Bulletin of the Iranian Mathematical Society (2020) 46:303–321
https://doi.org/10.1007/s41980-019-00258-8

ORIGINAL PAPER

Ergodic Shadowing of Semigroup Actions

Zahra Shabani[1]

Received: 13 February 2019 / Revised: 13 May 2019 / Accepted: 20 May 2019 / Published online: 13 June 2019

Abstract
The present work is concerned with the ergodic shadowing property of semigroup actions. We prove that any finitely generated semigroup with the ergodic shadowing property has the ordinary shadowing property, and some properties for semigroup actions with the ergodic shadowing property such as topologically mixing, chain mixing, and weakly mixing are investigated. Finally, we define some kind of specification property and clarify its relation to the ergodic shadowing property.

Keywords Ergodic shadowing property · Shadowing property · Iterated function system · Mixing

Mathematics Subject Classification 35B41 · 37E30

1 Introduction

In this work, we aim at an extension of ergodic shadowing to the realm of finitely generated free semigroup actions.

Theory of shadowing is a classical notion, which has originated from the works of Anosov [1] and Bowen [6]. A dynamical system has the shadowing property if any pseudo-trajectory is close to some exact trajectory. Many authors obtained results about chaos and stability by studying the various type of shadowing, see, for example, [5,6,16]. Ergodic shadowing property for a continuous map was introduced in [8]. The authors proved that any mapping with the ergodic shadowing property is chaotic in the sense of Li–Yorke and Auslander–Yorke [3,9]. Moreover, in [8], the equivalence of the ergodic shadowing to being topologically mixing and having the shadowing property for a continuous onto map was obtained.

Communicated by Fatemeh Helen Ghane.

✉ Zahra Shabani
zshabani@math.usb.ac.ir

1 Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

The specification property for a continuous map on a compact metric space X was introduced by Bowen in [6] to study the ergodic property of Axiom A diffeomorphisms. A mapping $f : X \to X$ has the specification property if one can approximate distinct pieces of orbits by a single periodic orbit with a certain uniformity. It has been shown that if f has the specification property, then it is topologically mixing and the set of all periodic points of f is dense in X. So, any mapping with the specification property is chaotic in the sense of Devaney [2]. Fakhari and Ghane in [8] defined some kind of specification property and investigated its relation with the ergodic shadowing property.

Recently, in 2014, Osipov and Thikhomirov [14] introduced the notion of shadowing property for the finitely generated group action. Later, Bahabadi [4] introduced the shadowing and average shadowing properties for free semigroup actions and proved that any semigroup with the average shadowing property is chain transitive. In [10], the authors proved that by the assumption of shadowing property for a semigroup, total transitivity, topologically weakly mixing, topologically mixing, the specification property and the average shadowing property are equivalent. For more recent results on the notion of shadowing and average shadowing properties, we refer the reader to [4,10,12,18] and references therein. Rodrigues and Varandas in [15] introduced the specification property for free semigroup actions. They showed that the semigroups of expanding maps satisfy the orbital specification property.

The concept of topological ergodic shadowing and its relation to topologically mixing for a dynamical system defined on Hausdorff uniform spaces were studied in [17], and the authors showed that any dynamical system with the topological ergodic shadowing property is topologically mixing.

In this paper, we extend the notion of ergodic shadowing property for finitely generated free semigroup actions and study its relations to topologically mixing, chain mixing, weakly mixing and the shadowing property. Moreover, we introduce some notions of strong and pseudo-orbital specification properties for continuous actions associated with finitely generated semigroup and show that the pseudo-orbital specification property is equivalent to the ergodic shadowing property. Also, in particular case of a semigroup of expanding maps, we prove that the strong pseudo-orbital specification property holds, which shows that the semigroup has the ergodic shadowing property.

The following theorems are the main results of this paper. The first theorem extends Theorem A of [8].

Theorem 1.1 *Let $f_1, f_2, \ldots, f_m$ be continuous maps on the compact metric space X, which one of them being an onto map. If Γ is the semigroup generated by the family $\Gamma_1 = \{id, f_1, f_2, \ldots, f_m\}$, then the following properties for the semigroup Γ are equivalent:*

(1) *ergodic shadowing,*
(2) *chain mixing and ordinary shadowing,*
(3) *topologically mixing and ordinary shadowing,*
(4) *pseudo-orbital specification.*

Theorem 1.2 *Any expanding semigroup has the ergodic shadowing property.*

This paper is organized as follows. In Sect. 2, we give an overview of the main concepts and discuss some preliminaries. In particular, the concepts of ergodic shadowing and pseudo-orbital specification are introduced for finitely generated semigroup actions. In Sect. 3, we show that any semigroup with the ergodic shadowing property has the ordinary shadowing property. In Sect. 4, it is proved that the ergodic shadowing property for semigroups implies topologically mixing. Then the equivalence of ergodic shadowing and pseudo-orbital specification is obtained in Sect. 5. In Sect. 6, we introduce the strong pseudo-orbital specification property for semigroup actions and show that the semigroups of expanding maps satisfy the strong pseudo-orbital specification property. Finally, in Sect. 7, we give some examples. The first and second examples illustrate semigroups with the ergodic shadowing property. The third example shows that the pseudo-orbital specification property does not result in the strong pseudo-orbital specification property, in general. In the last example, we build a semigroup that does not have the ergodic shadowing property.

2 Preliminaries

In this section, we describe the free semigroup actions, which we are interested in and state our major conclusions on the ergodic shadowing and specification within this context.

Throughout this paper, let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$. Consider the compact metric space X endowed with the metric d. Given a finite set of continuous maps $f_i : X \to X$, $i = 1, 2, \ldots, m$, $m > 1$, and the finitely generated semigroup Γ (under function composition) with the finite set of generators $G_1 = \{id, f_1, \ldots, f_m\}$, we will write

$$\Gamma = \bigcup_{n\in\mathbb{Z}^+} G_n,$$

where $G_0 = \{id\}$ and $f \in G_n$ if and only if $f = f_{i_n} \circ \cdots \circ f_{i_1}$, with $f_{i_j} \in G_1$.

We will assume that the generator set G_1 is minimal, meaning that no function f_j, for $j = 1, \ldots, m$, can be expressed as a composition from the remaining generators. Take $G_1^* = G_1 \setminus \{id\}$, and let G_n^* denote the space of concatenations of n elements in G_1^*.

If f is the concatenation of n elements of G_1^*, we will write $|f| = n$. Notice that different concatenations may generate the same element in Γ. Nevertheless, we shall consider different concatenations instead of the elements in Γ that they create. One way to interpret this statement is to consider the coding map $\varrho : \Gamma_{\mathcal{A}} \to \Gamma$ given by

$$w = w_1 \ldots w_n \mapsto f_w^n = f_{w_n} \circ \cdots \circ f_{w_1},$$

where $\Gamma_{\mathcal{A}}$ is the free semigroup with m generators $\mathcal{A}^m = \{1, \ldots, m\}$, and to regard concatenations on Γ as images by ϱ of paths on $\Gamma_{\mathcal{A}}$. This coding is injective if and only if Γ is a free semigroup.

