Bulletin of the Iranian Mathematical Society (2020) 46:1455–1463
https://doi.org/10.1007/s41980-019-00335-y

ORIGINAL PAPER

Checkable Criteria for the M-Positive Definiteness of Fourth-Order Partially Symmetric Tensors

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Received: 28 July 2019 / Revised: 24 October 2019 / Accepted: 28 November 2019 / Published online: 20 December 2019

Abstract
Based on the matrix unfolding technique of a tensor, three easily checkable sufficient conditions for the M-positive definiteness of fourth-order partially symmetric tensors are given. Numerical examples show that the proposed results are efficient.

Keywords Partially symmetric tensors · M-positive definiteness · Unfolding matrix

Mathematics Subject Classification 15A69 · 15A18 · 65F15

1 Introduction

The equilibrium equations [1,2]

$$c_{i_1 i_2 i_3 i_4}(1 + \nabla u)u_{i_3,i_4 i_2} = 0 \tag{1.1}$$

are of great importance in the theory of elasticity [3], where $u_i(X)(i = 1, 2, 3)$ is the displacement field (X is the coordinate of a material point in the reference configuration), $c_{i_1 i_2 i_3 i_4}$ is the component of elastic modulus tensor $\mathcal{C} = (c_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{3\times3\times3\times3}$ and has the following property:

$$c_{i_1 i_2 i_3 i_4} = c_{i_2 i_1 i_3 i_4} = c_{i_1 i_2 i_4 i_3} = c_{i_3 i_4 i_1 i_2}, \ \forall i_1, i_2, i_3, i_4 \in \langle 3\rangle = \{1, 2, 3\}.$$

Communicated by Abbas Salemi.

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and Eq. (1.1) is strongly elliptic if and only if

$$\mathcal{C}xyxy = \sum_{i_1,i_2,i_3,i_4=1}^{3} c_{i_1i_2i_3i_4}x_{i_1}y_{i_2}x_{i_3}y_{i_4} > 0 \tag{1.2}$$

holds for all unit vector $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$. For common usage, (1.2) is called the strong ellipticity condition. In the past several decades, considerable effort has been made to seek the sufficient or necessary criteria for the strong ellipticity condition such as literatures [4–9]. However, easily verifiable criteria are few because $\mathcal{C}xyxy$ in (1.2) can be equivalently written as $\mathcal{A}xyxy$, where $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{3\times3\times3\times3}$ is a partially symmetric tensor, that is, $a_{i_1i_2i_3i_4} = a_{i_3i_2i_1i_4} = a_{i_1i_4i_3i_2} = a_{i_3i_4i_1i_2}$ and $a_{i_1i_2i_3i_4} = \frac{1}{4}(c_{i_1i_2i_3i_4} + c_{i_3i_2i_1i_4} + c_{i_1i_4i_3i_2} + c_{i_3i_4i_1i_2})$. In 2009, Qi et al. [10] presented that the strong ellipticity condition holds if and only if the partially symmetric tensor $\mathcal{A}$ is M-positive definite which is defined as follows. Without loss of generality, in this paper, we consider a more general partially symmetric tensor, that is, $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{m\times n\times m\times n}$ with

$$a_{i_1i_2i_3i_4} = a_{i_3i_2i_1i_4} = a_{i_1i_4i_3i_2} = a_{i_3i_4i_1i_2}, \forall i_1, i_3 \in \langle m\rangle = \{1, 2, \ldots, m\},\ \forall i_2, i_4 \in \langle n\rangle.$$

Definition 1.1 [10] A partially symmetric tensor $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{m\times n\times m\times n}$ is called an M-positive definite tensor, if

$$\mathcal{A}xyxy = \sum_{i_1,i_3=1}^{m}\sum_{i_2,i_4=1}^{n} a_{i_1i_2i_3i_4}x_{i_1}y_{i_2}x_{i_3}y_{i_4} > 0 \tag{1.3}$$

holds for any unit vectors $x = (x_i) \in \mathbb{R}^m$ and $y = (y_i) \in \mathbb{R}^n$.

