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ORIGINAL PAPER

A Fixed Point Approach to Stability of k-th Radical Functional Equation in Non-Archimedean (n, β)-Banach Spaces

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Abstract

In this work, we prove a simple fixed point theorem in non-Archimedean (n, β)-Banach spaces, by applying this fixed point theorem, we will study the stability and the hyperstability of the kth radical-type functional equation:

$$f\left(\sqrt[k]{x^k + y^k}\right) = f(x) + f(y),$$

where f is a mapping on the set of real numbers and k is a fixed positive integer. Furthermore, we give some important consequences from our main results.

Keywords Fixed point theorem · Functional equation · Stability · Non-Archimedean (n, β)-normed spaces

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1 Introduction

In 1940, during a conference at Wisconsin University, Ulam [35] presented the following question concerning stability of group homomorphisms: *Let* $(G_1, \star)$, $(G_2, *)$ *be two groups and* $d : G_2 \times G_2 \to [0, \infty)$ *be a metric. Given* $\epsilon > 0$, *does there exist* $\delta > 0$, *such that if a function* $g : G_1 \to G_2$ *satisfies the inequality* $d(g(x \star y), g(x) * g(y)) \leq \delta$ *for all* $x, y \in G_1$, *then there is a homomorphism* $h : G_1 \to G_2$ *with* $d(g(x), h(x)) \leq \epsilon$ *for all* $x \in G_1$?

When the homomorphisms are stable? Therefore, we are interested in this question, that is, if a mapping is almost a homomorphism, then there exists an exact homomorphism that must be close. In following year, Hyers [24] was the first to give an affirmative answer to Ulam's question for the case where G_1 and G_2 are Banach spaces. The famous Hyers stability result that appeared in [24] was generalized in the stability involving a sum of powers of norms by Aoki [3]. In 1978, Rassias [32] provided a generalization of Hyers' theorem that allows the Cauchy difference to become unbounded. For the last decades, stability problems of various functional equations have been extensively investigated and generalized by many mathematicians [6,11,13,15,22,30,33,34,37]. The theory of 2-normed spaces was first developed by Gähler [20] in the mid-1960s, while that of 2-Banach spaces was studied later by Gähler [21] and White [36]. In 1897, Hensel [23] introduced a normed space which does not have the Archimedean property. It turns out that non-Archimedean spaces have many nice applications (see [4,25,28,31]).

The first hyperstability result appears to be due to Bourgin [5]. However, the term hyperstability was used for the first time in [29]. Quite often, hyperstability is confused with superstability, which admits also bounded functions. Numerous papers on this subject have been published and we refer to [7,12,14,16,18,19]. Recently, the stability problem and hyperstability results for the functional equations of the radical type in 2-Banach spaces and in some other generalized spaces have been also studied; for example, see [1,2,8,10,16–18,26,27].

The functional equation:

$$f\left(\sqrt{x^2 + y^2}\right) = f(x) + f(y) \tag{1.1}$$

is called a radical quadratic functional equation. Kim et al. [27] investigated the generalized Hyers–Ulam–Rassias stability of Eq. (1.1) in quasi-β-Banach spaces using the direct method.

In the whole paper, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all positive integers and real numbers, respectively; we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_0 := \mathbb{R} \backslash \{0\}$, and $\mathbb{R}_+ = [0, \infty)$, and we write $\mathcal{B}^{\mathcal{A}}$ to mean the family of all functions mapping from a nonempty set $\mathcal{A}$ into a nonempty set $\mathcal{B}$.

This work is organized as follows: in Sect. 2, we discuss some basic definitions and lemmas used in later sections to prove the stabilities on non-Archimedean (n, β)-Banach spaces. In Sect. 3, we introduce and solve the kth radical-type functional equation:

$$f\left(\sqrt[k]{x^k + y^k}\right) = f(x) + f(y), \quad x, y \in \mathbb{R}, \tag{1.2}$$

in the class of functions f from $\mathbb{R}$ into a vector space, where $k \in \mathbb{N}$ is fixed. In Sect. 4, we prove the fixed point theorem [9, Theorem 1] in non-Archimedean (n, β)-Banach space. In Sect. 5, we will apply the fixed point method to study the stability and the hyperstability of (1.2) in non-Archimedean (n, β)-Banach space. In Sect. 6, we will give some consequences from our main results. Our results are improvements and generalizations of many main results referred to in [1,2,16–18,26] on non-Archimedean (n, β)-Banach spaces.

2 Preliminaries

In this section, we will introduce some basic concepts concerning the non-Archimedean (n, β)-normed space.

Definition 2.1 By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|_* : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(1) $|r|_* = 0$ if and only if $r = 0$;
(2) $|rs|_* = |r|_*|s|_*$;
(3) $|r + s|_* \leq \max(|r|_*, |s|_*)$ for all $r, s \in \mathbb{K}$.

Clearly, $|1|_* = |-1|_* = 1$ and $|n|_* \leq 1$ for all $n \in \mathbb{N}$. The function $|\cdot|_*$ is called the trivial valuation if $|r|_* = 1, \forall r \in \mathbb{K}, r \neq 0$, and $|0|_* = 0$.

Definition 2.2 Let E be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|_*$. A function $\|\cdot\|_* : E \to \mathbb{R}_+$ is non-Archimedean norm (valuation) if it satisfies the following conditions:

(1) $\|x\|_* = 0$ if and only if $x = 0$;
(2) $\|rx\|_* = |r|_*\|x\|_*$ for all $r \in \mathbb{K}$ and $x \in E$;
(3) $\|x + y\|_* \leq \max(\|x\|_*, \|y\|_*)$ for all $x, y \in E$.

Then, $(E, \|\cdot\|_*)$ is called a non-Archimedean space or an ultrametric normed space. Due to the fact that:

$$\|x_m - x_n\|_* \leq \max\{\|x_{j+1} - x_j\|_* : m \leq j \leq n - 1\},$$

in which $n > m$, the sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. In a complete non-Archimedean space, every Cauchy sequence is convergent.

Example 2.3 Fix a prime number p. For any nonzero rational number x, there exists a unique positive integer n_x, such that $x = \frac{a}{b} p^{n_x}$, where a and b are positive integers not divisible by p. Then, $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$ (the set of rational numbers). The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$

is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $\left| \sum_{k \geq n_x}^{\infty} a_k p^k \right| = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and $\mathbb{Q}_p$ is a locally compact field.

Definition 2.4 Let X be a real vector space with $dim\ X \geq n$ over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|_*$, where $n \in \mathbb{N}$ and $\beta \in (0, 1]$ is a fixed number. A function $\|\cdot, \dots, \cdot\|_{*,\beta} : X^n \to \mathbb{R}_+$ is called a non-Archimedean (n, β)-norm on X if and only if it satisfies:

(N1) $\|x_1, x_2, \dots, x_n\|_{*,\beta} = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent;
(N2) $\|x_1, x_2, \dots, x_n\|_{*,\beta}$ is invariant under permutations of $x_1, x_2, \dots, x_n$;
(N3) $\|\lambda x_1, x_2, \dots, x_n\|_{*\beta} = |\lambda|_*^{\beta} \|x_1, x_2, \dots, x_n\|_{\beta}$;
(N4) $\|x + y, x_2, \dots, x_n\|_{*,\beta} \leq \max \left\{ \|x, x_2, \dots, x_n\|_{*,\beta}, \|y, , x_2, \dots, x_n\|_{*\beta} \right\}$

for all $x, y, x_1, x_2, \dots, x_n \in X$ and $\lambda \in \mathbb{K}$. Then, the pair $(X, \|\cdot, \dots, \cdot\|_{*,\beta})$ is called a *non-Archimedean* (n, β)*-normed space*.

