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ORIGINAL PAPER

Area Problem for Univalent Functions in the Unit Disk with Quasiconformal Extension to the Plane

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Abstract
Let $\Delta(r, f)$ denote the area of the image of the subdisk $|z| < r$, $0 < r \le 1$, under an analytic function f in the unit disk $|z| < 1$. Without loss of generality, in this context, we consider only the analytic functions f in the unit disk with the normalization $f(0) = 0 = f'(0) - 1$. We set $F_f(z) = z/f(z)$. Our objective in this paper is to obtain a sharp upper bound of $\Delta(r, F_f)$, when f varies over the class of normalized analytic univalent functions in the unit disk with quasiconformal extension to the entire complex plane.

Keywords Univalent functions · Area problem · Quasiconformal mappings · Quasiconformal extension

Mathematics Subject Classification Primary 30C55; Secondary 30C62

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1 Introduction and Preliminaries

In response to the classical Grötzsch problem raised in 1928, Ahlfors introduced the notion so-called "quasiconformal mappings" in 1935. Quasiconformal mappings are nothing but generalizations of conformal mappings. There are several equivalent definitions of quasiconformal mappings in the literature (see instance [1,8]). In this paper, we adopt the following definition of Ahlfors. Let $K \geq 1$. A homeomorphism f is called K*-quasiconformal* if f has locally L^2-derivative and it satisfies the Beltrami differential equation $f_{\overline{z}}(z) = \mu(z) f_z(z)$ a.e., where μ satisfies

$$|\mu(z)| \leq \frac{K-1}{K+1} = k < 1; \tag{1.1}$$

$f_{\overline{z}} = \partial f / \partial \overline{z}$ and $f_z = \partial f / \partial z$. The function μ is called the *complex dilatation* of f. Note that f is conformal if and only if μ vanishes identically. Therefore, 1-quasiconformal mappings are nothing but conformal. For basic properties of quasi-conformal mappings, we refer to [8].

By Σ, we denote the class of functions of the form:

$$g(z) = z + b_0 + \frac{b_1}{z} + \cdots \tag{1.2}$$

that are analytic and univalent in the domain $\Omega := \{z : |z| > 1\}$, except for simple pole at infinity with residue 1. The class Σ' denotes the collection of functions g in Σ, such that $g(z) \neq 0$ in Ω. Using a simple geometric argument, Gronwall [6] in 1914 proved the classical area theorem, which says that the coefficients of $g \in \Sigma$ satisfy the sharp inequality $\sum n|b_n|^2 \leq 1$. Furthermore, Lehto [7] generalized the area theorem by assuming the additional hypothesis that g admits a quasiconformal extension to the closed unit disk, where the resultant inequality is sharp. For updated research work related to the area theorem, readers can refer to [2,3]. Closely related to the class Σ is the class $\mathcal{S}$ of all analytic univalent functions f defined in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ with the normalization $f(0) = 0$ and $f'(0) = 1$. Note that functions in $\mathcal{S}$ have power series representation of the forms:

$$f(z) = z + a_2 z^2 + \cdots . \tag{1.3}$$

It is easy to verify that each $f \in \mathcal{S}$ is associated with a function $g \in \Sigma'$ through the relation $g(z) = \{f(1/z)\}^{-1}$. Therefore, there exists a one-to-one correspondence between $\mathcal{S}$ and Σ' (see [4, p 28]).

For an analytic function f in $\mathbb{D}_r := \{z : |z| < r,\ 0 < r \leq 1\}$, we set

$$\Delta(r, f) = \iint_{\mathbb{D}_r} |f'(z)|^2 \mathrm{d}x\mathrm{d}y, \quad z = x + iy, \tag{1.4}$$

which is called the Dirichlet integral of f. Geometrically, this describes the area of the image of $\mathbb{D}_r$ under f. One of the classical problems in univalent function theory is to obtain the class of functions f having finite Dirichlet integral $\Delta(1, f)$; we call such functions f as Dirichlet finite. In the recent years, such problems for various subclasses of $\mathcal{S}$ have been studied by Ponnusamy and his co-authors; see, for instance, [9–13]. The motivation to study these problems comes from a conjecture of Yamashita [14] which is settled in [9]. In this paper, we extend the problem of Yamashita to the functions in the family $\mathcal{S}$ having quasiconformal extension to the entire complex plane.

