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ORIGINAL PAPER

Weighted Space and Bloch-Type Space on the Unit Ball of an Infinite Dimensional Complex Banach Space

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Abstract

Let $\mathbf{B}_{\mathbb{X}}$ be the open unit ball of a complex Banach space $\mathbb{X}$, which may be infinite dimensional. The weighted composition operator and weighted space defined on $\mathbf{B}_{\mathbb{X}}$ are introduced. We obtain the boundedness and compactness of the weighted composition operator from the Bloch-type spaces to the weighted spaces, and some properties with the Bloch-type spaces are given. Our main results generalize the previous works on the Euclidean unit ball $\mathbb{B}^n$ to the case of $\mathbf{B}_{\mathbb{X}}$.

Keywords Boundedness · Complex Banach space · Compactness · Weighted composition operator · Weighted Bloch-type space

Mathematics Subject Classification 47B38 · 32A37 · 46B15

1 Introduction

Let $\mathbb{C}^n$ be the space of n-dimensional complex variables $z = (z_1, z_2, \ldots, z_n)$. The unit ball

$$\mathbb{B}^n = \left\{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \|z\|^2 = \sum_{k=1}^{n} |z_k|^2 < 1 \right\},$$

and $\mathbb{B}^1 \equiv \mathbb{U}$ denotes the unit disk in $\mathbb{C}$. Let $H(\mathbb{B}^n)$ be the family of holomorphic function from $\mathbb{B}^n$ to $\mathbb{C}$.

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A positive continuous function μ on $[0, 1)$ is called normal if there is $\delta \in [0, 1)$ and $0 < a < b < \infty$ such that

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\mu(r)}{(1-r)^a} = 0;$$
$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\mu(r)}{(1-r)^b} = \infty.$$

Then a normal function μ is strictly decreasing on $[\delta, 1)$ and $\mu(r) \to 0$ as $r \to 1$. Denote by $\mathrm{Aut}(\mathbb{B}^n)$ the holomorphic automorphism group of $\mathbb{B}^n$. If $u \in H(\mathbb{B}^n)$ and $\varphi \in \mathrm{Aut}(\mathbb{B}^n)$, then the corresponding weighted composition operator is defined by

$$uC_\varphi(f)(z) = u(z)f(\varphi(z)), \ \ z \in \mathbb{B}^n. \tag{1.1}$$

The boundedness and the compactness of the operators uC_φ on Bloch-type spaces, Zygmund spaces, Hardy spaces and the weighted Bergman space attract a lot of attentions (see, e.g., [6,8,12,14,16,18,19,22,23]).

The classical Bloch functions on the open unit disk $\mathbb{U}$ have been widely studied (see, e.g., [1,17]), and the corresponding notion in higher dimension was first introduced by Hahn [11]. Timoney [20] studied in depth the Bloch functions on bounded homogeneous domain in $\mathbb{C}^n$ with using Bergman metric (also see, Allen and Colonna [2]). Furthermore, Blasco et al. [3,4] extended the Bloch space on $\mathbb{B}^n$ to the case of unit ball $\mathbb{B}_{\mathbb{H}}$ of an infinite dimensional complex Hilbert space $\mathbb{H}$. The bounded symmetric domains in complex Banach spaces are exactly the open unit balls of JB*-triples which are complex Banach spaces equipped with a Jordan triple structure. Moreover, a complex Banach space is a JB*-triple if and only if its open unit ball is homogeneous (see, e.g., Deng and Ouyang [7], Kaup [13]). Recently, Chu et al. [5] generalized the Bloch space on $\mathbb{B}^n$ to the case of an infinite dimensional bounded symmetric domain realized as the open unit ball of a JB*-triple $\mathbb{X}$ by taking the place of the Bergman metric with the Kobayashi metric (compare with definition in Timoney [20]). In addition, they obtain the criteria for boundedness and compactness on composition operator between the Bloch spaces on infinite dimensional bounded symmetric domain. By [5], Hamada [10] continued to study the weighted composition operators from the Hardy space H^∞ to the Bloch space on bounded symmetric domains. Hamada [9] obtained the boundedness and compactness of the extended Cesàro operators between the Bloch-type spaces, which extended the results in Tang [21] to the case of unit ball of a infinite dimensional complex Banach space.

In this paper, we conform to the definitions of Bloch-type spaces and little Bloch-type spaces as [9], which generalize the corresponding spaces on $\mathbb{B}^n$ to the case of the open unit ball $\mathbf{B}_{\mathbb{X}}$ of an infinite dimensional complex Banach space $\mathbb{X}$ with arbitrary norm $\|\cdot\|$. All the weighted composition operator and weighted space are extended to $\mathbf{B}_{\mathbb{X}}$ (see, Sect. 2). We study the boundedness and compactness of the weighted composition operator from ω-Bloch space $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ (resp. little ω-Bloch space $\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$) to the weighted space on $H^\infty_\mu(\mathbf{B}_{\mathbb{X}})$ (resp. little weighted space $H^\infty_{\mu_0}(\mathbf{B}_{\mathbb{X}})$) (see, Sect. 4). In Sect. 3, we give the relations between the Bloch-type spaces $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ and the little Bloch-type spaces $\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$. Using the work of Hamada [9], we successfully construct two test functions (see, Lemma 2.4), which play a key role in the proof of our main results. Since the Bloch-type spaces $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ coincide with the Bloch-type spaces $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{H}})$ when $\mathbb{X}$ is a complex Hilbert space $\mathbb{H}$ [see Remark 2.1 in (iii)], our main results extend the corresponding works on $\mathbb{B}^n$ (see, e.g., Krantz and Stević [12]) to the case of $\mathbf{B}_{\mathbb{X}}$ of an infinite dimensional complex Banach space.

2 Preliminaries and Auxiliary Results

Let $\mathbf{B}_{\mathbb{X}}$ be the unit ball of a complex Banach space $\mathbb{X}$ with arbitrary norm $\|\cdot\|$. Let $H(\mathbf{B}_{\mathbb{X}})$ denote the set of holomorphic mappings from $\mathbf{B}_{\mathbb{X}}$ into $\mathbb{C}$. For $x \in \mathbb{X}\backslash\{0\}$, we define

$$T(x) = \{l_x \in \mathbb{X}^* : l_x(x) = \|x\|, \ \|l_x\| = 1\}.$$

Then $T(x) \neq \emptyset$ in view of the Hahn–Banach theorem.

Let ω be a normal function on $[0, 1)$, and ω can be extended to a function on $\mathbf{B}_{\mathbb{X}}$ by $\omega(z) = \omega(\|z\|)$. A function $f \in H(\mathbf{B}_{\mathbb{X}})$ is called a Bloch-type function with respect to ω if

$$\|f\|_{\mathcal{B}_{\mathcal{R},\omega}} = \sup\{\omega(z)|\mathcal{R}f(z)| : z \in \mathbf{B}_{\mathbb{X}}\} < +\infty, \tag{2.1}$$

where $\mathcal{R}f(z) = Df(z)z$ is the radial derivative of f and $Df(z)$ is the Fréchet derivative of f at z.

The class of all Bloch-type functions with respect to ω on $\mathbf{B}_{\mathbb{X}}$ is called a Bloch-type space on $\mathbf{B}_{\mathbb{X}}$ and is denoted by $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$. With the norm

$$\|f\|_{\mathcal{R},\omega} = |f(0)| + \|f\|_{\mathcal{B}_{\mathcal{R},\omega}},$$

the Bloch-type space $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ becomes a Banach space (see, Proposition 2.5 in Hamada [9]).