Qi et al. [10] proved that a fourth-order real partially symmetric tensor is M-positive definite if and only if its smallest M-eigenvalue is positive, whereas the computation for the M-eigenvalues of a partially symmetric tensor is difficult. To derive checkable criteria for the M-positive definiteness of partially symmetric tensors, we use another method to seek these criteria in this paper. Specifically, using the matrix unfolding technique of a tensor, we give three easily verifiable sufficient conditions for the M-positive definiteness of fourth-order partially symmetric tensors. Numerical examples show that the proposed results are efficient.

2 Main Results

We first give one checkable criterion to identify the M-positive definiteness of fourth-order real partially symmetric tensors.

Theorem 2.1 *Let $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{m\times n\times m\times n}$ be a partially symmetric tensor with positive diagonal entries $\{a_{ijij}\}_{i,j=1}^{m,n}$. Then $\mathcal{A}$ is M-positive definite, if*

$$a_{ijij} > \sum_{\substack{i_3 \in \langle m \rangle, i_4 \in \langle n \rangle, \\ \varsigma_{iji_3i_4}=0}} |a_{iji_3i_4}|, \ \forall i \in \langle m \rangle, \ \forall j \in \langle n \rangle, \tag{2.1}$$

where

$$\varsigma_{iji_3i_4} = \begin{cases} 1, & i_3 = i \text{ and } i_4 = j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof Suppose that $\mathcal{A}$ is not M-positive definite, by Definition 1.1, there exist at least one $x \in \mathbb{R}^m \backslash \{0\}$ and one $y \in \mathbb{R}^n \backslash \{0\}$ such that

$$0 \geq \mathcal{A}xyxy = \sum_{\substack{i_1, i_3 \in \langle m \rangle, \\ i_2, i_4 \in \langle n \rangle}} a_{i_1i_2i_3i_4} x_{i_1} y_{i_2} x_{i_3} y_{i_4}. \tag{2.2}$$

To derive contradiction, we unfold the tensor $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{m \times n \times m \times n}$ into a matrix $A = (a_{kh}) \in \mathbb{R}^{mn \times mn}$ by the following bijective mapping:

$$a_{kh} = a_{i_1i_2i_3i_4}, \tag{2.3}$$

where

$$k = i_1 + (i_2 - 1)m, \ h = i_3 + (i_4 - 1)m,$$

and we unfold the nonzero matrix $xy^{\mathrm{T}} = (x_i y_j) \in \mathbb{R}^{m \times n}$ into a vector $w = (w_k) \in \mathbb{R}^{mn} \backslash \{0\}$, where $w_k = x_{i_1} y_{i_2}$. Obviously, the unfolding matrix A is symmetric since the partial symmetry of $\mathcal{A}$, and $\mathcal{A}xyxy$ can be equivalently rewritten as $w^{\mathrm{T}}Aw$, i.e. $\mathcal{A}xyxy = w^{\mathrm{T}}Aw$. From (2.2), we know that $0 \geq w^{\mathrm{T}}Aw$, which means that the unfolding matrix A is not positive definite. By the properties of positive definite matrices [11], we know that A is not a strictly diagonal dominant matrix, i.e. there is at least one index $k' = i'_1 + (i'_2 - 1)m$ such that

$$a_{k'k'} \leq \sum_{\substack{h \neq k', \\ h=1}}^{mn} |a_{k'h}|. \tag{2.4}$$

Note that (2.4) is equivalent to

$$a_{i'_1i'_2i'_1i'_2} \leq \sum_{\substack{i_3 \in \langle m \rangle, i_4 \in \langle n \rangle, \\ \varsigma_{i'_1i'_2i_3i_4}=0}} |a_{i'_1i'_2i_3i_4}|,$$

which contradicts with (2.1). Hence, $\mathcal{A}$ is M-positive definite. □

Remark 2.2 From the proof of Theorem 2.1, we conclude that the positive definiteness of the unfolding matrix A implies the M-positive definiteness of the partially symmetric tensor $\mathcal{A}$. Hence, using other properties of the positive definite matrices, we can give other criteria for the M-positive definiteness of fourth-order real partially symmetric tensors.

