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ORIGINAL PAPER

Connectedness in a Category

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Abstract

A new definition of connectedness of an object in a category with respect to a closure operator is given. It is shown that many of the classical results about connectedness of topological spaces, under mild conditions, hold in an arbitrary category. In particular it is shown that the image of a connected object is connected; that the union and the product of connected objects are connected. Several illustrative examples are provided.

Keywords Closure operator · (Pseudo-)quasi-complement · Connectedness · (Quasi)factorization structure

Mathematics Subject Classification 06A15 · 18A20 · 18B30 · 54D05

1 Introduction and Preliminaries

Closure operators and connectedness of objects in a category with respect to a closure operator have been investigated by several authors, [2–8,12,13], among others.

In the current paper, we give yet another definition of connectedness of an object in a category with respect to a closure operator.

In [13], the definition of connectedness is based on the idea that a topological space is connected if there is no nontrivial closed subset A of X, such that $X\backslash A$ is closed too.

The definition given in this paper is based on the idea that a topological space is connected if there are no two non-empty closed subsets A and B of X, such that $A \cap B = \emptyset$ and $X = A \cup B$ (or equivalently $X\backslash(A \cup B) = \emptyset$).

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It turns out that in a category, the connectedness (given in this paper) is generally stronger than that given in [13], and that under certain conditions the two coincide.

In [13], the category is assumed to be finitely complete and the class $\mathcal{M}$ of monomorphisms, on which the closure operator is defined, is assumed to be the right part of a proper and stable factorization structure; while in this paper, there is no assumption on the category and the class $\mathcal{M}$ is not as restricted.

We give several classical results about connectedness under mild hypotheses. In particular in Sect. 2, we give the definition of connectedness of an object in a category, providing several examples; in Sect. 3, we prove that the image of a connected object under a morphism is connected; in Sect. 4, we define two kind of unions (joins) and we prove that the union of mutually disjoint connected objects is connected; and in Sect. 5, we show that the product of two connected objects is connected.

Under conditions that the connectedness here and the one given in [13] coincide, the results not provided in [13], such as connectedness of union and connectedness of product of objects, can be deduced.

To this end, in the rest of this section, we establish some notation and give some preliminaries needed in the subsequent sections.

For a class $\mathcal{M}$ of morphisms, we denote by $\mathcal{M}/X$ the class of isomorphism classes of monomorphisms in $\mathcal{M}$ with codomain X.

Denoting by $a \wedge b$ the diagonal of the pullback of a along b and calling it meet of a and b, we have:

Definition 1.1 A class $\mathcal{M}$ of monomorphisms in a category $\mathcal{C}$ is called a domain, if:

(1) it contains all the identities;
(2) it is stable under pullbacks, i.e., for all $m \in \mathcal{M}/X$ and $f : Y \longrightarrow X$ in $\mathcal{C}$, the pullback $f^{-1}(m)$ of m along f exists and is in $\mathcal{M}/Y$;
(3) for all $X \in \mathcal{C}$, $\mathcal{M}/X$ is closed under binary meets;
(4) for all $X \in \mathcal{C}$, $\mathcal{M}/X$ has a minimum (also called zero).

A class $\mathcal{M}$ that satisfies only (1)–(3) is called a weak domain; and a domain that is closed under composition is called a strong domain.

We remark that conditions 1 and 2 yield that $\mathcal{M}$ contains all the isomorphisms and is closed under composition with isomorphisms, on both sides and that the meet induces a preorder, denoted by $\leq$, on $\mathcal{M}/X$ which is up to isomorphism a partial order.

Denoting the minimum of $\mathcal{M}/X$ by $o_X : O_X \longrightarrow X$, we have:

Lemma 1.2 *Let $\mathcal{M}$ be a strong domain.*

(a) *Suppose that in the commutative triangle:*

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "j"'] & & C \\ & B \arrow[ur, "m"'] & \end{tikzcd}$$

i, j, and m are in $\mathcal{M}$. Then, $i = o_C$ if and only if $j = o_B$. In this case, $O_B = O_C$.

(b) *For all $C \in \mathcal{C}$, $o_{O_C} = 1_{O_C}$.*

Proof (*a*) Suppose $i = o_C$. Given $k \in \mathcal{M}/B$, we have $mk \in \mathcal{M}/C$. Therefore, $i \leq mk$, implying $mj \leq mk$. Therefore, $j \leq k$. Hence, $j = o_B$.

Now, suppose that $j = o_B$. Given $k \in \mathcal{M}/C$, we have $m^{-1}(k) \in \mathcal{M}/B$. Therefore, $j \leq m^{-1}(k)$, implying that $mj \leq mm^{-1}(k)$. Therefore, $i = mj \leq mm^{-1}(k) = kk^{-1}(m) \leq k$.

(*b*) Given $C \in \mathcal{C}$, let $B = O_C$ and $m = o_C : B \longrightarrow C$. By part (*a*), we have $mo_B = o_C$. Therefore, $mo_B = m$, implying that $o_B = 1_B$, as required. □

For $a, b \in \mathcal{M}/X$, defining the joint negation $\neg\{a, b\}$ to be the largest element of $\mathcal{M}/X$ whose meet with both a and b is o_X and the negation $\neg a$ to be the largest element of $\mathcal{M}/X$ whose meet with a is o_X, we have:

Lemma 1.3 *Let $\mathcal{M}$ be a strong domain in $\mathcal{C}$ and $a, b \in \mathcal{M}/X$. Then:*

(a) *for all $f \in \mathcal{C}$, $f^{-1}(a \wedge b) = f^{-1}(a) \wedge f^{-1}(b)$.*
(b) *for all $m : M \longrightarrow X$ in $\mathcal{M}$, $m^{-1}(o_X) = o_M$.*
(c) *for all $m \in \mathcal{M}$, $m^{-1}(\neg\{a, b\}) = \neg\{m^{-1}(a), m^{-1}(b)\}$.*

Proof All parts can be proved by straightforward computations. Parts (*b*) and (*c*) use Lemma 1.2. That $\mathcal{M}$ be a strong domain is needed only in (*c*). □

2 Connectedness

Recall that, [8]:

Definition 2.1 Let $\mathcal{M}$ be a weak domain in $\mathcal{C}$. A closure operator $\mathfrak{c}$, on $\mathcal{C}$, relative to $\mathcal{M}$ is a family $\mathfrak{c} = (c_X)_{X\in\mathcal{C}}$ of maps $c_X : \mathcal{M}/X \to \mathcal{M}/X$, such that for every $X \in \mathcal{C}$:

(1) (Extension) $m \leq c_X(m)$ for all $m \in \mathcal{M}/X$;
(2) (Monotonicity) if $m \leq m'$ in $\mathcal{M}/X$, then $c_X(m) \leq c_X(m')$;
(3) (Continuity) $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$ for all $f : X \longrightarrow Y$ and $n \in \mathcal{M}/Y$.

A morphism $m \in \mathcal{M}/X$ is called $\mathfrak{c}$-closed if $c_X(m) = m$ and $\mathfrak{c}$-dense if $c_X(m) = 1_X$.

Note that for any morphism $f : X \longrightarrow Y$, if $n \in \mathcal{M}/Y$ is $\mathfrak{c}$-closed, then so is $f^{-1}(n) \in \mathcal{M}/X$; and that the conditions on $\mathcal{M}$ are weaker than those in [8], where $\mathcal{M}$ is assumed to be part of a factorization structure.