The little Bloch-type space $\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$ is a subspace of $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ consisting of all f such that

$$\lim_{\|z\|\to 1} \omega(z)|\mathcal{R}f(z)| = 0. \tag{2.2}$$

Remark 2.1 (i) The Bloch-type space $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ and little Bloch-type space $\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$ were first introduced by Hamada [9], which generalize the corresponding spaces defined on the Euclidean unit ball $\mathbb{B}^n$ or on the unit disk $\mathbb{U}$.

(ii) By choosing different functions ω, we have the following special spaces:

- If $\omega(z) = 1-\|z\|^2$ in (2.1) and (2.2), respectively, then we obtain the Bloch space $\mathcal{B}_{\mathcal{R}}(\mathbf{B}_{\mathbb{X}})$ and little Bloch space $\mathcal{B}_{\mathcal{R},0}(\mathbf{B}_{\mathbb{X}})$ in the unit ball of a complex Banach space (the case in $\mathbb{B}^n$, see, e.g., [15]).
- If $\omega(z) = (1-\|z\|^2)^\alpha$ with $\alpha \in (0, \infty)$ in (2.1) and (2.2), respectively, then we obtain the α-Bloch space $\mathcal{B}^\alpha_{\mathcal{R}}(\mathbf{B}_{\mathbb{X}})$ and little α-Bloch space $\mathcal{B}^\alpha_{\mathcal{R},0}(\mathbf{B}_{\mathbb{X}})$ in the unit ball of a complex Banach space (the case in $\mathbb{B}^n$, see, e.g., [16]).

- If $\omega(z) = (1 - \|z\|^2)\big(\prod_{j=1}^{k} \ln^{[j]} \frac{e^k}{1-\|z\|^2}\big)$ in (2.1) and (2.2), respectively, then we obtain the iterated logarithmic Bloch space $\mathcal{B}_{\log,\mathcal{R}}(\mathbf{B}_{\mathbb{X}})$ and little Bloch space $\mathcal{B}_{\log,\mathcal{R},0}(\mathbf{B}_{\mathbb{X}})$ in the unit ball of a complex Banach space (the case in $\mathbb{B}^n$, see, e.g., [12]).

(iii) In the case $\mathbf{B}_{\mathbb{X}} = \mathbf{B}_{\mathbb{H}}$ is the unit ball of a complex Hilbert space $\mathbb{H}$, Hamada [9] proved that the condition (2.1) is equal to

$$\|f\|_{\mathcal{B}_{\mathcal{R},\omega}} = \sup\{\omega(z)\|Df(z)\| : z \in \mathbf{B}_{\mathbb{X}}\} < +\infty, \tag{2.3}$$

and the condition (2.2) is equal to

$$\lim_{\|z\|\to 1} \omega(z)\|Df(z)\| = 0. \tag{2.4}$$

The same situation holds with (2.3) and (2.4) when $\mathbf{B}_{\mathbb{X}} = \mathbb{B}^n$ (see, Tang [21]). In fact, if $f \in H(\mathbf{B}_{\mathbb{X}})$, then the relation $|\mathcal{R}f(z)| \leqslant \|Df(z)\|$ make sure that

$$\sup\{\omega(z)\|Df(z)\| : z \in \mathbf{B}_{\mathbb{X}}\} < +\infty \Rightarrow f \in \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}),$$

but the converse is not true.

The weighted space $H_\omega^\infty(\mathbf{B}_{\mathbb{X}})$ consisting of all $f \in H(\mathbf{B}_{\mathbb{X}})$ such that

$$\|f\|_{H_\omega^\infty} = \sup\{\omega(z)|f(z)| : z \in \mathbf{B}_{\mathbb{X}}\} < +\infty,$$

where ω is normal.

The little weighted space $H_{\omega 0}^\infty(\mathbf{B}_{\mathbb{X}})$ is a subspace of $H_\omega^\infty(\mathbf{B}_{\mathbb{X}})$ consisting of all f such that

$$\lim_{\|z\|\to 1} \omega(z)|f(z)| = 0.$$

If $u \in H(\mathbf{B}_{\mathbb{X}})$, and $\varphi \in \mathrm{Aut}(\mathbf{B}_{\mathbb{X}})$, then the operator uC_φ is defined by

$$uC_\varphi(f)(z) = u(z)f(\varphi(z)),\ f \in H(\mathbf{B}_{\mathbb{X}}), z \in \mathbf{B}_{\mathbb{X}}. \tag{2.5}$$

Remark 2.2 We note that (2.5) extends the corresponding weighted composition operator in (1.1) on $\mathbb{B}^n$ to the case of unit ball $\mathbf{B}_{\mathbb{X}}$ of a complex Banach space $\mathbb{X}$.

Next, we formulate and prove several auxiliary results which are used in the main theorems. Lemma 2.3 was proved by Hamada [9], (the corresponding results in $\mathbb{B}^n$, see Tang [21]). Lemma 2.5 is a generalization of the result on $\mathbb{B}^n$ (see, Krantz and Stević [12]) to the case of unit ball $\mathbf{B}_{\mathbb{X}}$ of a complex Banach space $\mathbb{X}$. In Lemma 2.4, we define two test functions, which play a key role in the proof of our main theorems.

Lemma 2.3 ([9], Lemma 2.1) *Let ω be a normal function. Denote $k_0 = \max(0, [\log_2 \frac{1}{\omega(\delta)}])$, $r_k = (\omega|_{[\delta,1)})^{-1}(\frac{1}{2^k})$ and $n_k = [\frac{1}{1-r_k}]$ for $k > k_0$, where the* *symbol* $[x]$ *means the greatest integer not more than* x. *Let*

$$G(\zeta)=1+\sum_{k>k_0}^{\infty}2^k\zeta^{n_k},\ \zeta\in\mathbb{U}.$$

Then

(i) G *is a holomorphic function on* $\mathbb{U}$ *such that* $G(r)$ *is increasing on* $[0,1)$ *and*

$$0<C_1=\inf_{r\in[0,1)}\omega(r)G(r)\leqslant\sup_{r\in[0,1)}\omega(r)G(r)=C_2<\infty;$$

(ii) *there exists a positive constant* C_3 *such that the inequality*

$$\int_0^r G(t)\mathrm{d}t\leqslant C_3\int_0^{r^2}G(t)\mathrm{d}t$$

holds for all $r\in[r_1,1)$, *where* $r_1\in(0,1)$ *is a constant such that*

$$\int_0^{r_1}G(t)\mathrm{d}t=1.$$

Lemma 2.4 *Let* $\mathbf{B}_{\mathbb{X}}$ *be the unit ball of a complex Banach space* $\mathbb{X}$. *For any* $\nu\in\mathbf{B}_{\mathbb{X}}\backslash\{0\}$ *and* $l_\nu\in T(\nu)$, *let*

$$f_{\nu,k}(z)=1+k\int_0^{\|\nu\|l_\nu(z)}G(\zeta)\mathrm{d}\zeta,\ \ z\in\mathbf{B}_{\mathbb{X}},$$

where G *is the function defined in Lemma 2.3 and* $0<k<+\infty$. *Then*

(a) $f_{\nu,k}\in\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$ *and* $\|f_{\nu,k}\|_{\mathcal{R},\omega}\leqslant 1+kC_2$, *where* C_2 *is the constant defined in Lemma 2.3.*

(b) *if* $\|\nu\|\geqslant r_1$ *and*

$$F_{\nu,k}(z)=\frac{1}{f_{\nu,k}(\nu)}(f_{\nu,k}(z))^2,\ \ z\in\mathbf{B}_{\mathbb{X}},$$

where r_1 *is defined in Lemma 2.3, then* $F_{\nu,k}\in\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$ *and*

$$\|F_{\nu,k}\|_{\mathcal{R},\omega}\leqslant\frac{1}{k}C_3+2(1+k)C_2C_3,$$

where C_2, C_3 *are the constants defined in Lemma 2.3. Moreover, if* $\int_0^1\frac{1}{\omega(t)}\mathrm{d}t=\infty$, *then* $F_{\nu,k}\to 0$ *uniformly on any closed ball strictly inside* $\mathbf{B}_{\mathbb{X}}$ *as* $\|\nu\|\to 1$.