Let k be defined as in (1.1). We denote $\Sigma(k)$ by the class of all functions $g \in \Sigma$ that admit K-quasiconformal extension to the unit disk $\mathbb{D}$, and $\Sigma_0(k)$ is obtained from $\Sigma(k)$ by assuming $g(0) = 0$. Similarly, let us denote $\mathcal{S}(k)$ by the class of all functions $f \in \mathcal{S}$ that admit K-quasiconformal extension to the plane. Clearly, $f \in \mathcal{S}(k)$ if and only if $1/f(1/\zeta) \in \Sigma_0(k)$.

Rest of the paper is organized as follows. Some well-known key results are collected in Sect. 2 followed by the proof of our main theorem. We observe that the modified Koebe function studied in [7] does not play extremal role in our problem. However, we construct a new function which also extends the Koebe function $z/(1-z)^2$ to the K-quasiconformal setting and show that it plays the extremal role in our problem. Section 3 is devoted to the comparison of the areas obtained in Sect. 2 for our extremal function with the modified Koebe function.

2 Main Result

Suppose that f is an analytic function in the disk $\mathbb{D}$ with the Taylor series expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.1}$$

and $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. Then, using Parseval–Gutzmer formula, the area $\Delta(r, f)$, of $f(\overline{\mathbb{D}}_r)$, as stated in (1.4) can be re-formulated as follows (see [5]):

$$\Delta(r, f) = \iint_{\mathbb{D}_r} |f'(z)|^2 \mathrm{d}x\mathrm{d}y = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}, \quad z = x + iy. \tag{2.2}$$

We concentrate particularly on this form of the area formula in this paper. Computing this area is called the *area problem for the functions of the type* f. However, area of $f(\mathbb{D})$ may not be bounded for all $f \in \mathcal{S}$. We remark that if $f \in \mathcal{S}$, then z/f is non-vanishing, and hence, $f \in \mathcal{S}$ may be expressed as follows:

$$f(z) = \frac{z}{F_f(z)}, \quad \text{where } F_f(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}.$$

Yamashita in [14] considered the area problem for functions of type F_f for $f \in \mathcal{S}$, and proved that the area of $F_f(\mathbb{D}_r)$ is bounded. Indeed, he proved.

Theorem A [14, Theorem 1] *We have*

$$\max_{f\in\mathcal{S}} \Delta(r, F_f) = 2\pi r^2(r^2+2),$$

for $0 < r \le 1$. *The maximum is attained only for a suitable rotation of the Koebe function.*

To consider the Yamashita problem for functions in $\mathcal{S}$ having quasiconformal extension to the entire complex plane, the following theorem of Lehto [7] is useful.

Theorem B *Let* $g \in \Sigma(k)$ *be of the form* (1.2). *Then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \le k^2. \tag{2.3}$$

The equality holds for the function:

$$g(z) = \frac{1}{z} + a_0 + a_1 z, \quad z \in \mathbb{D},$$

with $|a_1| = k$. *Moreover, its k-quasiconformal extension is given by setting:*

$$g(z) = \frac{1}{z} + a_0 + \frac{a_1}{\overline{z}}, \quad z \in \overline{\Omega}.$$

We also need an immediate consequence of Theorem B, proved by Lehto in the same paper, which gives the sharp bound for second coefficient of functions in $\mathcal{S}$ having quasiconformal extension to the plane. The consequence is stated as follows:

Theorem C [7, Corollary 3] *For a function* $f \in \mathcal{S}(k)$ *of the form* (1.3) *with* $f(\infty) = \infty$, *we have* $|a_2| \le 2k$.

Using Theorem B and Theorem C, we now state and prove our main result.