Example 2.5 Let $\mathbb{K}$ be a non-Archimedean field equipped with a non-trivial valuation $|\cdot|_*$. For $n = 2$, $\lambda \in \mathbb{K}$ and $x = (x_1, x_2),\ \ y = (y_1, y_2) \in X = \mathbb{K}^2$ with $x + y = (x_1 + y_1, x_2 + y_2)$ and $\lambda x = (\lambda x_1, \lambda x_2)$, the non-Archimedean $(2, \beta)$-norm on X is defined by:

$$\|x, y\|_{*,\beta} = |x_1 y_2 - x_2 y_1|_*^{\beta},$$

where $\beta \in (0, 1]$ is a fixed number.

It follows from the preceding definition that the non-Archimedean (n, β)-normed space is a non-Archimedean n-normed space if $\beta = 1$, and a non-Archimedean β-normed space if $n = 1$, respectively.

Lemma 2.6 *Let* $(X, \|\cdot, \dots, \cdot\|_{*,\beta})$ *be a non-Archimedean* (n, β)*-normed space, such that* $n \geq 2$ *and* $0 < \beta \leq 1$*. Then:*

(1) *if* $x \in X$ *and* $\|x, x_2, \dots, x_n\|_{*,\beta} = 0$ *for all* $x_2, \dots, x_n \in X$*, then* $x = 0$*;*
(2) *a sequence* $\{x_m\}$ *in a non-Archimedean* (n, β)*-normed space* X *is a Cauchy sequence if and only if* $\{x_{m+1} - x_m\}$ *converges to zero in* X*.*

Proof For (1), suppose that $x \neq 0$. Since dim $X \geq n$, choose $x_2, \dots, x_n \in X$, such that $\{x, x_2, \dots, x_n\}$ is linearly independent and so by (N1) in Definition 2.4, we have:

$$\|x, x_2, \dots, x_n\|_{*,\beta} \neq 0.$$

This is a contradiction and thus x should be a zero vector. For (2), it follows from (N4) that:

$$\|x_m - x_k, x_2, \dots, x_n\|_{*,\beta} \leq \max \left\{ \|x_{j+1} - x_j, x_2, \dots, x_n\|_{*,\beta} : k \leq j \leq m - 1 \right\}, \quad (m > k)$$

for all $x_2, \dots, x_n \in X$. Therefore, a sequence $\{x_m\}$ is a Cauchy sequence in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in X. □

Definition 2.7 (a) A sequence $\{x_m\}$ in a non-Archimedean (n, β)-normed space X is called a *convergent sequence* if there exists an element $x \in X$, such that $\lim_{m\to\infty} \|x_m - x, x_2, \ldots, x_n\|_{*,\beta} = 0$ for all $x_2, \ldots, x_n \in X$. In this case, we write $\lim_{m\to\infty} x_m := x$, and we have

$$\lim_{m\to\infty} \|x_m, x_2, \ldots, x_n\|_{*,\beta} = \|\lim_{m\to\infty} x_m, x_2, \ldots, x_n\|_{*,\beta}$$

for all $x_2, \ldots, x_n \in X$.

(b) A non-Archimedean (n, β)-normed space in which every Cauchy sequence is a convergent sequence is called a non-Archimedean (n, β)-*Banach space*.

3 Solution of Eq. (1.2)

In this section, we give the general solution of functional equation (1.2).

Theorem 3.1 *Let $\mathcal{Y}$ be a linear space. A function $f : \mathbb{R} \to \mathcal{Y}$ satisfies Eq. (1.2) if and only if there exists an additive function $T : \mathbb{R} \to \mathcal{Y}$, such that:*

$$f(x) = T(x^k), \quad x \in \mathbb{R}, \tag{3.1}$$

for each fixed $k \in \mathbb{N}$.

Proof (See [8, page 127]). □

Remark 3.2 (i) The function $f(x) = cx^k$ satisfies Eq. (1.2) for all $x \in \mathbb{R}$, where $k \in \mathbb{N}$ and $c \in \mathbb{R}$ are fixed numbers.

(ii) If f satisfies Eq. (1.2), then $f(r^{p/k}x) = r^p f(x)$ for all $x \in \mathbb{R}$ and integers p, where $r \in \mathbb{Q}\backslash\{0\}$ ($\mathbb{Q}$:=, the set rational numbers) if k is odd and $r \in \mathbb{Q}^+$ ($\mathbb{Q}^+$:= the set of positive rational numbers) if k is even.

(iii) If f satisfies Eq. (1.2) and continuous, then $f(x) = x^k f(1)$ for all $x \in \mathbb{R}$ if k is odd and $f(x) = x^k f(1)$ for all $x \in \mathbb{R}_+$ if k is even.

4 Fixed Point Theorem

In this section, we rewrite the fixed point theorem [9, Theorem 1] in non-Archimedean (n, β)-Banach space. For it, we need to introduce the following hypotheses.

(H1) W is a nonempty set and X is a non-Archimedean (n, β)-Banach space.

(H2) $f_1, \ldots, f_j : W \to W$ and $K_1, \ldots, K_j : W \times X^{n-1} \to \mathbb{R}_+$ are given maps.

(H3) $\Lambda : {\mathbb{R}_+}^{W\times X^{n-1}} \to \mathbb{R}_+^{W\times X^{n-1}}$ is a non-decreasing operator defined by:

$$(\Lambda\delta)(x, x_2, \ldots, x_n) := \max_{1\le i\le j} K_i(x, x_2, \ldots, x_n)\delta(f_i(x), x_2, \ldots, x_n)$$

for all $\delta \in \mathbb{R}_+^{W\times X^{n-1}}$, $(x, x_2, \ldots, x_n) \in W \times X^{n-1}$.

(H4) $\mathcal{T} : X^W \to X^W$ is an operator satisfying the inequality:

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), x_2, \ldots, x_n\|_{*,\beta} \le \max_{1\le i\le j} K_i(x, x_2, \ldots, x_n) \, \|\xi(f_i(x)) \\ -\mu(f_i(x)), x_2, \ldots, x_n\|_{*,\beta}$$

for all $\xi, \mu \in X^W$ and $(x, x_2, \ldots, x_n) \in W \times X^{n-1}$.

The basic tool in this paper is the following fixed point theorem.

