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ORIGINAL PAPER

On the Cofiniteness of Small-Level Generalized Local Cohomology Modules

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Abstract
Let R be a commutative Noetherian ring and M, N be two finitely generated R-modules. In this note, we prove the cofiniteness of generalized local cohomology modules $H_I^i(M, N)$ with respect to I for all $i < t$ and the finiteness of $(0 :_{H_I^t(M,N)} I)$ provided $H_I^i(M, N)_{\mathfrak{p}}$ is finitely generated for all $i < t$ and all $\mathfrak{p} \in \bigcup_{j<t} \operatorname{Supp}_R(H_I^j(M, N))$ with $\dim R/\mathfrak{p} > 1$, where t is a given non-negative integer. This extends the results of Bahmanpour and Naghipour (J Algebra 321:1997–2011, 2009), Bahmanpour et al. (Commun Algebra 41:2799–2814, 2013), and Cuong et al. (Kyoto J Math 55(1):169–185, 2015). This also provides a partially affirmative answer to Hartshorne's question in Hartshorne (Invent Math 9:145–164, 1970) for the case of generalized local cohomology modules.

Keywords Cofinite · Minimax · Artinian · Generalized local cohomology

Mathematics Subject Classification 13D45 · 13E99 · 18G60

1 Introduction

Throughout this note the ring R is commutative Noetherian. Let M and N be two finitely generated R-modules and I be an ideal of R. In [10], A. Grothendieck conjectured that if I is an ideal of R and N is a finitely generated R-module, then $(0 :_{H_I^j(N)} I)$ is finitely generated for all $j \geq 0$. R. Hartshorne provided a counterex-

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ample to this conjecture in [11]. He also defined an R-module K to be I-cofinite if $\operatorname{Supp}_R(K) \subseteq V(I)$ and $\operatorname{Ext}^j_R(R/I, K)$ is finitely generated for all $j \geq 0$ and he asked a question:

Question. For which rings R and ideals I are the modules $H^j_I(N)$ is I-cofinite for all j and all finitely generated modules N?

Hartshorne showed that if N is a finitely generated R-module and I a prime ideal with $\dim R/I = 1$, where R is a complete regular local ring, then $H^j_I(N)$ is I-cofinite (see [11, Corollary 7.7]). Yoshida [21, Theorem 1.1] refined this result to more general situation that if N is a finitely generated module over a commutative Noetherian local ring R and I is an ideal of R such that $\dim(R/I) = 1$, then $H^j_I(N)$ are I-cofinite for all $j \geq 0$. In 2009, Bahmanpour–Naghipour have extended this result to the case of non-local ring; more precisely, they showed that *if t is a non-negative integer such that* $\dim \operatorname{Supp}_R(H^j_I(N)) \leq 1$ for all $j < t$ then $H^j_I(N)$ *is I-cofinite for all $j < t$ and* $(0 :_{H^t_I(N)} I)$ *is finitely generated* (see [3, Theorem 2.6], or [6]). In 2013, Bahmanpour–Naghipour–Sedghi improved this result by replacing the condition that "$\dim \operatorname{Supp}_R(H^j_I(N)) \leq 1$ for all $j < t$" with the condition "$H^j_I(N)_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$ for all $j < t$ and all $\mathfrak{p} \in \operatorname{Supp}_R(N/IN)$ with $\dim R/\mathfrak{p} > 1$" (see [2, Proposition 3.1]).

There are some generalizations of the theory of local cohomology modules. The following generalization of local cohomology theory is given by J. Herzog in [12]: let j be a non-negative integer and M, N finitely generated R-modules. Then the jth generalized local cohomology module of M, N with respect to I is defined by $H^j_I(M, N) = \varinjlim_n \operatorname{Ext}^j_R(M/I^nM, N)$. These modules were studied further in many research papers such as [4–8,13,15,19,20],.... Note that $H^j_I(R, N)$ is just the ordinary local cohomology module $H^j_I(N)$.

The purpose of this paper is to investigate a similar question as above for the theory of generalized local cohomology. Our main result is the following theorem.

Theorem 1.1 *Let R be a commutative Noetherian ring and I an ideal of R. Let M, N be finitely generated R-modules such that $H^i_I(M, N)_{\mathfrak{p}}$ is a finitely generated module over $R_{\mathfrak{p}}$ for all $i < t$ and all $\mathfrak{p} \in \bigcup_{j<t} \operatorname{Supp}_R(H^j_I(M, N))$ with $\dim R/\mathfrak{p} > 1$, where t is a given non-negative integer. Then $H^i_I(M, N)$ is I-cofinite for all $i < t$ and $\operatorname{Hom}_R(R/I, H^t_I(M, N))$ is finitely generated.*

