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ORIGINAL PAPER

Some Notes on *b*-Weakly Compact Operators

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Abstract

The main aim of this paper is studying the family $W_b(E, F)$ of b-weakly compact operators between two Banach lattices. For an order dense sublattice G of a vector lattice E, if $T : G \to F$ is a b-weakly compact operator between two Banach lattices, then $T \in W_b(E, F)$ whenever the norm of E is order continuous and $T : E \to F$ is a positive operator. We also investigate the relationship between $W_b(E, F)$ and some other classes of operators like $L_c^{(1)}(E, F)$ and $L_c^{(2)}(E, F)$.

Keywords Banach lattice · Order continuous norm · b-weakly compact operator

Mathematics Subject Classification 46B42 · 47B60

1 Introduction and Preliminaries

An operator T from a Banach lattice E to a Banach space X is said to be b-weakly compact, if the image of every b-order bounded subset of E (that is, order bounded in the topological bidual E'' of E) under T is relatively weakly compact. The authors in [8] proved that an operator T from a Banach lattice E into a Banach space X is b-weakly compact if and only if $\{Tx_n\}_n$ is norm convergent for every positive increasing sequence $\{x_n\}_n$ of the closed unit ball B_E of E. The class of all b-weakly compact operators between E and X will be denoted by $W_b(E, X)$. The class of b-weakly compact operators was firstly introduced by Alpay et al. [3]. One of the interesting properties of the class of b-weakly compact operators is that it satisfies the domination property. Some more investigations on $W_b(E, X)$ were done by [3–5,7,8].

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In this paper, we continue the investigation on $W_b(E, X)$. In the first section, we provide some prerequisites. The second section is devoted to the main results. We mainly focus on the inclusion relationship between $W_b(E, X)$ with some known class of operators. We also study those Banach lattices for which the modulus of an order bounded operator is b-weakly compact.

1.1 Some Basic Definitions

Let E be a vector lattice. An element $e > 0$ in E is said to be an order unit whenever for each $x \in E$ there exists a $\lambda > 0$ with $|x| \leq \lambda e$. A sequence (x_n) in a vector lattice is said to be disjoint whenever $|x_n| \wedge |x_m| = 0$ holds for $n \neq m$. A vector lattice is called Dedekind complete whenever every nonempty bounded above subset has a supremum. For an operator $T : E \to F$ between two vector lattices, we shall say that its modulus $|T|$ exists whenever $|T| := T \vee (-T)$ exists in the sense that $|T|$ is the supremum of the set $\{-T, T\}$. An operator $T : E \to F$ between two vector lattices is called order bounded if it maps order bounded subsets of E into order bounded subsets of F. An operator $T : E \to F$ is said to be positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Note that each positive linear operator on a Banach lattice is continuous and order bounded. A Banach lattice E has order continuous norm if $\|x_\alpha\| \to 0$ for every decreasing net (x_α) with $\inf_\alpha x_\alpha = 0$. If E is a Banach lattice, its topological dual E', endowed with the dual norm and dual order, is also a Banach lattice. A Banach lattice E is said to be an AM-space if for each $x, y \in E^+$ such that $x \wedge y = 0$, we have $\|x \vee y\| = max(\|x\|, \|y\|)$. A Banach lattice E is said to be a KB-space whenever each increasing norm bounded sequence of E^+ is norm convergent. It is known that every reflexive Banach lattice is a KB-spaces. Moreover, each KB-space has order continuous norm. Recall that an operator T from a Banach lattice E into a Banach space X is said to be order weakly compact, if it maps each order bounded subset of E into a relatively weakly compact subset of X, i.e., $T[-x, x]$ is relatively weakly compact in X for each $x \in E^+$.

A positive linear operator $T : E \to F$ is called almost interval preserving if $T[0, x]$ is dense in $[0, Tx]$ for every $x \in E^+$. Let E be a vector lattice. A sequence $(x_n) \subset E$ is called order convergent to x if there exists a sequence (y_n) such that $y_n \downarrow 0$ and for some n_0, $|x_n - x| \leq y_n$ for all $n \geq n_0$. We will write $x_n \xrightarrow{o_1} x$ when (x_n) is order convergent to x. A sequence (x_n) in a vector lattice E is strongly order convergent to $x \in E$, denoted by $x_n \xrightarrow{o_2} x$ whenever there exists a net (y_β) in E such that $y_\beta \downarrow 0$ and that for every β, there exists a n_0 such that $|x_n - x| \leq y_\beta$ for every $n \geq n_0$. It is clear that every order convergent sequence is strongly order convergent.