Theorem 4.1 *Assume that hypotheses **(H1)–(H4)** are satisfied. Suppose that there are functions* $\varepsilon : W \times X^{n-1} \to \mathbb{R}_+$ *and* $\varphi : W \to X$*, such that:*

$$\left\|\mathcal{T}\varphi(x) - \varphi(x), x_2, \ldots, x_n\right\|_{*,\beta} \le \varepsilon(x, x_2, \ldots, x_n), \quad (x, x_2, \ldots, x_n) \in W \times X^{n-1}, \tag{4.1}$$

and

$$\lim_{m\to\infty} \Lambda^m \varepsilon(x, x_2, \ldots, x_n) = 0, \quad (x, x_2, \ldots, x_n) \in W \times X^{n-1}, \tag{4.2}$$

then, for every $x \in W$*, the limit:*

$$\psi(x) := \lim_{m\to\infty} (\mathcal{T}^m \varphi)(x)$$

exists and the function $\psi \in X^W$*, defined in this way, is a fixed point of* $\mathcal{T}$ *with:*

$$\|\varphi(x) - \psi(x), x_2, \ldots, x_n\|_{*,\beta} \le \sup_{m\in\mathbb{N}_0} (\Lambda^m \varepsilon)(x, x_2, \ldots, x_n) \tag{4.3}$$

for all $(x, x_2, \ldots, x_n) \in W \times X^{n-1}$*. Moreover, if*

$$\Lambda\big(\sup_{m\in\mathbb{N}_0} (\Lambda^m \varepsilon)\big)(x, x_2, \ldots, x_n) \le \sup_{m\in\mathbb{N}_0} (\Lambda^{m+1} \varepsilon)(x, x_2, \ldots, x_n)$$

for all $(x, x_2, \ldots, x_n) \in W \times X^{n-1}$*, then* ψ *is the unique fixed point of* $\mathcal{T}$ *satisfying* (4.3).

Proof First, we show by induction that, for any $m \in \mathbb{N}_0$:

$$\left\|(\mathcal{T}^{m+1}\varphi)(x) - (\mathcal{T}^m\varphi)(x), x_2, \ldots, x_n\right\|_{*,\beta} \le (\Lambda^m \varepsilon)(x, x_2, \ldots, x_n), \\ (x, x_2, \ldots, x_n) \in W \times X^{n-1}. \tag{4.4}$$

Clearly, by (4.1), the case $m = 0$ is trivial. Now, fix $m \in \mathbb{N}_0$ and suppose that (4.4) is valid. Then, using **(H3)** and **(H4)**, for any $(x, x_2, \ldots, x_n) \in W \times X^{n-1}$, we obtain:

$$\begin{aligned}&\left\|(\mathcal{T}^{m+2}\varphi)(x)-(\mathcal{T}^{m+1}\varphi)(x),x_2,\ldots,x_n\right\|_{*,\beta}=\left\|\mathcal{T}(\mathcal{T}^{m+1}\varphi)(x)-\mathcal{T}(\mathcal{T}^{m}\varphi)(x),x_2,\ldots,x_n\right\|_{*,\beta}\\&\quad\le\max_{1\le i\le j}K_i(x,x_2,\ldots,x_n)\left\|\mathcal{T}^{m+1}\varphi(f_i(x))-\mathcal{T}^{m}\varphi(f_i(x)),x_2,\ldots,x_n\right\|_{*,\beta}\\&\quad\le\max_{1\le i\le j}K_i(x,x_2,\ldots,x_n)(\Lambda^m\varepsilon)(f_i(x),x_2,\ldots,x_n)\\&\quad=(\Lambda^{m+1}\varepsilon)(x,x_2,\ldots,x_n),\end{aligned}$$

and therefore, (4.4) holds for every $m\in\mathbb{N}_0$.

By (4.2), (4.4) and Lemma 2.6, we get $\left\{(\mathcal{T}^m\varphi)(x)\right\}_{m\in\mathbb{N}}$ is a Cauchy sequence in X. Thus, the fact that X is a non-Archimedean (n,β)-Banach space implies that the limit $\psi(x)$ exists for every $x\in W$, i.e., $\psi(x):=\lim_{m\to\infty}(\mathcal{T}^m\varphi)(x)$ for any $x\in W$. Moreover, (4.4) shows that, for any $k\in\mathbb{N}$, $m\in\mathbb{N}_0$ and $(x,x_2,\ldots,x_n)\in W\times X^{n-1}$:

$$\begin{aligned}&\left\|(\mathcal{T}^{m}\varphi)(x)-(\mathcal{T}^{m+k}\varphi)(x),x_2,\ldots,x_n\right\|_{*,\beta}\\&\quad\le\max_{\ell\in\{0,\cdots k-1\}}\left\|(\mathcal{T}^{m+\ell}\varphi)(x)-(\mathcal{T}^{m+\ell+1}\varphi)(x),x_2,\ldots,x_n\right\|_{*,\beta}\\&\quad\le\max_{\ell\in\{0,\cdots k-1\}}(\Lambda^{m+\ell}\varepsilon)(x,x_2,\ldots,x_n)\\&\quad\le\sup_{\ell\ge m}(\Lambda^{\ell}\varepsilon)(x,x_2,\ldots,x_n).\end{aligned}$$

Letting now $k\to\infty$, we see that for any $m\in\mathbb{N}_0$ and $(x,x_2,\ldots,x_n)\in W\times X^{n-1}$, we have:

$$\left\|(\mathcal{T}^{m}\varphi)(x)-\psi(x),x_2,\ldots,x_n\right\|_{*,\beta}\le\sup_{\ell\ge m}(\Lambda^{\ell}\varepsilon)(x,x_2,\ldots,x_n).\tag{4.5}$$

Putting $m=0$ in (4.5), we see that inequality (4.3) holds. Moreover, using **(H3)–(H4)** and (4.5), we obtain:

$$\begin{aligned}&\left\|(\mathcal{T}\psi)(x)-(\mathcal{T}^{m+1}\varphi)(x),x_2,\ldots,x_n\right\|_{*,\beta}\\&\qquad\le\max_{1\le i\le j}K_i(x,x_2,\ldots,x_n)\left\|\psi(f_i(x))-\mathcal{T}^{m}\varphi(f_i(x)),x_2,\ldots,x_n\right\|_{*,\beta}\\&\qquad\le\max_{1\le i\le j}K_i(x,x_2,\ldots,x_n)\sup_{\ell\ge m}(\Lambda^{\ell}\varepsilon)(f_i(x),x_2,\ldots,x_n)\\&\qquad=\Lambda\big(\sup_{\ell\ge m}(\Lambda^{\ell}\varepsilon)\big)(x,x_2,\ldots,x_n)\end{aligned}\tag{4.6}$$

for all $(x,x_2,\ldots,x_n)\in W\times X^{n-1}$. From (4.2) and (4.6), we get:

$$\mathcal{T}(\psi)(x)=\lim_{m\to\infty}(\mathcal{T}^{m+1}\varphi)(x)=\psi(x),\quad x\in W.$$

To prove the statement on the uniqueness of ψ, suppose that $\psi_1,\ \psi_2\in X^W$ are two fixed points of $\mathcal{T}$ satisfies (4.3). Then, for each $(x,x_2,\ldots,x_n)\in W\times X^{n-1}$, we have:

$$\|\psi_1(x)-\psi_2(x),x_2,\ldots,x_n\|_{*,\beta}\le\sup_{m\in\mathbb{N}_0}(\Lambda^{m}\varepsilon)(x,x_2,\ldots,x_n),$$

and as in the proof of (4.4), for any $(x, x_2, \dots, x_n) \in W \times X^{n-1}$ and $k \in \mathbb{N}_0$, we get:

$$\begin{aligned}\|\psi_1(x) - \psi_2(x), x_2, \dots, x_n\|_{*,\beta} &= \|(\mathcal{T}^k\psi_1)(x) - (\mathcal{T}^k\psi_2)(x), x_2, \dots, x_n\|_{*,\beta} \\ &\le \sup_{m\in\mathbb{N}_0} (\Lambda^{m+k}\varepsilon)(x, x_2, \dots, x_n). \end{aligned} \tag{4.7}$$

Letting $m \to \infty$ in (4.7) and from (4.2), we finally get $\psi_1 = \psi_2$. □

5 A New Stability Result for Eq. (1.2)

The following theorem is the main result of this paper. It has been motivated by the issue of Ulam stability, which concerns approximate solutions of a functional equation (1.2) in non-Archimedean (n, β)-Banach spaces by applying the fixed point theorem 4.1.