Theorem 2.1 *For* $0 < r \le 1$, *we have*

$$\max_{f\in\mathcal{S}(k)} \Delta(r, F_f) = 2\pi r^2 k^2 (2 + r^2).$$

The maximum is attained only for a suitable rotation of the function:

$$f(z) = \begin{cases} \dfrac{z}{1 - 2kz + kz^2}, & \text{for } |z| < 1, \\[2ex] \dfrac{z\overline{z}}{\overline{z} - 2kz\overline{z} + kz}, & \text{for } |z| \ge 1. \end{cases} \tag{2.4}$$

Proof Let $f \in \mathcal{S}(k)$ be of the form (1.3). Then

$$\frac{1}{f(\frac{1}{z})} = z - a_2 + (a_2^2 - a_3)\frac{1}{z} + \cdots = z + b_1 + \frac{b_2}{z} + \cdots \text{ (say)}.$$

Substituting $1/z$ by z and multiplying z, we obtain

$$F_f(z) = \frac{z}{f(z)} = 1 - a_2 z + (a_2^2 - a_3)z^2 + \cdots = 1 + b_1 z + b_2 z^2 + \cdots$$

It is clear that $b_1 = -a_2$. Now, we compute

$$\begin{aligned}\frac{1}{\pi}\Delta(r, F_f) &= \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}\\ &= |b_1|^2 r^2 + \sum_{n=2}^{\infty} n|b_n|^2 r^{2n}\\ &= |-a_2|^2 r^2 + 2r^4 \sum_{n=1}^{\infty} \frac{n+1}{2}|b_{n+1}|^2 r^{2n-2}.\end{aligned}$$

Using the estimate for a_2 from Theorem C, we obtain

$$\frac{1}{\pi}\Delta(r, F_f) \le 4r^2k^2 + 2r^4 \sum_{n=1}^{\infty} n|b_{n+1}|^2.$$

Then, by Theorem B, we have

$$\frac{1}{\pi}\Delta(r, F_f) \le 4r^2k^2 + 2r^4k^2 = 2r^2k^2(r^2 + 2).$$

Now, it remains to consider the sharpness part. For $|z| < 1$, consider the function $f(z) = z/(1 - 2kz + kz^2)$. Therefore, $f_{\overline{z}} = 0$. That is, f is conformal in $\mathbb{D}$. Since $F_f(z) = 1 - 2kz + kz^2$, by (2.2), we obtain

$$\frac{1}{\pi}\Delta(r, F_f) = \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} = 4r^2k^2 + 2r^4k^2 = 2r^2k^2(r^2 + 2).$$

For $|z| \ge 1$, let

$$f(z) = \frac{z\overline{z}}{\overline{z} - 2kz\overline{z} + kz}.$$

An easy calculation shows that

$$f_{\overline{z}} = \frac{z(\overline{z} - 2kz\overline{z} + kz) - z\overline{z}(1 - 2kz)}{(\overline{z} - 2kz\overline{z} + kz)^2} = \frac{kz^2}{(\overline{z} - 2kz\overline{z} + kz)^2}$$

and

$$f_z = \frac{\overline{z}(\overline{z} - 2kz\overline{z} + kz) - z\overline{z}(-2k\overline{z} + k)}{(\overline{z} - 2kz\overline{z} + kz)^2} = \frac{\overline{z}^2}{(\overline{z} - 2kz\overline{z} + kz)^2}.$$

Thus, $|f_{\overline{z}}/f_z| = k$.

Both the functions defined in (2.4) agree on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. The proof is complete. □

Remark 2.2 Observe that Theorem 2.1 is a natural extension of Theorem A. In fact, for $k = 1$, Theorem 2.1 is equivalent to Theorem A.

Remark 2.3 It is easy to check that for $f \in \mathcal{S}(k)$, $\Delta(1, F_f) \leq 6\pi k^2$, and hence, F_f is Dirichlet finite.