Definition 2.2 (a) Let $\mathcal{M}$ be a domain in $\mathcal{C}$ and $a, b \in \mathcal{M}/X$. b is said to be a quasi-complement of a if $a \wedge b = o_X$ and $\neg\{a, b\} = o_X$. In this case, we also say that a and b are quasi-complements.
(b) With a closure operator $\mathfrak{c}$ on $\mathcal{C}$ relative to $\mathcal{M}$, we say that a and b are $\mathfrak{c}$-closed quasi-complements, if they are $\mathfrak{c}$-closed and quasi-complements.

Remark 2.3 The pseudo-complement $\neg a$ of a (if it exists), see [13], is a quasi-complement of a.

Definition 2.4 Let $\mathcal{M}$ be a domain in $\mathcal{C}$ and $\mathfrak{c}$ be a closure operator on $\mathcal{C}$ relative to $\mathcal{M}$. An object X of $\mathcal{C}$ is said to be $\mathfrak{c}$-connected if whenever a and b in $\mathcal{M}/X$ are $\mathfrak{c}$-closed quasi-complements, then $a = o_X$ or $b = o_X$.

Remark 2.5 If an object X is connected, then the only closed quasi-complement of a non-zero closed subobject of X must be o_X. Therefore, by Remark 2.3, the closed pseudo-complement of a non-zero closed subobject of X (if it exists) is o_X. Therefore, X is connected in the sense of [13]. The converse holds whenever for all non-zero closed subobjects a of X, if a non-zero closed quasi-complement of a exists, then the pseudo-complement of a exists and is closed.

Therefore, if the category $\mathcal{C}$ satisfies the conditions of [13], and the above condition holds, then the connectedness here and the one given in [13] coincide.

Example 2.6 Let $\mathcal{C}$ be the category Top of topological spaces.

(1) Let $\mathcal{M}$ be the collection of initial monos and $\mathfrak{c}$ be the Kuratowski closure operator. Then, a topological space is $\mathfrak{c}$-connected if and only if it is connected in the classical sense.
(2) Let $\mathcal{M}$ be the collection of all monos. For each $X \in \mathcal{C}$ and a in $\mathcal{M}/X$, define $c_X(a)$ to be the inclusion $\bar{I_a} \hookrightarrow X$ with induced topology, where I_a is the image of A under a and $\bar{I_a}$ is the Kuratowski closure of I_a. One can verify that $\mathfrak{c} = \{c_X(a) : X \in \mathcal{C}, a \in \mathcal{M}/X\}$ is a closure operator on $\mathcal{C}$ relative to $\mathcal{M}$. Since $a \in \mathcal{M}/X$ is $\mathfrak{c}$-closed if and only if it is initial and closed with respect to Kuratowsky closure operator, the $\mathfrak{c}$-connectedness coincides with the classical one.

Recalling that a preradical r in the category R-mod of R-modules is a subfunctor of the identity functor of R-mod, any preradical r gives a closure operator c^r of R-mod, such that for a mono $m : B \longrightarrow A$, $c^r_A(B) = \pi^{-1}(r(A/I_m))$, where I_m is the image and π is the cokernel of m, [8].

Example 2.7 Let $\mathcal{C}$ be the category Ab of abelian groups and $\mathcal{M}$ be the collection of all monos.

(1) For the preradical t defined by the torsion subgroup,

$$t(A) = \{a \in A : (\exists n \in Z)n > 0 \text{ and } na = 0\}$$

the corresponding closure operator c^t, for $m \in \mathcal{M}/A$ is given by:

$$c^t_A(m) = \{a \in A : (\exists n \in Z)n > 0 \text{ and } na \in I_m\}.$$

   (a) Any abelian group A for which $t(A) \neq 0$ is c^t-connected because for any c^t-closed m and n in $\mathcal{M}/A$, $m \wedge n \neq 0$. Therefore, by Remark 2.5, it is connected in the sense of [13]. While for $A = \mathbb{Z}_2 \oplus \mathbb{Z}$, for which $t(A) \neq 0$, it is not connected in the sense of [7].
   (b) The free abelian group $\mathbb{Z}$ of integers is c^t-connected, because every non-zero $m \in \mathcal{M}/\mathbb{Z}$ is c^t-dense. Therefore, it is connected in the sense of [13]. It can be verified that it is not in the sense of [7].
   (c) The group $A = \mathbb{Z} \oplus \mathbb{Z}$, where $\oplus$ denotes the direct sum, is c^t-connected, because if b and c are any two non-zero c^t-closed members of $\mathcal{M}/A$, such that $b \wedge c = 0$, then for some integers $m, n, r, s \in \mathbb{Z}$, $I_b = \langle (m, n) \rangle$ and $I_c = \langle (r, s) \rangle$, and so for $d : \langle (m + r, n + s) \rangle \hookrightarrow A$, we have $d \wedge b =$

$d \wedge c = 0$, implying that $(\{b, c\} \Rightarrow 0_A) \neq 0_A$ if it exists.

Connectedness in the sense of [13] follows. It can be shown that it is not connected in the sense of [7].

(2) For the preradical f defined by the Frattini subgroup:

$$f(A) = \cap\{M : M \text{ is a maximal (proper) subgroup of } A\}$$

the corresponding closure operator c^f, for $m \in \mathcal{M}/A$ is given by:

$$c^f_A(m) = \cap\{M : M \text{ is a maximal subgroup of } A \text{ containing } I_m\}.$$

(a) $\mathbb{Z}_p$ is c^f-connected, because it does not have any nontrivial subgroup. Therefore, it is connectedness in the sense [13]. Simple verification shows that it is not in the sense of [7].

(b) Consider the abelian group $A = \mathbb{Z}_p \oplus \mathbb{Z}_q$. Since $(\mathbb{Z}_p \oplus \mathbb{Z}_q)/(\mathbb{Z}_p \oplus 0) \equiv \mathbb{Z}_q$, $f((\mathbb{Z}_p \oplus \mathbb{Z}_q)/(\mathbb{Z}_p \oplus 0)) \equiv f(\mathbb{Z}_q) = 0$, implying that $a : \mathbb{Z}_p \oplus 0 \hookrightarrow A$ is c^f-closed. Similarly, $b : 0 \oplus \mathbb{Z}_q \hookrightarrow A$ is c^f-closed. Since A is cyclic, the only nontrivial subgroups of A are $\mathbb{Z}_p \oplus 0$ and $0 \oplus \mathbb{Z}_q$. Therefore, if for $m \in \mathcal{M}/A$, $m \wedge a = 0$ and $m \wedge b = 0$, then $m = 0$. This shows (a, b) is a c^f-closed partition of A. Therefore, A is not c^f-connected in the sense of [13], and therefore not c^f-connected. It can be shown that it is not connected in the sense of [7].

(c) Let $A = \mathbb{Z}_{p^n}$ and $n \geq 2$. The subgroups of $\mathbb{Z}_{p^n}$ are $\langle p^i \rangle$ for $i = 0, ..., n$. Since $\mathbb{Z}_{p^n}/\langle p^i \rangle = Z_{p^i}$, we have $f(\mathbb{Z}_{p^n}/\langle p^i \rangle) = f(\mathbb{Z}_{p^i})$. If $i \geq 2$, then $f(\mathbb{Z}_{p^i}) \neq 0$. Therefore, the only c^f-closed subgroup of $\mathbb{Z}_{p^n}$ is $\langle p \rangle$. Thus, $\mathbb{Z}_{p^n}$ is c^f-connected. Therefore, it is c^f-connected in the sense of [13]. It is not connected in the sense of [7].