Symbolic dynamic is a way to represent the elements of Γ. Indeed, consider the product space $\Sigma^m = \{1, \ldots, m\}^{\mathbb{N}}$. For any sequence $\omega = \omega_1\omega_2 \ldots \omega_n \cdots \in \Sigma^m$, take $f_\omega^0 := id$ and

$$f_\omega^n(x) = f_{\omega_1 \ldots \omega_n}^n(x) = f_{\omega_n} \circ f_\omega^{n-1}(x) \quad \text{for all } \; n \in \mathbb{N}.$$

Obviously, $f_\omega^n = f_{\omega_1 \ldots \omega_n}^n = f_{\omega_n} \circ f_\omega^{n-1} \in \Gamma$ for every $n \in \mathbb{N}$.

For $k \in \mathbb{N}$, consider

$$\mathcal{A}_k^m = \{w \in \Gamma_{\mathcal{A}} : |w| = k\}.$$

We now consider the product semigroup $\Gamma \times \Gamma$ generated by

$$\{f_i \times f_j : i, j \in \mathcal{A}^m\},$$

where $(f_i \times f_j)(x, y) = (f_i(x), (f_j(y))$ for $x, y \in X$. The metric on the space $X \times X$ is given by

$$D((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}.$$

Clearly, each $f_i \times f_j$ is a continuous map on $X \times X$. For $\omega^{1,2} := (\omega^1, \omega^2) \in (\Sigma^m)^2$, the $\omega^{1,2}$-orbit of $(x, y) \in X \times X$ (under $\Gamma \times \Gamma$) is the sequence $\{f_{\omega^{1,2}}^n(x, y)\}_{n \geq 0}$, where

$$f_{\omega^{1,2}}^n(x, y) := f_{\omega^1}^n \times f_{\omega^2}^n(x, y).$$

For any two open subsets U and V of X, put

$$N_\Gamma(U, V) = \left\{n \in \mathbb{N} : f_\omega^n(U) \cap V \neq \emptyset, \text{ for some } \omega \in \Sigma^m\right\}.$$

Definition 2.1 (*See* [11,15]) We say that the semigroup Γ is

(1) topologically transitive if $N_\Gamma(U, V) \neq \emptyset$ for any pair of nonempty open subsets $U, V \in X$;
(2) topologically mixing if $N_\Gamma(U, V)$ is a co-finite set for any pair of nonempty open subsets $U, V \in X$;
(3) weakly mixing if the product semigroup $(\Gamma \times \Gamma)$ is topologically transitive.

Notice that for a finite family G of continuous self-maps on a compact metric space, the action of the semigroup generated by G is also called the iterated function system associated with G. For iterated function systems, Bahabadi [4] introduced the concept of shadowing. Let $\delta > 0$ be given. The sequence $\{x_i\}_{i \geq 0}$ is called a (δ, ω)-*pseudo-orbit* for the semigroup Γ and for some sequence $\omega = \omega_1\omega_2 \ldots \in \Sigma^m$, if for any $i \geq 1$,

$$d(f_{\omega_i}(x_{i-1}), x_i) < \delta.$$

The semigroup Γ is said to have the *shadowing property*, provided that for every $\epsilon > 0$, there exists $\delta > 0$, such that for any (δ, ω)-pseudo-orbit $\{x_i\}_{i\geq 0}$, there is a point $z \in X$ satisfying the following property:

$$d(f_{\omega}^{i}(z), x_i) < \epsilon, \quad i \geq 0.$$

A finite sequence $\{x_i\}$, $i \in [0, n] \subset \mathbb{Z}^+$, is said to be a finite δ-pseudo-orbit(δ-*chain*) for Γ provided that there exists a finite word $w = w_1 w_2 \ldots w_n \in \Gamma_{\mathcal{A}}$ such that

$$d(f_{w_i}(x_{i-1}), x_i) < \delta, \quad 1 \leq i \leq n.$$

We say that the semigroup Γ satisfies *the finite shadowing property*, if for any $\epsilon > 0$, there exists $\delta > 0$, such that for any finite (δ, w)-pseudo-orbit $\{x_i\}_{0\leq i\leq n}$, there is a point $z \in X$ with

$$d(f_{w}^{i}(z), x_i) < \epsilon, \quad 0 \leq i \leq n.$$

A semigroup Γ is called *chain transitive* if for any two points $x, y \in X$ and any $\delta > 0$ there exists a δ-chain from x to y. The semigroup Γ is called *chain mixing* if for any two points $x, y \in X$ and any $\delta > 0$, there is a positive integer N such that for any integer $n \geq N$ there is a δ-chain from x to y of length n.

Here, we generalize the notions of ergodic shadowing and pseudo-orbital specification for semigroup actions. For a sequence $\eta = \{x_i\}_{i\geq 0}$, $\delta > 0$ and $\omega = \omega_1\omega_2 \ldots \in \Sigma^m$, put

$$Np(\eta, \Gamma, \omega, \delta) = \big\{i \in \mathbb{N} : d(f_{\omega_i}(x_{i-1}), x_i) \geq \delta\big\}$$

and

$$Np_n(\eta, \Gamma, \omega, \delta) = Np(\eta, \mathcal{F}, \omega, \delta) \cap \{1, \ldots, n\}.$$

To simplify the notation, we denote them by $Np(\eta, \omega, \delta)$ and $Np_n(\eta, \omega, \delta)$, respectively. Given a sequence $\eta = \{x_i\}_{i\geq 0}$ and a point $z \in X$, consider

$$Ns(\eta, \omega, z, \delta) = \Big\{i \in \mathbb{Z}^+ : d(f_{\omega}^{i}(z), x_i) \geq \epsilon\Big\}$$

and

$$Ns_n(\eta, \omega, z, \delta) = Ns(\eta, \omega, z, \delta) \cap \{1, \ldots, n\}.$$

Definition 2.2 A sequence $\eta = \{x_i\}_{i\geq 0}$ is called a (δ, ω)-*ergodic pseudo-orbit* of Γ for some sequence $\omega = \omega_1\omega_2 \ldots \in \Sigma^m$ provided that the set $Np(\eta, \omega, \delta)$ has zero density, that is,

$$\lim_{n\to\infty}\frac{\#Np_n(\eta,\omega,\delta)}{n}=0.$$

Note that if $\gamma \in \Sigma^m$ and $\gamma_i = \omega_i$, for all i except for $i \in Np(\eta,\omega,\delta)$, then a (δ,ω)-ergodic pseudo-orbit $\eta = \{x_i\}_{i\geq 0}$ of Γ is also a (δ,γ)-ergodic pseudo-orbit for Γ.

A (δ,ω)-ergodic pseudo-orbit η is said to be ϵ-ergodic shadowed by a point $z \in X$ if there exists a sequence $\gamma \in \Sigma^m$ with $\gamma_i = \omega_i$ for $i \in \mathbb{N}\setminus Np(\eta,\omega,\delta)$ such that

$$\lim_{n\to\infty}\frac{\#Ns_n(\eta,\gamma,z,\epsilon)}{n}=0.$$

A semigroup Γ has the *ergodic shadowing property* if for each $\epsilon > 0$ there exists $\delta > 0$ such that any (δ,ω)-ergodic pseudo-orbit of Γ can be ϵ-ergodic shadowed by some point z in X.