For instance, based on the fact that double strictly diagonal dominant symmetric matrices [11] with positive diagonal entries are positive definite, the second criterion for the M-positive definiteness of fourth-order real partially symmetric tensors can be easily obtained.

Corollary 2.3 *Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{m\times n\times m\times n}$ be a partially symmetric tensor with positive diagonal entries $\{a_{ijij}\}_{i,j=1}^{m,n}$. Then $\mathcal{A}$ is M-positive definite, if*

$$a_{ijij} \cdot a_{klkl} > \left(\sum_{\substack{i_3\in\langle m\rangle, i_4\in\langle n\rangle,\\ \varsigma_{iji_3i_4}=0}} |a_{iji_3i_4}| \right) \cdot \left(\sum_{\substack{i_3\in\langle m\rangle, i_4\in\langle n\rangle,\\ \varsigma_{kli_3i_4}=0}} |a_{kli_3i_4}| \right) \tag{2.5}$$

holds for all $i, k \in \langle m\rangle$, $j, l \in \langle n\rangle$ and $\varsigma_{ijkl} = 0$.

Proof Assume that $\mathcal{A}$ is not M-positive definite, then similar to the proof of Theorem 2.1, we know that the unfolding matrix A is not positive definite. From the properties of positive definite matrices, we confirm that A is not a double strictly diagonal dominant symmetric matrix, i.e. there are at least two different indexes $j' = i_1' + (i_2' - 1)m$, $k' = l_1' + (l_2' - 1)m$, where $i_1', l_1' \neq k' \in \langle m\rangle$ and $i_2', l_2' \in \langle n\rangle$, such that

$$a_{j'j'} \cdot a_{k'k'} \leq \left(\sum_{\substack{h\neq j',\\ h=1}}^{mn} |a_{j'h}| \right) \cdot \left(\sum_{\substack{h\neq k',\\ h=1}}^{mn} |a_{k'h}| \right). \tag{2.6}$$

By (2.3), we know that (2.6) is equivalent to

$$a_{i_1'i_2'i_1'i_2'} \cdot a_{l_1'l_2'l_1'l_2'} \leq \left(\sum_{\substack{i_3\in\langle m\rangle, i_4\in\langle n\rangle,\\ \varsigma_{i_1'i_2'i_3i_4}=0}} |a_{i_1'i_2'i_3i_4}| \right) \cdot \left(\sum_{\substack{i_3\in\langle m\rangle, i_4\in\langle n\rangle,\\ \varsigma_{l_1'l_2'i_3i_4}=0}} |a_{l_1'l_2'i_3i_4}| \right),$$

which contradicts with (2.5). Therefore, $\mathcal{A}$ is M-positive definite. □

Remark 2.4 Given a partially symmetric tensor $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{m\times n\times m\times n}$, it is not difficult to see that the criterion (2.1) holds implies that the criterion (2.5) must hold, but the converse is not necessarily true.

Besides, according to the position of the eigenvalue inclusion sets [12] of the unfolding matrix in the complex plane, we can judge whether the corresponding partially symmetric tensor $\mathcal{A}$ is M-positive definite or not.

Corollary 2.5 *Let* $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{m\times n\times m\times n}$ *be a partially symmetric tensor. Then* $\mathcal{A}$ *is M-positive definite, if*

$$\Gamma(\mathcal{A}) \bigcap \mathbb{C}^- = \emptyset \quad \text{or} \quad \mathcal{K}(\mathcal{A}) \bigcap \mathbb{C}^- = \emptyset, \tag{2.7}$$

where

$$\Gamma(\mathcal{A}) = \bigcup_{i\in\langle m\rangle, j\in\langle n\rangle} \left\{ z \in \mathbb{C} : |z - a_{ijij}| \le \sum_{\substack{i_3\in\langle m\rangle, i_4\in\langle n\rangle,\\ \varsigma_{iji_3i_4}=0}} |a_{iji_3i_4}| \right\},$$

