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ORIGINAL PAPER

On Classification of $(n + 5)$-Dimensional Nilpotent n-Lie Algebras of Class Two

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Abstract
In this paper, we classify $(n + 5)$-dimensional nilpotent n-Lie algebras of class two over an arbitrary field, when $n \geq 3$.

Keywords Nilpotent n-Lie algebra · Classification · Nilpotent n-Lie algebra of class two

Mathematics Subject Classification Primary 17B05 · 17B30; Secondary 17D99

1 Introduction

The classification of low-dimensional Lie algebras is one of the fundamental issues in Lie algebras theory. The classification of Lie algebras can be found in many books and papers. In 1950 Morozov [11] proposed a classification of six-dimensional nilpotent Lie algebras over fields of characteristic zero. The classification of the six-dimensional Lie algebras on the arbitrary field was shown by Cicalo et al. [4]. Moreover, the seven-dimensional nilpotent Lie algebras over algebraically closed fields and real number field were classified in [9]. In 1985, Filippov [8] introduced the notion of n-Lie algebras. A nonsymmetrical linear vector space A is called an n-Lie algebra if it satisfies the following Jacoby identity:

$$[[x_1, x_2, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_2, \ldots, y_n], x_{i+1}, \ldots, x_n]$$

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for all $x_i, y_j \in A$, $1 \leq i \leq n$, $2 \leq j \leq n$. He also classified n-Lie algebras of dimensions n and $n+1$ on the algebraically closed field with characteristic zero.

In 2008, Bai et al. [2] classified n-Lie algebras of dimension $n+1$ on fields of characteristic two. Then, Bai et al. [1] classified n-Lie algebras of dimension $n+2$ on the algebraically closed fields with characteristic zero.

Assume that $A_1, \ldots, A_n$ are subalgebras of an n-Lie algebra A. Then, the subalgebra of A generated by all vectors $[x_1, \ldots, x_n](x_i \in A_i)$ will be represented by the symbol $[A_1, \ldots, A_n]$. The subalgebra $A^2 = [A, \ldots, A]$ is called derived n-Lie algebra of A. The center of n-Lie algebra A is defined as follows:

$$Z(A) = \{x \in A : [x, A, \ldots, A] = 0\}.$$

Assume that $Z_0(A) = 0$; then the ith center of A is defined inductively as

$$Z_i(A)/Z_{i-1}(A) = Z(A/Z_{i-1}(A)) \qquad \text{for all} \quad i \geq 1.$$

The notion of nilpotent n-Lie algebra was defined by Kasymov [10] as follows. We say that an n-Lie algebra A is nilpotent if $A^s = 0$, where s is a non-negative integer number. Note that A^i is defined as induction by $A^1 = A$, $A^{i+1} = [A^i, A, \ldots, A]$. The n-Lie algebra A is nilpotent of class c if $A^{c+1} = 0$ and $A^i \neq 0$ for each $i \leq c$, (for more information see [3,12]).

An important category of n-Lie algebras of class 2, which plays an important role in nilpotent n-Lie algebras, is the Heisenberg n-Lie algebras. We call n-Lie algebra A, generalized Heisenberg of rank k, if $A^2 = Z(A)$ and $\dim A^2 = k$. In [6], the authors studied the case when $k = 1$, which is called later special Heisenberg n-Lie algebras.

The rest of our paper is organized as follows. Section 2 includes the results that are used frequently in the next section. In Sect. 3, we classify $(n+5)$-dimensional n-Lie algebras of class two.

2 Preliminaries

In this section, we introduce some known and necessary results.

Theorem 2.1 [6] *Every special Heisenberg n-Lie algebra has dimension $mn+1$ for some natural number m, and it is isomorphic to*

$$H(n, m) = \langle x, x_1, \ldots, x_{nm} : [x_{n(i-1)+1}, x_{n(i-1)+2}, \ldots, x_{ni}] = x, i = 1, \ldots, m\rangle.$$

Theorem 2.2 [5] *Let A be a d-dimensional nilpotent n-Lie algebra, and let* $\dim A^2 = 1$. *Then, for some $m \geq 1$,*

$$A \cong H(n, m) \oplus F(d - mn - 1).$$

in which $F(d-n-1)$ is the abelian n-Lie algebra of dimension $d-n-1$.