Definition 2.3 We say that a semigroup Γ has the *pseudo-orbital specification property* if for any $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ and $K = K(\epsilon) > 0$ such that for any nonnegative integers

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$$

with $a_{j+1} - b_j \geq K$ and (δ, w^j)-pseudo-orbits ξ_j, with $\xi_j = \{x_{(j,i)}\}, i \in I_j = [a_j, b_j] \subset \mathbb{N}$ and $1 \leq j \leq n$, and $w^j = w^j_{a_j}\dots w^j_{b_j-1} \in \Gamma_{\mathcal{A}}$, there exist a point $z \in X$ and $\omega \in \Sigma^m$, with $\omega_i = w^j_i$ for $i \in [a_j, b_j - 1]$ and $1 \leq j \leq n$, such that

$$d(f^i_\omega(z), x_{(j,i)}) < \epsilon, \quad i \in I_j, \quad 1 \leq j \leq n.$$

3 Shadowing Property

In this section, we show that ergodic shadowing for semigroup actions implies ordinary shadowing. We first prove that any semigroup with the ergodic shadowing property is chain transitive.

Lemma 3.1 *Suppose that $f_1, f_2, \dots, f_m$ are continuous maps on a compact metric space X such that one of them is surjective. If the semigroup Γ generated by these maps has the ergodic shadowing property, then it is chain transitive.*

Proof Consider a semigroup Γ that satisfies the assumptions of the lemma, and take any $x, y \in X$. Given $\epsilon > 0$, let $\delta > 0$ be provided by the ergodic shadowing property. Let $a_0 = 0$, $b_0 = 1$ and for any $k \geq 1$, put $a_k = b_{k-1} + k$ and $b_k = a_k + k + 1$. Fix a sequence $\omega = \omega_1\omega_2\dots \in \Sigma^m$. Suppose that the map f_ℓ, $\ell \in \{1,\dots,m\}$ is surjective.

Hence, for any $k \geq 1$, there is a point $z_k \in X$ with $f_\ell^k(z_k) = y$. For each $k \geq 0$, let us take

$$x_i := \begin{cases} f_\omega^{i-a_k}(x), & a_k \leq i < b_k, \\ f_\ell^{i-b_k}(z_k), & b_k \leq i < a_{k+1}; \end{cases}$$

hence,

$$x_{a_k} = x, \quad x_{a_k+1} = f_{\omega_1}(x), \quad x_{a_k+2} = f_{\omega_2} \circ f_{\omega_1}(x), \quad \ldots, x_{b_k-1} = f_\omega^{b_k-a_k-1}(x),$$

and

$$x_{b_k} = z_k, \quad x_{b_k+1} = f_\ell(z_k), \ldots, x_{a_{k+1}-1} = f_\ell^k(z_k) = y.$$

By the choice, on $[a_k, b_k)$ we use $f_\omega^i(x)$, $i = 0, \ldots, k$, which is the piece of the orbit of x beginning at x and of length k, and on $[b_k, a_{k+1})$ we use a piece of an orbit of length k which ends at y, that is, $f_\ell^i(z_k)$, $i = 0, \ldots, k$. By this observation, if we take a sequence η of the form

$$\eta = ss\omega_1 sls\omega_1\omega_2 slls\omega_1\omega_2\omega_3 slll \ldots \in \Sigma^m,$$

for some $s \in \{1, 2, \ldots, m\}$, then the sequence

$$\{x_i\}_{i\geq 0} = \left\{x_0 = x, y, x, f_{\omega_1}(x), z_1, y, x, f_{\omega_1}(x), f_{\omega_2} \circ f_{\omega_1}(x), z_2, f_\ell(z_2), y, \ldots\right\}$$

is a (δ, η)-ergodic pseudo-orbit.

Indeed, $d(f_{\eta_i}(x_i), x_{i+1}) < \delta$ except for the set $Np(\{x_i\}_{i\geq 0}, \eta, \delta) = \{i : \eta_i = s\}$, which has zero density. Since by construction for any arbitrary large positive integer L, there exists an interval $I \subset \mathbb{N}$ of length L such that for any $i \in I$, $\eta_i \neq s$ and so $I \cap Np(\{x_i\}_{i\geq 0}, \eta, \delta) = \emptyset$. Therefore, by the definition of ergodic shadowing, there are a sequence $\gamma \in \Sigma^m$ with $\gamma_i = \eta_i$, for each i except for $i \in Np(\{x_i\}_{i\geq 0}, \eta, \delta)$, and a point $z \in X$ such the set $Ns(\{x_i\}_{i\geq 0}, \gamma, z, \epsilon)$ has zero density, so its complement must intersect infinitely many intervals $[a_k, b_k)$ and infinitely many intervals $[b_k, a_{k+1})$. Therefore, there exist positive integers i_0, j_0, i_1, j_1 with $j_0 < i_1$ such that

$$d\left(f_\gamma^{i_0}(z), f_\omega^{j_0}(x)\right) < \epsilon, \quad d\left(f_\gamma^{i_1}(z), t\right) < \epsilon, \quad t \in f_\ell^{-j_1}(y).$$

Thus,

$$\Big\{x, f_{\omega_1}(x), \ldots, f_\omega^{j_0}(x), f_\gamma^{i_0}(z), f_\gamma^{i_0+1}(z), \ldots, f_\gamma^{i_1-1}(z), t,$$
$$f_\ell(t), \ldots, y = f_\ell^{j_1}(t)\Big\},$$

is an (ϵ, w)-chain from x to y for

$$w = \omega_1\omega_2\ldots\omega_{j_0}\gamma_{i_0}\gamma_{i_0+1}\ldots\gamma_{i_1-1}\underbrace{\ell\ldots\ell}_{j_1 \text{ times}}.$$

□

Remark 3.2 In Lemma 3.1, we assume that one of the generators of the semigroup Γ maps the space X onto itself. We can replace this assumption by the following:

$$\bigcup_{i=1}^{m} f_i(X) = X.$$

It is well known that the finite shadowing property for a continuous map implies the shadowing property, see [2]. In the following lemma, we show that it is true for semigroup actions.

Lemma 3.3 *Let Γ be a semigroup generated by a finite collection of continuous maps on a compact metric space X. If any finite pseudo-orbit of Γ can be shadowed by a true orbit, then Γ has the shadowing property.*

Proof Let $\epsilon > 0$ be given. Suppose that $\delta > 0$ is the ϵ-modulus of the finite shadowing property for Γ. Let $\{x_i\}_{i\geq 0}$ be an infinite δ-pseudo-orbit for Γ. Then there exists a sequence $\omega = \omega_1\omega_2\ldots \in \Sigma^m$ with

$$d(f_{\omega_i}(x_{i-1}), x_i) < \delta, \quad i \geq 1.$$

For any $k > 0$, there exists a point $z_k \in X$ which shadows (δ, η)-pseudo-orbit $\{x_i\}_{0\leq i\leq k}$ of the semigroup Γ with $\eta = \omega_1\omega_2\ldots\omega_k \in \mathcal{A}_k^m$, that is,

$$d(f_\eta^i(z_k), x_i) = d(f_\omega^i(z_k), x_i) < \epsilon, \quad 0 \leq i \leq k.$$

Since X is compact, there exist a subsequence $\{z_{k_n}\}_{n\geq 0}$ and a point $z \in X$ such that $\{z_{k_n}\} \to z$ as $n \to \infty$. By continuity of mapping f_j, $j = 1, \ldots, m$, for any $i \geq 0$, there is an integer $k_n > i$ such that $d(f_\omega^i(z_{k_n}), f_\omega^i(z)) < \epsilon$. Therefore, for any $i \geq 0$,

$$d(f_\omega^i(z), x_i) \leq d(f_\omega^i(z), f_\omega^i(z_{k_n})) + d(f_\omega^i(z_{k_n}), x_i) < 2\epsilon.$$

