generator W, there is a cardinal l such that $\eta : W \to C^l$ is an embedding map. Since ψ is surjective, the induced comodule homomorphism $\psi^l : Q^l \to C^l$ is surjective. By the projectivity of W, there exists a $\theta : W \to Q^l$ such that $\psi^l\theta = \eta$. Therefore, W can be embedded into Q^l, and Q is faithful. Since ψ is surjective, there exists an exact sequence

$$0 \to K \to Q \to C \to 0, \tag{5.10}$$

where $Q \in \mathcal{T}$, and K, C are in $\mathcal{T}^{\perp}$. Applying $\mathrm{Hom}_C(\mathcal{T}, -)$ to (5.10), we obtain the following long exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}_C(\mathcal{T}, K) \to \mathrm{Hom}_C(\mathcal{T}, Q) \to \mathrm{Hom}_C(\mathcal{T}, C) \to \\ \mathrm{Ext}^1_C(\mathcal{T}, K) \to \mathrm{Ext}^1_C(\mathcal{T}, Q) \to \mathrm{Ext}^1_C(\mathcal{T}, C) \to \cdots. \end{aligned}$$

Since $\mathrm{Ext}^1_C(\mathcal{T}, K) = 0$ and $\mathrm{Ext}^1_C(\mathcal{T}, C) = 0$, we get $Q \in \mathcal{T}^{\perp}$. Take an exact sequence

$$0 \to X \xrightarrow{\theta} Y \to G \to 0, \tag{5.11}$$

where θ is injective and $G = \mathrm{Coker}\theta \in \mathcal{T}$. Applying $\mathrm{Hom}_C(-, Q)$ to (5.11), we derive the long exact sequence

$$0 \to \mathrm{Hom}_C(G, Q) \to \mathrm{Hom}_C(Y, Q) \to \mathrm{Hom}_C(X, Q) \to \mathrm{Ext}^1_C(G, Q) \to \mathrm{Ext}^1_C(Y, Q) \to \mathrm{Ext}^1_C(X, Q) \to \cdots .$$

Since $Q \in \mathcal{T}^{\perp}$, we get $\mathrm{Ext}^1_C(G, Q) = 0$. Therefore, $\mathrm{Hom}_C(-, Q)$ is exact on any monomorphism whose cokernel is in $\mathcal{T}$. If X, Y, G are all in $\mathcal{T}$, then Q is $\mathcal{T}$-injective.

(3) $\Rightarrow$ (1) Assume that $\pi : Q \to C$ satisfies the conditions of (3). By (2) $\Rightarrow$ (3) of Corollary 4.4, we obtain that $\mathcal{T} = \mathrm{Cogen}(Q)$, and Q is cofinendo. It follows from Theorem 5.6 that $\mathcal{T}$ is a tilting torsion-free class. □

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