$$\mathcal{K}(\mathcal{A}) = \bigcup_{\substack{i,k\in\langle m\rangle, j,l\in\langle n\rangle,\\ \varsigma_{ijkl}=0}} \left\{ z \in \mathbb{C} : |z - a_{ijij}||z - a_{klkl}| \le \left(\sum_{\substack{i_3\in\langle m\rangle, i_4\in\langle n\rangle,\\ \varsigma_{iji_3i_4}=0}} |a_{iji_3i_4}| \right) \cdot \left(\sum_{\substack{i_3\in\langle m\rangle, i_4\in\langle n\rangle,\\ \varsigma_{kli_3i_4}=0}} |a_{kli_3i_4}| \right) \right\},$$

and $\mathbb{C}^- = \{z \in \mathbb{C} : \mathbf{Re}(z) \le 0\}$.

Proof Provided that $\mathcal{A}$ is not M-positive definite, by the proof of Theorem 2.1, we know that the unfolding matrix A is not positive definite. According to the eigenvalue properties of positive definite matrices, we confirm that both regions of the Geršgorin set [12] $\Gamma(A)$ and the Brauer set [12] $\mathcal{K}(A)$ of A cannot be located only in the right half complex plane, i.e.

$$\Gamma(A) \bigcap \mathbb{C}^- \ne \emptyset \quad \text{and} \quad \mathcal{K}(A) \bigcap \mathbb{C}^- \ne \emptyset, \tag{2.8}$$

where

$$\Gamma(A) = \bigcup_{k\in\langle mn\rangle} \left\{ z \in \mathbb{C} : |z - a_{kk}| \le \sum_{\substack{j\in\langle mn\rangle,\\ j\ne k}} |a_{kj}| \right\},$$

and

$$\mathcal{K}(A) = \bigcup_{\substack{k,h\in\langle mn\rangle,\\ h\ne k}} \left\{ z \in \mathbb{C} : |z - a_{kk}||z - a_{hh}| \le \left(\sum_{\substack{j\in\langle mn\rangle,\\ j\ne k}} |a_{kj}| \right) \cdot \left(\sum_{\substack{j\in\langle mn\rangle,\\ j\ne h}} |a_{hj}| \right) \right\}.$$

Using (2.3) and replacing A with $\mathcal{A}$ in (2.8), we have

$$\Gamma(\mathcal{A})\bigcap\mathbb{C}^-\neq\emptyset \quad\text{and}\quad \mathcal{K}(\mathcal{A})\bigcap\mathbb{C}^-\neq\emptyset,$$

which contradicts with (2.7). Hence, $\mathcal{A}$ is M-positive definite. □

Next we use two examples to illustrate that above criteria are efficient.

Example 2.6 Consider the partially symmetric tensor $\mathcal{A}_1=(a_{i_1i_2i_3i_4})\in\mathbb{R}^{3\times3\times3\times3}$ with

$$\begin{aligned}
&a_{1111}=6,\ a_{2211}=0.2,\ a_{3211}=1.5,\ a_{3311}=2,\ a_{1221}=0.2,\ a_{2121}=1,\\
&a_{1231}=1.5,\ a_{1331}=2,\ a_{3131}=4,\ a_{1212}=3,\ a_{2112}=0.2,\ a_{3112}=1.5,\\
&a_{3212}=0.2,\ a_{1122}=0.2,\ a_{2222}=1,\ a_{3222}=-0.2,\ a_{3322}=-0.2,\\
&a_{1132}=1.5,\ a_{1232}=0.2,\ a_{2232}=-0.2,\ a_{2332}=-0.2,\ a_{3232}=4.3,\\
&a_{1313}=3,\ a_{3113}=2,\ a_{2323}=2,\ a_{3223}=-0.2,\ a_{1133}=2,\ a_{2233}=-0.2,
\end{aligned}$$

$a_{3333}=5$ and $a_{i_1i_2i_3i_4}=0$, otherwise.