Theorem 5.1 *Let X be a non-Archimedean (n, β)-Banach space. Let $f : \mathbb{R} \to X$, $c : \mathbb{N} \to \mathbb{R}_+$ and $L : \mathbb{R}_0 \times \mathbb{R}_0 \times X^{n-1} \to \mathbb{R}_+$ be functions satisfying the following three conditions:*

$$\mathcal{M} := \{m \in \mathbb{N} \mid a_m := \max\left\{c(m^k), c(1 + m^k)\right\} < 1\} \neq \emptyset, \tag{5.1}$$

$$L\big(tx^k, ty^k, x_2, \dots, x_n\big) \le c(t)L\big(x^k, y^k, x_2, \dots, x_n\big),$$
$$t \in \left\{m^k, 1 + m^k\right\}, \quad m \in \mathcal{M}, \tag{5.2}$$

$$\left\| f\left(\sqrt[k]{x^k + y^k}\right) - f(x) - f(y), x_2, \dots, x_n \right\|_{*,\beta} \le L\big(x^k, y^k, x_2, \dots, x_n\big) \tag{5.3}$$

for all $x, y \in \mathbb{R}_0$ and $x_2, \dots, x_n \in X$, with $k \in \mathbb{N}$ is fixed. Then, there exists a unique additive function $T : \mathbb{R} \to X$ (i.e., $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}$), such that:

$$\left\| f(x) - T(x^k), x_2, \dots, x_n \right\|_{*,\beta} \le \phi_L(x, x_2, \dots, x_n), \quad x \in \mathbb{R}_0, \quad x_2, \dots, x_n \in X, \tag{5.4}$$

where:

$$\phi_L(x, x_2, \dots, x_n) := \inf_{m\in\mathcal{M}} L\big(x^k, m^k x^k, x_2, \dots, x_n\big). \tag{5.5}$$

Proof Taking $y = mx$ in (5.3), we get:

$$\begin{aligned}\left\| f\left(\sqrt[k]{(1 + m^k)x^k}\right) - f(mx) - f(x), x_2, \dots, x_n \right\|_{*,\beta} &\le L\big(x^k, m^k x^k, x_2, \dots, x_n\big) \\ &=: \varepsilon_m(x, x_2, \dots, x_n) \end{aligned} \tag{5.6}$$

for all $x \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$, with $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, we will define an operator $\mathcal{T}_m : X^{\mathbb{R}} \to X^{\mathbb{R}}$ by:

$$\mathcal{T}_m \xi(x) := \xi\left(\sqrt[k]{(1+m^k)x^k}\right) - \xi(mx), \quad \xi \in X^{\mathbb{R}}, \quad x \in \mathbb{R}.$$

Then:

$$\mathcal{T}_m^{\ell} f(0) = 0, \quad \ell, m \in \mathbb{N}, \tag{5.7}$$

and inequality (5.6) takes the form:

$$\|\mathcal{T}_m f(x) - f(x), x_2, \ldots, x_n\|_{*,\beta} \leq \varepsilon_m(x, x_2, \ldots, x_n)$$

for all $x \in \mathbb{R}_0$, $x_2, \ldots, x_n \in X$, and $m \in \mathbb{N}$.

Let $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_0 \times X^{n-1}} \to \mathbb{R}_+^{\mathbb{R}_0 \times X^{n-1}}$ be an operator which is defined by:

$$\Lambda_m \delta(x, x_2, \ldots, x_n) = \max\left\{\delta\left(\sqrt[k]{(1+m^k)x^k}, x_2, \ldots, x_n\right), \delta(mx, x_2, \ldots, x_n)\right\}$$

for all $\delta \in \mathbb{R}_+^{\mathbb{R}_0}$, $x \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$. Then, it is easily seen that, for each $m \in \mathbb{N}$, the operator $\Lambda := \Lambda_m$ has the form described in (**H3**), with $j = 2$, $W = \mathbb{R}_0$ and:

$$f_1(x) = \sqrt[k]{(1+m^k)x^k}, \quad f_2(x) = mx, \quad K_1(x, x_2, \ldots, x_n) = K_2(x, x_2, \ldots, x_n) = 1$$

for all $x \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$. Moreover, for every $\xi, \mu \in X^{\mathbb{R}_0}$, $m \in \mathbb{N}$, $x \in \mathbb{R}_0$, and $x_2, \ldots, x_n \in X$, we obtain:

$$\begin{aligned}
&\left\|\mathcal{T}_m \xi(x) - \mathcal{T}_m \mu(x), x_2, \ldots, x_n\right\|_{*,\beta} \\
&\quad = \left\|\xi\left(\sqrt[k]{(1+m^k)x^k}\right) - \xi(mx) - \mu\left(\sqrt[k]{(1+m^k)x^k}\right) + \mu(mx), x_2, \ldots, x_n\right\|_{*,\beta} \\
&\quad \leq \max\left\{\left\|\xi(f_1(x)) - \mu(f_1(x)), x_2, \ldots, x_n\right\|_{*,\beta}, \left\|\xi(f_2(x)) - \mu(f_2(x)), x_2, \ldots, x_n\right\|_{*,\beta}\right\} \\
&\quad = \max_{1 \leq i \leq 2} K_i(x, x_2, \ldots, x_n)\left\|(\xi - \mu)(f_i(x)), x_2, \ldots, x_n\right\|_{*,\beta},
\end{aligned}$$

where $(\xi - \mu)(x) \equiv \xi(x) - \mu(x)$. Therefore, **(H4)** is valid for $\mathcal{T}_m$ with $m \in \mathbb{N}$. Note that, in view of (5.2), we have:

$$\Lambda_m \varepsilon_l(x, x_2, \ldots, x_n) \leq a_m \varepsilon_l(x, x_2, \ldots, x_n), \quad l, m \in \mathbb{N}, \quad x \in \mathbb{R}_0, \quad x_2, \ldots, x_n \in X. \tag{5.8}$$