Proof (a) Using Lemma 2.3, we obtain

$$\omega(z)|\mathcal{R}f_{\nu,k}(z)|=k\omega(z)|G(\|\nu\|l_\nu(z))|\|\nu\||l_\nu(z)|\leqslant k\omega(\|z\|)G(\|z\|)\leqslant kC_2.$$

Therefore, $f_{\nu,k} \in \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ and $\|f_{\nu,k}\|_{\mathcal{R},\omega} \leqslant 1 + kC_2$. Moreover, since $\mathcal{R}f_{\nu,k}$ is bounded on $\mathbf{B}_{\mathbb{X}}$, we have $\lim_{\|z\|\to 1}\omega(z)|\mathcal{R}f_{\nu,k}(z)| = 0$, which implies $f_{\nu,k} \in \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$.

(b) By some simple estimates and using the Lemma 2.3, we have

$$\begin{aligned}
\omega(z)|\mathcal{R}F_{\nu,k}(z)| &= 2k\cdot\omega(z)\frac{1}{f_{\nu,k}(\nu)}|f_{\nu,k}(z)|G(\|\nu\|l_\nu(z))|\|\nu\||l_\nu(z)| \\
&\leqslant 2k\cdot\omega(\|z\|)\frac{1}{f_{\nu,k}(\nu)}|f_{\nu,k}(z)|G(\|z\|) \\
&\leqslant 2kC_2\frac{1+k\int_0^{\|\nu\|}G(\zeta)\mathrm{d}\zeta}{1+k\int_0^{\|\nu\|^2}G(\zeta)\mathrm{d}\zeta} \\
&\leqslant 2kC_2\frac{1+kC_3\int_0^{\|\nu\|^2}G(\zeta)\mathrm{d}\zeta}{k\int_0^{\|\nu\|^2}G(\zeta)\mathrm{d}\zeta} \\
&= 2kC_2\left(\frac{1}{k}\frac{1}{\int_0^{\|\nu\|^2}G(\zeta)\mathrm{d}\zeta}+C_3\right) \\
&= 2kC_2\left(\frac{1}{k}\frac{\int_0^{r_1}G(\zeta)\mathrm{d}\zeta}{\int_0^{\|\nu\|^2}G(\zeta)\mathrm{d}\zeta}+C_3\right) \\
&\leqslant 2kC_2\left(\frac{1}{k}\frac{\int_0^{\|\nu\|}G(\zeta)\mathrm{d}\zeta}{\int_0^{\|\nu\|^2}G(\zeta)\mathrm{d}\zeta}+C_3\right) \\
&\leqslant 2kC_2\left(\frac{1}{k}C_3+C_3\right) = 2(1+k)C_2C_3.
\end{aligned} \tag{2.6}$$

By (2.6), we obtain $F_{\nu,k} \in \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ and

$$\begin{aligned}
\|F_{\nu,k}\|_{\mathcal{R},\omega} &\leqslant |F_{\nu,k}(0)| + 2(1+k)C_2C_3 \\
&= \left|\frac{1}{f_{\nu,k}(\nu)}\right| + 2(1+k)C_2C_3 \\
&= \frac{1}{1+k\int_0^{\|\nu\|^2}G(\zeta)\mathrm{d}\zeta} + 2(1+k)C_2C_3 \\
&\leqslant \frac{\int_0^{\|\nu\|}G(\zeta)\mathrm{d}\zeta}{k\int_0^{\|\nu\|^2}G(\zeta)\mathrm{d}\zeta} + 2(1+k)C_2C_3 \\
&= \frac{1}{k}C_3 + 2(1+k)C_2C_3.
\end{aligned} \tag{2.7}$$

Moreover, since $\mathcal{R}F_{\nu,k}$ is bounded on $\mathbf{B}_{\mathbb{X}}$, we have $\lim_{\|z\|\to 1}\omega(z)|\mathcal{R}F_{\nu,k}(z)| = 0$, which implies $F_{\nu,k} \in \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$.

Next, if $\int_0^1 \frac{1}{\omega(t)}\mathrm{d}t = \infty$, then we can prove that $F_{\nu,k} \to 0$ uniformly for $\|z\| \leqslant r$ $(0 < r < 1)$ as $\|\nu\| \to 1$ by a similar way in ([9], Lemma 2.7). The proof is finished. □

Lemma 2.5 *Suppose that $u \in H(\mathbf{B}_{\mathbb{X}})$, μ is normal and $\varphi \in Aut(\mathbf{B}_{\mathbb{X}})$. Then the operator $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H^\infty_\mu(\mathbf{B}_{\mathbb{X}})$ is compact if and only if uC_φ: $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H^\infty_\mu(\mathbf{B}_{\mathbb{X}})$ is bounded, and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ converging to zero uniformly on compact subset of $\mathbf{B}_{\mathbb{X}}$ as $k \to \infty$, we have that* $\lim_{k\to\infty} \|uC_\varphi(f_k)\|_{H^\infty_\mu} = 0$.

Proof It is similar to the proofs of the corresponding results [cf. ([4], Lemma 4.4) and ([12], Lemma 5)]. We omit the proof here.

Throughout this paper, the notation $A \asymp B$ means that there is a positive constant C such that $\frac{A}{C} \leqslant B \leqslant CA$. □

3 Some Properties with Spaces $\mathcal{B}_{\mathcal{R},\mu}(\mathbf{B}_{\mathbb{X}})$ and $\mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$

In this section, we prove $\mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$ is a closed subset of $\mathcal{B}_{\mathcal{R},\mu}(\mathbf{B}_{\mathbb{X}})$ in Theorem 3.1, and the transform relationship between of them is given by delay function in Theorem 3.2, which generalize the previous works on $\mathbb{B}^n$ to the case of unit ball $\mathbf{B}_{\mathbb{X}}$ (see, Theorem 2 and Theorem 3 in Krantz and Stević [12]).

Theorem 3.1 *Let $\mathbf{B}_{\mathbb{X}}$ be the unit ball of a complex Banach space $\mathbb{X}$. Then $\mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$ is a closed subset of $\mathcal{B}_{\mathcal{R},\mu}(\mathbf{B}_{\mathbb{X}})$, where μ is normal on* $[0, 1)$.