This implies that the point z, 2ϵ-shadows the sequence $\{x_i\}_{i\geq 0}$. □

Lemma 3.4 *Let Γ be a finitely generated semigroup satisfying the assumption of Lemma 3.1. If Γ has the ergodic shadowing property, then it enjoys the ordinary shadowing property.*

Proof Consider a semigroup Γ satisfies the assumptions of the lemma having the ergodic shadowing property. By the previous lemma, it is enough to show that any finite pseudo-orbit of Γ can be shadowed by a true orbit. Given $\epsilon > 0$, let $\delta > 0$ be an ϵ-modulus of ergodic shadowing. Suppose that $\{x_i\}_{0\le i\le n}$ is a finite δ-pseudo-orbit for Γ of finite length. Hence, there exists a finite word $w = w_1 w_2 \dots w_n \in \Gamma_{\mathcal{A}}$, with

$$d(f_{w_i}(x_{i-1}), x_i) < \delta, \quad 1 \le i \le n.$$

By Lemma 3.1, there is a δ-chain from x_n to x_0, that is, there exist a sequence $\{y_i\}_{i=0}^{\ell}$ with $y_0 = x_n$ and $y_\ell = x_0$ and a word $\gamma = \gamma_1 \dots \gamma_\ell \in \Gamma_{\mathcal{A}}$ such that

$$d(f_{\gamma_i}(y_{i-1}), y_i) < \delta, \quad 1 \le i \le \ell.$$

Let us take

$$z_i := \begin{cases} x_{i-m_j}, & i \in [m_j, m_j + n], \\ y_{i-m_j-n-1}, & i \in [m_j + n + 1, m_j + n + \ell + 1], \end{cases}$$

for $m_j := j(n + \ell + 2)$ and $i, j \ge 0$, that is,

$$\{z_i\}_{i\ge 0} := \{x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_\ell, x_0, x_1, \dots, x_n, \dots\}.$$

Consider a sequence $\omega := wk\gamma kwk\gamma kw \dots \in \Sigma^m$, for some symbol $k \in \mathcal{A}^m$, where w and γ are finite words as described above. Thus, $d(f_{\omega_i}(z_{i-1}), z_i) < \delta$, except for a subset of positive integers with zero density. Therefore, $\{z_i\}_{i\ge 0}$ is a (δ, ω)-ergodic pseudo-orbit and so can be ρ-ergodic shadowed by some point z and for some sequence $\rho \in \Sigma^m$ with $\rho_i = \omega_i$ for all i except for $i \in Np(\{z_i\}_{i\ge 0}, \omega, \delta)$. Since z ergodic shadows $\{z_i\}_{i\ge 0}$, $Ns(\{z_i\}_{i\ge 0}, \rho, z, \epsilon)$ cannot meet every interval $[m_j, m_j + n] \subset \mathbb{N}$; therefore, it would have positive density. Hence, at least one n interval is entirely ε shadowed by a piece of the z orbit. □

4 Topologically Mixing

Chain mixing, weakly mixing, and topologically mixing are formal ways to get some special dynamical properties and also chaos in global sense. Here, we show that the ergodic shadowing implies the mentioned properties at the same time.

Fix $n \in \mathbb{N}$, and consider a finitely generated semigroup Γ with the finite set of generators $G_1 = \{id, f_1, \dots, f_m\}$.

Take $G_1^* = G_1 \setminus \{id\}$, and let G_n^* denote the space of compositions of n elements in G_1^*. Thus, $f \in G_n^*$ if $f = f_w^n$ for some finite word $w \in \Gamma_{\mathcal{A}}$ with length n. Denote Γ_n the semigroup generated by G_n^*.

Notice that a sequence $\{x_i\}_{i\ge 0}$ is a (δ, ω)-ergodic pseudo-orbit of Γ_n, if we can write $\omega := w^1 w^2 \dots \in \Sigma^m$, where $w^i = w_1^i w_2^i \dots w_n^i \in \mathcal{A}_n^m$, $i \ge 1$, such that the set

$$Np_n(\{x_i\}_{i\ge 0}, \omega, \delta) := \left\{i \in \mathbb{N} : d\left(f_{w^i}^n(x_{i-1}), x_i\right) \ge \delta\right\}$$

has zero density. The semigroup Γ_n has the ergodic shadowing property if for each $\epsilon > 0$, there are $\delta > 0$ and $\gamma = \gamma^1 \gamma^2 \dots \in \Sigma^m$, where $\gamma^i \in \mathcal{A}_n^m$, such that $\gamma^i = w^i$

for all i except for $i \in Np_n(\{x_i\}_{i\geq 0}, \omega, \delta)$ and the subset

$$Ns_n(\{x_i\}_{i\geq 0}, \gamma, z, \epsilon) := \left\{i \in \mathbb{N} : d\left(f^n_{\gamma^i})(z), x_i\right) \geq \epsilon\right\}$$

has zero density.

Moreover, if $\{x_i\}_{i\geq 0}$ is a (δ, ω)-ergodic pseudo-orbit of Γ_n, then the sequence $\{y_i\}_{i\geq 0}$ of the form

$$\begin{aligned}\{y_i\}_{i\geq 0} = \Big\{&x_0, f_{\omega_1}(x_0), f_{\omega_2} \circ f_{\omega_1}(x_0), \ldots, f_{\omega_n} \circ \cdots \circ f_{\omega_1}(x_0),\\ &x_1, f_{\omega_{n+1}}(x_1), f_{\omega_{n+2}} \circ f_{\omega_{n+1}}(x_1), \ldots, f_{\omega_{2n}} \circ \ldots \circ f_{\omega_{n+1}}(x_1),\\ &\vdots\\ &x_k, f_{\omega_{nk+1}}(x_k), f_{\omega_{nk+2}} \circ f_{\omega_{nk+1}}(x_k), \ldots, f_{\omega_{nk+n}} \circ \ldots \circ f_{\omega_{nk+1}}(x_k), \ldots\Big\}\end{aligned}$$

is a (δ, ω)-ergodic pseudo-orbit of Γ, if we can rewrite ω of the form $\omega = \omega_1\omega_2\ldots$, where $w^i = \omega_{n(i-1)+1}\ldots\omega_{n(i-1)+n}$ for each $i \geq 1$.

By these observations, the next result can be followed.

Lemma 4.1 *If the semigroup Γ has the ergodic shadowing property, then for any $n \in \mathbb{N}$, the semigroup Γ_n has also the ergodic shadowing property.*

Corollary 4.2 *Let Γ be a semigroup generated by a finite family of continuous self-maps on a compact metric space X such that one of the generators is a surjective map. If Γ has the ergodic shadowing property, then it is chain mixing.*

Proof By Lemma 4.1, for any $n \in \mathbb{N}$, the semigroup Γ_n has the ergodic shadowing property, so by Lemma 3.1, it is chain transitive. This implies that the semigroup Γ is chain mixing, see [18, Theorem 2.3]. □

In the following, we prove that any semigroup with the ergodic shadowing property is topologically mixing. For this purpose, we need the following auxiliary lemma.