This theorem has some consequences as follows. First, if we replace $M = R$ in Theorem 1.1, then we have that *if $H^i_I(N)_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$ for all $i < t$ and all $\mathfrak{p} \in \bigcup_{i<t} \operatorname{Supp}_R(H^i_I(N))$ then $H^i_I(N)$ is I -cofinite for all $i < t$ and $\operatorname{Hom}_R(R/I, H^t_I(N))$ is finitely generated* (see Corollary 2.8). This corollary is a slight improvement of Bahmanpour–Naghipour (see [3, Theorem 2.6]) and of Bahmanpour–Naghipour–Sedghi (see [2, Proposition 3.1]). In [3] and [2], they had used a basic property of local cohomology that $H^j_I(N) \cong H^j_I(N/\Gamma_I(N))$ for all $j > 0$; then it is easy to reduce to the case when $\Gamma_I(N) = 0$. But, for the case of generalized local cohomology, it is not true that $H^j_I(M, N) \cong H^j_I(M, N/\Gamma_{I_M}(N))$ for all $j > 0$ in general, where $I_M = \operatorname{ann}_R(M/IM)$. Hence, to prove Theorem 1.1, we need to establish Lemma 2.5 and Corollary 2.6 (R is not a necessary local) which implies that $\bigcup_{j<t}\operatorname{Supp}_R(H_I^j(M, N/\Gamma_{I_M}(N))) \subseteq \bigcup_{j<t}\operatorname{Supp}_R(H_I^j(M, N))$. Also, one of the ingredients in the proof of Theorem 1.1 is a lemma of Cuong–Goto–Hoang [6] (cf. Lemma 2.4); using this lemma, instead of studying the cofiniteness of $H_I^j(M, N)$ with respect to I, we need only to prove the cofiniteness of these modules with respect to ideal I_M. Second, as an other application of Theorem 1.1, we obtain a Corollary 2.10 which says that if $t := h_I^2(M, N)$ is the least integer such that $H_I^i(M, N)$ is not in dimension < 2 then $H_I^i(M, N)$ is I-cofinite for all $i < t$ and $\operatorname{Hom}_R(R/I, H_I^t(M, N))$ is finitely generated, where the notion of "in dimension < 2 module" is mentioned in Remark 2.2.

The paper is divided into two sections. In Sect. 2, we first prove some auxiliary lemmas which will be used in the sequel. The rest of Sect. 2 is devoted to prove the main result (Theorem 2.7) and its consequences.

2 Main Result

The following result seems to be well known but we cannot find exactly a reference for it, so for the sake of the completeness we still give a clear proof.

Lemma 2.1 *Let K be an R-module and I an ideal of R. If $(0:_K I)$ is finitely generated R-module then so is $(0:_K I^n)$ for all n.*

Proof We first prove by induction on $n \geq 1$ that $\operatorname{Hom}(R/I^n, K)$ is a finitely generated R-module. The case $n = 1$ is nothing. Assume that $n > 1$ and the result is true for $n-1$. From the short exact sequence $0 \to I^{n-1}/I^n \to R/I^n \to R/I^{n-1} \to 0$, we obtain the following exact sequence:

$$0 \to \operatorname{Hom}_R(R/I^{n-1}, K) \to \operatorname{Hom}_R(R/I^n, K) \to \operatorname{Hom}_R(I^{n-1}/I^n, K). \quad (*)$$

Note that by the hypothesis $(0:_K I)$ is also a finitely generated R/I-module. Since I is an ideal in a Noetherian ring, I^{n-1}/I^n is also a finitely generated R/I-module. Assume that I^{n-1}/I^n is generated by m elements. Then we have an exact sequence $(R/I)^m \to I^{n-1}/I^n \to 0$. Hence, we get an exact sequence $0 \to \operatorname{Hom}_R(I^{n-1}/I^n, K) \to \operatorname{Hom}_R((R/I)^m, K)$. Note that $\operatorname{Hom}_R((R/I)^m, K) = \operatorname{Hom}_R(R/I, K)^m$ is finitely generated over R/I by the hypothesis of $(0:I)_K$. Thus, $\operatorname{Hom}_R(I^{n-1}/I^n, K)$ is a finitely generated R/I-module, and so $\operatorname{Hom}_R(I^{n-1}/I^n, K)$ is also finitely generated as an R-module. Moreover, we get by the inductive assumption that $\operatorname{Hom}_R(R/I^{n-1}, K)$ is finitely generated over R. Therefore, we obtain by the exact sequence (*) that the R-module $\operatorname{Hom}_R(R/I^n, K)$ is finitely generated, as required. □

Remark 2.2 Let $0 \leq n \in \mathbb{Z}$. We recall that an R-module T is called in dimension $< n$ if there exists a finitely submodule K of T such that $\dim \operatorname{Supp}(T/K) < n$ (see [1, Definition 2.1]). It is clear that the class of in dimension $< n$ modules consists of class of finitely generated modules (this is the case of $n = 0$). Moreover, an R-module T is said to be minimax, if there exists a finitely generated submodule K of T such that T/K is Artinian (cf. [22] and [1]).

We next recall some results which will be used in the proof of our main result.