Theorem 2.3 [5] *Let A be a nilpotent n-Lie algebra of dimension $d = n + k$ for $3 \le k \le n + 1$ such that $A^2 = Z(A)$ and $\dim A^2 = 2$. Then,*

$$A \cong \langle e_1, \ldots, e_{n+k} : [e_{k-1}, \ldots, e_{n+k-2}] = e_{n+k}, [e_1, \ldots, e_n] = e_{n+k-1} \rangle.$$

Theorem 2.4 [5] *Let A be a nonabelian nilpotent n-Lie algebra of dimension $d \le n + 2$. Then A is isomorphic to $H(n, 1)$, $H(n, 1) \oplus F(1)$, or $A_{n,n+2,1}$, where $A_{n,n+2,1} = \langle e_1, \ldots, e_{n+2} : [e_1, \ldots, e_n] = e_{n+1}, [e_2, \ldots, e_{n+1}] = e_{n+2} \rangle$.*

For unification of notation, in what follows the tth d-dimensional n-Lie algebra will be denoted by $A_{n,d,t}$.

Theorem 2.5 [4]

(1) *Over a field F of characteristic different from 2, the following is the list of the isomorphisms types of six-dimensional nilpotent Lie algebras:*
   - $L_{5,k} \oplus F$ *with* $k \in \{1, \ldots, 9\}$;
   - $L_{6,k}$ *with* $k \in \{10, \ldots, 18, 20, 23, 25, \ldots, 28\}$;
   - $L_{6,k}(\varepsilon_1)$ *with* $k \in \{19, 21\}$ *and* $\varepsilon_1 \in F^*/(\overset{*}{\sim})$;
   - $L_{6,k}(\varepsilon_2)$ *with* $k \in \{22, 24\}$ *and* $\varepsilon_2 \in F/(\overset{*}{\sim})$.

(2) *Over a field F of characteristic 2, the isomorphism types of six-dimensional nilpotent Lie algebras are*
   - $L_{5,k} \oplus F$ *with* $k \in \{1, \ldots, 9\}$,
   - $L_{6,k}$ *with* $k \in \{10, \ldots, 18, 20, 23, 25, \ldots, 28\}$,
   - $L_{6,k}(\varepsilon_1)$ *with* $k \in \{19, 21\}$ *and* $\varepsilon_1 \in F^*/(\overset{*}{\sim})$,
   - $L_{6,k}(\varepsilon_2)$ *with* $k \in \{22, 24\}$ *and* $\varepsilon_2 \in F/(\overset{*+}{\sim})$,
   - $L^{(2)}_{6,k}$ *with* $k \in \{1, 2, 5, 6\}$,
   - $L^{(2)}_{6,k}(\varepsilon_3)$ *with* $k \in \{3, 4\}$ *and* $\varepsilon_3 \in F^*/(\overset{*+}{\sim})$,
   - $L^{(2)}_{6,k}(\varepsilon_4)$ *with* $k \in \{7, 8\}$ *and* $\varepsilon_4 \in \{0, \omega\}$.

Eshrati et al. [7] classified $(n + 3)$-dimensional nilpotent n-Lie algebras for $n > 2$. Additionally, they proved the following theorem for $(n + 4)$-dimensional n-Lie algebras.