Lemma 4.3 *If Γ has the chain mixing and shadowing properties, then it is topologically mixing.*

Proof Given two open sets $U, V \subset X$, with $x \in U$ and $y \in V$, choose $\epsilon > 0$ such that $N_\epsilon(x) \subset U$ and $N_\epsilon(y) \subset V$. Let $\delta > 0$ be an ϵ-modulus of shadowing. Since Γ is chain mixing, so for sufficiently large integer n, there is a (δ, w)-chain $\{x = x_0, x_1, \ldots, x_{n-1} = y\}$ from x to y of length n, where w is a finite word of length n. Using shadowing property, one can find a point $z \in X$ such that $z \in N_\epsilon(x_0)$ and $f^n_w(z) \in N_\epsilon(y)$. Thus, $f^n_w(U) \cap V \neq \emptyset$. □

Corollary 4.4 *If Γ has the ergodic shadowing property, then it is topologically mixing.*

Proof Suppose that Γ has the ergodic shadowing property. By Lemma 3.4, it has shadowing property and so by Corollary 4.2 and Lemma 4.3, it is topologically mixing. □

Lemma 4.5 *If Γ has the ergodic shadowing property, then the product semigroup $\Gamma \times \Gamma$ has also the ergodic shadowing property.*

Proof Given $\epsilon > 0$, and let $\delta > 0$ be an ϵ-modulus of shadowing for Γ. Suppose that $\eta = \{(x_i, y_i)\}_{i \geq 0}$ is a δ-ergodic pseudo-orbit for the semigroup $\Gamma \times \Gamma$. Therefore, there are sequences $\omega^1, \omega^2 \in \Sigma^m$ such that

$$Np(\eta, \omega^{1,2}, \delta) := \left\{ i \in \mathbb{Z}^+ : D\left(f_{\omega_i^{1,2}}(x_{i-1}, y_{i-1}), (x_i, y_i) \right) \geq \delta \right\}, \tag{4.1}$$

has zero density, where $f_{\omega_i^{1,2}}(x, y) = f_{\omega_i^1}(x) \times f_{\omega_i^2}(y)$. The definition of the metric D yields that

$$\begin{aligned} D\left(f_{\omega_i^{1,2}}(x_{i-1}, y_{i-1}), (x_i, y_i) \right) &= D\left((f_{\omega_i^1}(x_{i-1}), f_{\omega_i^2}(y_{i-1})), (x_i, y_i) \right) \\ &= \max\left\{ d\left(f_{\omega_i^1}(x_{i-1}), x_i \right), d\left(f_{\omega_i^2}(y_{i-1}), y_i \right) \right\}. \end{aligned}$$

Therefore,

$$\#Np(\{x_i\}_{i \geq 0}, \omega^1, \delta) \leq \#Np(\eta, \omega^{1,2}, \delta)$$

and

$$\#Np(\{y_i\}_{i \geq 0}, \omega^2, \delta) \leq \#Np(\eta, \omega^{1,2}, \delta).$$

These facts imply that both sets $Np(\{x_i\}_{i \geq 0}, \omega^1, \delta)$ and $Np(\{y_i\}_{i \geq 0}, \omega^2, \delta)$ have zero density. Thus, the sequences $\{x_i\}_{i \geq 0}$ and $\{y_i\}_{i \geq 0}$ are (δ, ω^1) and (δ, ω^2)-ergodic pseudo-orbits for Γ, respectively. Hence, there are two points $z_1, z_2 \in X$ and sequences $\gamma^1, \gamma^2 \in \Sigma^m$ with $\gamma_i^1 = \omega_i^1$ for all i except for $i \in Np(\{x_i\}_{i \geq 0}, \omega^1, \delta)$ and $\gamma_i^2 = \omega_i^2$ for all i except for $i \in Np(\{y_i\}_{i \geq 0}, \omega^2, \delta)$ such that the subsets $Ns(\{x_i\}_{i \geq 0}, \gamma^1, z_1, \epsilon)$ and $Ns(\{y_i\}_{i \geq 0}, \gamma^2, z_2, \epsilon)$ have zero density. It is easy to see that

$$\begin{aligned} Ns(\eta, \gamma^{1,2}, (z_1, z_2), \epsilon) &:= \left\{ i \in \mathbb{Z}^+ : D\left(f^i_{\omega^{1,2}}(z_1, z_2), (x_i, y_i) \right) \geq \epsilon \right\} \\ &\subseteq Ns(\{x_i\}_{i \geq 0}, \gamma^1, z_1, \epsilon) \cup Ns(\{y_i\}_{i \geq 0}, \gamma^2, z_2, \epsilon); \end{aligned}$$

hence, $Ns(\eta, \gamma^{1,2}, (z_1, z_2), \epsilon)$ has also zero density. □

Lemma 4.6 *Let Γ be a semigroup generated by a finite family of continuous self-maps on the compact metric space X such that one of them is a surjective map. If Γ has the ergodic shadowing property, then it is weakly mixing.*

Proof By Lemma 4.5, the semigroup $\Gamma \times \Gamma$ has the ergodic shadowing property. Using Lemmas 3.1 and 3.4 implies that $\Gamma \times \Gamma$ is chain transitive and has the shadowing property and hence it is topologically transitive, see [10, Lemma 3.10]. Thus, $\Gamma \times \Gamma$ is weakly mixing. □

5 Pseudo-orbital Specification Property

The aim of this section is to show that the ergodic shadowing property for a semigroup Γ is equivalent to the pseudo-orbital specification property.

Lemma 5.1 *If semigroup Γ has shadowing and topologically mixing properties, then it has the pseudo-orbital specification property and the ergodic shadowing property.*

Proof Take a semigroup Γ generated by a finite family of continuous self-maps $\{f_i : i = 1, \dots, m\}$ having the shadowing and topologically mixing properties. Let $\epsilon > 0$ be given and let $\delta > 0$ be an ϵ-modulus of shadowing property for Γ. By uniform continuity of the mappings f_i, there exists $\eta < \delta$ such that $d(x, y) < \eta$ implies that $d(f_i(x), f_i(y)) < \delta$ for all $1 \le i \le m$ and $x, y \in X$. Since X is compact, we can find a finite open cover $\mathcal{U} = \{U_1, \dots, U_M\}$ of X composed of open balls of radius $\eta/2$. Since Γ is topologically mixing, hence for any two open sets $U_i, U_j \in \mathcal{U}$, there exist an integer $M_{ij} > 0$ and sequence $\omega^{ij} \in \Sigma^m$ such that $f^n_{\omega^{ij}}(U_i) \cap U_j \ne \emptyset$ for $n \ge M_{ij}$. Take

$$K(\epsilon) = \max\{M_{ij} : 1 \le i, j \le M\}.$$

Let $\xi_j = \{x_{(j,i)}\}$ be (δ, ρ^j)-pseudo-orbits in X defined on subinterval $I_j = [a_j, b_j], 1 \le j \le n$ with $a_{j+1} - b_j > K(\epsilon)$, where $\rho^j = \rho^j_{a_j} \dots \rho^j_{b_j-1} \in \Gamma_{\mathcal{A}}$. Notice that for any $x \in X$, there is an open set $U \in \mathcal{U}$ such that $x \in U$, which is denoted by $U(x)$. Set $m_j := a_{j+1} - b_j$, by the choice of $K(\epsilon)$ there exists a sequence $\gamma^j \in \{\omega^{ij} : 1 \le i, j \le M\} \subset \Sigma^m$ such that

$$f^{m_j}_{\gamma^j}(U(x_{(j,b_j)}) \cap U(x_{(j+1,a_{j+1})}) \ne \emptyset.$$

Therefore, there is $y_j \in U(x_{(j,b_j)})$ such that $f^{m_j}_{\gamma^j}(y_j) \in U(x_{(j+1,a_{j+1})})$. Now consider the sequence $\omega = \omega_0\omega_1 \dots \in \Sigma^m$ with

$$\omega_\ell = \begin{cases} \rho^j_\ell, & a_j \le \ell \le b_j - 1, \\ \gamma^j_{\ell-b_j}, & b_j \le \ell \le a_{j+1} - 1, \end{cases}$$

and $\zeta_j = \{f_{\gamma^j}(y_j), \dots, f^{m_j}_{\gamma^j}(y_j)\}$. One can see that $\{\xi_1, \zeta_1, \xi_2, \zeta_2, \dots, \zeta_{n-1}, \xi_n\}$ is a (δ, ω)-pseudo-orbit and can be ϵ-shadowed by a point $z \in X$. This shows that the semigroup Γ has pseudo-orbital specification property.