Lemma 2.3 (see [14, Lemma 2.1]) *Let t be a nonnegative integer such that $H_I^j(M, N)$ is in dimension $< n$ for all $j < t$. Then $\operatorname{Hom}_R(R/I, H_I^t(M, N))$ is in dimension $< n$.*

Lemma 2.4 (see [6, Lemma 4.2]) *Let t be a non-negative integer. Then*

(i) *The R-module $H_I^t(M, N)$ is I-cofinite if and only if $H_{I_M}^t(M, N)$ is I_M-cofinite, where $I_M = \operatorname{ann}_R(M/IM)$.*
(ii) *The R-module $\operatorname{Hom}_R(R/I, H_I^t(M, N))$ is finitely generated if and only if so is $\operatorname{Hom}_R(R/I_M, H_I^t(M, N))$.*

Next we need establish a lemma on support of generalized local cohomology module which is an extension of [8, Lemma 2.8] for the case of arbitrary commutative Noetherian ring R.

Lemma 2.5 *Let $t \in \mathbb{N} \cup \{\infty\}$. Then we have*

$$\bigcup_{j \leq t} \operatorname{Supp}_R \left(H_I^j(M, N) \right) = \bigcup_{j \leq t} \operatorname{Supp}_R \left(\operatorname{Ext}_R^j(M/IM, N) \right. .$$

Proof Let $\mathfrak{p} \in \bigcup_{j \leq t} \operatorname{Supp}_R(H_I^j(M, N))$. Then there exists an integer $0 \leq t_0 \leq t$ such that $\mathfrak{p} \in \operatorname{Supp}(H_I^{t_0}(M, N))$ and $\mathfrak{p} \notin \bigcup_{j < t_0} \operatorname{Supp}_R(H_I^j(M, N))$. Hence, $H_I^j(M, N)_\mathfrak{p} = 0$ for all $j < t_0$ and $H_I^{t_0}(M, N)_\mathfrak{p} \neq 0$. Thus, we get by [4, Proposition 5.5] that $\operatorname{depth}(I_{M_\mathfrak{p}}, N_\mathfrak{p}) = t_0$. Thus, $\operatorname{Ext}_{R_\mathfrak{p}}^{t_0}(M_\mathfrak{p}/IM_\mathfrak{p}, N_\mathfrak{p}) \neq 0$. It implies that $\mathfrak{p} \in \bigcup_{j \leq t} \operatorname{Supp}_R(\operatorname{Ext}_R^j(M/IM, N)$. Conversely, for each $\mathfrak{p} \in \bigcup_{j \leq t} \operatorname{Supp}_R(\operatorname{Ext}_R^j(M/IM, N)$, there exists $0 \leq t' \leq t$ such that $\mathfrak{p} \in \operatorname{Supp}_R (\operatorname{Ext}_R^{t'}(M/IM, N))$ and $\mathfrak{p} \notin \bigcup_{j < t'} \operatorname{Supp}_R(\operatorname{Ext}_R^j(M/IM, N))$. It yields that $\operatorname{Ext}_R^j (M/IM, N)_\mathfrak{p} = 0$ for all $j < t'$ and $\operatorname{Ext}_R^{t'}(M/IM, N)_\mathfrak{p} \neq 0$. Thus, $\operatorname{depth}(I_{M_\mathfrak{p}}, N_\mathfrak{p}) = t'$. So we get by [4, Proposition 5.5] that $H_I^{t'}(M, N)_\mathfrak{p} \neq 0$, and hence $\mathfrak{p} \in \operatorname{Supp}_R(H_I^{t'}(M, N)) \subseteq \bigcup_{j \leq t} \operatorname{Supp}_R(H_I^j(M, N))$. □

By similar arguments as in the proof of Lemma 2.5, we obtain the following corollary.

Corollary 2.6 *We have $\bigcup_{j \leq t} \operatorname{Supp}_R(H_I^j(M, N)) = \bigcup_{j \leq t} \operatorname{Supp}_R(H_{I_M}^j(N)$, where $t \in \mathbb{N} \cup \{\infty\}$.*

Before proving Theorem 1.1, we recall some known facts on the theory of secondary representation. In [16], Macdonald has developed the theory of attached prime ideals and secondary representation of a module. A non-zero R-module K is called secondary if for each $r \in R$ multiplication by r on K is either surjective or nilpotent. Then $\mathfrak{p} = \sqrt{\operatorname{ann}_R(K)}$ is a prime ideal, and K is called $\mathfrak{p}$-secondary. We say that K has a secondary representation if there is a finite number of secondary submodules $K_1, \ldots, K_n$ such that $K = K_1 + \ldots + K_n$. One may assume that $\mathfrak{p}_i = \sqrt{\operatorname{ann}_R(K_i)}$, for $i = 1, 2, \ldots, n$, are all distinct and, by omitting redundant summands, that the representation is minimal. Then the set of prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ does not depend on the representation, and it is called the set of attached prime ideals of K and denoted by $\mathrm{Att}_R(K)$. Note that if A is an Artinian R-module, then A has a secondary representation; moreover, the set of minimal prime ideals of $\mathrm{ann}_R(A)$ is just the set of minimal elements of $\mathrm{Att}_R(A)$ (see [16]). The basic properties on the set $\mathrm{Att}_R(A)$ of attached primes of A are referred in a paper by Macdonald [16].

The following is our main result in this note (that is Theorem 1.1 as mentioned in the part Introduction).