Proof Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$ such that

$$\lim_{j\to+\infty} \|f_j - f\|_{\mathcal{R},\mu} = 0, \quad f \in \mathcal{B}_{\mathcal{R},\mu}(\mathbf{B}_{\mathbb{X}}). \tag{3.1}$$

Using the (3.1), we have that, for every $\varepsilon > 0$, there is an $j_0 \in \mathbb{N}$ such that

$$\|f_j - f\|_{\mathcal{R},\mu} < \varepsilon \tag{3.2}$$

for $j \geqslant j_0$. In particular, taking $j = j_0$ in (3.2), it gives that

$$\begin{aligned}\sup_{z\in\mathbf{B}_{\mathbb{X}}} \mu(\|z\|)|\mathcal{R}f_{j_0}(z) - \mathcal{R}f(z)| &\leqslant |f_{j_0}(z) - f(0)| \\ &\quad + \sup_{z\in\mathbf{B}_{\mathbb{X}}} \mu(\|z\|)|\mathcal{R}f_{j_0}(z) - \mathcal{R}f(z)| < \varepsilon.\end{aligned} \tag{3.3}$$

On the other hand, $f_{j_0} \in \mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$ makes sure that, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mu(\|z\|)|\mathcal{R}f_{j_0}(z)| < \varepsilon \tag{3.4}$$

for $\delta < \|z\| < 1$. Thus, from (3.3) and (3.4), we have

$$\begin{aligned}\mu(\|z\|)|\mathcal{R}f(z)| &= \mu(\|z\|)|\mathcal{R}f(z) - \mathcal{R}f_{j_0}(z) + \mathcal{R}f_{j_0}(z)| \\ &\leqslant \mu(\|z\|)|\mathcal{R}f(z) - \mathcal{R}f_{j_0}(z)| + \mu(\|z\|)|\mathcal{R}f_{j_0}(z)| \\ &\leqslant 2\varepsilon\end{aligned}$$

for $\delta < \|z\| < 1$, which implies that $f \in \mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$. The proof of this theorem is completed. □

Theorem 3.2 *Let $\mathbf{B}_{\mathbb{X}}$ be the unit ball of a complex Banach space $\mathbb{X}$. Assume that $f \in \mathcal{B}_{\mathcal{R},\mu}(\mathbf{B}_{\mathbb{X}})$ and $f_r(z) = f(rz)$, $r \in [0, 1)$, $z \in \mathbf{B}_{\mathbb{X}}$. Then $f \in \mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$ if and only if*

$$\lim_{r\to 1^-} \|f - f_r\|_{\mathcal{R},\mu} = 0, \tag{3.5}$$

where μ is normal decreasing function of $\|z\|$ with $\mu(0) < +\infty$.

Proof After some simple computations, it is easy to see that

$$\mathcal{R}f_r(z) = \mathcal{R}f(rz), z \in \mathbf{B}_{\mathbb{X}}. \tag{3.6}$$

Since μ is normal decreasing function of $\|z\|$, by (3.6), it follows that

$$\mu(\|z\|)|\mathcal{R}f_r(z)| = \mu(\|z\|)|\mathcal{R}f(rz)| \leqslant \|f\|_{\mathcal{B}_{\mathcal{R},\mu}} \frac{\mu(\|z\|)}{\mu(\|rz\|)} \leqslant \|f\|_{\mathcal{B}_{\mathcal{R},\mu}} \frac{\mu(\|z\|)}{\mu(r)}. \tag{3.7}$$

Assume that (3.5) holds. Let $f \in \mathcal{B}_{\mathcal{R},\mu}(\mathbf{B}_{\mathbb{X}})$, then by (3.7), we have

$$\lim_{\|z\|\to 1^-} \mu(\|z\|)|\mathcal{R}f_r(z)| = 0.$$

This implies that $f_r \in \mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$. Furthermore, $\mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$ is a closed subset of $\mathcal{B}_{\mathcal{R},\mu}(\mathbf{B}_{\mathbb{X}})$ (see, Theorem 3.1), then (3.5) implies that $f \in \mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$.

Now assume $f \in \mathcal{B}_{\mathcal{R},\mu_0}(\mathbf{B}_{\mathbb{X}})$. Then, for every $\varepsilon > 0$, there is a $\delta \in (0, 1)$ such that

$$\mu(z)|\mathcal{R}f(z)| < \varepsilon, \tag{3.8}$$

as $\delta^2 < \|z\| < 1$. By (3.6), we have

$$\begin{aligned}\|f - f_r\|_{\mathcal{R},\mu} &= \sup_{z\in\mathbf{B}_{\mathbb{X}}} \mu(z)|\mathcal{R}f(rz) - \mathcal{R}f(z)| \\ &\leqslant \sup_{\|z\|\leqslant\delta} \mu(z)|\mathcal{R}f(rz) - \mathcal{R}f(z)| \\ &\quad + \sup_{\|z\|>\delta} \mu(z)|\mathcal{R}f(rz) - \mathcal{R}f(z)|.\end{aligned} \tag{3.9}$$

It is obviously

$$\lim_{r\to 1^-} \sup_{\|z\|\leqslant\delta} |\mathcal{R}f(rz) - \mathcal{R}f(z)| = 0 \tag{3.10}$$

and

$$\sup_{\|z\|\leqslant\delta} \mu(z) = \sup_{\|z\|\leqslant\delta} \mu(\|z\|) \leqslant \mu(0) < +\infty. \tag{3.11}$$

Thus, using (3.10) and (3.11), it follows that

$$\lim_{r\to 1^-}\sup_{\|z\|\leqslant\delta}\mu(z)|\mathcal{R}f(rz)-\mathcal{R}f(z)|=0. \tag{3.12}$$

On the other hand, by (3.8), for all $z\in\mathbf{B}_{\mathbb{X}}$ and $r\in[0,1)$ such that $\delta<\|z\|<1$, $\delta<r<1$, we have

$$\mu(z)|\mathcal{R}f(rz)|<\mu(rz)|\mathcal{R}f(rz)|<\varepsilon. \tag{3.13}$$

Hence, by (3.8) and (3.13), it follows that

$$\begin{aligned}\sup_{\delta<\|z\|<1}\mu(z)|\mathcal{R}f(rz)-\mathcal{R}f(z)|\leqslant&\sup_{\delta<\|z\|<1}\mu(z)|\mathcal{R}f(rz)|\\&+\sup_{\delta<\|z\|<1}\mu(z)|\mathcal{R}f(z)|<2\varepsilon\end{aligned} \tag{3.14}$$

for every $r\in(\delta,1)$. Using (3.12) and (3.14) in (3.9), we obtain (3.5). The proof is finished. □

4 $uC_\varphi:\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$(or $\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$) $\to H^\infty_\mu(\mathbf{B}_{\mathbb{X}})$(or $H^\infty_{\mu_0}(\mathbf{B}_{\mathbb{X}})$)

In this section, we study the boundedness and compactness of operator $uC_\varphi:\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})\to H^\infty_\mu(\mathbf{B}_{\mathbb{X}})$ (resp. $\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})\to H^\infty_{\mu_0}(\mathbf{B}_{\mathbb{X}})$), which generalize the corresponding results on $\mathbb{B}^n$ to the case of unit ball $\mathbf{B}_{\mathbb{X}}$ of a infinite dimensional complex Banach space $\mathbb{X}$ (see, Theorems 16–19 in Krantz and Stević [12]). The following set $E_{\varepsilon,\rho}$ is needed when consider the compactness of the operator uC_φ. For $\forall\varepsilon>0$ and $\rho\in(0,1)$, we define

$$E_{\varepsilon,\rho}=\left\{z\in\mathbf{B}_{\mathbb{X}}:\|z\|\leqslant\rho,\exists s\in\left[1,\frac{1}{\rho}\right],\text{s.t. }\mu(sz)|u(sz)|\geqslant\varepsilon\right\}. \tag{4.1}$$