For more information concerning the Banach lattice and the related topics, we refer the reader to [2,10].

2 Results

We commence with the following result showing that if an operator is b-weakly compact on some order dense sublattice, then it will be b-weakly compact on the whole space.

Theorem 2.1 *Let E and F be two Banach lattices such that the norm of E is order continuous and let $T : E \to F$ be a positive operator. Then, for an order dense sublattice G of E, if $T \in W_b(G, F)$ then $T \in W_b(E, F)$.*

Proof. Let (x_n) be a bounded positive increasing sequence in E. Since G is order dense in E, from [2, Theorem 1.34] we have $\{y \in G : 0 \leq y \leq x_n\} \uparrow x_n$, for every n. Let $(y_{mn})_m \subset G$ with $0 \leq y_{mn} \uparrow x_n$ for every n and set $z_{mn} = \vee_{i=1}^{n} y_{mi}$. It follows that $z_{mn} \uparrow_m x_n$ and $\sup_{m,n} \|z_{mn}\| \leq \sup_n \|x_n\| < \infty$. Now, if $T \in W_b(G, F)$, then (Tz_{mn}) is norm convergent to some $y \in F$. Then, from

$$\|Tx_n - Tz_{mn}\| \leq \|T\| \|x_n - z_{mn}\| \leq \|T\| \|x_n - y_{mn}\| \to 0,$$

we get

$$\|Tx_n - y\| \leq \|Tx_n - Tz_{mn}\| + \|Tz_{mn} - y\|; \qquad \square$$

and this completes the proof.

Definition 2.2 Let E and F be two vector lattices. We define $L_c^{(1)}(E, F)$ (resp. $L_c^{(2)}(E, F)$) as the collection of all order bounded operators T for which $x_n \xrightarrow{o_1} 0$ (resp. $x_n \xrightarrow{o_2} 0$) implies $Tx_{n_k} \xrightarrow{o_1} 0$ (resp. $Tx_{n_k} \xrightarrow{o_2} 0$) for some subsequence (x_{n_k}) of (x_n).

It should be noted that $L_c^{(1)}(E, F) = L_c^{(2)}(E, F)$ when F is Dedekind complete, see for example [1], in which there are also examples showing that these two collections can be different.

Theorem 2.3 *Let E and F be two Banach lattices such that the norm of E is order continuous. Then*

(1) $W_b(E, F)^+ \subseteq L_c^{(2)}(E, F)$.
(2) *If $W_b(E, F)$ is a vector lattice and F is Dedekind complete, then $W_b(E, F)$ is an order ideal of $L_c^{(1)}(E, F) = L_c^{(2)}(E, F)$.*

Proof. (1) Let $T \in W_b(E, F)^+$ and let $(x_n) \subset E$ be a strongly order convergent sequence. Without lose of generality, we assume that $0 \leq x_n \xrightarrow{o_2} 0$, which follows that (x_n) is norm convergent to zero. Set (x_{n_j}) as a subsequence with $\sum_{j=1}^{\infty} \|x_{n_j}\| < \infty$. Define $y_m = \sum_{j=1}^{m} x_{n_j}$. Then, (y_m) is bounded and $0 \leq y_m \uparrow$. Since T is a b-weakly compact operator, (Ty_m) is norm convergent to some point $z \in F$. Using [9, Lemma 3.11], (Ty_m) has a subsequence (Ty_{m_k}) strongly order convergent to $z \in F$. Thus, there exists a net $(z_\beta) \subset F^+$ with the property that for each β there exists some n_0 such that if $k \geq n_0$, then $|Ty_{m_k} - z| \leq z_\beta \downarrow 0$. Consequently

$$
\begin{aligned}
0 \le Tx_{n_{m_k}} &\le |Ty_{m_k} - Ty_{m_{k'}}| \\
&\le |Ty_{m_k} - z| + |Ty_{m_{k'}} - z| \\
&\le z_\beta + z_\beta \downarrow 0,
\end{aligned}
$$

for every $k \ge k' \ge n_0$, which confirms that $T \in L_c^{(2)}(E, F)$ as required.

(2) By [2, Corollary 4.10], E is Dedekind complete, so by Dedekind completeness of F, $L_c^{(2)}(E, F) = L_c^{(1)}(E, F)$. Furthermore, since $W_b(E, F)$ is a vector lattice, it follows from part (1) that $W_b(E, F)$ is a subspace of $L_c^{(1)}(E, F)$. Now proof follows from the fact that $W_b(E, F)$ satisfies the domination property. □

We need the following elementary lemma in some of the forthcoming results in this section.