Theorem 2.7 *Let R be a Noetherian commutative ring and I an ideal of R. Let M, N be finitely generated R-modules such that $H^i_I(M, N)_\mathfrak{p}$ is a finitely generated module over $R_\mathfrak{p}$ for all $i < t$ and all $\mathfrak{p} \in \bigcup_{j<t} \mathrm{Supp}_R(H^j_I(M, N))$ with $\dim R/\mathfrak{p} > 1$, where t is a given non-negative integer. Then $H^i_I(M, N)$ is I-cofinite for all $i < t$ and $\mathrm{Hom}_R(R/I, H^t_I(M, N))$ is finitely generated.*

Proof By Lemma 2.4, we need only to claim that $H^i_I(M, N)$ is I_M-cofinite for all $i < t$ and $\mathrm{Hom}_R(R/I_M, H^t_I(M, N))$ is finitely generated, provided $H^i_I(M, N)_\mathfrak{p}$ is finitely generated for all $i < t$ and all $\mathfrak{p} \in \bigcup_{j<t} \mathrm{Supp}_R(H^j_I(M, N))$ with $\dim R/\mathfrak{p} > 1$, where t is a given integer.

We process by induction on $t \geq 0$. The case $t = 0$ is trivial. If $t = 1$ then it is clear that $H^0_I(M, N)$ is I_M-cofinite; moreover, $\mathrm{Hom}(R/I_M, H^1_I(M, N))$ is a finitely generated R-module by Lemma 2.3. We now suppose that $t > 1$, and the result has been proved for $t - 1$. From the short exact sequence $0 \to \Gamma_{I_M}(N) \to N \to \overline{N} \to 0$, we obtain exact sequences

$$\mathrm{Ext}^i_R\left(M, \Gamma_{I_M}(N)\right) \xrightarrow{f_i} H^i_I(M, N) \xrightarrow{g_i} H^i_I\left(M, \overline{N}\right) \xrightarrow{h_i} \mathrm{Ext}^{i+1}_R\left(M, \Gamma_{I_M}(N)\right)$$

for all i, where $\overline{N} = N/\Gamma_{I_M}(N)$. Note that $\mathrm{Ext}^i_R(M, \Gamma_{I_M}(N))$ is finitely generated for all i. For each $i \geq 0$, we split the above exact sequence into the following two exact sequences:

$$0 \to \mathrm{Im}\, f_i \to H^i_I(M, N) \to \mathrm{Im}\, g_i \to 0 \text{ and}$$
$$0 \to \mathrm{Im}\, g_i \to H^i_I(M, \overline{N}) \to \mathrm{Im}\, h_i \to 0.$$

Note that $\mathrm{Im}\, f_i$ and $\mathrm{Im}\, h_i$ are finitely generated for all $i \geq 0$. Then, for any $i < t$, we obtain that $H^i_I(M, N)$ is I_M-cofinite if and only if so is $H^i_I(M, \overline{N})$. On the other hand, we get by Corollary 2.6 that

$$\bigcup_{j<t} \mathrm{Supp}_R\left(H^j_I(M, \overline{N})\right) = \bigcup_{j<t} \mathrm{Supp}_R\left(H^j_{I_M}(\overline{N})\right) \text{ and}$$
$$\bigcup_{j<t} \mathrm{Supp}_R\left(H^j_I(M, N)\right) = \bigcup_{j<t} \mathrm{Supp}_R\left(H^j_{I_M}(N)\right).$$

Thus, we obtain that $\bigcup_{j<t} \mathrm{Supp}_R(H^j_I(M,\overline{N})) \subseteq \bigcup_{j<t} \mathrm{Supp}_R(H^j_I(M,N))$ since $H^j_{I_M}(N) \cong H^j_{I_M}(\overline{N})$ for all $1 \leq j < t$. Therefore, we get by the hypothesis that $H^i_I(M,\overline{N})_{\mathfrak{p}}$ is finitely generated for all $i < t$ and all $\mathfrak{p} \in \bigcup_{j<t} \mathrm{Supp}_R(H^j_I(M,\overline{N}))$ with $\dim(R/\mathfrak{p}) > 1$. Therefore, to prove the theorem for the case of $t > 1$, we may assume without loss of generality that $\Gamma_{I_M}(N) = 0$. Thus, $I_M \nsubseteq \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(N)} \mathfrak{p}$.

For each $0 < n \in \mathbb{Z}$ and $0 \leq i < t$, we set $H_{i,n} = (0 :_{H^i_I(M,N)} I^n_M)$. Recall that $H^i_I(M,N)$ is I_M-cofinite for all $i < t-1$ and $(0 :_{H^{t-1}_I(M,N)} I_M)$ is finitely generated. Hence, $(0 :_{H^i_I(M,N)} I_M)$ is finitely generated for all $0 \leq i < t$. Thus, we obtain by Lemma 2.1 that $H_{i,n}$ is finitely generated for all $0 \leq i < t$ and all $n > 0$. Hence, $\mathrm{Supp}_R(H_{i,n+1}/H_{i,n}) = V(\mathrm{ann}_R(H_{i,n+1}/H_{i,n}))$. Note that for each $n > 0$, we have $\mathrm{ann}_R(H_{i,n+1}/H_{i,n}) \subseteq \mathrm{ann}_R(H_{i,n+2}/H_{i,n+1})$ by [2, Lemma 2.1]. Hence, $V(\mathrm{ann}_R(H_{i,n+1}/H_{i,n})) \supseteq V(\mathrm{ann}_R(H_{i,n+2}/H_{i,n+1}))$ for all $n > 0$. Thus, we obtain the following sequence:

$$V(\mathrm{ann}_R(H_{i,n+1}/H_{i,n})) \supseteq V(\mathrm{ann}_R(H_{i,n+2}/H_{i,n+1})) \supseteq \ldots.$$