Using mathematical induction, we will show that for each $x \in \mathbb{R}$ and $x_2, \ldots, x_n \in X$, we have:

$$\Lambda_m^{\ell}\varepsilon_l(x, x_2, \ldots, x_n) \le a_m^{\ell}\varepsilon_l(x, x_2, \ldots, x_n) \tag{5.9}$$

for all $\ell \in \mathbb{N}$ and $m \in \mathcal{M}$. From (5.8), we obtain that the inequality (5.9) holds for $\ell = 1$. Next, we will assume that (5.9) holds for $\ell = r$, where $r \in \mathbb{N}$. Then, we have:

$$\begin{aligned}
\Lambda_m^{r+1}\varepsilon_l(x, x_2, \ldots, x_n) &= \Lambda_m\left(\Lambda_m^{r}\varepsilon_l(x, x_2, \ldots, x_n)\right)\\
&= \max\left\{\Lambda_m^{r}\varepsilon_l\left(\sqrt[k]{(m^k+1)x^k}, x_2, \ldots, x_n\right), \Lambda_m^{r}\varepsilon_l(mx, x_2, \ldots, x_n)\right\}\\
&\le a_m^{r}\max\left\{\varepsilon_l\left(\sqrt[k]{(m^k+1)x^k}, x_2, \ldots, x_n\right), \varepsilon_l(mx, x_2, \ldots, x_n)\right\}\\
&\le a_m^{r+1}\varepsilon_l(x, x_2, \ldots, x_n).
\end{aligned}$$

This shows that (5.9) holds for $\ell = r + 1$. Now, we can conclude that the inequality (5.9) holds for all $\ell \in \mathbb{N}$. Therefore, by (5.9), we obtain that:

$$\lim_{\ell\to\infty}\Lambda_m^{\ell}\varepsilon_m(x, x_2, \ldots, x_n) = 0$$

for all $x \in \mathbb{R}_0$ and $m \in \mathcal{M}$. Furthermore, for each $\ell \in \mathbb{N}_0$, $m \in \mathcal{M}$, $x \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$, we have:

$$\begin{aligned}
&\sup_{\ell\in\mathbb{N}_0}\Lambda_m^{\ell}\varepsilon_m(x, x_2, \ldots, x_n) = \varepsilon_m(x, x_2, \ldots, x_n),\\
&\sup_{\ell\in\mathbb{N}_0}\Lambda_m^{\ell+1}\varepsilon_m(x, x_2, \ldots, x_n) = \Lambda_m\varepsilon_m(x, x_2, \ldots, x_n).
\end{aligned}$$

Consequently, by Theorem 4.1 (with $W = \mathbb{R}_0$ and $\varphi = f$), for each $m \in \mathcal{M}$ the mapping $F'_m : \mathbb{R}_0 \to X$, given by $F'_m(x) = \lim_{\ell\to\infty}\mathcal{T}_m^{\ell}f(x)$ for $x \in \mathbb{R}_0$, is a fixed point of $\mathcal{T}_m$, that is:

$$F'_m(x) = F'_m\left(\sqrt[k]{(m^k+1)x^k}\right) - F'_m(mx), \quad x \in \mathbb{R}_0, \quad m \in \mathcal{M}.$$

Moreover:

$$\left\| f(x) - F'_m(x), x_2, \ldots, x_n \right\|_{*,\beta} \le \sup_{\ell\in\mathbb{N}_0}\Lambda_m^{\ell}\varepsilon_m(x, x_2, \ldots, x_n)$$

for all $x \in \mathbb{R}_0$, $x_2, \ldots, x_n \in X$, and $m \in \mathcal{M}_0$.

Define $F_m : \mathbb{R} \to X$ by $F_m(0) = F'_m(0)$ and $F_m(x) = F'_m(x)$ for $x \in \mathbb{R}_0$ and $m \in \mathcal{M}$. Then, it is easily seen that, by (5.7):

$$F_m(x) = \lim_{\ell\to\infty}\mathcal{T}_m^{\ell}f(x), \quad x \in \mathbb{R}, \quad m \in \mathcal{M}.$$

Next, we show that:

$$\left\|\mathcal{T}_m^{\ell}f\left(\sqrt[k]{x^k+y^k}\right) - \mathcal{T}_m^{\ell}f(x) - \mathcal{T}_m^{\ell}f(y), x_2, \ldots, x_n\right\|_{*,\beta} \le a_m^{\ell}L\left(x^k, y^k, x_2, \ldots, x_n\right) \tag{5.10}$$

for every $x, y \in \mathbb{R}_0, x_2, \ldots, x_n \in X, \ell \in \mathbb{N}_0$, and $m \in \mathcal{M}$.

Clearly, if $\ell = 0$, then (5.10) is simply (5.3). Therefore, fix $\ell \in \mathbb{N}_0$ and suppose that (5.10) holds for n and every $x, y \in \mathbb{R}_0$ and $x_2, \ldots, u_n \in X$. Then, for every $x, y \in \mathbb{R}_0$ and $x_2, \ldots, u_n \in X$:

$$\begin{aligned}
&\left\| \mathcal{T}_m^{\ell+1} f\left(\sqrt[k]{x^k + y^k}\right) - \mathcal{T}_m^{\ell+1} f(x) - \mathcal{T}_m^{\ell+1} f(y), x_2, \ldots, x_n \right\|_{*,\beta} \\
&= \Big\| \mathcal{T}_m^{\ell} f\Big(\sqrt[k]{(m^k+1)(x^k+y^k)}\Big) - \mathcal{T}_m^{\ell} f\Big(m\sqrt[k]{x^k+y^k}\Big) - \mathcal{T}_m^{\ell} f\Big(\sqrt[k]{(m^k+1)x^k}\Big) + \mathcal{T}_m^{\ell} f(mx) \\
&\quad - \mathcal{T}_m^{\ell} f\Big(\sqrt[k]{(m^k+1)y^k}\Big) + \mathcal{T}_m^{n} f(my), x_2, \ldots, x_n \Big\|_{*,\beta} \\
&\le \max\Big\{ \Big\| \mathcal{T}_m^{\ell} f\Big(\sqrt[k]{(m^k+1)(x^k+y^k)}\Big) - \mathcal{T}_m^{\ell} f\Big(\sqrt[k]{(m^k+1)x^k}\Big) \\
&\quad - \mathcal{T}_m^{\ell} f\Big(\sqrt[k]{(m^k+1)y^k}\Big), x_2, \ldots, x_n \Big\|_{*,\beta}, \Big\| \mathcal{T}_m^{\ell} f\Big(m\sqrt[k]{x^k+y^k}\Big) - \mathcal{T}_m^{\ell} f(mx) \\
&\quad - \mathcal{T}_m^{\ell} f(my), x_2, \ldots, x_n \Big\|_{*,\beta} \Big\} \\
&\le \max\Big\{ a_m^{\ell} L\big((1+m^k)x^k, (1+m^k)y^k, x_2, \ldots, x_n\big), a_m^{\ell} L\big(m^k x^k, m^k y^k, x_2, \ldots, x_n\big) \Big\} \\
&\le a_m^{\ell+1} L\big(x^k, y^k, x_2, \ldots, x_n\big).
\end{aligned}$$

Thus, by induction, we have shown that (5.10) holds for all $x, y \in \mathbb{R}_0, x_2, \ldots, x_n \in X$ and for all $\ell \in \mathbb{N}_0$. Letting $\ell \to \infty$ in (5.10), we obtain that:

$$F_m\left(\sqrt[k]{x^k + y^k}\right) = F_m(x) + F_m(y), \quad x, y \in \mathbb{R}_0, \quad m \in \mathcal{M}. \tag{5.11}$$

Therefor, we have proved that for each $m \in \mathcal{M}$, there exists a function $F_m : \mathbb{R} \to X$ satisfying (1.2) for $x, y \in \mathbb{R}_0$, such that:

$$\left\| f(x) - F_m(x), x_2, \ldots, x_n \right\|_{*,\beta} \le \sup_{\ell \in \mathbb{N}_0} \Lambda_m^{\ell} \varepsilon_m(x, x_2, \ldots, x_n) = \varepsilon_m(x, x_2, \ldots, x_n) \tag{5.12}$$

for all $x \in \mathbb{R}_0, x_2, \ldots, x_n \in X$, and $m \in \mathcal{M}$.