The following example shows connectedness and connectedness in the sense of [13] are not the same, due to non-existence of pseudo-complements.

Example 2.8 Let $\mathcal{C}$ be the category generated by the following preordered set:

$$\begin{tikzcd}
& A_1 \arrow[rr] \arrow[dr, "a_1"] & & A_2 \arrow[r] \arrow[dl, "a_2"'] & A_3 \arrow[dll, "a_3"] \\
O \arrow[ur] \arrow[dr] & & X & & \\
& B_1 \arrow[rr] \arrow[ur, "b_1"] & & B_2 \arrow[r] \arrow[ul, "b_2"] & B_3 \arrow[ull, "b_3"']
\end{tikzcd}$$

$\mathcal{M}$ be the collection of all morphisms and consider the identity closure operator $\mathfrak{c}$ on $\mathcal{C}$, relative to $\mathcal{M}$.

One can verify that $a_n \wedge b_m = o_X$ and $\neg\{a_n, b_m\} = o_X$. Since a_n and b_m are $\mathfrak{c}$-closed and non-zero, X is not $\mathfrak{c}$-connected. However. for all n, A_n and B_n are $\mathfrak{c}$-connected.

Since neither a_n nor b_n has a pseudo-complement, there is no $\mathfrak{c}$-closed partition of X. Thus. X is $\mathfrak{c}$-connected in the sense of [13]. One can verify that it is $\mathfrak{c}$-connected in the sense of [7].

In the following example, connectedness and connectedness in the sense of [13] are not the same, this time due to non-closedness of pseudo-complement.

Example 2.9 Let $\mathcal{C}$ be the category generated by the following preordered set:

$$\begin{tikzcd}
& A \arrow[r, "b"] \arrow[drr, "g"] & B \arrow[dr, "e"] & \\
O \arrow[ur, "a"] \arrow[dr, "c"'] & & & X \\
& C \arrow[r, "d"'] \arrow[urr, "h"] & D \arrow[ur, "f"'] &
\end{tikzcd}$$

where the two triangles commute.

With $\mathcal{M}$ the collection of all morphisms, we define the closure operator $\mathfrak{c}$ as follows:

In $\mathcal{C}/O, \mathcal{C}/A, \mathcal{C}/B, \mathcal{C}/C$, and $\mathcal{C}/D$, all morphisms are $\mathfrak{c}$-closed, while in $\mathcal{C}/X$, e and f are $\mathfrak{c}$-dense and the rest are $\mathfrak{c}$-closed.

The only $\mathfrak{c}$-closed subobjects of X are o_X, g, h and 1_X. One can easily verify that $g \wedge h = o_X$ and $\neg\{g, h\} = o_X$, so that g and h are non-zero $\mathfrak{c}$-closed quasi-complements. Hence, X is not $\mathfrak{c}$-connected. However, $\neg g = f$ and $\neg h = e$. Therefore, in the sense of [13], X is $\mathfrak{c}$-connected.

3 Image of a Connected Object is Connected

In this section, we introduce the notion of fine epi and use it to show that the image of a connected object under a fine epi is connected.

Definition 3.1 Let $\mathcal{M}$ be a domain in $\mathcal{C}$. A morphism $f : X \longrightarrow Y$ is said to be $\mathcal{M}$-fine epi, if for any $i : I \rightarrowtail Y$ in $\mathcal{M}$, $f^{-1}(i) = o_X$ implies $i = o_Y$.

When the domain $\mathcal{M}$ is the collection of all the monos whose pullback along every morphism exist, then $\mathcal{M}$-fine epi is also called fine epi.

Example 3.2 Let $\mathcal{M}$ be a domain and $f : X \longrightarrow Y$ be a morphism in $\mathcal{C}$.

(1) If $f^{-1}(o_Y) \neq o_X$, then f is $\mathcal{M}$-fine epi.

This follows from the fact that for $i \in \mathcal{M}/Y$, $f^{-1}(o_Y) \leq f^{-1}(i)$.

(2) If f is a retraction, then it is $\mathcal{M}$-fine epi.

Let $i : B \longrightarrow Y$ in $\mathcal{M}/Y$ be given, such that the following square is a pullback.

$$\begin{tikzcd}
O_X \arrow[r, "i^{-1}(f)"] \arrow[d, "f^{-1}(i)=o_X"'] & B \arrow[d, "i"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Since $i^{-1}(f)$ is a retraction, there is $s : B \longrightarrow O_X$, such that $i^{-1}(f)s = 1_B$. To show $i = o_Y$, let $a : A \longrightarrow Y$ be given in $\mathcal{M}/Y$. Since $o_X \leq f^{-1}(a)$, there is h, such that $o_X = f^{-1}(a)h$. We have $aa^{-1}(f)hs = ff^{-1}(a)hs = fo_Xs = ii^{-1}(f)s = i$, implying that $i \leq a$, as desired.

(3) Suppose that all morphisms in $\mathcal{C}$ with domain O_X are mono. If f is stable regular epi, then f is $\mathcal{M}$-fine epi.

With i as in part 2, $i^{-1}(f)$ is regular epi. Since it is by hypothesis mono, it is an isomorphism. An argument similar to part 2 shows that $i = o_Y$.

(4) Suppose that $\mathcal{C}$ is balanced and all morphisms in $\mathcal{C}$ with domain O_X are mono. If f is stable epi, then f is $\mathcal{M}$-fine epi.

(5) With $\mathcal{C}$ the following poset, $O \to A$ is epi but not $\mathcal{M}$-fine epi, while $A \to B$ is both epi and $\mathcal{M}$-fine epi.

$$O \longrightarrow A \longrightarrow B \ .$$

Extending the concept of essential module, [9], to essential homomorphism, by calling an R-module homomorphism $f : X \longrightarrow Y$ essential if Y is an essential extension of $f(X)$, which is equivalent to, for all $y \in Y$, $f(X) \cap Ry = 0$ yields $y = 0$, we have:

Example 3.3 In the category $Rmod$ of R-modules, with R a commutative ring with identity, a morphism $f : X \longrightarrow Y$ is fine epi, if and only if it is not mono or it is essential mono. In particular, every essential, hence epi, is fine epi.

Suppose that $f : X \longrightarrow Y$ is fine epi and mono. Let $y \in Y$, such that $f(X) \cap Ry = 0$. Since f is mono, we get $f^{-1}(Ry) = 0$. Since f is fine epi, $Ry = 0$ and so $y = 0$. Hence, f is essential mono.

Conversely if f is not mono, then $f^{-1}(0) \neq 0$. Therefore, by part (1) of Example 3.2, f is fine epi and if it is essential mono, then let $i : B \longrightarrow Y$ be mono, such that $f^{-1}(i) = 0$. For $b \in B$, set $y = i(b)$. Since $f^{-1}(Ry) = 0$, we get $f(X) \cap Ry = 0$. Therefore, $y = 0$ and thus, $b = 0$. Hence, $B = 0$ as desired.

Example 3.4 In the category Set of sets, a morphism is fine epi if and only if it is epi.