From this we get by the Noetherianness of space $\mathrm{Spec}\, R$ that there exists an integer $k > 0$ such that

$$\mathrm{Supp}_R(H_{i,k+1}/H_{i,k}) = \mathrm{Supp}_R(H_{i,n+1}/H_{i,n}) \tag{2.1}$$

for all $n \geq k+1$ and all $0 \leq i < t$. Suppose that $\{\mathfrak{p} \in \mathrm{Supp}_R(H_{i,k+1}/H_{i,k}) \mid \dim R/\mathfrak{p} > 1\} \neq \emptyset$. Then we take $\mathfrak{p} \in \mathrm{Supp}_R(H_{i,k+1}/H_{i,k})$ such that $\dim R/\mathfrak{p} > 1$. Thus, $\mathfrak{p} \in \bigcup_{j<t} \mathrm{Supp}_R(H^j_I(M,N))$ since $\mathrm{Supp}_R(H_{i,k+1}/H_{i,k}) \subseteq \mathrm{Supp}_R(H^i_I(M,N))$. Hence, we get by the hypothesis that $H^i_I(M,N)_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module for all $0 \leq i < t$. Thus, for each $0 \leq i < t$, there exists a finitely generated submodule L of $H^i_I(M,N)$ such that $H^i_I(M,N)_{\mathfrak{p}} = L_{\mathfrak{p}}$. Since L is I_M-torsion by [6, Lemma 2.1], there is an integer $n \geq k+1$ such that $I^n_M L = 0$. It follows that

$$L_{\mathfrak{p}} = H^i_I(M,N)_{\mathfrak{p}} \supseteq (H_{i,n+1})_{\mathfrak{p}} \supseteq (H_{i,n})_{\mathfrak{p}} \supseteq L_{\mathfrak{p}}.$$

Hence, $(H_{i,n+1}/H_{i,n})_{\mathfrak{p}} = 0$. From this, we get by the fact in formula (2.1) that $\mathfrak{p} \notin \mathrm{Supp}(H_{i,k+1}/H_{i,k})$, this is a contradiction. Hence,

$$\{\mathfrak{p} \in \mathrm{Supp}_R(H_{i,k+1}/H_{i,k}) \mid \dim R/\mathfrak{p} > 1\} = \emptyset,$$

and hence $\dim R/\mathfrak{p} \leq 1$ for all $\mathfrak{p} \in \mathrm{Supp}_R(H_{i,k+1}/H_{i,k})$. We now take $\mathfrak{p} \in \mathrm{Supp}_R(H^i_I(M,N)/H_{i,k})$. If $\mathfrak{p} \notin \mathrm{Supp}_R(H_{i,k+1}/H_{i,k})$ then we get by (2.1) that $\mathfrak{p} \notin \mathrm{Supp}_R(H_{i,n+1}/H_{i,n})$ for all $n \geq k$. Thus, $(H_{i,n+1})_{\mathfrak{p}} = (H_{i,k})_{\mathfrak{p}}$ for all $n \geq k$, and so $H^i_I(M,N)_{\mathfrak{p}} = (H_{i,n})_{\mathfrak{p}}$ for all $n \geq k$. Hence, $\mathfrak{p} \notin \mathrm{Supp}_R(H^i_I(M,N)/H_{i,n})$ for all $n \geq k$, which is a contradiction. Thus,

$$\mathrm{Supp}_R(H^i_I(M,N)/H_{i,k}) = \mathrm{Supp}_R(H_{i,k+1}/H_{i,k}),$$

and hence

$$\dim \operatorname{Supp}_R(H_I^i(M,N)/H_{i,k}) \leq 1. \tag{2.2}$$

From the short exact sequence

$$0 \to H_{i,k} \to H_I^i(M,N) \to H_I^i(M,N)/H_{i,k} \to 0,$$

we get the following exact sequence:

$$\left(0 :_{H_I^i(M,N)} I_M\right) \to \left(0 :_{H_I^i(M,N)/H_{i,k}} I_M\right) \to \operatorname{Ext}_R^1\left(R/I_M, H_{i,k}\right).$$

Hence, the R-module $(0 :_{H_I^i(M,N)/H_{i,k}} I_M)$ is finitely generated for all $0 \leq i < t$ because of the finiteness of $(0 :_{H_I^i(M,N)} I_M)$ and $H_{i,k}$ for all $0 \leq i < t$. Thus, $\operatorname{Ass}_R(H_I^i(M,N)/H_{i,k})$ is a finite set for all $0 \leq i < t$. For each $0 \leq i < t$, we set

$$T_i = \{\mathfrak{p} \in \operatorname{Supp}_R(H_I^i(M,N)/H_{i,k}) \mid \dim R/\mathfrak{p} = 1\}.$$

Then we obtain by formula (2.2) that $T_i \subseteq \operatorname{Ass}_R(H_I^i(M,N)/H_{i,k})$, and hence the set $T = \bigcup_{i=0}^{t-1} T_i$ is finite. Assume that $T = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_v\}$.