Now, we show that $F_m = F_l$ for all $m, l \in \mathcal{M}$. Therefore, fix $m, l \in \mathcal{M}$. Note that F_l satisfies (5.11) with m replaced by l. Hence, taking $y = mx$ in (5.11), we get $\mathcal{T}_m F_j = F_j$ for $j = m, l$ and:

$$\left\| F_m(x) - F_l(x), x_2, \ldots, x_n \right\|_{*,\beta} \le \max\{\varepsilon_m(x, x_2, \ldots, x_n), \varepsilon_l(x, x_2, \ldots, x_n)\}$$

for all $x \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$. Whence, by (5.9):

$$\begin{aligned}\left\| F_m(x) - F_l(x), x_2, \dots, x_n \right\|_{*,\beta} &= \left\| \mathcal{T}_m^\ell F_m(x) - \mathcal{T}_m^\ell F_l(x), x_2, \dots, x_n \right\|_{*,\beta} \\ &\le \max \left\{ \Lambda_m^\ell \varepsilon_m(x, x_2, \dots, x_n), \Lambda_m^\ell \varepsilon_l(x, x_2, \dots, x_n) \right\} \\ &\le a_m^\ell \max \left\{ \varepsilon_m(x, x_2, \dots, x_n), \varepsilon_l(x, x_2, \dots, x_n) \right\}\end{aligned}$$

for all $x \in \mathbb{R}_0$, $x_2, \dots, x_n \in X$ and $\ell \in \mathbb{N}_0$. Letting $\ell \to \infty$, we get $F_m = F_l =: F$. Thus, in view of (5.12), we have proved that:

$$\left\| f(x) - F(x), x_2, \dots, x_n \right\|_{*,\beta} \le \varepsilon_m(x, x_2, \dots, x_n), \quad x \in \mathbb{R}_0, \quad x_2, \dots, x_n \in X, \quad m \in \mathcal{M}.$$

Since, in view of (5.11), it is easy to notice that F is a solution to (1.2) and, by Theorem 3.1, the function $F : \mathbb{R} \to X$ has the form $F(x) = T\left(x^k\right)$ with some additive function T. Therefore, we derive (5.4).

It remains to prove the statement concerning the uniqueness of F. Therefore, let $G : \mathbb{R} \to X$ be also a solution of (1.2) and:

$$\left\| f(x) - G(x), x_2, \dots, x_n \right\|_{*,\beta} \le \phi_L(x, x_2, \dots, x_n), \quad x \in \mathbb{R}_0, \quad x_2, \dots, x_n \in X.$$

Then:

$$\left\| F(x) - G(x), x_2, \dots, x_n \right\|_{*,\beta} \le \phi_L(x, x_2, \dots, x_n), \quad x \in \mathbb{R}_0, \quad x_2, \dots, x_n \in X.$$

Further $\mathcal{T}_m G = G$ for each $m \in \mathbb{N}$. Consequently, with a fixed $m \in \mathcal{M}$:

$$\begin{aligned}\left\| F(x) - G(x), x_2, \dots, x_n \right\|_{*,\beta} &= \left\| \mathcal{T}_m^\ell F(x) - \mathcal{T}_m^\ell G(x), x_2, \dots, x_n \right\|_{*,\beta} \\ &\le \Lambda_m^\ell \phi_L(x, x_2, \dots, x_n) \\ &\le \Lambda_m^\ell \varepsilon_m(x, x_2, \dots, x_n) \\ &\le a_m^\ell \varepsilon_m(x, x_2, \dots, x_n)\end{aligned}$$

for all $x \in \mathbb{R}_0$, $x_2, \dots, x_n \in X$ and $\ell \in \mathbb{N}_0$. Letting $\ell \to \infty$, we get $F = G$. This also confirms the uniqueness of T. The proof of the theorem is complete. □

The following hyperstability result can be deduced from Theorem 5.1. This result is a generalization of many works referenced in [16–18].

Corollary 5.2 *Let X be a non-Archimedean (n, β)-Banach space. Let $f : \mathbb{R} \to X$, $c : \mathbb{N} \to \mathbb{R}_+$ and $L : \mathbb{R}_0 \times \mathbb{R}_0 \times X^{n-1} \to \mathbb{R}_+$ be functions and the conditions (5.1), (5.2), and (5.3) be valid. Assume that:*

$$\inf_{m \in \mathcal{M}} L\left(x^k, m^k x^k, x_2, \dots, x_n\right) = 0 \tag{5.13}$$

for all $x \in \mathbb{R}_0$ and $x_2, \dots, x_n \in X$, where $k \in \mathbb{N}$ is fixed. Then, f satisfies (1.2) for all $x, y \in \mathbb{R}$.

Proof In view of (5.13), $\phi_L(x, x_2, \cdots, x_n) = 0$ for each $x \in \mathbb{R}_0,\ x_2, \ldots, x_n \in X$, where ϕ_L is defined by (5.5). Hence, from Theorems 3.1 and 5.1, we easily derive that f is a solution of (1.2) for all $x, y \in \mathbb{R}$. □

Corollary 5.3 *Let X be a non-Archimedean (n, β)-Banach space. Let $c : \mathbb{N} \to \mathbb{R}_+$ and $L : \mathbb{R}_0 \times \mathbb{R}_0 \times X^{n-1} \to \mathbb{R}_+$ be functions and the conditions (5.1), (5.2) and (5.13) be valid. Let $f : \mathbb{R} \to X$ and $F : \mathbb{R}^2 \to X$ be two functions, such that:*

$$\left\| f\left(\sqrt[k]{x^k + y^k}\right) - f(x) - f(y) - F(x, y), x_2, \ldots, x_n \right\|_{*,\beta} \le L\big(x^k, y^k, x_2 \ldots, x_n\big),$$
$$x, y \in \mathbb{R}_0,\ x_2, \ldots, x_n \in X,$$

where $k \in \mathbb{N}$ is fixed. Assume that the functional equation:

$$h\left(\sqrt[k]{x^k + y^k}\right) = h(x) + h(y) + F(x, y), \quad x, y \in \mathbb{R}_0 \tag{5.14}$$

admits a solution $f_0 : \mathbb{R} \to X$ for $x, y \in \mathbb{R}_0$, with $F(0, 0) = -f_0(0)$. Then, f is a solution of (5.14) for all $x, y \in \mathbb{R}$.