Suppose f is fine epi. Let $i : Y - f(X) \longrightarrow Y$ be the inclusion. Then, $f^{-1}(i) = \emptyset$. Thus, $Y - f(X) = \emptyset$, and so, $Y = f(X)$. Hence, f is epi. The converse is true by part 4 of Example 3.2.

Example 3.5 Let $(\mathcal{C}, |\ |)$ be a concrete category over $\mathcal{X}$, [1], $\mathcal{M}_\mathcal{C}$ be a domain in $\mathcal{C}$ and $\mathcal{M}$ be a domain in $\mathcal{X}$. Suppose that $|\ |$ takes $\mathcal{M}_\mathcal{C}$ elements to $\mathcal{M}$ elements, preserves pullbacks, and preserves and reflects zeros. If $|f|$ is $\mathcal{M}$-fine epi, then f is $\mathcal{M}_\mathcal{C}$-fine epi. If in addition structured $\mathcal{M}$ elements lift to $\mathcal{M}_\mathcal{C}$ elements, then the converse is true.

In particular, in the construct Top of topological spaces, a morphism is fine epi if and only if it is surjective.

Let $i : B \longrightarrow Y$ be in $\mathcal{M}_\mathcal{C}/Y$, such that $f^{-1}(i) = o_X$. Since $|\ |$ preserves pullbacks and zeros, $|f|^{-1}(|i|)) = o_{|X|}$. Now, $|f|$ is $\mathcal{M}$-fine epi, so $|i| = o_{|Y|}$. Since $|\ |$ reflects zeros, we get $i = o_Y$.

For the converse, let $i' : B' \longrightarrow |Y|$ be in $\mathcal{M}/|Y|$, such that $|f|^{-1}(i') = o_{|X|}$. Let $i : B \longrightarrow Y$ be a lift in $\mathcal{M}_{\mathcal{C}}$, of i'. By preservation of pullbacks, we have $|f^{-1}(i)| = |f|^{-1}(i') = o_{|X|}$, and so, $f^{-1}(i) = o_X$. Since f is $\mathcal{M}_{\mathcal{C}}$-fine epi, $i = o_Y$. Therefore, $i' = |i| = o_{|Y|}$.

Definition 3.6 A (weak, strong) domain $\mathcal{M}$ in $\mathcal{C}$ is said to have images, if for all $f : X \longrightarrow Y$, the map $f^{-1}() : \mathcal{M}/Y \longrightarrow \mathcal{M}/X$ has a left adjoint $f()$.

Example 3.7 Let $\mathcal{M}$ be a strong domain with images. If $f : X \to Y$ is a morphism, such that for any $i : B \to Y$ in $\mathcal{M}/Y$ for which $f^{-1}(i) = o_X$, $(i^{-1}(f))(1_{O_X}) = 1_B$, then f is $\mathcal{M}$-fine epi.

Let the morphism $i : B \longrightarrow Y$ in $\mathcal{M}/Y$ be such that $f^{-1}(i) = o_X$. The map $i^{-1}(f)() : \mathcal{M}/O_X \longrightarrow \mathcal{M}/B$ being a left adjoint, preserves zeros and by part (2) of Lemma 1.2, $o_{O_X} = 1_{O_X}$. Therefore, $o_B = i^{-1}(f)(o_{O_X}) = i^{-1}(f)(1_{O_X}) = 1_B$. Thus, by part 1 of Lemma 1.2, $o_Y = io_B = i1_B = i$.

Recalling that a collection $\mathcal{M}$ of morphisms is called a quasi-right factorization structure in $\mathcal{C}$, [11], if, for every $f \in \mathcal{C}$, there is a smallest $m \in \mathcal{M}$, called a quasi-right part of f, such that $f \leq m$; and that when $\mathcal{M}$ is a collection of monos whose pullbacks along every morphism exits, then a quasi-right factorization structure is a right factorization structure, we have:

Proposition 3.8 *A weak domain $\mathcal{M}$ is a (quasi) right factorization structure in $\mathcal{C}$ if and only if it is a weak domain with images.*

Proof The proof for quasi-right factorization structure follows from the fact that for each $f \in \mathcal{C}$ and $i \in \mathcal{M}$, such that fi is defined, a quasi-right part of fi corresponds to $f(i)$, [11]. Since, when $\mathcal{M}$ consists of monos, as is the case here, quasi-right factorization structures are right factorization structures, the result follows. $\square$

Example 3.9 Let the domain $\mathcal{M}$ be a quasi-right factorization structure in $\mathcal{C}$ that is closed under composition. If $f : X \to Y$ is a morphism, such that for any $i : B \to Y$ in $\mathcal{M}/Y$ for which $f^{-1}(i) = o_X$, a quasi,right part of $i^{-1}(f)$ is 1_B, then by Example 3.7, f is $\mathcal{M}$-fine epi.

Also $\mathcal{M}$ is a right factorization structure, and since it is closed under composition, by [8], there is a collection $\mathcal{E}$, such that $(\mathcal{E}, \mathcal{M})$ is a factorization structure. Therefore, if $f : X \to Y$ is a morphism, such that for any $i : B \to Y$ in $\mathcal{M}/Y$ for which $f^{-1}(i) = o_X$, the map $i^{-1}(f)$ is in $\mathcal{E}$, then f is $\mathcal{M}$-fine epi. In particular, if $\mathcal{E}$ is pullback stable, then every $\mathcal{E}$-morphism is $\mathcal{M}$-fine epi.

Saying a morphism f reflects (closed) quasi-complements, if whenever a and b are (closed) quasi-complements, then so are $f^{-1}(a)$ and $f^{-1}(b)$, we have:

Theorem 3.10 *Let $\mathcal{M}$ be a domain in $\mathcal{C}$ and $\mathfrak{c}$ be a closure operator on $\mathcal{C}$ relative to $\mathcal{M}$. Suppose that $f : X \longrightarrow Y$ is $\mathcal{M}$-fine epi and f reflects closed quasi-complements. If X is $\mathfrak{c}$-connected, then so is Y.*

Proof Let a and b in $\mathcal{M}/Y$ be closed quasi-complements. Since X is $\mathfrak{c}$-connected and $f^{-1}(a)$ and $f^{-1}(b)$ are $\mathfrak{c}$-closed, $f^{-1}(a) = o_X$ or $f^{-1}(b) = o_X$. f being a fine epi, yields $a = o_Y$ or $b = o_Y$. Therefore, Y is $\mathfrak{c}$-connected. □

Example 3.11 Since in the category Top and in every Topos, [10], implications and zeros are stable under pullbacks, every morphism reflects quasi-complements. Therefore, by Theorem 3.10, every fine epi preserves connectedness.

4 Union of Connected Objects is Connected

In this section, we first define two different joins (unions), and then, we show that the union of connected objects is connected in both cases, each under certain conditions.

Definition 4.1 Let $\mathcal{M}$ be a domain in $\mathcal{C}$. For i and j in $\mathcal{M}/X$, define $i \preceq j$ if for any $k \in \mathcal{M}/X$, $k \wedge j = o_X$ implies $k \wedge i = o_X$. We write $i \sim j$, whenever $i \preceq j$ and $j \preceq i$.