For any $\mathfrak{p} \in T$ and any $0 \leq i < t$, we have by the previous paragraph that $(0 :_{H_I^i(M,N)/H_{i,k}} I_M)_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module. Moreover, since either $\mathfrak{p}$ is a minimal element of $\operatorname{Supp}_R(H_I^i(M,N)/H_{i,k})$ or $\mathfrak{p}$ is not in $\operatorname{Supp}_R(H_I^i(M,N)/H_{i,k})$, we obtain that

$$V(\operatorname{ann}_{R_\mathfrak{p}}((0 :_{H_I^i(M,N)/H_{i,k}} I_M)_\mathfrak{p}) \subseteq \{\mathfrak{p}R_\mathfrak{p}\}.$$

Hence, $(0 :_{H_I^i(M,N)/H_{i,k}} I_M)_\mathfrak{p}$ is an Artinian $R_\mathfrak{p}$-module. Thus, we get by the Melkersson's criterion on the Artinianness (see [17, Theorem 1.3]) that $(H_I^i(M,N)/H_{i,k})_\mathfrak{p}$ is an Artinian $R_\mathfrak{p}$-module. It implies by [6, Lemma 4.1 (ii)] that

$$V((I_M)_\mathfrak{p}) \cap \operatorname{Att}_{R_\mathfrak{p}}((H_I^i(M,N)/H_{i,k})_\mathfrak{p}) \subseteq \{\mathfrak{p}R_\mathfrak{p}\}. \tag{2.3}$$

Now we set

$$S = \bigcup_{i=0}^{t-1} \bigcup_{j=1}^{v} \left\{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}R_{\mathfrak{p}_j} \in \operatorname{Att}_{R_{\mathfrak{p}_j}}((H_I^i(M,N)/H_{i,k})_{\mathfrak{p}_j})\right\}.$$

Then it is easy to see that $S \cap V(I_M) \subseteq T$. We now choose an element $x \in I_M$ such that

$$x \notin \left(\bigcup_{\mathfrak{p} \in S \setminus V(I_M)} \mathfrak{p}\right) \cup \left(\bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(N)} \mathfrak{p}\right).$$

We then have the short exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$. It implies the following exact sequences:

$$\ldots \to H_I^i(M,N) \xrightarrow{x} H_I^i(M,N) \to H_I^i(M,N/xN) \to H_I^{i+1}(M,N) \xrightarrow{x} H_I^{i+1}(M,N) \to \ldots$$

for all $i \geq 0$. Thus, we obtain exact sequences

$$0 \to H_I^i(M,N)/xH_I^i(M,N) \xrightarrow{\varphi_i} H_I^i(M,N/xN) \to (0:x)_{H_I^{i+1}(M,N)} \to 0 \tag{2.4}$$

for all $i \geq 0$. Note that

$$\bigcup_{j<t-1} \mathrm{Supp}_R\left(H_I^j(M,N/xN)\right) \subseteq \bigcup_{j<t} \mathrm{Supp}_R\left(H_I^j(M,N)\right)$$

by the long exact sequence as above mentioned. Hence, we get by the hypothesis that $H_I^j(M,N/xN)_\mathfrak{p}$ is finitely generated $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \bigcup_{j<t-1} \mathrm{Supp}_R (H_I^j(M,N/xN))$ with $\dim R/\mathfrak{p} > 1$. Therefore, we get by the inductive assumption that the R-modules

$$H_I^0(M,N/xN), H_I^1(M,N/xN), \ldots, H_I^{t-2}(M,N/xN)$$

are I_M-cofinite and module $(0:_{H_I^{t-1}(M,N/xN)} I_M)$ is finitely generated (*). For each $0 \leq i < t$, we set

$$N_i = \varphi_i\left(\frac{xH_I^i(M,N) + H_{i,k}}{xH_I^i(M,N)}\right).$$

Hence, N_i is a finitely generated submodule of $H_I^i(M,N/xN)$ since

$$\frac{xH_I^i(M,N) + H_{i,k}}{xH_I^i(M,N)} \cong \frac{H_{i,k}}{H_{i,k} \cap xH_I^i(M,N)}$$

is a finitely generated submodule of $H_I^i(M,N)/xH_I^i(M,N)$. Therefore, we obtain the following exact sequences:

$$0 \to \frac{H_I^i(M,N)}{xH_I^i(M,N) + H_{i,k}} \to \frac{H_I^i(M,N/xN)}{N_i} \to (0:_{H_I^{i+1}(M,N)} x) \to 0$$

for all $0 \leq i < t$. For each $0 \leq i < t$, set

$$U_i = \frac{H_I^i(M,N)}{xH_I^i(M,N) + H_{i,k}} \text{ and } V_i = \frac{H_I^i(M,N/xN)}{N_i}.$$