Theorem 2.6 *[7] The only $(n + 4)$-dimensional nilpotent n-Lie algebras of class two are $H(n, 1) \oplus F(3)$, $A_{n,n+4,1}$, $A_{n,n+4,2}$, $A_{n,n+4,3}$, $H(2, 2) \oplus F(1)$, $H(3, 2)$, $L_{6,22}(\varepsilon)$, and $L^2_{6,7}(\eta)$.*

Theorem 2.7 [9] *The seven-dimensional nilpotent Lie algebras of class two over algebraically closed fields and real number field are*

$$H(2, 1) \oplus F(4), \quad H(2, 2) \oplus F(2), \quad H(2, 3), \quad \text{and} \quad L_{7,i}, \quad 1 \le i \le 10.$$

3 Main Results

In this section, we classify $(n+5)$-dimensional nilpotent n-Lie algebras of class two. An n-Lie algebra A is nilpotent of class two, when A is nonabelian and $A^2 \subseteq Z(A)$. The nilpotent n-Lie algebra of class two plays an essential role in some geometry problems such as the commutative Riemannian manifold. Additionally, the classification of nilpotent Lie algebras of class two is one of the most important issues in Lie algebras.

We first prove a lemma for three-Lie algebras.

Lemma 3.1 *Let A be a three-Lie algebra of dimension eight such that $A^2 = Z(A)$ and* $\dim A^2 = 2$. *Then,*

$$A = \langle e_1, \ldots, e_8 : [e_1, e_2, e_3] = e_7, [e_4, e_5, e_6] = e_8 \rangle .$$

Proof Let $A = \langle e_1, \ldots, e_8 \rangle$, and let $A^2 = Z(A) = \langle e_7, e_8 \rangle$. We may assume that $[e_4, e_5, e_6] = e_8$. So, there are α_0, β_0, $\alpha_{i,j,k}$, and $\beta_{i,j,k}$ of the field F such that

$$\begin{cases} [e_1, e_2, e_3] = \alpha_0 e_8 + \beta_0 e_7 \\ \left[e_i, e_j, e_k\right] = \alpha_{i,j,k} e_8 + \beta_{i,j,k} e_7, \ 1 \le i < j < k \le 6, (i, j, k) \neq (1, 2, 3), (4, 5, 6) . \end{cases}$$

Taking $I = \langle e_7 \rangle$, $\dim (A/I)^2 = 1$. Therefore, by Theorem 2.2, A/I is isomorphic to $H(3, 1) \oplus F(3)$ or $H(3, 2)$.

(i) Assume that $A/I \cong H(3, 1) \oplus F(3)$. In this case, according to the structure of A/I, we have $\alpha_0 = \alpha_{i,j,k} = 0$. Therefore, the brackets of A are as follows:

$$\begin{cases} [e_4, e_5, e_6] = e_8, \qquad [e_1, e_2, e_3] = \beta_0 e_7, \\ [e_i, e_j, e_k] = \beta_{i,j,k} e_7, \qquad 1 \le i < j < k \le 6, (i, j, k) \neq (1, 2, 3), (4, 5, 6). \end{cases}$$

Now, by choosing $J = \langle e_8 \rangle$, and by taking into account $\dim (A/J)^2 = 1$, we have

$$A/J \cong \langle \bar{e}_1, \ldots, \bar{e}_7 \ : [\bar{e}_1, \bar{e}_2, \bar{e}_3] = \beta_0 \bar{e}_7, \ [\bar{e}_i, \bar{e}_j, \bar{e}_k] = \beta_{i,j,k} \bar{e}_7 \rangle$$

for $1 \le i < j < k \le 6$, $(i, j, k) \neq (1, 2, 3), (4, 5, 6)$.

According to the above brackets and special Heisenberg n-Lie algebra, A/J is isomorphic to $H(3, 1) \oplus F(3)$. So, only one of β_0 and $\beta_{i,j,k}$ is equal to one and the other coefficients are zero.

If one of the coefficients $\beta_{i,j,k}$ is equal to one, then the condition $A^2 = Z(A)$ will be false. So, we conclude $\beta_{i,j,k} = 0$ for each $1 \le i < j < k \le 6$, $(i, j, k) \neq (1, 2, 3), (4, 5, 6)$. Thus,

$$A = \langle e_1, ..., e_8 : [e_1, e_2, e_3] = e_8, [e_4, e_5, e_6] = e_7 \rangle. \tag{1}$$