Lemma 4.2 *$(\mathcal{M}/X, \preceq)$ is a preordered class.*

Proof Obvious. □

Definition 4.3 Let $\mathcal{M}$ be a domain in $\mathcal{C}$ and $\{a_\alpha : \alpha \in I\} \subseteq \mathcal{M}/X$. $a \in \mathcal{M}/X$ is called:

(a) a join of $\{a_\alpha\}$, denoted by $\vee a_\alpha$, if it is a join relative to the preorder $\leq$.
(b) a prime join of $\{a_\alpha\}$, denoted by $\overset{p}{\curlyvee} a_\alpha$, if for any $b \in \mathcal{M}/X$, $b \wedge a = o_X$ if and only if for all $\alpha \in I$, $b \wedge a_\alpha = o_X$.

Proposition 4.4 *Let $\mathcal{M}$ be a domain in $\mathcal{C}$ and $\{a_\alpha : \alpha \in I\} \subseteq \mathcal{M}/X$.*

(a) *Both join and prime join, if they exist, are unique up to $\sim$.*
(b) *For any $b \in \mathcal{M}/X$, $b \wedge \overset{p}{\curlyvee} a_\alpha \sim \overset{p}{\curlyvee}(b \wedge a_\alpha)$.*
(c) *If $\vee a_\alpha$ exists and commutes with meet, then $\vee a_\alpha$ is a prime join.*

Proof Follows from straightforward computations. □

Example 4.5 Let $\mathcal{C}$ be the category Top of topological spaces.

(1) Let $\mathcal{M}$ be the collection of all monos. for $a : A \rightarrowtail X$ in $\mathcal{M}/X$, let $I_a = a(A)$ be the set image of A under a. One can easily verify that:

   (a) $a \wedge b$ is the inclusion $I_a \cap I_b \hookrightarrow X$, where for $i = a^{-1}(b)$ and $j = b^{-1}(a)$, the topology of $I_a \cap I_b$ is generated by the subbase $\{i^{-1}(G), j^{-1}(H) : G$ open in A and H open in $B\}$.
   (b) $a \leq b$ if and only if $I_a \subseteq I_b$ and that the inclusion $I_a \hookrightarrow I_b$ is continuous, where I_a and I_b have topologies induced by the isomorphisms $I_a \cong A$ and $I_b \cong B$, respectively.
   (c) $a \preceq b$ if and only if $I_a \subseteq I_b$.

   (d) The negations of a and $\{a, b\}$ in $\mathcal{M}/X$ are respectively the inclusions $\neg a : X - I_a \hookrightarrow X$ and $\neg\{a, b\} : X - (I_a \cup I_b) \hookrightarrow X$ with induced topologies.
   (e) $\vee a_\alpha$ is the inclusion $\cup I_{a_\alpha} \hookrightarrow X$, with the coinduced topology from all I_{A_α}.
   (f) $\overset{p}{\curlyvee} a_\alpha$ is the inclusion $\cup I_{a_\alpha} \hookrightarrow X$, with any topology containing the induced topology.

(2) Let $\mathcal{M}$ be the collection of all initial monos, i.e., monos with induced topology.

   (a) $a \wedge b$ is the inclusion $I_a \cap I_b \hookrightarrow X$, with induced topology.
   (b) $a \leq b$ if and only if $I_a \subseteq I_b$. In this case, the inclusion $I_a \hookrightarrow I_b$ is initial, where I_a and I_b have topologies induced by the isomorphisms $I_a \cong A$ and $I_b \cong B$, respectively.
   (c) $a \preceq b$ if and only if $I_a \subseteq I_b$, i.e., $a \preceq b$ if and only if $a \leq b$.
   (d) The negations of a and $\{a, b\}$ in $\mathcal{M}/X$ are, respectively, the inclusions $\neg a : X - I_a \hookrightarrow X$ and $\neg\{a, b\} : X - (I_a \cup I_b) \hookrightarrow X$ with induced topologies.
   (e) $\vee a_\alpha = \overset{p}{\curlyvee} a_\alpha$ is $\cup I_{a_\alpha} \hookrightarrow X$, with induced topology.

Example 4.6 Let $\mathcal{C}$ be the category $Rmod$ of R-modules over a commutative ring R with identity and $\mathcal{M}$ be the collection of all monos. for $a : A \rightarrowtail X$ in $\mathcal{M}/X$, let $I_a = a(A)$ be the set image of A under a. One can easily verify that:

(1) $a \wedge b$ is $I_a \cap I_b \hookrightarrow X$, with submodule structure.
(2) $a \leq b$ if and only if $I_a \subseteq I_b$ and that the inclusion $I_a \hookrightarrow I_b$ is a module homomorphism, where I_a and I_b have module structures induced by the isomorphisms $I_a \cong A$ and $I_b \cong B$, respectively.
(3) $a \preceq b$ if and only if $I_a \cap I_b \hookrightarrow I_a$ is essential.
(4) The negations, $\neg a$ and $\neg\{a, b\}$, do not exist in general.
(5) $\vee a_\alpha$ is the inclusion $\Sigma I_{a_\alpha} \hookrightarrow X$ with submodule structure, where for each α, I_{a_α} has the module structure induced by the isomorphism $I_{a\alpha} \cong A_\alpha$.
(6) The existence of $\overset{p}{\curlyvee} a_\alpha$, in general, is not known.

Example 4.7 Let $\mathcal{C}$ be the category generated by the following preordered set:

$$\begin{tikzcd}
 & A_1 \arrow[rr] \arrow[dr, "a_1"] & & A_2 \arrow[rr] \arrow[dl, "a_2"'] & & A_3 \arrow[dlll, "a_3"'] \\
O \arrow[ur] \arrow[dr] & & X & & & \\
 & B_1 \arrow[rr] \arrow[ur, "b_1"] & & B_2 \arrow[rr] \arrow[ul, "b_2"] & & B_3 \arrow[ulll, "b_3"]
\end{tikzcd}$$

which has pullbacks. Let $\mathcal{M}$ be the collection of all morphisms, $\mathcal{A} = \{a_i : i \in \mathbb{N}\} \subseteq \mathcal{M}/X$ and $\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}$.

Every a_i is a prime join of $\mathcal{A}'$. If $\mathcal{A}'$ is finite, then the join of $\mathcal{A}'$ is the $a_i \in \mathcal{A}'$ with maximum index, which is a stable join as well; and if $\mathcal{A}'$ is infinite, then the join of $\mathcal{A}'$ is 1_X, which is not a stable join. Since for any b_j, $\bigvee_{a_i \in \mathcal{A}'}(a_i \wedge b_j) = O$, but $(\bigvee_{a_i \in \mathcal{A}'} a_i) \wedge b_j = 1_X \wedge b_j = b_j$, we conclude that 1_X is not a prime join of $\mathcal{A}'$.

For any a_i and b_j, $a_i \vee b_j = 1_X$, which is not stable because $(a_i \vee b_j) \wedge a_{i+1} = 1_X \wedge a_{i+1} = a_{i+1}$, but $(a_i \wedge a_{i+1}) \vee (b_j \wedge a_{i+1}) = a_i \vee 0 = a_i$. One can verify that 1_X is also a prime join of a_i and b_j

Example 4.8 Let $\mathcal{C}$ be the category generated by the preordered set:

$$\begin{tikzcd} & & B \arrow[dr, "b"] & \\ 0 \arrow[r] \arrow[drr] & A \arrow[ur, "a"] \arrow[r, "a'"] & C \arrow[r, "c"] & X \\ & & D \arrow[ur, "d"'] & \end{tikzcd}$$

which has pullbacks. Let $\mathcal{M}$ be the collection of all morphisms. One can easily verify that in $\mathcal{M}/X$, 1_X is the join of b and c, and that ca', b and c are prime joins of b and c.