Proof Let $g(x) := f(x) - f_0(x)$ for $x \in \mathbb{R}$. Then:

$$\begin{aligned}
&\left\| g\left(\sqrt[k]{x^k + y^k}\right) - g(x) - g(y), x_2, \ldots, x_n \right\|_{*,\beta} \\
&\quad = \left\| f\left(\sqrt[k]{x^k + y^k}\right) - f_0\left(\sqrt[k]{x^k + y^k}\right) - f(x) + f_0(x) - f(y) \right. \\
&\qquad \left. - F(x, y) + f_0(y) + F(x, y), x_2, \ldots, x_n \right\|_{*,\beta} \\
&\quad \le \max\left\{ \left\| f\left(\sqrt[k]{x^k + y^k}\right) - f(x) - f(y) - F(x, y), x_2, \ldots, x_n \right\|_{*,\beta}, \right. \\
&\qquad \left. \left\| f_0\left(\sqrt[k]{x^k + y^k}\right) - f_0(x) - f_0(y) - F(x, y), x_2, \ldots, x_n \right\|_{*,\beta} \right\} \\
&\quad = \left\| f\left(\sqrt[k]{x^k + y^k}\right) - f(x) - f(y) - F(x, y), x_2, \ldots, x_n \right\|_{*,\beta} \\
&\quad \le L\big(x^k, y^k, x_2, \ldots, x_n\big), \quad x_2, \ldots, x_n \in X,\ x, y \in \mathbb{R}_0.
\end{aligned}$$

It follows from Corollary 5.2 that g satisfies the functional equation (1.2) for all $x, y \in \mathbb{R}$. Therefore:

$$\begin{aligned}
f\left(\sqrt[k]{x^k + y^k}\right) - f(x) - f(y) - F(x, y) &= g\left(\sqrt[k]{x^k + y^k}\right) - g(x) - g(y) \\
&\quad + f_0\left(\sqrt[k]{x^k + y^k}\right) - f_0(x) - f_0(y) - F(x, y) = 0
\end{aligned}$$

for all $x, y \in \mathbb{R}$. □

6 Some Consequences

According to Theorem 5.1 and Corollaries 5.2, 5.3, we derive three natural examples of functions L and c satisfying the conditions (5.1) and (5.2). Namely, for:

(i) $L(x, y, x_2, \ldots, x_n) := \varepsilon|x|^p|y|^q\|z, x_2, \ldots, x_n\|_{*,\beta}$;
(ii) $L(x, y, x_2, \ldots, x_n) := \varepsilon\big(|x|^p|y|^q + |x|^{p+q} + |y|^{p+q}\big)\|z, x_2, \ldots, x_n\|_{*,\beta}$;
(iii) $L(x, y, x_2, \ldots, x_n) := \varepsilon\big(\alpha_1|x|^{s_1} + \alpha_2|y|^{s_2}\big)^w\|z, x_2, \ldots, x_n\|_{*,\beta}$

for all $x, y \in \mathbb{R}_0$, $x_2, \ldots, x_n \in X$, and for some arbitrary element $z \in X$ and $\varepsilon, p, q, s_i, w \in \mathbb{R}$, such that $\varepsilon \geq 0$, $p + q < 0$, $\alpha_i > 0$ and $ws_i < 0$ for $i = 1, 2$.

Corollary 6.1 *Let X be a non-Archimedean (n, β)-Banach space and $\varepsilon, p, q \in \mathbb{R}$, with $\varepsilon \geq 0$ and $p + q < 0$. Suppose that $f : \mathbb{R} \to X$ satisfies the inequality:*

$$\left\| f\left(\sqrt[k]{x^k + y^k}\right) - f(x) - f(y), x_2, \ldots, x_n \right\|_{*,\beta} \leq \varepsilon|x|^p|y|^q\|z, x_2, \ldots, x_n\|_{*,\beta}$$

for all $x, y \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$ and for some arbitrary element $z \in X$, with $k \in \mathbb{N}$ is fixed. Then, the following two statements are valid.

(a) *If $q \geq 0$, then there exists a unique additive function $T : \mathbb{R} \to X$ for all $x, y \in \mathbb{R}$, such that:*

$$\left\| f(x) - T(x^k), x_2, \ldots, x_n \right\|_{*,\beta} \leq \varepsilon|x|^{p+q}\|z, x_2, \ldots, x_n\|_{*,\beta},$$
$$x \in \mathbb{R}_0, \quad x_2, \ldots, x_n \in X.$$

(b) *If $q < 0$, then f satisfies (1.2) for all $x, y \in \mathbb{R}$.*

Proof Let $L\big(x^k, y^k, x_2, \ldots, x_n\big) := \varepsilon|x|^p|y|^q\|z, x_2, \ldots, x_n\|_{*,\beta}$ and $c(t) = t^{(p+q)/k}$ in Theorem 5.1 for all $x, y \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$ and for some arbitrary element $z \in X$, where $t \in \mathbb{N}$ and $p, q, \varepsilon \in \mathbb{R}$, such that $\varepsilon \geq 0$ and $p + q < 0$, then we get that the condition (5.2) is valid. Obviously, (5.13) holds if $q < 0$, but if $q \geq 0$, then $\inf_{m\in\mathcal{M}} L\big(x^k, m^kx^k, x_2, \ldots, x_n\big) = L\big(x^k, x^k, x_2, \ldots, x_n\big)$. On the other hand, there exists $m_0 \in \mathbb{N}$, such that:

$$\max\{c(m^k), c(m^k + 1)\} = m^{p+q} < 1, \quad m \geq m_0.$$

Therefore, we obtain (5.1), as well. Then, by Theorem 5.1 and Corollary 5.2, we get the desired results. □

Corollary 6.2 *Let X be a non-Archimedean (n, β)-Banach space and $\varepsilon, p, q \in \mathbb{R}$, such that $\varepsilon \geq 0$, $p + q < 0$ and $q < 0$. Let $f : \mathbb{R} \to X$ and $F : \mathbb{R}^2 \to X$ be two functions, such that:*

$$\left\| f\left(\sqrt[k]{x^k + y^k}\right) - f(x) - f(y) - F(x, y), x_2, \ldots, x_n \right\|_{*,\beta} \leq \varepsilon|x|^p|y|^q\|z, x_2, \ldots, x_n\|_{*,\beta}$$

for all $x, y \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$ and for some arbitrary element $z \in X$, with $k \in \mathbb{N}$ is fixed. Assume that the functional equation:

$$g\left(\sqrt[k]{x^k + y^k}\right) = g(x) + g(y) + F(x, y), \quad x, y \in \mathbb{R}_0 \tag{6.1}$$

admits a solution $g_0 : \mathbb{R} \to X$ for $x, y \in \mathbb{R}_0$ with $F(0, 0) = -g_0(0)$. Then, f is a solution of (6.1) for all $x, y \in \mathbb{R}$.

Corollary 6.3 *Let X be a non-Archimedean (n, β)-Banach space and $\varepsilon, p, q \in \mathbb{R}$, such that $\varepsilon \geq 0$ and $p + q < 0$. Suppose that $f : \mathbb{R} \to X$ satisfies the inequality:*

$$\left\| f\left(\sqrt[k]{x^k + y^k}\right) - f(x) - f(y), x_2, \ldots, x_n \right\|_{*,\beta} \leq \varepsilon\big(|x|^p|y|^q + |x|^{p+q} + |y|^{p+q}\big)\|z, x_2, \ldots, x_n\|_{*,\beta}$$

for all $x, y \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$ and for some arbitrary element $z \in X$, with $k \in \mathbb{N}$ is fixed. Then, the following two statements are valid.