Theorem 4.1 *Let $\mathbf{B}_{\mathbb{X}}$ be the unit ball of a complex Banach space. Assume that $u\in H(\mathbf{B})$, $\varphi\in Aut(\mathbf{B}_{\mathbb{X}})$, ω and μ are normal on $[0,1)$. Then $uC_\varphi:\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})\to H^\infty_\mu(\mathbf{B}_{\mathbb{X}})$ is bounded if and only if*

$$\sup_{z\in\mathbf{B}_{\mathbb{X}}}\mu(z)|u(z)|\left(1+\int_0^{\|\varphi(z)\|}\frac{1}{\omega(t)}\mathrm{d}t\right)<\infty. \tag{4.2}$$

Moreover, if $uC_\varphi:\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})\to H^\infty_\mu(\mathbf{B}_{\mathbb{X}})$ is bounded, then

$$\|uC_\varphi\|_{\mathcal{R},\mu}\asymp\sup_{z\in\mathbf{B}_{\mathbb{X}}}\mu(z)|u(z)|\left(1+\int_0^{\|\varphi(z)\|}\frac{1}{\omega(t)}\mathrm{d}t\right). \tag{4.3}$$

Proof Assume that (4.2) holds and $f \in \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$, $z \in \mathbf{B}_{\mathbb{X}}$. Using Proposition 2.4 in [9], we have

$$\begin{aligned}\|uC_\varphi(f)(z)\|_{\mathcal{B}_{\mathcal{R},\omega}\to H_\mu^\infty} &= \sup_{z\in\mathbf{B}_{\mathbb{X}}} \mu(z)|uC_\varphi(f)(z)| = \sup_{z\in\mathbf{B}_{\mathbb{X}}} \mu(z)|f(\varphi(z))u(z)| \\ &= \sup_{z\in\mathbf{B}_{\mathbb{X}}} \mu(z)|u(z)||f(\varphi(z))| \\ &\leqslant C_4\|f\|_{\mathcal{R},\omega} \sup_{z\in\mathbf{B}_{\mathbb{X}}} \mu(z)|u(z)|\left(1+\int_0^{\|\varphi(z)\|}\frac{1}{\omega(t)}\mathrm{d}t\right).\end{aligned} \tag{4.4}$$

From (4.2) and (4.4), it follows that

$$\|uC_\varphi\|_{H_\mu^\infty} \leqslant C_4 \sup_{z\in\mathbf{B}_{\mathbb{X}}} \mu(z)|u(z)|\left(1+\int_0^{\|\varphi(z)\|}\frac{1}{\omega(t)}\mathrm{d}t\right) < \infty \tag{4.5}$$

and $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is bounded.

Conversely, assume that $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is bounded. For $\nu \in \mathbf{B}_{\mathbb{X}}$, $k \geqslant 0$ and $l_\nu \in T(\nu)$, we give the test function

$$f_{\nu,k}(z) = 1 + k\int_0^{\|\nu\| l_\nu(z)} G(\zeta)\mathrm{d}\zeta,\ z \in \mathbf{B}_{\mathbb{X}}, \tag{4.6}$$

where G is defined as Lemma 2.3. By Lemma 2.4, then $f_{\nu,k} \in \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}}) \subset \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ and $\|f_{\nu,k}\|_{\mathcal{R},\omega} \leqslant C_2$. Let r_1 be the constant in Lemma 2.3. Thus, for $\nu \in \mathbf{B}_{\mathbb{X}}$ with $\|\varphi(\nu)\| \geqslant r_1$, we get

$$\begin{aligned}&\mu(\nu)|u(\nu)|\left(1+\int_0^{\|\varphi(\nu)\|}\frac{1}{\omega(t)}\mathrm{d}t\right) \leqslant \mu(\nu)|u(\nu)|\left(1+\int_0^{\|\varphi(\nu)\|}\frac{G(t)}{C_1}\mathrm{d}t\right) \\ &\leqslant \mu(\nu)|u(\nu)|\left(1+\frac{C_3}{C_1}\int_0^{\|\varphi(\nu)\|^2} G(t)\mathrm{d}t\right) \\ &\leqslant \mu(\nu)|u(\nu)|f_{\varphi(\nu),\frac{C_3}{C_1}}(\varphi(\nu))| \leqslant \sup_{\nu\in\mathbf{B}_{\mathbb{X}}} \mu(\nu)|u(\nu)||f_{\varphi(\nu),\frac{C_3}{C_1}}(\varphi(\nu))| \\ &= \sup_{\nu\in\mathbf{B}_{\mathbb{X}}} \mu(\nu)|uC_\varphi[f_{\varphi(\nu),\frac{C_3}{C_1}}](\nu)| = \|uC_\varphi[f_{\varphi(\nu),\frac{C_3}{C_1}}]\|_{\mathcal{B}_{\mathcal{R},\omega}\to H_\mu^\infty} \\ &\leqslant \|uC_\varphi\|\|f_{\varphi(\nu),\frac{C_3}{C_1}}\|_{\mathcal{R},\omega} \leqslant C_2\|uC_\varphi\| < \infty.\end{aligned} \tag{4.7}$$

If $\|\varphi(\nu)\| < r_1$, then by Lemma 2.3, we have

$$\begin{aligned}\mu(\nu)|u(\nu)|\left(1+\int_0^{\|\varphi(\nu)\|}\frac{1}{\omega(t)}\mathrm{d}t\right) &\leqslant \mu(\nu)|u(\nu)|\left(1+\int_0^{\|\varphi(\nu)\|}\frac{G(t)}{C_1}\mathrm{d}t\right)\\ &\leqslant \mu(\nu)|u(\nu)|\left(1+\int_0^{r_1}\frac{G(t)}{C_1}\mathrm{d}t\right)=\left(1+\frac{1}{C_1}\right)\mu(\nu)|u(\nu)|\\ &\leqslant \left(1+\frac{1}{C_1}\right)\sup_{\nu\in\mathbf{B}_{\mathbb{X}}}\mu(\nu)|u(\nu)| \leqslant \left(1+\frac{1}{C_1}\right)\|uC_\varphi(1)\|_{\mathcal{B}_{\mathcal{R},\omega}\to H_\mu^\infty}\\ &\leqslant (1+\frac{1}{C_1})\|uC_\varphi\|<\infty. \end{aligned} \tag{4.8}$$

The inequalities (4.7) and (4.8) yield (4.2), as desired. Moreover, from (4.4), (4.7) and (4.8), we obtain (4.3). This completes the proof. □

Theorem 4.2 *Let $\mathbf{B}_{\mathbb{X}}$ be the unit ball of a complex Banach space $\mathbb{X}$. Assume $u \in H(\mathbf{B})$, $\varphi \in Aut(\mathbf{B}_{\mathbb{X}})$, ω and μ are normal functions on $[0, 1)$. Let the set $E_{\varepsilon,\rho}$ is relatively compact in $\mathbf{B}_{\mathbb{X}}$ for any $\varepsilon > 0$ and $\rho \in (0, 1)$. The following statements are true.*

(a) *If $\int_0^1 \frac{1}{\omega(t)}\mathrm{d}t < \infty$, then $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is compact if and only if $u \in H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$.*
(b) *If $\int_0^1 \frac{1}{\omega(t)}\mathrm{d}t = \infty$, then $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is compact if and only if $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is bounded and*

$$\lim_{\|\varphi(z)\|\to 1}\mu(z)|u(z)|\left(1+\int_0^{\|\varphi(z)\|}\frac{1}{\omega(t)}\mathrm{d}t\right)=0. \tag{4.9}$$

Proof (a) Suppose that $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is compact. Then it is clear that uC_φ is bounded. Take the test function $\widehat{f}(z) \equiv 1$, $z \in \mathbf{B}_{\mathbb{X}}$. It is easy to know that $\widehat{f} \in \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$. We have

$$\sup_{z\in\mathbf{B}_{\mathbb{X}}}\mu(z)uC_\varphi[\widehat{f}](z)=\sup_{z\in\mathbf{B}_{\mathbb{X}}}\mu(z)|u(z)|<\infty,$$

which implies $u \in H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$.