Example 4.9 Let $\mathcal{C}$ be the category generated by the preordered set:

$$\begin{tikzcd} & A \arrow[drr] \arrow[drrr, bend left] & & & \\ 0 \arrow[ur] \arrow[r] \arrow[dr] & B \arrow[rrr, bend left] \arrow[rr, bend right] & \dots \arrow[r] & D_2 \arrow[r] & D_1 \\ & C \arrow[urr] \arrow[urrr, bend right] & & & \end{tikzcd}$$

which has pullbacks. Let $\mathcal{M}$ be the collection of all morphisms. Denoting the maps from A to D_i by a_i, etc, and the maps from D_i to D_1 by d_i, consider a_1 and b_1 in $\mathcal{M}/D_1$. Neither a join nor a prime join of a_1 and b_1 exists.

Lemma 4.10 *Let $\mathcal{M}$ be a strong domain in $\mathcal{C}$. Then, every $m : M \longrightarrow X$ in $\mathcal{M}/X$ reflects quasi-complements.*

Proof Let $a, b \in \mathcal{M}/X$ be quasi-complements. Using Lemma 1.3, we have $m^{-1}(a) \wedge m^{-1}(b) = m^{-1}(a \wedge b) = m^{-1}(o_X) = o_M$, and $\neg\{m^{-1}(a), m^{-1}(b)\} = m^{-1}(\neg\{a, b\}) = m^{-1}(o_X) = o_M$. Hence, $m^{-1}(a)$ and $m^{-1}(b)$ are quasi-complements. □

Theorem 4.11 *Let $\mathcal{M}$ be a strong domain in $\mathcal{C}$ and $\mathfrak{c}$ be a closure operator on $\mathcal{C}$ relative to $\mathcal{M}$. Suppose that $m : M \to X$ in $\mathcal{M}$ is $\mathfrak{c}$-dense and for any $k \in \mathcal{M}/X$, $m \preceq k$ implies $c_X(m) \leq c_X(k)$. If M is $\mathfrak{c}$-connected, then so is X.*

Proof Let i and j in $\mathcal{M}/X$ be closed quasi-complements. Using Lemma 4.10, $m^{-1}(i)$ and $m^{-1}(j)$ are closed quasi-complements. Since M is $\mathfrak{c}$-connected, $m^{-1}(i) = o_M$ or $m^{-1}(j) = o_M$. In case $m^{-1}(i) = o_M$, by Lemma 1.2, we get $m \wedge i = o_X$. Now, $m \preceq j$, because if $k \wedge j = o_X$, then $k \wedge m \wedge i = o_X$ and $k \wedge m \wedge j = o_X$ implies $k \wedge m = o_X$. Now, by hypothesis, $c_X(m) \leq c_X(j)$. Since m is dense and j is closed, $1_X \leq j$, implying that $j = 1_X$. Since $i \wedge j = o_X$, we get $i = o_X$. Similarly, in case $m^{-1}(j) = o_M$, we get $j = o_X$. Hence, X is $\mathfrak{c}$-connected. □

Calling a collection $\{a_\alpha\} \subseteq \mathcal{M}/X$ mutually intersecting if for all α and β, $a_\alpha \wedge a_\beta \neq o_X$, we have:

Theorem 4.12 *Let $\mathcal{M}$ be a strong domain in $\mathcal{C}$ and $\mathfrak{c}$ be a closure operator on $\mathcal{C}$ relative to $\mathcal{M}$. Suppose that the mutually intersecting collection $\{a_\alpha : A_\alpha \to X\} \subseteq \mathcal{M}/X$ has a join $a : A \to X$ that commutes with meet. If for every α, A_α is $\mathfrak{c}$-connected, then so is A.*

Proof Without loss of generality, we assume for all α, $a_\alpha \neq o_X$. Let i and j in $\mathcal{M}/A$ be closed quasi-complements. Since a is a join of $\{a_\alpha\}$, for each α, a morphism $b_\alpha : A_\alpha \longrightarrow A$ exists, such that $a_\alpha = ab_\alpha$. By the fact that for every α, b_α is in $\mathcal{M}$, Lemma 4.10 implies $b_\alpha^{-1}(i)$ and $b_\alpha^{-1}(j)$ are closed quasi-complements. Since A_α is connected, for each α, $b_\alpha^{-1}(i) = o_{A_\alpha}$ or $b_\alpha^{-1}(j) = o_{A_\alpha}$.

If there is α_0, such that $b_{\alpha_0}^{-1}(i) \neq o_{A_{\alpha_0}}$, and there is α_1, such that $b_{\alpha_1}^{-1}(j) \neq o_{A_{\alpha_1}}$, then by above $b_{\alpha_0}^{-1}(j) = o_{A_{\alpha_0}}$ and $b_{\alpha_1}^{-1}(i) = o_{A_{\alpha_1}}$. By Lemma 1.2, it follows that $(b_{\alpha_0} \wedge b_{\alpha_1}) \wedge i = o_A$ and $(b_{\alpha_0} \wedge b_{\alpha_1}) \wedge j = o_A$. Therefore, $b_{\alpha_0} \wedge b_{\alpha_1} = o_A$. Thus, $a_{\alpha_0} \wedge a_{\alpha_1} = a(b_{\alpha_0} \wedge b_{\alpha_1}) = o_X$, which is a contradiction whether $\alpha_0 = \alpha_1$ or not. Therefore, either for every α, $b_\alpha^{-1}(i) = o_{A_\alpha}$ or for every α, $b_\alpha^{-1}(j) = o_{A_\alpha}$.

If for every α, $b_\alpha^{-1}(i) = o_{A_\alpha}$, then $a_\alpha \wedge (ai) = (ab_\alpha) \wedge (ai) = a(b_\alpha \wedge i) = o_X$. Since a commutes with binary meet, $a \wedge (ai) = o_X$. Therefore, $ai = o_X$, implying that $i = o_A$. Similarly if for every α, $b_\alpha^{-1}(j) = o_{A_\alpha}$, then $j = o_A$. Hence, either $i = o_A$ or $j = o_A$. $\square$

Example 4.13 Let Top be the category of topological spaces, $\mathcal{M}$ be the collection of initial monos, and $\mathfrak{c}$ be the Kuratowski closure operator. By Example 2.6 and the fact that all the requirements of Theorem 4.12 are met, the theorem gives the known classic result about topological spaces.

Definition 4.14 Let $\mathcal{M}$ be a (weak) domain in $\mathcal{C}$. A closure operator $\mathfrak{c}$ on $\mathcal{C}$ relative to $\mathcal{M}$ is said to be strongly continuous if for $a : A \to X$ in $\mathcal{M}/X$ satisfying $a \sim 1_X$, we have $ac_A(i) = c_X(ai)$, for every $i \in \mathcal{M}/A$.

Remark 4.15 If the preorders $\preceq$ and $\leq$ on $\mathcal{M}/X$ coincide, then every closure operator relative to $\mathcal{M}$ is strongly continuous.

Example 4.16 Let $\mathcal{C}$ be the category Top of topological spaces.