By computing, we find that $\mathcal{A}_1$ satisfies both the condition (2.1) of Theorem 2.1 and the condition (2.5) of Corollary 2.3; therefore, $\mathcal{A}_1$ is M-positive definite. Additionally, we draw the regions generated by the set $\mathcal{K}(\mathcal{A}_1)$ and the set $\Gamma(\mathcal{A}_1)$ in Fig. 1. From Fig. 1, we know that

$$\mathcal{K}(\mathcal{A}_1)\subset\Gamma(\mathcal{A}_1)\text{ and }\Gamma(\mathcal{A}_1)\bigcap\mathbb{C}^-=\emptyset,$$

which means that $\mathcal{A}_1$ satisfies the condition (2.7) of Corollary 2.5; so, we also conclude that $\mathcal{A}_1$ is M-positive definite.

In fact, for all $x, y\in\mathbb{R}^3$ with $x^{\mathrm{T}}x=1$ and $y^{\mathrm{T}}y=1$, we have

$$\begin{aligned}
\mathcal{A}_1xyxy=\ &6x_1^2y_1^2+0.8x_1x_2y_1y_2+6x_1x_3y_1y_2+8x_1x_3y_1y_3+x_2^2y_1^2+4x_3^2y_1^2\\
&+3x_1^2y_2^2+0.4x_1x_3y_2^2+x_2^2y_2^2-0.4x_2x_3y_2^2-0.8x_2x_3y_2y_3+4.3x_3^2y_2^2\\
&+3x_1^2y_3^2+2x_2^2y_3^2+5x_3^2y_3^2
\end{aligned}$$

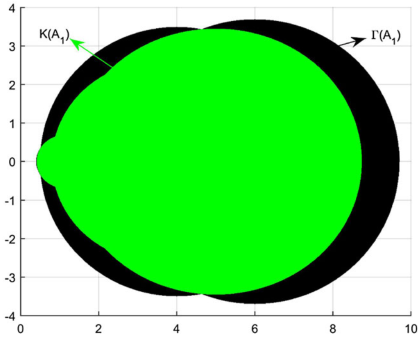

Fig. 1 $\mathcal{K}(\mathcal{A}_1)\subset\Gamma(\mathcal{A}_1)$ and $\Gamma(\mathcal{A}_1)\bigcap\mathbb{C}^-=\emptyset$

$$
\begin{aligned}
&= (0.8x_1y_1 + 0.5x_2y_2)^2 + (1.6x_1y_2 + 1.875x_3y_1)^2 + (2x_1y_1 + 2x_3y_3)^2 \\
&\quad +(0.5x_1y_2 + 0.4x_3y_2)^2 + (0.2x_2y_2 - x_3y_2)^2 + (0.8x_2y_2 - 0.5x_3y_3)^2 \\
&\quad +(1.36x_1^2 + x_2^2 + 0.484375x_3^2)y_1^2 + (0.19x_1^2 + 0.07x_2^2 + 3.14x_3^2)y_2^2 \\
&\quad +(3x_1^2 + 2x_2^2 + 0.75x_3^2)y_3^2 \\
&> 0.
\end{aligned}
$$

By Definition 1.1, we know that $\mathcal{A}_1$ is M-positive definite, which also illustrates that the proposed results are efficient.

Example 2.7 Consider the elastic modulus tensor $\mathcal{C} = (c_{i_1i_2i_3i_4}) \in \mathbb{R}^{3\times3\times3\times3}$ in equilibrium Eq. (1.1), where $\mathcal{C}$ belongs to the rhombic system with nine elasticities [3], that is,

$$
\begin{aligned}
&c_{1111} = 6,\ c_{2222} = 8,\ c_{3333} = 10,\ c_{1122} = 1,\ c_{2211} = 1,\ c_{2233} = 2,\ c_{3322} = 2, \\
&c_{1133} = 3,\ c_{3311} = 3,\ c_{2323} = 4,\ c_{3223} = 4,\ c_{2332} = 4,\ c_{3232} = 4,\ c_{1212} = 5, \\
&c_{2112} = 5, c_{1221} = 5,\ c_{2121} = 5,\ c_{1313} = 6,\ c_{3113} = 6,\ c_{1331} = 6,\ c_{3131} = 6
\end{aligned}
$$

and $c_{i_1i_2j_1j_2} = 0$, otherwise.