To verify that the semigroup Γ has the ergodic shadowing property, we use the approach used in the proof of Lemma 4.5 in [8]. Let $\epsilon > 0$ be given. Choose $\delta > 0$ and $K(\epsilon)$ as above. Let $\xi = \{x_i\}_{i \ge 0}$ be a (δ, ω)-ergodic pseudo-orbit for Γ. There exists a sequence

$$a_1 < b_1 < a_2 < b_2 < \cdots$$

of natural numbers such that the following conditions hold:

(1) $[a_n, b_n] \subset \mathbb{N} \setminus Np(\xi, \omega, \delta)$ for each n,
(2) $a_{n+1} - b_n \geq K$ for each n,
(3) $\lim_{n\to\infty} \frac{(\sum_{k=1}^{n} a_{k+1}-b_k)}{b_n} \to 0$.

Set $\xi_j = \{x_{(j,i)}\} := \{x_i\}$ for $i \in I_j$, $j \in \mathbb{N}$. Then ξ_j is a (δ, ρ^j)-pseudo-orbit in X defined on subinterval $I_j = [a_j, b_j]$, $j \in \mathbb{N}$ with $\rho^j = \omega_{a_j} \ldots \omega_{b_j-1} \in \Gamma_{\mathcal{A}}$. Using a similar argument as above, implies that $\{\xi_1, \zeta_1, \xi_2, \zeta_2, \ldots\}$ is a (δ, η)-pseudo-orbit, where $\eta = \eta_0\eta_1 \ldots \in \Sigma^m$ and

$$\eta_\ell = \begin{cases} \rho^j_\ell, & a_j \leq \ell \leq b_j - 1, \\ \gamma^j_{\ell-b_j}, & b_j \leq \ell \leq a_{j+1} - 1. \end{cases}$$

Thus, there is a point $z \in X$, which ϵ-shadows this sequence. Therefore, the semigroup Γ has the ergodic shadowing property. □

Lemma 5.2 *If the semigroup Γ has the pseudo-orbital specification property, then it is topologically mixing and has the shadowing property.*

Proof First, we show that Γ is topologically mixing. Given two open sets $U, V \subset X$. Choose $\epsilon > 0$ such that $N_\epsilon(x) \subset U$ and $N_\epsilon(y) \subset V$. Let $K = K(\epsilon) > 0$, $\delta = \delta(\epsilon) > 0$ be as in the definition of pseudo-orbital specification property. Put $a_1 = 0 < b_1$ and $a_2 = K + b_1 < b_2$. Let $x_{(1,i)} = f^{i-a_1}_{w^1}(x)$ and $x_{(2,i)} = f^{i-a_2}_{w^2}(y)$ for some $w^j = w^j_{a_j} \ldots w^j_{b_j-1} \in \Gamma_{\mathcal{A}}$, $i \in I_j = [a_j, b_j]$, $j = 1, 2$. Clearly, $\xi_j = \{x_{(j,i)}\}$, $i \in I_j = [a_j, b_j]$, $j = 1, 2$ are (δ, w^j)-pseudo-orbits for Γ; therefore, there exist a point $z \in X$ and $\omega \in \Sigma^m$, with $\omega_i = w^j_i$ for $i \in [a_j, b_j - 1]$ and $j = 1, 2$, such that

$$d(f^i_\omega(z), x_{(j,i)}) < \epsilon, \quad i \in I_j, \quad j = 1, 2.$$

In particular,

$$d(f^{a_1}_\omega(z), x_{(1,a_1)}) = d(z, x) < \epsilon$$

and

$$d(f^{a_2}_\omega(z), x_{(2,a_2)}) = d(f^{a_2}_\omega(z), y) < \epsilon.$$

So $z \in N_\epsilon(x) \subset U$ and $f^{a_2}_\omega(z) \in N_\epsilon(y) \subset V$, which implies that $f^{b_1+K}_\omega(U) \cap V \neq \emptyset$. Similar arguments show that for any $n > b_1 + K$ there exists a sequence $\omega \in \Sigma^m$, such that $f^n_\omega(U) \cap V \neq \emptyset$.

Now, we show that the semigroup Γ has the shadowing property. By Lemma 3.3, it is enough to show that it has the finite shadowing property. Given $\epsilon > 0$, let $\delta = \delta(\epsilon)$ be an ϵ-modulus pseudo-orbital specification property. Suppose that $\{x_i\}_{0\leq i\leq n}$ is a finite δ-pseudo-orbit for Γ. Hence, there exists a finite word $w = w_0w_2 \ldots w_{n-1} \in \Gamma_{\mathcal{A}}$, with

$$d(f_{w_i}(x_i), x_{i+1}) < \delta, \quad 0 \leq i \leq n - 1.$$

Set $a_1 = 0, b_1 = n$, and $x_{(1,i)} = x_i, i \in [0, n]$. Then $\xi_1 = \{x_{(1,i)}\}$ is a (δ, w)-pseudo-orbit for Γ. By pseudo-orbital specification property, there is a point $z \in X$ and $\omega \in \Sigma^m$, with $\omega_i = w_i$ for $i \in [0, n-1]$ such that

$$d(f^i_\omega(z), x_i) < \epsilon, \quad i \in [0, n].$$

□

Collecting the lemmas which have been proved in the previous sections, we can state the proof of Theorem 1.1.

Proof of Theorem 1.1 (1) ⇒ (2) is followed by Lemma 3.4 and Corollary 4.2, (2) ⇒ (3) is Lemma 4.3, (3) ⇒ (4) is Lemma 5.1. (4) ⇒ (1) is followed from Lemma 3.4, Corollary 4.4 and Lemma 5.2. □

6 Expanding Semigroups and Strong Pseudo-orbital Specification

In this section, we introduce a stronger notion of pseudo-orbital specification. Our main goal here is to prove that semigroups of expanding maps satisfy the strong pseudo-orbital specification property and, therefore, they have the ergodic shadowing property.

Assume that X is a compact Riemannian manifold. Consider a finitely generated semigroup Γ with the finite set of generators $G_1 = \{id, f_1, \ldots, f_m\}$. We say that the semigroup Γ is *expanding*, if for each $i \in \{1, \ldots, m\}$, the mapping f_i is an expanding C^1 map, that is, there are constants $C > 0$ and $0 < \lambda_i < 1$ such that

$$\|(Df_i^n(x))^{-1}\| \leq C\lambda_i^n,$$

for every $n \geq 1$ and $x \in X$.