(1) Let $\mathcal{M}$ be the collection of all initial monos, [1]. Then, any closure operator relative to $\mathcal{M}$, in particular the Kuratowski closure operator, is strongly continuous, because for $a \in \mathcal{M}/X$, $a \sim 1_X$ if and only if $a \cong 1_X$.
(2) Let $\mathcal{M}$ be the collection of all monos and $\mathfrak{c}$ be as in Example 2.6. $\mathfrak{c}$ is not strongly continuous, because for $a = 1_X : (X, \mathcal{T}_1) \longrightarrow (X, \mathcal{T}_2)$, where $\mathcal{T}_1 \supseteq \mathcal{T}_2$, $a \sim 1_{(X,\mathcal{T}_1)}$, but it is not necessarily closed.

Example 4.17 Let $\mathcal{C}$ be the category Ab of abelian groups.

(1) The closure operator c^t introduced in Example 2.7 is not strongly continuous, because for $a : \{\bar{0}, \bar{2}\} \hookrightarrow \mathbb{Z}_4$, $a \sim 1_{\mathbb{Z}_4}$, but it is not c^t-closed.

(2) Also the closure operator c^f introduced in Example 2.7 is not strongly continuous because for $a : 4\mathbb{Z} \hookrightarrow \mathbb{Z}$, $a \sim 1_{\mathbb{Z}}$, but it is not c^f-closed.

Lemma 4.18 *Let $\mathcal{M}$ be a strong domain in $\mathcal{C}$. Consider the following pullback square:*

$$\begin{tikzcd} D \arrow[r, "b^{-1}(a)"] \arrow[d, "a^{-1}(b)"'] & B \arrow[d, "b"] \\ A \arrow[r, "a"'] & X \end{tikzcd}$$

If $a \preceq b$, then

(a) $a^{-1}(b) \sim 1_A$.
(b) *Let $\mathfrak{c}$ be a strongly continuous closure operator on $\mathcal{C}$ relative to $\mathcal{M}$. If $j \in \mathcal{M}/B$ is $\mathfrak{c}$-closed, then so is $a^{-1}(bj)$.*

Proof (a) Let $k \wedge a^{-1}(b) = o_A$. We have $(ak) \wedge b = a(k \wedge a^{-1}(b)) = ao_A = o_X$. Since $a \preceq b$, $(ak) \wedge a = o_X$. Therefore, $ak = o_X$, and thus, by Lemma 1.2, $k = o_A$.
(b) Let $d = b^{-1}(a)$. Since j is $\mathfrak{c}$-closed, $d^{-1}(j)$ is $\mathfrak{c}$-closed. By part (a), we have $a^{-1}(bj) = a^{-1}(b)d^{-1}(j) = a^{-1}(b)c_D(d^{-1}(j)) = c_A(a^{-1}(b)d^{-1}(j)) = c_A(a^{-1}(bj))$, as required.

□

Lemma 4.19 *Let $\mathcal{M}$ be a strong domain in $\mathcal{C}$, and let $a : A \longrightarrow X$ and $b : B \longrightarrow X$ be in $\mathcal{M}/X$, such that $a \preceq b$. If $i, j \in \mathcal{M}/B$ are quasi-complements, then so are $a^{-1}(bi)$ and $a^{-1}(bj)$.*

Proof Using Lemma 1.3, we have $a^{-1}(bi) \wedge a^{-1}(bj) = a^{-1}(b(i \wedge j)) = a^{-1}(o_X) = o_A$. To show $\neg\{a^{-1}(bi), a^{-1}(bj)\} = o_A$, let $k \in \mathcal{M}/A$, such that $k \wedge a^{-1}(bi) = o_A$ and $k \wedge a^{-1}(bj) = o_A$. The former yields $b(b^{-1}(ak) \wedge i) = ak \wedge bi = a(k \wedge a^{-1}(bi)) = o_X$ and so by Lemma 1.2, $b^{-1}(ak) \wedge i = o_B$; and the latter gives $b^{-1}(ak) \wedge j = o_B$. We get $b^{-1}(ak) = o_B$. Therefore, $ak \wedge b = bb^{-1}(ak) = o_X$. Since $a \preceq b$, $ak \wedge a = o_X$, and since $ak \leq a$, $ak = o_X$. Thus, $k = o_A$. □

Note that the above Lemma holds, when the inequality $a \preceq b$ is replaced by the stronger inequality $a \leq b$.

Theorem 4.20 *Let $\mathcal{M}$ be a strong domain in $\mathcal{C}$ and $\mathfrak{c}$ be a strongly continuous closure operator on $\mathcal{C}$ relative to $\mathcal{M}$. Suppose that the mutually intersecting collection $\{a_\alpha : A_\alpha \to X\} \subseteq \mathcal{M}/X$ has a prime join $a : A \to X$. If for every α, A_α is $\mathfrak{c}$-connected, then so is A.*

Proof Without loss of generality, we assume for all α, $a_\alpha \neq o_X$. Let $i, j \in \mathcal{M}/A$ be closed quasi-complements. Now suppose $i \neq o_A$ and $j \neq o_A$. By Lemma 1.2, we have $ai \wedge a = ai \neq o_X$ and $aj \wedge a = aj \neq o_X$. Since a is a prime join, there exist α_0 and α_1, such that $ai \wedge a_{\alpha_0} \neq o_X$ and $aj \wedge a_{\alpha_1} \neq o_X$. The former yields $a_{\alpha_0}a_{\alpha_0}^{-1}(ai) \neq o_X$, implying that $a_{\alpha_0}^{-1}(ai) \neq o_{A_{\alpha_0}}$ and the latter gives $a_{\alpha_1}^{-1}(aj) \neq o_{A_{\alpha_1}}$. Since $a_{\alpha_0} \preceq a$ and

$a_{\alpha_1} \preceq a$, by Lemma 4.19, $a_{\alpha_0}^{-1}(ai)$ and $a_{\alpha_0}^{-1}(aj)$ in $\mathcal{M}/A_{\alpha_0}$ and $a_{\alpha_1}^{-1}(ai)$ and $a_{\alpha_1}^{-1}(aj)$ in $\mathcal{M}/A_{\alpha_1}$, are closed quasi-complements.

Since A_{α_0} and A_{α_1} are $\mathfrak{c}$-connected, $a_{\alpha_0}^{-1}(ai) \neq o_{A_{\alpha_0}}$ and $a_{\alpha_1}^{-1}(aj) \neq o_{A_{\alpha_1}}$, we get $a_{\alpha_0}^{-1}(aj) = o_{A_{\alpha_0}}$ and $a_{\alpha_1}^{-1}(ai) = o_{A\alpha_1}$. We have $a(a^{-1}(a_{\alpha_1}) \wedge i) = a_{\alpha_1} \wedge ai = a_{\alpha_1}a_{\alpha_1}^{-1}(ai) = a_{\alpha_1}o_{A_{\alpha_1}} = o_X$, implying that $a^{-1}(a_{\alpha_1}) \wedge i = o_A$. Similarly, $a^{-1}(a_{\alpha_0}) \wedge j = o_A$. So $a^{-1}(a_{\alpha_0} \wedge a_{\alpha_1}) \wedge i = o_A$ and $a^{-1}(a_{\alpha_0} \wedge a_{\alpha_1}) \wedge j = o_A$, and thus, $a^{-1}(a_{\alpha_0} \wedge a_{\alpha_1}) = o_A$. Therefore, $a \wedge (a_{\alpha_0} \wedge a_{\alpha_1}) = o_X$, implying that $a_{\alpha_0} \wedge a_{\alpha_1} = o_X$, which is a contradiction whether $\alpha_0 = \alpha_1$ or not. Therefore, either $i = o_A$ or $j = o_A$, as desired. $\square$

Remark 4.21 If the preorders $\preceq$ and $\leq$ on $\mathcal{M}/X$ coincide, then Theorems 4.12 and 4.20 coincide.