3 Comparison of Areas

Recall the modified Koebe function from [7] which is defined by the following:

$$g(z) = \begin{cases} \dfrac{z}{(1 + ke^{i\phi}z)^2}, & \text{for } |z| < 1, \\ \dfrac{z\overline{z}}{(\sqrt{\overline{z}} + ke^{i\phi}\sqrt{z})^2}, & \text{for } |z| \geq 1. \end{cases} \tag{3.1}$$

A simple computation yields

$$\Delta(r, F_g) = 2r^2k^2(k^2r^2 + 2)\pi,$$

which geometrically describes the area of $F_g(\mathbb{D})$. Note that

$$2r^2k^2(k^2r^2 + 2)\pi = \Delta(r, F_g) < \Delta(r, F_f) = 2r^2k^2(r^2 + 2).$$

To see the graphical and numerical comparisons of the Dirichlet finites $\Delta(1, F_g)$ and $\Delta(1, F_f)$, we end this section with the following observations (Table 1; Figs. 1, 2, 3, 4). First, we show the graphs of F_f and F_g, where f and g are defined

Table 1 Comparison of areas of $F_f(\mathbb{D})$ and $F_g(\mathbb{D})$

k	$\Delta(1, F_g)$	$\Delta(1, F_f)$
0.2	0.1632π	0.24π
0.5	1.125π	1.5π
0.7	2.4402π	2.94π
0.9	4.5522π	4.86π
1	6π	6π