Note that

$$U_i \cong \frac{H_I^i(M,N)/H_{i,k}}{x\left(H_I^i(M,N)/H_{i,k}\right)}.$$

By the hypothesis of x and the fact in (2.3), we have that

$$x \notin \mathfrak{q}R_{\mathfrak{p}_j} \text{ for all } \mathfrak{q}R_{\mathfrak{p}_j} \in \operatorname{Att}_{R_{\mathfrak{p}_j}}\left((H_I^i(M,N)/H_{i,k})_{\mathfrak{p}_j}\right)\backslash\{\mathfrak{p}_j R_{\mathfrak{p}_j}\}$$

for all $0 \le i < t$ and all $1 \le j \le v$. Hence, we get by [6, Lemma 4.1 (i)] that $\ell_{R_{\mathfrak{p}_j}}((U_i)_{\mathfrak{p}_j}) < +\infty$ for all $0 \le i < t$ and all $j = 1, 2, \ldots, v$. Thus, for each $0 \le i < t$ and $1 \le j \le v$ there exists a finitely generated R-submodule $B_{i,j}$ of U_i such that $(U_i)_{\mathfrak{p}_j} = (B_{i,j})_{\mathfrak{p}_j}$. For each $0 \le i < t$, we set $B_i = B_{i,1} + B_{i,2} + \ldots + B_{i,l}$; it is clear that B_i is a finitely generated R-submodule of U_i. Then there is a submodule K_i of $H_I^i(M,N)$ such that $K_i \supseteq xH_I^i(M,N) + H_{i,k}$ and $\dfrac{K_i}{xH_I^i(M,N) + H_{i,k}} = B_i$. Thus,

$$\operatorname{Supp}_R(U_i/B_i) \subseteq \operatorname{Supp}_R(H_I^i(M,N)/H_{i,k})\backslash T \subseteq \operatorname{Max}(R).$$

On the other hand, from the short exact sequence

$$0 \to N_i \to H_I^i(M, N/xN) \to V_i \to 0,$$

we get the following exact sequence:

$$\operatorname{Hom}_R\left(R/I_M, H_I^i(M, N/xN)\right) \to \operatorname{Hom}_R(R/I_M, V_i) \to \operatorname{Ext}_R^1(R/I_M, N_i)$$

for all $0 \le i < t$. Hence, it implies that the R-module $\operatorname{Hom}_R(R/I_M, V_i)$ is finitely generated for all $0 \le i < t$. Hence, since the sequence

$$0 \to \operatorname{Hom}_R(R/I_M, U_i) \to \operatorname{Hom}_R(R/I_M, V_i)$$

is exact, it yields that $\operatorname{Hom}_R(R/I_M, U_i)$ is also a finitely generated R-module. Thus, $\operatorname{Hom}_R(R/I_M, U_i/B_i)$ is a finitely generated R-module because of the exactness of sequence

$$\operatorname{Hom}_R(R/I_M, U_i) \to \operatorname{Hom}_R(R/I_M, U_i/B_i) \to \operatorname{Ext}_R^1(R/I_M, B_i)$$

and the finiteness of B_i. Then we obtain by the fact $\operatorname{Supp}(U_i/B_i) \subseteq \operatorname{Max}(R)$ that $\operatorname{Hom}_R(R/I_M, U_i/B_i)$ is Artinian for all $0 \le i < t$. Therefore, we get by [17, Theorem 1.3] that U_i/B_i is Artinian for all $0 \le i < t$ since U_i/B_i is I_M-torsion for all $0 \le i < t$. Therefore, U_i is a minimax R-module for all $0 \le i < t$. We also have

$$U_i = \frac{H_I^i(M,N)/xH_I^i(M,N)}{\left(xH_I^i(M,N) + H_{i,k}\right)/xH_I^i(M,N)},$$

where

$$\frac{xH_I^i(M,N)+H_{i,k}}{xH_I^i(M,N)} \cong \frac{H_{i,k}}{xH_I^i(M,N)\cap H_{i,k}}$$

is a finitely generated R-module for all $0 \leq i < t$. Therefore, R-module $H_I^i(M,N)/xH_I^i(M,N)$ also is a minimax R-module by [18, Sect. 4] for all $0 \leq i < t$. Moreover, by (2.4) and the inductive assumption, we obtain that $\operatorname{Hom}_R\left(R/I_M, H_I^i(M,N)/xH_I^i(M,N)\right)$ is a finitely generated R-module for all $0 \leq i < t$. Hence, we obtain that $H_I^i(M,N)/xH_I^i(M,N)$ is I_M-cofinite for all $0 \leq i < t$ by [18, Proposition 4.3]. We keep in mind the fact as mentioned in (*). Therefore, we get by the sequence (2.4) again that $(0:_{H_I^i(M,N)} x)$ is I_M-cofinite for all $0 \leq i < t$ and $\operatorname{Hom}_R\left(R/I_M, (0:_{H_I^t(M,N)} x)\right)$ is a finitely generated R-module. From the I_M-cofiniteness of $H_I^i(M,N)/xH_I^i(M,N)$ and $(0:_{H_I^i(M,N)} x)$ for all $0 \leq i < t$, we get that $H_I^i(M,N)$ is I_M-cofinite for all $0 \leq i < t$ by [6, Lemma 3.1]. Moreover, note that

$$\operatorname{Hom}_R\left(R/I_M, (0:x)_{H_I^t(M,N)}\right) = \operatorname{Hom}_R\left(R/I_M, H_I^t(M,N)\right)$$

since $x \in I_M$. Hence, $\operatorname{Hom}_R(R/I_M, H_I^t(M,N))$ is finitely generated, as required. □