(ii) Consider $A/I \cong H(3,2)$. According to the structure of A/I, we have $\alpha_0 = 1$ and $\alpha_{i,j,k} = 0$. Therefore, the brackets of A

$$\begin{cases} [e_4, e_5, e_6] = e_8, \qquad [e_1, e_2, e_3] = e_8 + \beta_0 e_7, \\ [e_i, e_j, e_k] = \beta_{i,j,k} e_7, \qquad 1 \le i < j < k \le 6,\ (i,j,k) \neq (1,2,3), (4,5,6). \end{cases}$$

Now, by choosing $J = \langle e_8 \rangle$, we have $\dim (A/J)^2 = 1$. Hence, with respect to the structure of special Heisenberg n-Lie algebras, this algebra is isomorphic to $H(3,1) \oplus F(3)$. Thus, only one of coefficients β_0 and $\beta_{i,j,k}$ is equal to one and the other coefficients are zero. Of course, if one of the coefficients $\beta_{i,j,k}$ is equal to one, we have a contradiction. So, we have $\beta_{i,j,k} = 0$ for each $1 \le i < j < k \le 6$, $(i,j,k) \neq (1,2,3)$, $(4,5,6)$. As a result, the brackets of A are as follows:

$$[e_4, e_5, e_6] = e_8, \qquad [e_1, e_2, e_3] = e_8 + e_7.$$

By interchanging

$$e'_i = e_i, \quad 1 \le i \le 8, i \neq 7, \qquad e'_7 = e_8 + e_7,$$

this algebra is isomorphic to A in relation (1). Consequently, the proof is completed. □

Now, we are going to classify $(n+5)$-dimensional nilpotent n-Lie algebras of class two.

Assume that A is an $(n+5)$-dimensional nilpotent n-Lie algebra of class two, where $n \ge 3$ and $A = \langle e_1, \ldots, e_{n+5} \rangle$ (see Theorem 2.7 for the case $n = 2$). If $\dim A^2 = 1$, then by Theorem 2.2, A is isomorphic to one of the following algebras:

$$H(n,1) \oplus F(4), \quad H(3,2) \oplus F(1), \quad H(4,2).$$

Now, assume that $\dim A^2 \ge 2$ and that $\langle e_{n+4}, e_{n+5} \rangle \subseteq A^2$. Ergo, $A/\langle e_{n+5} \rangle$ is an $(n+4)$-dimensional nilpotent n-Lie algebra of class 2. It follows from Theorem 2.6 that $A/\langle e_{n+5} \rangle$ is one of the following forms:

$$H(n,1) \oplus F(3), \quad A_{n,n+4,1}, \quad A_{n,n+4,2}, \quad A_{n,n+4,3}, \quad H(3,2).$$

Case 1 $A/\langle e_{n+5} \rangle \cong \langle \bar{e}_1, \ldots, \bar{e}_{n+4} : [\bar{e}_1, \ldots, \bar{e}_n] = \bar{e}_{n+4} \rangle \cong H(n,1) \oplus F(3)$.

The brackets of A are as follows:

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+4} + \alpha e_{n+5}, & \\ [e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+1}] = \alpha_i e_{n+5}, & 1 \le i \le n, \\ [e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+2}] = \beta_i e_{n+5}, & 1 \le i \le n, \\ [e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+3}] = \gamma_i e_{n+5}, & 1 \le i \le n, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}] = \theta_{ij} e_{n+5}, & 1 \le i < j \le n, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+3}] = \mu_{ij} e_{n+5}, & 1 \le i < j \le n, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+2}, e_{n+3}] = \lambda_{ij} e_{n+5}, & 1 \le i < j \le n, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, \hat{e}_k, \ldots, e_n, e_{n+1}, e_{n+2}, e_{n+3}] = \zeta_{ijk} e_{n+5}, & 1 \le i < j < k \le n. \end{cases}$$

By changing the base, we can have $\alpha = 0$. Since $\dim A^2 \geq 2$, we obtain $\dim Z(A) \leq 4$.