Conversely, if $u \in H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$, then $\int_0^1 \frac{1}{\omega(t)}\mathrm{d}t < \infty$ makes sure that (4.2) holds. Thus, $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is bounded by Theorem 4.1. For any $\varepsilon > 0$, there exists $\rho \in (\frac{1}{2}, 1)$ such that

$$\mu(z)|u(z)|\left(1+\int_\rho^{\|\varphi(z)\|}\frac{1}{\omega(t)}\mathrm{d}t\right)<\frac{\varepsilon}{3}, \tag{4.10}$$

when $z \in \mathbf{B}_{\mathbb{X}}$ with $\rho < \|\varphi(z)\| < 1$. Let $\{f_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}})$ which converges to 0 uniformly on any compact subset of $\mathbf{B}_{\mathbb{X}}$. We may assume that

$\|f_j\|_{\mathcal{R},\omega} \leqslant 1$. Then $|f_j| \leqslant C_\rho$ for all j and $\|z\| \leqslant \rho$ by Proposition 2.4 in [9], where

$$C_\rho = C_4\Big(1 + \int_0^\rho \frac{1}{\omega(t)}\mathrm{d}t\Big).$$

There exists a positive N such that

$$|f_j(w)| \leqslant \frac{\varepsilon}{3\|u\|_{H_\mu^\infty} + 1},\ j > N,\ \ w \in E_{\varepsilon/(3C_\rho),\rho}.$$

Therefore, for $\|\varphi(z)\| \leqslant \rho$ and $t = 1$ or for $\rho < \|\varphi(z)\| < 1$ and $t = \frac{\rho}{\|\varphi(z)\|}$, we have

$$\mu(z)|u(z)||f_j(t\varphi(z))| < \frac{\varepsilon}{3},\ \ j > N. \tag{4.11}$$

Next, we will finish the proof through a case by case check:

Case I: if $z \in \mathbf{B}_{\mathbb{X}}$ with $\|\varphi(z)\| \leqslant \rho$, by (4.11), then we have

$$\mu(z)|u(z)||f_j(\varphi(z))| < \frac{\varepsilon}{3},\ \ j > N \tag{4.12}$$

holds.

Case II: if $z \in \mathbf{B}_{\mathbb{X}}$ with $\rho < \|\varphi(z)\| < 1$, by (4.10), (4.11) and some simple calculation, we have

$$\begin{aligned}
&\mu(z)|u(z)||f_j(\varphi(z))| \\
&\leqslant \mu(z)|u(z)|\Big|f_j(\varphi(z)) - f_j\Big(\rho\frac{\varphi(z)}{\|\varphi(z)\|}\Big)\Big| + \mu(z)|u(z)|\Big|f_j\Big(\rho\frac{\varphi(z)}{\|\varphi(z)\|}\Big)\Big| \\
&\leqslant \mu(z)|u(z)\int_{\frac{\rho}{\|\varphi(z)\|}}^1 |\mathcal{R}f_j(t\varphi(z))|\frac{1}{t}\mathrm{d}t + \mu(z)|u(z)|\Big|f_j\Big(\rho\frac{\varphi(z)}{\|\varphi(z)\|}\Big)\Big| \\
&\leqslant \mu(z)|u(z)|\frac{\|\varphi(z)\|}{\rho}\int_{\frac{\rho}{\|\varphi(z)\|}}^1 |\mathcal{R}f_j(t\varphi(z))|\mathrm{d}t + \mu(z)|u(z)|\Big|f_j\Big(\rho\frac{\varphi(z)}{\|\varphi(z)\|}\Big)\Big| \\
&\leqslant 2\mu(z)|u(z)|\|\varphi(z)\|\int_{\frac{\rho}{\|\varphi(z)\|}}^1 |\mathcal{R}f_j(t\varphi(z))|\mathrm{d}t + \mu(z)|u(z)|\Big|f_j\Big(\rho\frac{\varphi(z)}{\|\varphi(z)\|}\Big)\Big| \\
&\leqslant 2\mu(z)|u(z)|\|\varphi(z)\|\int_{\frac{\rho}{\|\varphi(z)\|}}^1 \frac{1}{\omega(t\|\varphi(z)\|)}\mathrm{d}t + \mu(z)|u(z)|\Big|f_j\Big(\rho\frac{\varphi(z)}{\|\varphi(z)\|}\Big)\Big| \\
&\leqslant 2\mu(z)|u(z)|\int_\rho^{\|\varphi(z)\|} \frac{1}{\omega(t)}\mathrm{d}t + \mu(z)|u(z)|\Big|f_j\Big(\rho\frac{\varphi(z)}{\|\varphi(z)\|}\Big)\Big| \\
&< \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon
\end{aligned} \tag{4.13}$$

for $j > N$.

For any $\varepsilon > 0$, employing the (4.12) and (4.13), we obtain

$$\|uC_\varphi f_j\|_{H_\mu^\infty} < \varepsilon, \;\; j > N. \tag{4.14}$$

Using Lemma 2.5 and (4.14), $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is compact.

(b) If $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is compact, then $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is bounded. Now, we assume that conditions (4.9) dose not hold, then there exist $\varepsilon > 0$ and a sequence $\{\varphi(z_k)\}_{k\in\mathbb{N}} \subset \mathbf{B}_{\mathbb{X}}$ such that

$$\lim_{k\to\infty} \|\varphi(z_k)\| = 1 \tag{4.15}$$

and

$$\mu(z_k)|u(z_k)|\left(1 + \int_0^{\|\varphi(z_k)\|} \frac{1}{\omega(t)}\mathrm{d}t\right) \geqslant \varepsilon, \;\; k = 1, 2, 3, \ldots \tag{4.16}$$

For $\nu \in \mathbf{B}_{\mathbb{X}}$ and $l_\nu \in T(\nu)$, we give the test function

$$F_{\varphi(z_k), \frac{C_3}{C_1}}(z) = \frac{1}{1 + \frac{C_3}{C_1}\int_0^{\|\varphi(z_k)\|^2} G(\zeta)\mathrm{d}\zeta}\left(1 + \frac{C_3}{C_1}\int_0^{\|\varphi(z_k)\| l_{\varphi(z_k)}(z)} G(\zeta)\mathrm{d}\zeta\right)^2, \tag{4.17}$$

where G is defined as Lemma 2.3 and $z \in \mathbf{B}_{\mathbb{X}}$. In fact, (4.15) implies that we can assume that $\|\varphi(z_k)\| > r_1$, where r_1 is the constant in Lemma 2.3. Furthermore, if we take $f_k(z) = F_{\varphi(z_k), \frac{C_3}{C_1}}(z)$ for $z \in \mathbf{B}_{\mathbb{X}}$, from Lemma 2.4, then $\{f_k\}_{k\in N}$ is a bounded sequence in $\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$ and $f_k \to 0$ uniformly on any compact subset of $\mathbf{B}_{\mathbb{X}}$. Therefore, we have