(a) *If $q > 0$, then there exists a unique additive function $T : \mathbb{R} \to X$ for all $x, y \in \mathbb{R}$, such that:*

$$\left\| f(x) - T(x^k), x_2, \ldots, x_n \right\|_{*,\beta} \leq 3\varepsilon|x|^{p+q}\|z, x_2, \ldots, x_n\|_{*,\beta}, \quad x \in \mathbb{R}_0, \quad x_2, \ldots, x_n \in X.$$

(b) *If $q = 0$, then there exists a unique additive function $T : \mathbb{R} \to X$ for all $x, y \in \mathbb{R}$, such that:*

$$\left\| f(x) - T(x^k), x_2, \ldots, x_n \right\|_{*,\beta} \leq 2\varepsilon|x|^{p+q}\|z, x_2, \ldots, x_n\|_{*,\beta}, \quad x \in \mathbb{R}_0, \quad x_2, \ldots, x_n \in X.$$

(c) *If $q < 0$, then there exists a unique additive function $T : \mathbb{R} \to X$ for all $x, y \in \mathbb{R}$, such that:*

$$\left\| f(x) - T(x^k), x_2, \ldots, x_n \right\|_{*,\beta} \leq \varepsilon|x|^{p+q}\|z, x_2, \ldots, x_n\|_{*,\beta}, \quad x \in \mathbb{R}_0, \quad x_2, \ldots, x_n \in X.$$

Proof Let $L\big(x^k, y^k, x_2, \ldots, x_n\big) := \varepsilon\big(|x|^p|y|^q + |x|^{p+q} + |y|^{p+q}\big)\|z, x_2, \ldots, x_n\|_{*,\beta}$ and $c(t) = t^{(p+q)/k}$ in Theorem 5.1 for all $x, y \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$ and for some arbitrary element $z \in X$, where $t \in \mathbb{N}$ and $\varepsilon, p, q \in \mathbb{R}$, such that $\varepsilon \geq 0$ and $p + q < 0$, then we get that the condition (5.2) is valid. Obviously:

$$\inf_{m \in \mathcal{M}} L\big(x^k, m^k x^k, x_2, \ldots, x_n\big) = \varepsilon|x|^{p+q}\|z, x_2, \ldots, x_n\|_{*,\beta} \text{ if } q < 0,$$
$$\inf_{m \in \mathcal{M}} L\big(x^k, m^k x^k, x_2, \ldots, x_n\big) = 2\varepsilon|x|^{p+q}\|z, x_2, \ldots, x_n\|_{*,\beta} \text{ if } q = 0$$

and

$$\inf_{m \in \mathcal{M}} L\big(x^k, m^k x^k, x_2, \ldots, x_n\big) = 3\varepsilon|x|^{p+q}\|z, x_2, \ldots, x_n\|_{*,\beta} \text{ if } q > 0.$$

On the other hand, there exists $m_0 \in \mathbb{N}$, such that:

$$\max\{c(m^k), c(m^k+1)\} = m^{p+q} < 1, \quad m \geq m_0.$$

Therefore, we obtain (5.1), as well. Then, by Theorem 5.1, we get the desired results. ◻

Corollary 6.4 *Let X be a non-Archimedean (n, β)-Banach space and $\varepsilon, s_i, w, \alpha_i \in \mathbb{R}$, such that $\varepsilon \geq 0$, $\alpha_i > 0$ and $ws_i < 0$ for $i = 1, 2$. Suppose that $f : \mathbb{R} \to X$ satisfies the inequality:*

$$\left\| f\left(\sqrt[k]{x^k + y^k}\right) - f(x) - f(y), x_2, \ldots, x_n \right\|_{*,\beta} \leq \varepsilon\big(\alpha_1|x|^{s_1} + \alpha_2|y|^{s_2}\big)^w \|z, x_2, \ldots, x_n\|_{*,\beta}$$

for all $x, y \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$ and for some arbitrary element $z \in X$, with $k \in \mathbb{N}$ is fixed. Then, the following two statements are valid.

(a) *If $w > 0$, then there exists a unique additive function $T : \mathbb{R} \to X$ for all $x, y \in \mathbb{R}$, such that:*

$$\left\| f(x) - T(x^k), x_2, \ldots, x_n \right\|_{*,\beta} \leq \varepsilon\alpha_1^w |x|^{s_1 w} \|z, x_2, \ldots, x_n\|_{*,\beta}.$$

(b) *If $w < 0$, then there exists a unique additive function $T : \mathbb{R} \to X$ for all $x, y \in \mathbb{R}$, such that:*

$$\left\| f(x) - T(x^k), x_2, \ldots, x_n \right\|_{*,\beta} \leq \varepsilon(\alpha_1 + \alpha_2)^w |x|^{s_0 w} \|z, x_2, \ldots, x_n\|_{*,\beta}$$

for all $x \in \mathbb{R}_0$, and all $x_2, \ldots, x_n \in X$ and for some arbitrary element $z \in X$, where:

$$s_0 := \begin{cases} \max\{s_1, s_2\} & \textit{if } w > 0; \\ \min\{s_1, s_2\} & \textit{if } w < 0. \end{cases}$$

Proof Let $L\big(x^k, y^k, x_2, \ldots, x_n\big) := \varepsilon\big(\alpha_1|x|^{s_1} + \alpha_2|y|^{s_2}\big)^w \|z, x_2, \ldots, x_n\|_{*,\beta}$ and $c(t) = t^{s_0 w/k}$ in Theorem 5.1 for all $x, y \in \mathbb{R}_0$ and $x_2, \ldots, x_n \in X$ and for some arbitrary element $z \in X$, where $t \in \mathbb{N}$ and $\varepsilon, \alpha_i, s_i, w \in \mathbb{R}$, such that $ws_i < 0$, $\alpha_i >$ and $\varepsilon \geq 0$ for $i \in \{0, 1, 2\}$, with:

$$s_0 := \begin{cases} \max\{s_1, s_2\} & \text{if } w > 0; \\ \min\{s_1, s_2\} & \text{if } w < 0. \end{cases}$$

Then, we get that the condition (5.2) is valid. Obviously:

$$\begin{cases} \inf_{m\in\mathcal{M}} L\big(x^k, m^k x^k, x_2, \ldots, x_n\big) = \varepsilon\alpha_1^w |x|^{s_1 w} \|z, x_2, \ldots, x_n\|_{*,\beta} & \text{if } w > 0; \\ \inf_{m\in\mathcal{M}} L\big(x^k, m^k x^k, x_2, \ldots, x_n\big) = \varepsilon(\alpha_1 + \alpha_2)^w |x|^{s_0 w} \|z, x_2, \ldots, x_n\|_{*,\beta} & \text{if } w < 0. \end{cases}$$

On the other hand, there exists $m_0 \in \mathbb{N}$, such that:

$$\max\{c(m^k), c(m^k + 1)\} = m^{ws_0} < 1, \quad m \geq m_0.$$

Therefore, we obtain (5.1), as well. Then, by Theorem 5.1, we get the desired results. □

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