First, we assume that $\dim Z(A) = 4$. In this case, without loss of generality, we can assume that $Z(A) = \langle e_{n+2}, e_{n+3}, e_{n+4}, e_{n+5} \rangle$. Consequently, the nonzero brackets of A are as follows:

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+4}, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+1}\right] = \alpha_i e_{n+5}, \quad 1 \leq i \leq n. \end{cases}$$

At least one of the α_is is nonzero. Without loss of generality, we can assume that $\alpha_1 \neq 0$. We replace e_1 with $e_1 + \sum_{i=2}^{n} (-1)^{i-1} \frac{\alpha_i}{\alpha_1} e_i$ and e_{n+5} with $\alpha_1 e_{n+5}$; we have

$$[e_1, \ldots, e_n] = e_{n+4}, \qquad [e_2, \ldots, e_{n+1}] = e_{n+5}.$$

We denote this algebra by $A_{n,n+5,1}$. Now, suppose that $\dim Z(A) = 3$. Without loss of generality, we assume that $Z(A) = \langle e_{n+3}, e_{n+4}, e_{n+5} \rangle$. Therefore, the brackets of A are as follows:

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+4}, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+1}\right] = \alpha_i e_{n+5}, & 1 \leq i \leq n, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+2}\right] = \beta_i e_{n+5}, & 1 \leq i \leq n, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}\right] = \theta_{ij} e_{n+5}, & 1 \leq i < j \leq n. \end{cases}$$

Since $\dim (A/\langle e_{n+4} \rangle)^2 = 1$, we have $A/\langle e_{n+4} \rangle \cong H(n, 1) \oplus F(3)$. According to the structure of n-Lie algebras, we conclude that one of the coefficients

$$\theta_{ij}\,(1 \leq i < j \leq n), \quad \beta_i\,(1 \leq i \leq n), \quad \alpha_i\,(1 \leq i \leq n)$$

is equal to one, and the others are zero. According to $Z(A) = \langle e_{n+3}, e_{n+4}, e_{n+5} \rangle$, the coefficients of $\beta_i\,(1 \leq i \leq n)$, and $\alpha_i\,(1 \leq i \leq n)$ cannot be equal to one. Without loss of generality, assume that

$$\begin{array}{llll} \theta_{12} = 1, & & \theta_{ij} = 0 & (1 \leq i < j \leq n, (i,j) \neq (1,2)) \\ \beta_i = 0 & (1 \leq i \leq n) & \alpha_i = 0 & (1 \leq i \leq n). \end{array}$$

So, the brackets of A are as follows:

$$[e_1, \ldots, e_n] = e_{n+4}, \qquad \left[e_3, \ldots, e_{n+2}\right] = e_{n+5}.$$

We denote this algebra by $A_{n,n+5,2}$.

Now, assume that $\dim Z(A) = 2$. Therefore, $Z(A) = A^2$. In the case $n \geq 4$, using Theorem 2.3, the brackets of A are as follows:

$$[e_1, \ldots, e_n] = e_{n+5}, \qquad \left[e_4, \ldots, e_{n+3}\right] = e_{n+4}.$$

We denote this algebra by $A_{n,n+5,3}$. In the case $n = 3$, according to Lemma 3.1, the desired algebra is

$$A = \langle e_1, \ldots, e_8 : [e_1, e_2, e_3] = e_8, [e_4, e_5, e_6] = e_7 \rangle ,$$

This algebra is the same as $A_{n,n+5,3}$ for $n = 3$.
Case 2 $A/\langle e_{n+5}\rangle \cong \langle \overline{e}_1, \ldots, \overline{e}_{n+4} : [\overline{e}_1, \ldots, \overline{e}_n] = \overline{e}_{n+3}, [\overline{e}_2, \ldots, \overline{e}_{n+1}] = \overline{e}_{n+4}\rangle$.