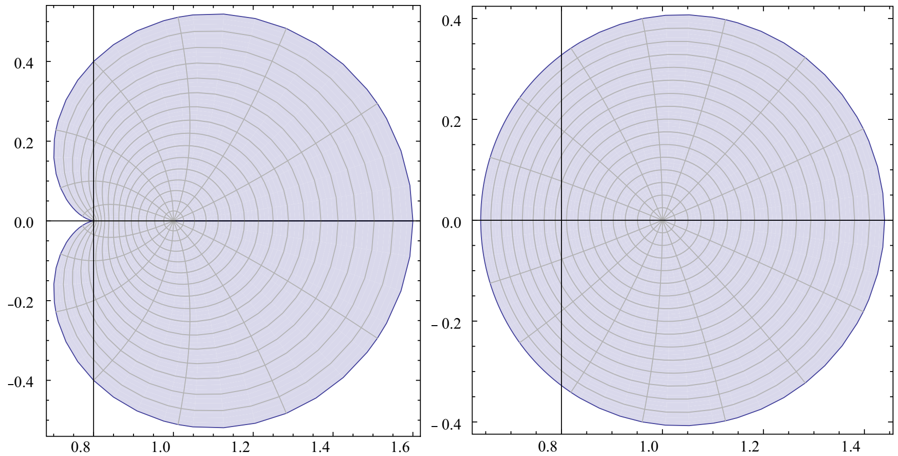

Fig. 1 Graphs of F_f and F_g for $k = 0.2$

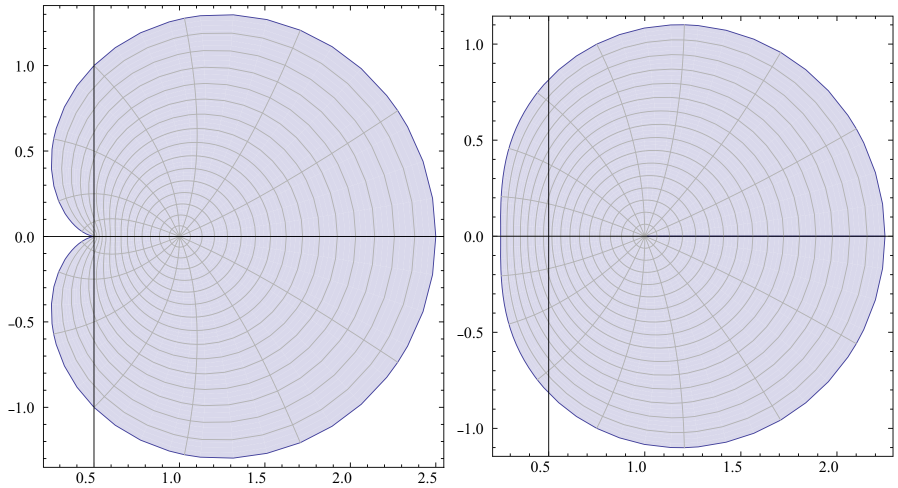

Fig. 2 Graphs of F_f and F_g for $k = 0.5$

by (2.4) and (3.1) respectively, for different values of k. Here, the terminology *the graph of* F_f means *the image domain* $F_f(\mathbb{D})$ and, similarly, for the graph of F_g. Observe that as $k \to 1$, the graphs of F_g are approaching to those of F_f.

Second, for these choices of k, Table 1 compares the area $\Delta(1, F_g)$, of the image of $\mathbb{D}$ under F_g, and the area $\Delta(1, F_f)$, of the image of $\mathbb{D}$ under F_f.

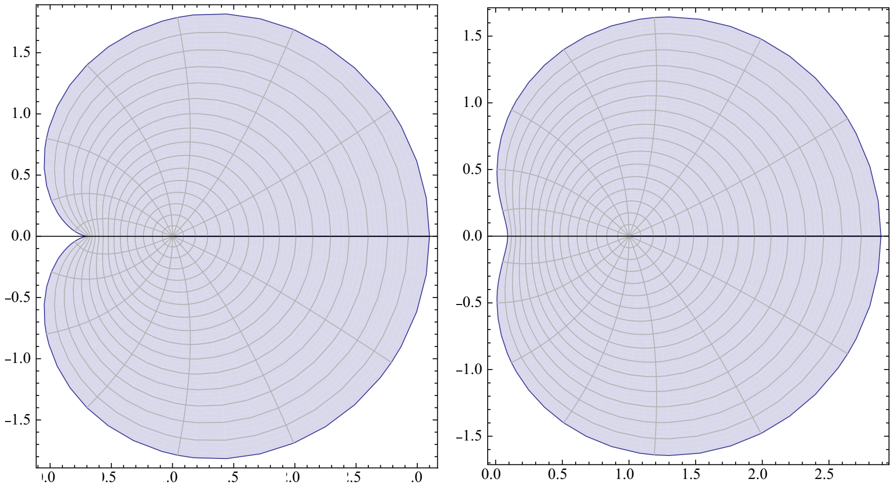

Fig. 3 Graphs of F_f and F_g for $k = 0.7$

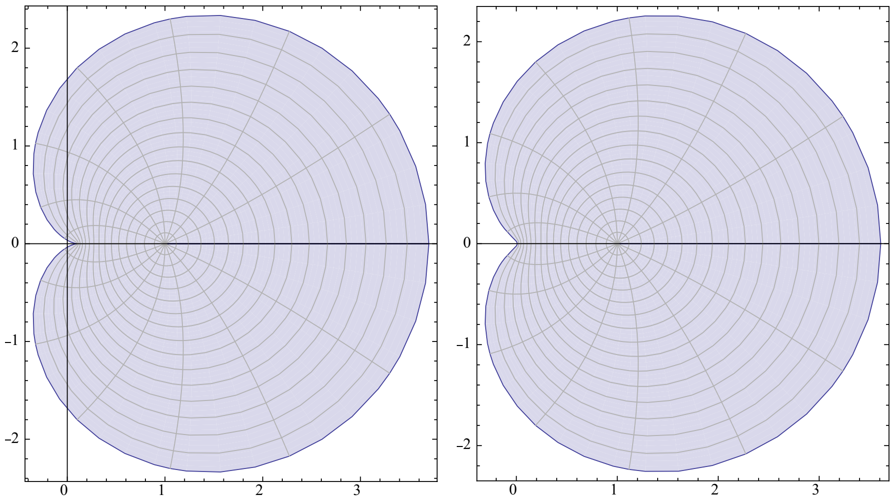

Fig. 4 Graphs of F_f and F_g for $k = 0.9$

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