To identify whether equilibrium Eq. (1.1) is strongly elliptic or not, we transfer $\mathcal{C}$ into a partially symmetric tensor $\mathcal{A}_2 = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{3\times3\times3\times3}$ by taking

$$
a_{i_1i_2i_3i_4} = \frac{1}{4}(c_{i_1i_2i_3i_4} + c_{i_3i_2i_1i_4} + c_{i_1i_4i_3i_2} + c_{i_3i_4i_1i_2}),\ \forall i_1, i_2, i_3, i_4 \in \langle 3\rangle.
$$

By computing, we have

$$
\begin{aligned}
&a_{1111} = 6, R_{11}(\mathcal{A}_2) = 7.5; a_{1212} = 5, R_{12}(\mathcal{A}_2) = 3; a_{1313} = 6, R_{13}(\mathcal{A}_2) = 4.5; \\
&a_{2121} = 5, R_{21}(\mathcal{A}_2) = 3; a_{2222} = 8, R_{22}(\mathcal{A}_2) = 6; a_{2323} = 4, R_{23}(\mathcal{A}_2) = 3; \\
&a_{3131} = 6, R_{31}(\mathcal{A}_2) = 4.5; a_{3232} = 4, R_{32}(\mathcal{A}_2) = 3; a_{3333} = 10, R_{33}(\mathcal{A}_2) = 7.5.
\end{aligned}
$$

It is obvious that $\mathcal{A}_2$ does not satisfy condition (2.1) of Theorem 2.1 but satisfies condition (2.5) of Corollary 2.3; hence, $\mathcal{A}_2$ is M-positive definite, which means that equilibrium Eq. (1.1) is strongly elliptic. Additionally, we draw the regions generated by $\mathcal{K}(\mathcal{A}_2)$ and $\Gamma(\mathcal{A}_2)$ in Fig. 2. From Fig. 2, we know that

$$
\mathcal{K}(\mathcal{A}_2) \subset \Gamma(\mathcal{A}_2),\ \Gamma(\mathcal{A}_2) \bigcap \mathbb{C}^- \neq \emptyset,\ \text{but}\ \mathcal{K}(\mathcal{A}_2) \bigcap \mathbb{C}^- = \emptyset,
$$

which means that $\mathcal{A}_2$ satisfies the condition (2.7) of Corollary 2.5.

Therefore, we conclude that $\mathcal{A}_2$ is M-positive definite, and then equilibrium Eq. (1.1) is strongly elliptic.

Remark 2.8 Examples 2.6 and 2.7 also illustrate the conclusion of Remark 2.4 is correct.

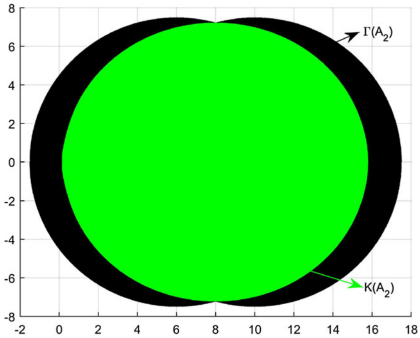

Fig. 2 $\mathcal{K}(\mathcal{A}_2) \subset \Gamma(\mathcal{A}_2)$, $\Gamma(\mathcal{A}_2) \bigcap \mathbb{C}^- \neq \emptyset$, but $\mathcal{K}(\mathcal{A}_2) \bigcap \mathbb{C}^- = \emptyset$

3 Conclusions

We give three easily verifiable sufficient conditions for the M-positive definiteness of fourth-order real partially symmetric tensors using the matrix unfolding technique of a tensor. Numerical examples show that the proposed results are efficient. Actually, except for the above three sufficient conditions, we can give many other checkable criteria for the M-positive definiteness by other subclasses of H-matrices and eigenvalue inclusion sets of matrices [12,13].

Acknowledgements Suhua Li's work is supported in part by Yunnan University's Research Innovation Fund for Graduate Students(Grant number 2018Z057); Yunnan Provincial Doctoral Graduate Academic Newcomer Award; China Scholarship Council (Grant number 201807030004).
Yaotang Li's work is supported by National Natural Science Foundations of China (Grant number 11861077).

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