In this case $\dim A^2 = 3$; so $A^2 = \langle e_{n+3}, e_{n+4}, e_{n+5}\rangle$. Thus $3 \leq \dim Z(A) \leq 4$. The brackets of A are as follows:

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3} + \alpha e_{n+5}, \\ \left[e_2, \ldots, e_{n+1}\right] = e_{n+4} + \beta e_{n+5}, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+1}\right] = \alpha_i e_{n+5}, & 2 \leq i \leq n, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+2}\right] = \beta_i e_{n+5}, & 1 \leq i \leq n, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}\right] = \theta_{ij} e_{n+5}, & 1 \leq i < j \leq n. \end{cases}$$

By changing the base, we can take $\alpha = \beta = 0$. First, we assume that $\dim Z(A) = 4$. It follows that $Z(A) = \langle e_{n+2}, e_{n+3}, e_{n+4}, e_{n+5}\rangle$. Therefore

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3}, \\ \left[e_2, \ldots, e_{n+1}\right] = e_{n+4}, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+1}\right] = \alpha_i e_{n+5}, \ \ 2 \leq i \leq n. \end{cases}$$

Due to the derived dimension, we conclude that at least one of the α_i's $(2 \leq i \leq n)$ is nonzero. Without loss of generality, we assume that $\alpha_2 \neq 0$. We replace e_2 with $e_2 + \sum_{i=3}^{n} (-1)^i \frac{\alpha_i}{\alpha_2} e_i$ and e_{n+5} with $\alpha_2 e_{n+5}$. Accordingly, the nonzero brackets of this algebra are as follows:

$$[e_1, \ldots, e_n] = e_{n+3}, \quad \left[e_2, \ldots, e_{n+1}\right] = e_{n+4}, \quad \left[e_1, e_3, \ldots, e_{n+1}\right] = e_{n+5}.$$

We denote this algebra by $A_{n,n+5,4}$.

Now, assume that $\dim Z(A) = 3$; thus $A^2 = Z(A) = \langle e_{n+3}, e_{n+4}, e_{n+5}\rangle$. Hence,

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3}, \\ \left[e_2, \ldots, e_{n+1}\right] = e_{n+4}, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+1}\right] = \alpha_i e_{n+5}, & 2 \leq i \leq n, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+2}\right] = \beta_i e_{n+5}, & 1 \leq i \leq n, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}\right] = \theta_{ij} e_{n+5}, & 1 \leq i < j \leq n. \end{cases}$$

Since $\dim (A/\langle e_{n+3}, e_{n+4}\rangle\ )^2 = 1$, we have $A/\langle e_{n+3}, e_{n+4}\rangle \cong H(n, 1) \oplus F(2)$. According to the structure of n-Lie algebras, we infer that only one of the coefficients θ_{ij} $(1 \leq i < j \leq n)$, β_i $(1 \leq i \leq n)$, and α_i $(2 \leq i \leq n)$ is equal to one and the others are zero. On account of $Z(A) = \langle e_{n+3}, e_{n+4}, e_{n+5}\rangle$, the coefficients α_i $(2 \leq i \leq n)$ cannot be equal to one. We have two cases.

(i) Only one of the coefficients β_i $(1 \le i \le n)$ is equal to one and the others are zero. Without loss of generality, we assume that $\beta_1 = 1$ and the others are zero. So the nonzero brackets of algebra are as follows:

$$[e_1, \ldots, e_n] = e_{n+3}, \qquad \left[e_2, \ldots, e_{n+1}\right] = e_{n+4}, \qquad \left[e_2, \ldots, e_n, e_{n+2}\right] = e_{n+5}.$$

We denote this algebra by $A_{n,n+5,5}$.
(ii) Only one of the coefficients θ_{ij} $(1 \le i < j \le n)$ is equal to one and the others are zero. Without loss of generality, we assume that $\theta_{12} = 1$, and the others are zero. So, the nonzero brackets of the algebra are as follows:

$$[e_1, \ldots, e_n] = e_{n+3}, \qquad \left[e_2, \ldots, e_{n+1}\right] = e_{n+4}, \qquad \left[e_3, \ldots, e_{n+2}\right] = e_{n+5}.$$

We denote this algebra by $A_{n,n+5,6}$.
Case 3 $A/\langle e_{n+5}\rangle \cong \langle \overline{e}_1, \ldots, \overline{e}_{n+4} : [\overline{e}_1, \ldots, \overline{e}_n] = \overline{e}_{n+3}, [\overline{e}_3, \ldots, \overline{e}_{n+2}] = \overline{e}_{n+4}\rangle$.