$$\lim_{k\to\infty} \|uC_\varphi(f_k)\|_{H_\mu^\infty} = 0. \tag{4.18}$$

On the other hand, using (4.16) and Lemma 2.3 with $\|\varphi(z_k)\| > r_1$, we have

$$\begin{aligned}
\|uC_\varphi(f_k)\|_{H_\mu^\infty} &= \sup_{z\in\mathbf{B}_{\mathbb{X}}} \mu(z)|u(z)||f_k(\varphi(z))| \geqslant \mu(z_k)|u(z_k)||f_k(\varphi(z_k))| \\
&= \mu(z_k)|u(z_k)|\left(1 + \frac{C_3}{C_1}\int_0^{\|\varphi(z_k)\|^2} G(t)\mathrm{d}t\right) \\
&\geqslant \mu(z_k)|u(z_k)|\left(1 + \frac{1}{C_1}\int_0^{\|\varphi(z_k)\|} G(t)\mathrm{d}t\right) \\
&\geqslant \mu(z_k)|u(z_k)|\left(1 + \int_0^{\|\varphi(z_k)\|} \frac{1}{\omega(t)}\mathrm{d}t\right) \geqslant \varepsilon,
\end{aligned} \tag{4.19}$$

(4.19) is a contradiction compare with (4.18). Thus, we obtain (4.9).

Conversely, assume that $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$ is bounded and (4.9) holds. By Theorem 4.1, we know that condition (4.2) holds. Furthermore, since $\int_0^1 \frac{1}{\omega(t)}\mathrm{d}t = \infty$, we get (4.2) implies $u \in H_\mu^\infty(\mathbf{B}_{\mathbb{X}})$. Moreover, according to (4.9), there exists

$\rho \in (\frac{1}{2}, 1)$ such that

$$\mu(z)|u(z)|\left(1 + \int_0^{\|\varphi(z)\|} G(t)\mathrm{d}t\right) < \frac{\varepsilon}{3}, \ \ \rho < \|\varphi(z)\| < 1,$$

where ε is any small positive number. The rest of the proof is similar to the case **(a)**, we omit it. This proof is completed. □

Theorem 4.3 *Let $\mathbf{B}_{\mathbb{X}}$ be the unit ball of a complex Banach space. Assume $u \in H(\mathbf{B}_{\mathbb{X}})$, $\varphi \in Aut(\mathbf{B}_{\mathbb{X}})$, ω and μ are normal functions on $[0, 1)$. Then $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}}) \to H^\infty_{\mu,0}(\mathbf{B}_{\mathbb{X}})$ is bounded if and only if $u \in H^\infty_{\mu_0}(\mathbf{B}_{\mathbb{X}})$ and*

$$M =: \sup_{z\in\mathbf{B}_{\mathbb{X}}} \mu(z)|u(z)|\left(1 + \int_0^{\|\varphi(z)\|} \frac{1}{\omega(t)}\mathrm{d}t\right) < \infty. \tag{4.20}$$

Proof First assume that $u \in H^\infty_{\mu,0}(\mathbf{B}_{\mathbb{X}})$ and (4.20) hold. By Theorem 4.1, we know that $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_{\mathbb{X}}) \to H^\infty_\mu(\mathbf{B}_{\mathbb{X}})$ is bounded. Therefore, we only need to prove that $uC_\varphi(f) \in H^\infty_{\mu,0}(\mathbf{B}_{\mathbb{X}})$ for any $f \in \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$. Now, let $f \in \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$. Applying (4.20), then there exists $\rho_0 \in (\frac{1}{2}, 1)$ such that

$$\omega(z)|\mathcal{R}f(z)| < \frac{\varepsilon}{4M}, \ \ \rho_0 \leqslant \|z\| < 1, \tag{4.21}$$

where ε is an arbitrarily small positive number.

Taking $\widetilde{z} = \rho_0 \frac{\varphi(z)}{\|\varphi(z)\|}$, then $\|\widetilde{z}\| = \rho_0 < 1$. For any $z \in \mathbf{B}_{\mathbb{X}}$ with $\rho_0 < \|z\| < 1$, using (4.21), we have

$$\begin{aligned} |f(\varphi(z)) - f(\widetilde{z})| &= \left|\int_{\frac{\rho_0}{\|\varphi(z)\|}}^1 \frac{\mathcal{R}f(t\varphi(z))}{t}\mathrm{d}t\right| \\ &\leqslant \frac{\|\varphi(z)\|}{\rho_0}\int_{\frac{\rho_0}{\|\varphi(z)\|}}^1 |\mathcal{R}f(t\varphi(z))|\mathrm{d}t \\ &\leqslant \frac{\varepsilon}{4M}\frac{\|\varphi(z)\|}{\rho_0}\int_{\frac{\rho_0}{\|\varphi(z)\|}}^1 \frac{1}{\omega(t\|\varphi(z)\|)}\mathrm{d}t \\ &\leqslant \frac{\varepsilon}{4\rho_0 M}\int_{\rho_0}^{\|\varphi(z)\|} \frac{1}{\omega(\tilde{t})}\mathrm{d}\tilde{t} \\ &\leqslant \frac{\varepsilon}{2M}\int_{\rho_0}^{\|\varphi(z)\|} \frac{1}{\omega(\tilde{t})}\mathrm{d}\tilde{t}. \end{aligned} \tag{4.22}$$

In the above equality (4.22), we use the fact that

$$\rho_0 = \frac{\rho_0}{\|\varphi(z)\|}\|\varphi(z)\| < \|t\varphi(z)\| < t < 1.$$

Set $K = \sup_{\|z\| \leqslant \rho_0} |f(z)|$. By Proposition 2.4 in [9], $K < \infty$. Since $u \in H^\infty_{\mu,0}(\mathbf{B}_\mathbb{X})$, we have, for any $\varepsilon > 0$, there is $\rho_1 \in (\rho_0, 1)$ such that

$$\mu(z)|u(z)| < \frac{\varepsilon}{2K}, \tag{4.23}$$

whenever $\rho_1 < \|z\| < 1$. Combining (4.22) and (4.23), for $\rho_1 < \|z\| < 1$, we obtain

$$\begin{aligned} \mu(z)|uC_\varphi(f)(z)| &= \mu(z)|u(z)||f(\varphi(z))| \\ &= \mu(z)|u(z)||f(\varphi(z)) - f(\widetilde{z}) + f(\widetilde{z})| \\ &\leqslant \mu(z)|u(z)||f(\varphi(z)) - f(\widetilde{z})| + \mu(z)|u(z)||f(\widetilde{z})| \\ &\leqslant \frac{\varepsilon}{2M}\mu(z)|u(z)| \int_{\rho_0}^{\|\varphi(z)\|} \frac{1}{\omega(\tilde{t})} \mathrm{d}\tilde{t} + \mu(z)|u(z)||f(\widetilde{z})| \\ &\leqslant \frac{\varepsilon}{2M} M + \frac{\varepsilon}{2K} K = \varepsilon. \end{aligned} \tag{4.24}$$

Hence, (4.24) implies that $uC_\varphi(f) \in H^\infty_{\mu,0}(\mathbf{B}_\mathbb{X})$.