Lemma 2.4 *Let E and X be Banach spaces and let $T : E \to X$ be a bounded linear operator with closed range. Then*

(1) *If T is compact, then T is of finite rank.*
(2) *If T is weakly compact, then $T(E)$ is reflexive.*

Proof. (1) If T is compact, then $T(U)$ is relatively norm compact, where U is the open unit ball in E. On the other hand, by the open mapping theorem, $T(U)$ is open. It follows that the Banach subspace $T(E)$ of X is locally compact, so it must be of finite dimensional, as claimed.

(2) If T is weakly compact, then $T(B)$ is relatively weakly compact, where B denotes the closed unit ball in E. This fact together with the equality $T(E) = \bigcup_{n\in\mathbb{N}} nT(B)$ implies that the unit closed ball in $T(E)$ is weakly compact. Consequently, $T(E)$ must be reflexive. □

Proposition 2.5 *Let E be a Banach lattice and let X be a non-reflexive Banach space. If $T : E \to X$ is a surjective b-weakly compact operator, then the norm of E' is not order continuous.*

Proof If the norm of E' is order continuous, then by [7, Theorem 2.2] T must be weakly compact and Lemma 2.4 implies that X is reflexive which is a contradiction.

Proposition 2.6 *Let E and F be two Banach lattices such that the norm of F' is order continuous. If $T : E \to F$ is an injective almost interval preserving and b-weakly compact operator with closed range, then E is reflexive.*

Proof Since T has closed range, we may assume, without loss of generality, that T is onto. Thus, $T : E \to F$ is a bijection between two Banach spaces and it follows that $T' : F' \to E'$ is also a bijection. On the other hand, since T is almost interval preserving, by [10, Theorem 1.4.19], T' is a lattice homomorphism, so by [2, Theorem 2.15], both T' and $(T')^{-1}$ are positive operators. Since the norm of F' is order continuous, by [2, Theorem 4.59], F' is a KB-space, so T' is b-weakly compact and the norm of E' is also order continuous. Since T is b-weakly compact, by [7, Theorem 2.2], T is weakly compact. It follows that T' is also weakly compact. Now Lemma 2.4 implies that E' must be reflexive, so E is reflexive, as claimed.

Proposition 2.7 *Let E be a Banach lattice, X be a Banach space and let $T : E \to X$ be an injective b-weakly compact operator with closed range. Then, E is finite-dimensional when either of the following conditions hold.*

(1) *E is an AM-space with order continuous norm.*
(2) *E is an AM-space and E' is discrete.*

Proof Similar to the proof of Proposition 2.6, we may assume that T is onto, so $T' : X' \to E'$ is a also onto. Then by [8, Proposition 2.3], T is a compact operator under either of the conditions (1) and (2). Thus, T' is compact. Now, by Lemma 2.4, E' is finite dimensional. Hence, E is finite dimensional.

Theorem 2.8 *For two Banach lattices E and F, if E has order unit and the norm of F is order continuous, then every order bounded operator $T : E \to F$ is b-weakly compact operator.*

Proof Let $T : E \to F$ be a bounded operator and take a bounded increasing sequence (x_n) in E. Let $e \in E^+$ be an order unit for E. For each $x \in E$, the norm $\|x\|_\infty = \inf\{\lambda > 0 : |x| \leq \lambda e\}$ on E is equivalent to the original norm, so it follows that $\sup_n \|x_n\|_\infty < \infty$. For each $n \in \mathbb{N}$, there exists a $\lambda_n > 0$ such that $\lambda_n \leq \|x_n\|_\infty + 1$ and $|x_n| \leq \lambda_n e$. Then, we get

$$0 < \lambda = \sup\lambda_n \leq \sup_n \|x_n\|_\infty + 1 < \infty.$$

In particular, $(x_n) \subseteq [-\lambda e, \lambda e]$. Now since F has order continuous norm, it is Dedekind complete, and so T^+ exists. It follows that $(T^+ x_n) \subseteq T^+[-\lambda e, \lambda e] \subseteq [-T^+\lambda e, T^+\lambda e]$. By [2, Theorem 4.9], $[-T^+\lambda e, T^+\lambda e]$ is weakly compact, and so there is a subsequence $(T^+ x_{n_j})$ of $(T^+ x_n)$ which is weakly convergent to some point $z \in F$. Since $(T^+ x_n)$ is an increasing sequence, $(T^+ x_n)$ is norm convergent to z, so $T^+ \in W_b(E, F)$. A similar argument reveals that $T^- \in W_b(E, F)$. We thus conclude that $T = T^+ - T^- \in W_b(E, F)$. □

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