The strong orbital specification property for semigroup actions was introduced in [15]. Here, we define the strong pseudo-orbital specification property for semigroup actions.

Definition 6.1 The semigroup Γ has *strong pseudo-orbital specification property* if for any $\epsilon > 0$, there exist $\delta = \delta(\epsilon) > 0$ and $K = K(\epsilon) > 0$ such that for any nonnegative integers

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$$

with $a_{j+1} - b_j \geq K$ and (δ, w^j)-pseudo-orbits ξ_j, with $\xi_j = \{x_{(j,i)}\}, i \in I_j = [a_j, b_j] \subset \mathbb{N}$ and $1 \leq j \leq n$, and $w^j = w^j_{a_j} \ldots w^j_{b_j-1} \in \Gamma_{\mathcal{A}}$, if $\omega \in \Sigma^m$ with $\omega_i = w^j_i$ for $i \in [a_j, b_j - 1]$ and $1 \leq j \leq n$, then there exists a point $z \in X$ such that

$$d(f^i_\omega(z), x_{(j,i)}) < \epsilon, \quad i \in I_j, \quad 1 \leq j \leq n.$$

Remark 6.2 Observe that for $m = 1$, this definition coincides with the definition of pseudo-orbital specification property of mapping f_1 in [8]. Therefore, if a semigroup Γ with the set of generators $G_1 = \{id, f_1 \dots, f_m\}$ has the strong pseudo-orbital specification property, then one can see that, for each $i = 1, \dots, m$, the mapping f_i has the pseudo-orbital specification property. Obviously, if the semigroup Γ has strong pseudo-orbital specification property, then it has pseudo-orbital specification property.

For any $w \in \Gamma_{\mathcal{A}}$, with $w = w_0 \dots w_{\ell-1}$ and $\epsilon > 0$, define a metric d_w on X by

$$d_w(x_1, x_2) = \max_{0 \le i \le \ell} d\left(f_w^i(x_1), f_w^i(x_2)\right) \quad \text{for all } x_1, x_2 \in X,$$

also we define the *dynamical ball* $B(x, w, \epsilon)$ by

$$B(x, w, \epsilon) = \{y \in X : d_w(x, y) \le \epsilon\}.$$

Lemma 6.3 (see [15]) *Let Γ be an expanding semigroup with the set of generators $G_1 = \{id, f_1, \dots, f_m\}$, then the following statements hold:*

(1) *There exists $\delta_0 > 0$ such that $f_w^\ell(B(x, w, \delta)) = B(f_w^\ell(x), \delta)$, for any $\delta \le \delta_0$, any $x \in X$, and any $w \in \Gamma_{\mathcal{A}}$, with $|w| = \ell \in \mathbb{N}$.*
(2) *For any $\delta > 0$, there exists $K = K(\delta) \in \mathbb{N}$ such that $f_w^K(B(x, \delta)) = X$ for every $x \in X$ and every $w \in \mathcal{A}_K^m$.*

Lemma 6.4 *Any expanding semigroup Γ has the strong pseudo-orbital specification property.*

Proof If a semigroup Γ is expanding, then it has the shadowing property, see [7,13]. Let $\epsilon > 0$ be given. For any nonnegative integers

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n,$$

since semigroup Γ has the shadowing property, there exists $\delta > 0$ such that any (δ, w^j)-pseudo-orbit $\xi_j = \{x_{(j,i)}\}$ with $w^j = w^j_{a_j} \dots w^j_{b_j-1} \in \Gamma_{\mathcal{A}}$, can be $\frac{\epsilon}{2}$-shadowed by some point $z_j \in X$, that is,

$$d\left(f_{w^j}^{i-a_j}(z_j), x_{(j,i)}\right) < \frac{\epsilon}{2} \quad \text{for} \quad i \in I_j = [a_j, b_j], \quad 1 \le j \le n.$$

Set $m_j := b_j - a_j$; then $|w^j| = m_j$. By Lemma 6.3, there exits $\eta_0 > 0$ such that for any $\eta \le \eta_0$,

$$f_{w^j}^{m_j}\left(B(z_j, w^j, \eta)\right) = B\left(f_{w^j}^{m_j}(z_j), \eta\right) \quad \text{for all} \quad 1 \le j \le n.$$

We assume that $\frac{\epsilon}{2} \le \eta_0$. Using Lemma 6.3, there exists a number $K = K(\epsilon)$ such that $f_{\gamma_i}^K(B(f_{w^i}^{m_i}(z_i), \frac{\epsilon}{2})) = X$, for any $\gamma_i \in \mathcal{A}_K^m$, $1 \le i \le n$.

Let $\overline{z}_n \in B(z_n, w^n, \frac{\epsilon}{2})$, and let $\gamma_i \in \mathcal{A}_K^m, 1 \le i \le n$ be arbitrary. So there exists $\overline{z}_{n-1} \in B(f_{w^{n-1}}^{m_{n-1}}(z_{n-1}), \frac{\epsilon}{2})$ such that $f_{\gamma_{n-1}}^K(\overline{z}_{n-1}) = \overline{z}_n$. Therefore, there exists $\overline{z}_{n-2} \in B(z_{n-1}, w^{n-1}, \frac{\epsilon}{2})$ such that $f_{w^{n-1}}^{m_{n-1}}(\overline{z}_{n-2}) = \overline{z}_{n-1}$, which implies that $f_{\gamma_{n-1}}^K \circ f_{w^{n-1}}^{m_{n-1}}(\overline{z}_{n-2}) = \overline{z}_n$. By induction, there exists $x \in B(z_1, w^1, \frac{\epsilon}{2})$ such that

$$f_{\gamma_{j-1}}^K \circ f_{w^{j-1}}^{m_{j-1}} \circ \cdots \circ f_{\gamma_1}^K \circ f_{w^1}^{m_1}(x) = \overline{z}_j,$$

and hence

$$f_{\gamma_{j-1}}^K \circ f_{w^{j-1}}^{m_{j-1}} \circ \cdots \circ f_{\gamma_1}^K \circ f_{w^1}^{m_1}(x) \in B\left(z_j, w^j, \frac{\epsilon}{2}\right),$$

which implies that, for any $i \in [a_j, b_j]$,

$$d\left(f_{w^j}^{i-a_j} \circ f_{\gamma_{j-1}}^K \circ f_{w^{j-1}}^{m_{j-1}} \circ \cdots \circ f_{\gamma_1}^K \circ f_{w^1}^{m_1}(x), f_{w^j}^{i-a_j}(z_j)\right) < \frac{\epsilon}{2}.$$