In this case, $\dim A^2 = 3$, and so $A^2 = Z(A) = \langle e_{n+3}, e_{n+4}, e_{n+5}\rangle$. The brackets of A are as follows:

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3} + \alpha e_{n+5}, \\ \left[e_3, \ldots, e_{n+2}\right] = e_{n+4} + \beta e_{n+5}, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+1}\right] = \alpha_i e_{n+5}, & 1 \le i \le n, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+2}\right] = \beta_i e_{n+5}, & 1 \le i \le n, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}\right] = \theta_{ij} e_{n+5}, & 1 \le i < j \le n \text{ and } (i, j) \ne (1, 2). \end{cases}$$

By changing the base, we can take $\alpha = \beta = 0$. So,

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3}, \\ \left[e_3, \ldots, e_{n+2}\right] = e_{n+4}, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+1}\right] = \alpha_i e_{n+5}, & 1 \le i \le n, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+2}\right] = \beta_i e_{n+5}, & 1 \le i \le n, \\ \left[e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}\right] = \theta_{ij} e_{n+5}, & 1 \le i < j \le n \text{ and } (i, j) \ne (1, 2). \end{cases}$$

Since $\dim\left(A/\langle e_{n+3}, e_{n+4}\rangle\right)^2 = 1$, we find $A/\langle e_{n+3}, e_{n+4}\rangle \cong H(n, 1) \oplus F(2)$. According to the structure of n-Lie algebras, we conclude that at least one of the coefficients θ_{ij} $(1 \le i < j \le n, (i, j) \ne (1, 2))$, β_i $(1 \le i \le n)$, and α_i $(1 \le i \le n)$ is equal to one and the others are zero. Since $Z(A) = \langle e_{n+3}, e_{n+4}, e_{n+5}\rangle$, we have three cases.
(i) Only one of the coefficients α_i $(1 \le i \le n)$ is equal to one and the others are zero. Without loss of generality, we assume that $\alpha_1 = 1$ and the others are zero. So, we have

$$[e_1, \ldots, e_n] = e_{n+3}, \qquad \left[e_3, \ldots, e_{n+2}\right] = e_{n+4}, \qquad \left[e_2, \ldots, e_n, e_{n+1}\right] = e_{n+5}.$$

One can check that this algebra is isomorphic to $A_{n,n+5,6}$.

(ii) Only one of the coefficients β_i ($1 \le i \le n$) is equal to one and the others are zero. Without loss of generality, we assume that $\beta_1 = 1$ and the others are zero. Hence, the nonzero brackets of the algebra are

$$[e_1, \ldots, e_n] = e_{n+3}, \qquad [e_3, \ldots, e_{n+2}] = e_{n+4}, \qquad [e_2, \ldots, e_n, e_{n+2}] = e_{n+5}.$$

One can easily see that, this algebra is isomorphic to $A_{n,n+5,6}$.
(iii) Only one of the coefficients of θ_{ij} ($1 \le i < j \le n$, $(i, j) \neq (1, 2)$) is equal to one and the others are zero. Without loss of generality, we assume that $\theta_{13} = 1$ and the others are zero. Hence, the nonzero brackets of the algebra are

$$[e_1, \ldots, e_n] = e_{n+3}, \qquad [e_2, \ldots, e_{n+1}] = e_{n+4}, \qquad [e_2, e_4, \ldots, e_{n+2}] = e_{n+5}.$$

Obviously, this algebra is isomorphic to $A_{n,n+5,6}$.
Case 4 $A/\langle e_{n+5}\rangle \cong \langle \overline{e}_1, \ldots, \overline{e}_{n+4} : [\overline{e}_1, \ldots, \overline{e}_n] = \overline{e}_{n+1}, [\overline{e}_2, \ldots, \overline{e}_n, \overline{e}_{n+2}] = \overline{e}_{n+3}\rangle$.