Conversely, assume that $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_\mathbb{X}) \to H^\infty_{\mu,0}(\mathbf{B}_\mathbb{X})$ is bounded. Taking $f(z) \equiv 1$, then $f \in \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_\mathbb{X})$. Thus,

$$\lim_{\|z\| \to 1} \mu(z)|uC_\varphi(1)| = \lim_{\|z\| \to 1} \mu(z)|u(z)| = 0,$$

which implies that $u \in H^\infty_{\mu,0}(\mathbf{B}_\mathbb{X})$. On the other hand, For $\nu \in \mathbf{B}_\mathbb{X}, k \geqslant 0$ and $l_\nu \in T(\nu)$, we give the test function

$$f_{\nu,k}(z) = 1 + k \int_0^{\|\nu\| l_\nu(z)} G(\zeta)\mathrm{d}\zeta, \ \ z \in \mathbf{B}_\mathbb{X}, \tag{4.25}$$

where G is defined as Lemma 2.3. Applying the facts $\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_\mathbb{X}) \subset \mathcal{B}_{\mathcal{R},\omega}(\mathbf{B}_\mathbb{X})$ and $H^\infty_{\mu,0}(\mathbf{B}_\mathbb{X}) \subset H^\infty_\mu(\mathbf{B}_\mathbb{X})$, we can obtain (4.20) by (4.7) and (4.8). The proof is finished. □

Theorem 4.4 *Let* $\mathbf{B}_\mathbb{X}$ *be the unit ball of a complex Banach space* $\mathbb{X}$. *Assume* $u \in H(\mathbf{B}_\mathbb{X})$, $\varphi \in Aut(\mathbf{B}_\mathbb{X})$, ω *and* μ *are normal functions on* $[0, 1)$. *The set* $E_{\varepsilon,\rho}$ *is relatively compact in* $\mathbf{B}_\mathbb{X}$ *for any* $\varepsilon > 0$ *and* $\rho \in (0, 1)$. *Then* $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_\mathbb{X}) \to H^\infty_{\mu_0}(\mathbf{B}_\mathbb{X})$ *is compact if and only if*

$$\lim_{\|z\| \to 1} \mu(z)|u(z)| \left(1 + \int_0^{\|\varphi(z)\|} \frac{1}{\omega(t)} \mathrm{d}t\right) = 0. \tag{4.26}$$

Proof Assume that (4.26) holds. Taking any $f \in \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$, by Proposition 2.4 in [9], we have

$$\begin{aligned}\mu(z)|uC_\varphi(f)(z)| &= \mu(z)|u(z)||f(\varphi(z))| \\ &\leqslant C_4\mu(z)|u(z)|\left(1+\int_0^{\|\varphi(z)\|}\frac{1}{\omega(t)}\mathrm{d}t\right)\|f\|_{\mathcal{R},\omega},\end{aligned} \tag{4.27}$$

which implies

$$\lim_{\|z\|\to 1}\sup\{\mu(z)|uC_\varphi(f)(z)| : \ \|f\|_{\mathcal{R},\omega} \leqslant 1\} = 0. \tag{4.28}$$

Therefore, $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}}) \to H^\infty_{\mu_0}(\mathbf{B}_{\mathbb{X}})$ is compact.

Conversely, assume that $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}}) \to H^\infty_{\mu_0}(\mathbf{B}_{\mathbb{X}})$ is compact, then uC_φ is bounded. Taking the test function $\widehat{f}(z) = 1$, we have

$$\lim_{\|z\|\to 1}\mu(z)uC_\varphi(f)(z) = \lim_{\|z\|\to 1}\mu(z)|u(z)| = 0. \tag{4.29}$$

Now we prove

$$\lim_{\|\varphi(z)\|\to 1}\mu(z)|u(z)|\left(1+\int_0^{\|\varphi(z)\|}\frac{1}{\omega(t)}\mathrm{d}t\right) = 0 \tag{4.30}$$

holds. In fact, if (4.30) does not hold, then there exist $\varepsilon > 0$ and a sequence $\{\varphi(z_j)\} \subset \mathbf{B}_{\mathbb{X}}$ such that $\lim_{j\to\infty}\|\varphi(z_j)\| = 1$ and

$$\mu(z_j)|u(z_j)|\left(1+\int_0^{\|\varphi(z_j)\|}\frac{1}{\omega(t)}\mathrm{d}t\right) \geqslant \varepsilon, \ \ j = 1, 2, 3, \ldots \tag{4.31}$$

Taking the same test functions $f_k(z)$ as (4.17) and using the similar ways in Theorem 4.2, we note that $\{f_k\}$ is a bounded sequence in $\mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}})$ and $f_k \to 0$ uniformly on any compact subset of $\mathbf{B}_{\mathbb{X}}$. Since $uC_\varphi : \mathcal{B}_{\mathcal{R},\omega_0}(\mathbf{B}_{\mathbb{X}}) \to H^\infty_{\mu_0}(\mathbf{B}_{\mathbb{X}})$ is compact, we may assume that there exists some $h \in H^\infty_{\mu_0}(\mathbf{B}_{\mathbb{X}})$ such that $\|uC_\varphi(f_k) - h\|_{H^\infty_{\mu,0}} \to 0$ as $k \to \infty$. Then for each $z \in \mathbf{B}_{\mathbb{X}}$, we have

$$h(z) = \lim_{k\to\infty}uC_\varphi(f_k)(z) = uC_\varphi(\lim_{k\to\infty}f_k)(z) = uC_\varphi(0)(z) = 0.$$

Thus, we have $\|uC_\varphi(f_k)\|_{H^\infty_{\mu,0}} \to 0$ as $k \to \infty$. This contradicts with (4.31). Thus, (4.30) holds. By (4.30) we have that for any $\varepsilon > 0$ there exists $r \in (0, 1)$ such that

$$\mu(z)|u(z)|\left(1+\int_0^{\|\varphi(z)\|}\frac{1}{\omega(t)}\mathrm{d}t\right) < \varepsilon \tag{4.32}$$

with $r < \|\varphi(z)\| < 1$. On the other hand, using (4.29), there exists $\rho \in (0, 1)$ such that

$$\mu(z)|u(z)| < \frac{\varepsilon}{1 + \int_0^r \frac{1}{\omega(t)} \mathrm{d}t} \tag{4.33}$$

with $\rho < \|z\| < 1$. Therefore, when $\rho < \|z\| < 1$ and $r < \|\varphi(z)\| < 1$, from (4.32) and (4.33), we have

$$\mu(z)|u(z)| \left(1 + \int_0^{\|\varphi(z)\|} \frac{1}{\omega(t)} \mathrm{d}t \right) < \varepsilon. \tag{4.34}$$

If $\rho < \|z\| < 1$ and $\|\varphi(z)\| \leqslant r$, from (4.33), then

$$\mu(z)|u(z)| \left(1 + \int_0^{\|\varphi(z)\|} \frac{1}{\omega(t)} \mathrm{d}t \right) < \mu(z)|u(z)| \int_0^r \frac{1}{\omega(t)} \mathrm{d}t < \varepsilon. \tag{4.35}$$

Combining (4.34) and (4.35), we obtain

$$\lim_{\|z\| \to 1} \mu(z)|u(z)| \int_0^{\|\varphi(z)\|} \frac{1}{\omega(t)} \mathrm{d}t = 0. \tag{4.36}$$

This proof is completed. □

Remark 4.5 This assumption that the set $E_{\varepsilon,\rho}$ is relatively compact in $\mathbf{B}_{\mathbb{X}}$ is automatically satisfied when $\dim(\mathbb{X}) < +\infty$ (see Theorems 4.2, 4.4).

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