Let $\omega \in \Sigma^m$ with $\omega_i = w_i^j, i \in [a_j, b_j - 1], 1 \le j \le n$. Let $K_j = a_{j+1} - b_j \ge K$. Since any map $f_i, 1 \le i \le m$ is surjective, choose $z \in (f_\omega^{a_1})^{-1}(x)$. Therefore, for any $i \in [a_j, b_j]$,

$$f_\omega^i(z) = f_{w^j}^{i-a_j} \circ f_{\gamma_{j-1}}^{K_{j-1}} \circ f_{w^{j-1}}^{m_{j-1}} \circ \cdots \circ f_{\gamma_1}^{K_1} \circ f_{w^1}^{m_1}\left(f_\omega^{a_1}(z)\right)$$

for some $\gamma_j \in \mathcal{A}_{K_j}^m$. Thus, for $i \in [a_j, b_j]$,

$$\begin{aligned} d(f_\omega^i(z), x_{(j,i)}) &= d\left(f_{w^j}^{i-a_j} \circ f_{\gamma_{j-1}}^{K_{j-1}} \circ f_{w^{j-1}}^{m_{j-1}} \circ \cdots \circ f_{\gamma_1}^{K_1} \circ f_{w^1}^{m_1}(x), x_{(j,i)}\right) \\ &\le d\left(f_{w^j}^{i-a_j} \circ f_{\gamma_{j-1}}^{K_{j-1}} \circ f_{w^{j-1}}^{m_{j-1}} \circ \cdots \circ f_{\gamma_1}^{K_1} \circ f_{w^1}^{m_1}(x), f_{w^j}^{i-a_j}(z_j)\right) \\ &\quad + d\left(f_{w^j}^{i-a_j}(z_j), x_{(j,i)}\right) < \epsilon. \end{aligned}$$

Therefore, the semigroup Γ has the strong pseudo-orbital specification property. □

Corollary 6.5 *Any expanding semigroup* Γ *has the ergodic shadowing property.*

7 Examples

Here, we give some examples investigating the ergodic shadowing property.

Example 7.1 Consider the space $X = \{a, b, c\}$ with the discrete metric d, where a, b, c are different points. Let $f_1(x) = a$, $f_2(x) = b$, and $f_3(x) = c$ for any $x \in X$. Then f_0, f_1, and f_2 are continuous maps on the complete metric space X, such that $\cup_{i=1}^3 f_i(X) = X$. Let $G_1 = \{id, f_0, f_1, f_2\}$ and let Γ be a finitely generated semigroup. We show that Γ has the ergodic shadowing property.

Let $0 < \epsilon < 1$ be arbitrary. Put $\delta = \epsilon$. Suppose that $\eta = \{x_i\}_{i\geq 0}$ is a δ-ergodic pseudo-orbit for Γ. So there is $\omega = \omega_1\omega_2\ldots \in \Sigma^3$ such that the set

$$Np(\eta, \omega, \delta) = \left\{i \in \mathbb{N} : d(f_{\omega_i}(x_{i-1}), x_i) \geq \delta\right\},$$

has zero density. Clearly, $f_{\omega_i}(x_{i-1}) = x_i$ for any $i \in \mathbb{N} \setminus Np(\eta, \omega, \delta)$. Since f_0, f_1, and f_2 are constant functions, $f_{\omega_i}(x) = x_i$ for any $i \in \mathbb{N} \setminus Np(\eta, \omega, \delta)$ and $x \in X$. Let $z \in X$ be arbitrary, and choose $\gamma \in \Sigma^m$ with $\gamma_i = \omega_i$ for any i except for the set $Np(\eta, \omega, \delta)$. For each $i \in \mathbb{N} \setminus Np(\eta, \omega, \delta)$, we have

$$\begin{aligned} d(f^i_\gamma(z), x_i) &= d(f_{\gamma_i} \circ f_{\gamma_{i-1}} \circ \cdots \circ f_{\gamma_1}(z), x_i) \\ &= d(f_{\omega_i}(f_{\gamma_{i-1}} \circ \cdots \circ f_{\gamma_1}(z)), x_i) \\ &= 0 < \epsilon. \end{aligned}$$

Since the set $Np(\eta, \omega, \delta)$ has density zero, so the point z, ϵ-ergodic shadows the sequence η. By Theorem 1.1, the semigroup Γ has shadowing and pseudo-orbital specification properties. Also, it is chain mixing, topologically mixing, and weakly mixing.

Example 7.2 Consider the mappings f_0, and f_1 on $S^1 = \frac{\mathbb{R}}{\mathbb{Z}}$ defined by

$$f_0(x) = 2x \pmod 1, \quad f_1(x) = 3x \pmod 1.$$

Then f_i are C^1 expanding maps. So by Corollary 6.5, the semigroup Γ with generators $G_1 = \{id, f_0, f_1\}$ has the ergodic shadowing property.

Example 7.3 In this example, we build a semigroup with the ergodic shadowing property which does not have the strong pseudo-orbital specification property. Let $X = \{a, b, c\}$ and let a, b, c be three different points of X. With the discrete metric d, X is a compact metric space. Let f_1 and f_2 be two cyclic permutations on X, that is,

$$f_1(a) = b,\ f_1(b) = c,\ f_1(c) = a, \quad f_2(a) = c,\ f_2(b) = a,\ f_2(c) = b.$$

Then f_0 and f_1 are homeomorphisms on X. Denote by Γ the semigroup generated by $\Gamma_1 = \{id, f_1, f_2\}$. Let $\xi = \{x_i\}_{i\geq 0}$ be a (δ, ω)-ergodic pseudo-orbit for Γ. Choose a sequence

$$a_1 < b_1 < a_2 < b_2 < \cdots$$

of natural numbers such that the following conditions hold:

(1) $[a_n, b_n] \subset \mathbb{N} \setminus Np(\xi, \omega, \delta)$ for each n,
(2) $a_{n+1} - b_n \geq 3$ for each n,
(3) $\lim_{n\to\infty} \frac{(\sum_{k=1}^{n} a_{k+1} - b_k)}{b_n} \to 0$.

It is easy to see that we can choose a point $z \in X$ and $\gamma \in \Sigma^2$ with $\gamma_i = \omega_i, a_j \le i \le b_j - 1$ such that

$$f_\gamma^{a_i}(z) = x_{a_i} \quad \text{for any} \quad i \in \mathbb{N}.$$

Since for any $a_j \le i \le b_j$, $j \in \mathbb{N}$, $f_{\omega_i}(x_{i-1}) = x_i$, we have

$$\begin{aligned} d(f_\gamma^i(z), x_i) &= d\left(f_{\sigma^{a_j}\gamma}^{i-a_j}(f_\gamma^{a_j}(z)),\, f_{\sigma^{a_j}\omega}^{i-a_j}(x_{a_j})\right) \\ &= d\left(f_\omega^{i-a_j}(x_{a_j}),\, f_{\sigma^{a_j}\omega}^{i-a_j}(x_{a_j})\right) = 0 < \epsilon. \end{aligned}$$

This implies that Γ has the ergodic shadowing property, and by Theorem 1.1, it has the pseudo-orbital specification property. Now, we show that Γ has not the strong pseudo-orbital specification property. Indeed, since the mapping f_i has the finite order, if $\omega \in \Sigma^2$ and $\omega_i = 1$ for $i \in \mathbb{N} \setminus [a_j, b_j]$, then there are arbitrary large iterates n such that $f_\omega^{n+b_j}(x_{b_j}) = x_{b_j}$ and choosing $x_{a_j+1} \ne x_{b_j}$ implies that semigroup Γ does not have the strong pseudo-orbital specification property.

Example 7.4 Let $f : S^1 \to S^1$ be a C^1-expanding map of the circle and let $R_\alpha : S^1 \to S^1$ be the rotation of the angle α. Consider the semigroup Γ generated by $G_1 = \{id, f, R_\alpha\}$. It is well known that R_α does not have the shadowing property, so the semigroup Γ does not have the shadowing property, see [4]. Therefore, by Theorem 1.1, the semigroup Γ does not have the ergodic shadowing property.

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