The following proposition is needed in the next section.

Proposition 4.22 *Let $\mathcal{M}$ be a strong domain in $\mathcal{C}$ and let $\mathfrak{c}$ be a strongly continuous closure operator on $\mathcal{C}$ relative to $\mathcal{M}$. For $a : A \to X$ and $b : B \to X$ in $\mathcal{M}/X$, such that $a \sim b$, if A is $\mathfrak{c}$-connected, then so is B.*

Proof Let $i, j \in \mathcal{M}/B$ be closed quasi-complements. Since $a \preceq b$, by Lemma 4.18, $a^{-1}(bi)$ and $a^{-1}(bj)$ are $\mathfrak{c}$-closed and by Lemma 4.19, they are closed quasi-complements. Since A is $\mathfrak{c}$-connected, $a^{-1}(bi) = o_A$ or $a^{-1}(bj) = o_A$. If $a^{-1}(bi) = o_A$, then by Lemma 1.2, $a \wedge bi = o_X$. Since $b \preceq a$, $b \wedge bi = o_X$ and since $bi \leq b$, $bi = o_X$. Thus, $i = o_B$. Similarly, if $a^{-1}(bj) = o_A$, then $j = o_B$. $\square$

5 Product of Connected Objects is Connected

In this section, we prove that under certain conditions, the product of connected objects is connected.

Definition 5.1 Suppose $\mathcal{C}$ has a terminal object T and $\mathcal{M}$ is a weak domain in $\mathcal{C}$. For morphisms $t : T \rightarrowtail X$ and $s : T \rightarrowtail Y$ in $\mathcal{M}$, an (s, t)-copy of X and Y is a prime join, in $\mathcal{M}/X \times Y$, of $1_X \times s$ and $t \times 1_Y$.

Definition 5.2 Let $\mathcal{C}$ be a category with a strict initial object O. We say that a domain $\mathcal{M}$ in $\mathcal{C}$ has a common zero if for each object X, the unique map $!_X : O \longrightarrow X$ belongs to $\mathcal{M}/X$.

Note that when $\mathcal{M}$ has a common zero, then for each object X, $O_X = O$ and $!_X = o_X : O \longrightarrow X$.

Proposition 5.3 *Suppose $\mathcal{C}$ has a strict initial object O and a terminal object $T \neq O$. Let $\mathcal{M}$ be a strong domain with a common zero in $\mathcal{C}$ and $\mathfrak{c}$ be a strongly continuous closure operator on $\mathcal{C}$ relative to $\mathcal{M}$. Suppose also that there is a non-empty collection $\{t_i : T \to X | i \in I\} \subseteq \mathcal{M}$, a morphism $s : T \to Y$ in $\mathcal{M}$ and for each $i \in I$, an (s, t_i)-copy, $w_i : W_i \rightarrowtail X \times Y$, of X and Y. If $w : W \rightarrowtail X \times Y$ is a prime join of all the w_i's and X and Y are $\mathfrak{c}$-connected, then so is W.*

Proof Since $X \times T \cong X$, by Theorem 3.10, $X \times T$ is $\mathfrak{c}$-connected. Similarly, $T \times Y$ is $\mathfrak{c}$-connected. For each i, the pullback square:

$$\begin{tikzcd} T \times T \arrow[r, "t_i \times 1_T"] \arrow[d, "1_T \times s"'] & X \times T \arrow[d, "1_X \times s"] \\ T \times Y \arrow[r, "t_i \times 1_Y"'] & X \times Y \end{tikzcd}$$

shows that $(1_X \times s) \wedge (t_i \times 1_Y) \neq o_{X\times Y}$. Therefore, by Theorem 4.20, W_i is $\mathfrak{c}$-connected. Now, if $w_i \wedge w_j = o_{X\times Y}$, then since $w_j = (1_X \times s) \overset{p}{\curlyvee} (t_j \times 1_Y)$, we get $w_i \wedge (1 \times s) = o_{X\times Y}$. This in turn implies $(1 \times s) \wedge (t_i \times 1) = o_{X\times Y}$, which is a contradiction. Hence, w_is are mutually intersecting. By Theorem 4.20, W is $\mathfrak{c}$-connected. □

Theorem 5.4 *Suppose $\mathcal{C}$ has a strict initial object O, a terminal object $T \neq O$, and for any object $A \neq O$, there is a morphism $T \longrightarrow A$ in $\mathcal{C}$. Let $\mathcal{M}$ be a strong domain with common zero in $\mathcal{C}$ and $\mathfrak{c}$ be a strongly continuous closure operator on $\mathcal{C}$ relative to $\mathcal{M}$. Suppose also that the collection $\mathcal{T} = \{t_i : T \to X | i \in I\}$ of all morphisms from T to X is contained in $\mathcal{M}$, there exists a morphism $s : T \to Y$ in $\mathcal{M}$, and that for each $i \in I$, there exists an (s, t_i)-copy of X and Y. If X and Y are $\mathfrak{c}$-connected, then so is $X \times Y$.*

Proof For each $i \in I$, let $w_i : W_i \longrightarrow X \times Y$ be an (s, t_i)-copy of X and Y. We show $\overset{p}{\curlyvee} w_i = 1_{X\times Y}$. Let $a : A \longrightarrow X \times Y$ be in $\mathcal{M}/X \times Y$, such that for all $i \in I$, $a \wedge w_i = o_{X\times Y}$. So for all i, $a \wedge (t_i \times 1_Y) = o_{X\times Y}$. If $a \neq o_{X\times Y}$, then there is $t : T \longrightarrow A$ in $\mathcal{C}$. Now, the composition $T \xrightarrow{t} A \xrightarrow{a} X \times Y \xrightarrow{\pi_1} X$ is in $\mathcal{T}$ and so equals, say, t_j. In the following diagram, all the squares are pullbacks:

$$\begin{tikzcd} O \arrow[r] \arrow[d] & O \arrow[r] \arrow[d] & T \times Y \arrow[r] \arrow[d, "t_j \times 1_Y"] & T \arrow[d, "t_j"] \\ T \arrow[r, "t"'] \arrow[rrr, bend right, "t_j"'] & A \arrow[r, "a"'] & X \times Y \arrow[r, "\pi_1"'] & X \end{tikzcd}$$

This is a contradiction, because pullback of t_j along itself is 1_T. Therefore, $a = o_{X\times Y}$. Thus, $1_{X\times Y}$ is a prime join of w_i's. By Proposition 5.3, $X \times Y$ is $\mathfrak{c}$-connected. □

Example 5.5 (1) For $\mathcal{C}$ the category Top of topological spaces, it can be easily verified that the collection $\mathcal{M}$ of all the monos whose pullbacks along every morphism exists is a strong domain with common zero. By Examples 4.5 and 4.16, Theorem 5.4 generalizes the classical result.

(2) For $\mathcal{C}$ a topos, [10], in which for any object $A \neq O$, there is a morphism $T \longrightarrow A$, by Theorem 5.4, product of connected objects is connected.

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