Note that if $\dim \operatorname{Supp}_R(H_I^i(M,N)) \leq 1$ for all $i < t$ then $H_I^i(M,N)_{\mathfrak{p}} = 0$ for all $i < t$ and all $\mathfrak{p} \in \bigcup_{j<t} \operatorname{Supp}_R(H_I^j(M,N))$ with $\dim R/\mathfrak{p} > 1$. Hence, as an immediate consequence of Theorem 2.7 we obtain again a result of Cuong–Goto–Hoang (see [6, Theorem 1.2]) which says that *Let t be a non-negative integer such that* $\dim \operatorname{Supp}_R(H_I^j(M,N)) \leq 1$ *for all $j < t$. Then $H_I^i(M,N)$ is I-cofinite for all $i < t$ and* $\operatorname{Hom}_R(R/I, H_I^t(M,N))$ *is finitely generated.*

By replacing $M = R$ in Theorem 2.7 we get the following result for the case of ordinary local cohomology modules $H_I^i(N)$.

Corollary 2.8 *Let t be a non-negative integer such that $H_I^i(N)_{\mathfrak{p}}$ is a finitely generated module over $R_{\mathfrak{p}}$ for all $i < t$ and all $\mathfrak{p} \in \bigcup_{j<t} \operatorname{Supp}_R(H_I^j(N))$ with* $\dim R/\mathfrak{p} > 1$. *Then $H_I^i(N)$ is I-cofinite for all $i < t$ and* $\operatorname{Hom}_R(R/I, H_I^t(N))$ *is finitely generated.*

The result [2, Proposition 3.1] says that *if $H_I^i(N)_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$ for all $i < t$* and all $\mathfrak{p} \in \operatorname{Supp}(N/IN)$ *with* $\dim R/\mathfrak{p} > 1$ *then $H_I^i(N)$ is I-cofinite for all $i < t$* and $\operatorname{Hom}_R(R/I, H_I^t(N))$ *is finitely generated*. Thus Corollary 2.8 is a slight improvement of [2, Proposition 3.1] of Bahmanpour–Naghipour–Sedghi; and so Corollary 2.8 is also an extension of Bahmanpour–Naghipour [3, Theorem 2.6].

Remark 2.9 For any $0 \leq n \in \mathbb{Z}$. Note that the notion of in dimension $< n$ module is mentioned in Remark 2.2. On the other hand, in the proof of [14, Theorem 2.4], we consider the number $h_I^n(M,N)$ defined by

$$h_I^n(M,N) = \inf\{0 \leq i \in \mathbb{Z} \mid H_I^i(M,N) \text{ is not in dimension } < n\}.$$

Now, as an application of Theorem 2.7, we get the following result on the cofiniteness of $H^i_I(M, N)$ whenever $i < h^2_I(M, N)$.

Corollary 2.10 *The R-module* $H^i_I(M, N)$ *is I-cofinite for all* $i < h^2_I(M, N)$ *and* $\operatorname{Hom}_R(R/I, H_I^{h^2_I(M,N)}(M, N))$ *is finitely generated, where*

$$h^2_I(M, N) = \inf\{0 \le i \in \mathbb{Z} \mid H^i_I(M, N) \textit{ is not in dimention } < 2\}.$$

Proof For any $0 \le n \in \mathbb{Z}$. We recall that the nth finiteness dimension $f^n_I(M, N)$ of M and N with respect to I was defined in [14, Definition 2.3] as follows:

$$f^n_I(M, N) = \inf\{f_{I_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(N/I_M N), \dim R/\mathfrak{p} \ge n\},$$

where $f_{I_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ is the least integer i such that $H^i_{I_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ is not a finitely generated $R_\mathfrak{p}$-module.

Hence, we obtain in the case of $n = 2$ that $H^i_I(M, N)_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module for all $i < f^2_I(M, N)$ and all $\mathfrak{p} \in \operatorname{Supp}_R(N/I_M N)$ with $\dim R/\mathfrak{p} \ge 2$. On the other hand, in [14, Theorem 2.4], we proved that $h^2_I(M, N) = f^2_I(M, N)$. Thus, $H^i_I(M, N)_\mathfrak{p}$ is finitely generated over $R_\mathfrak{p}$ for all $i < h^2_I(M, N)$ and all $\mathfrak{p} \in \bigcup_{j<h^2_I(M,N)} \operatorname{Supp}_R(H^j_I(M, N))$ with $\dim R/\mathfrak{p} > 1$. Therefore, we obtain by Theorem 2.7 that the R-module $H^i_I(M, N)$ is I-cofinite for all $i < h^2_I(M, N)$, and $\operatorname{Hom}_R(R/I, H_I^{h^2_I(M,N)}(M, N))$ is finitely generated, as required. □

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