In this case, $\dim A^2 = 4$; thus $A^2 = Z(A) = \langle e_{n+1}, e_{n+3}, e_{n+4}, e_{n+5}\rangle$. The brackets of this algebra are as follows:

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+1} + \alpha e_{n+5}, \\ [e_2, \ldots, e_n, e_{n+2}] = e_{n+3} + \beta e_{n+5}, \\ [e_1, e_3, \ldots, e_n, e_{n+2}] = e_{n+4} + \gamma e_{n+5}, \\ [e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+2}] = \beta_i e_{n+5}, \qquad 3 \le i \le n. \end{cases}$$

We change the basis to obtain $\alpha = \beta = \gamma = 0$. Thus, the brackets are

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+1}, \\ [e_2, \ldots, e_n, e_{n+2}] = e_{n+3}, \\ [e_1, e_3, \ldots, e_n, e_{n+2}] = e_{n+4}, \\ [e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+2}] = \beta_i e_{n+5}, \qquad 3 \le i \le n. \end{cases}$$

Due to the derived dimension, we conclude that at least one of the β_i ($3 \le i \le n$) is nonzero. Without loss of generality, we assume that $\beta_3 \neq 0$. We replace e_3 with $e_3 + \sum_{i=4}^{n} (-1)^{i-1} \frac{\beta_i}{\beta_3} e_i$ and e_{n+5} with $\beta_3 e_{n+5}$. the nonzero brackets of algebra are as follows:

$$[e_1, \ldots, e_n] = e_{n+1}, \qquad [e_2, \ldots, e_n, e_{n+2}] = e_{n+3},$$
$$[e_1, e_3, \ldots, e_n, e_{n+2}] = e_{n+4}, \qquad [e_1, e_2, e_4, \ldots, e_n, e_{n+2}] = e_{n+5}.$$

We denote this algebra by $A_{n,n+5,7}$.
Case 5 $A/\langle e_{n+5}\rangle \cong H(3, 2) \cong \langle \overline{e}_1, \ldots, \overline{e}_7 : [\overline{e}_1, \overline{e}_2, \overline{e}_3] = [\overline{e}_4, \overline{e}_5, \overline{e}_6] = \overline{e}_7\rangle$.

In this case, according to the structure of this algebra, we get $A^2 = Z(A) = \langle e_7, e_8\rangle$. According to Lemma 3.1, we have $A = \langle e_1, \ldots, e_8 : [e_1, e_2, e_3] = e_7, [e_4, e_5, e_6] = e_8\rangle$. For this algebra $A/\langle e_8\rangle \ncong H(3, 2)$. Therefore, this algebra does not satisfy our conditions.

Theorem 3.2 *The $(n+5)$-dimensional nilpotent n-Lie algebras of class two, for $n > 2$, over an arbitrary field are*

$$H(n,1) \oplus F(4), \quad H(3,2) \oplus F(1), \quad H(4,2), \quad A_{n,n+5,i}, \quad 1 \le i \le 7.$$

According to the above theorem and Theorem 2.7, the main theorem of this paper is as follows.

Corollary 3.3 *The $(n+5)$-dimensional nilpotent n-Lie algebras of class two are as following*

$$\begin{array}{llll} H(n,1) \oplus F(4), & H(2,2) \oplus F(2), & H(3,2) \oplus F(1), & H(2,3), \\ H(4,2), & A_{n,n+5,i}, \quad 1 \le i \le 7, & & L_{7,i}, \quad 1 \le i \le 10. \end{array}$$

These algebras are valid for the case $n = 2$ on the integers field and algebraically closed field and for $n > 2$ on the arbitrary field.

Table 1 The list of algebras which are presented in this article

Nilpotent n-Lie algebra	Nonzero multiplications
$A_{n,n+4,1}$	$[e_1,\dots,e_n] = e_{n+3}, [e_2,\dots,e_{n+1}] = e_{n+4}$
$A_{n,n+4,2}$	$[e_1,\dots,e_n] = e_{n+3}, [e_3,\dots,e_{n+2}] = e_{n+4}$ $(n \ge 3)$
$A_{n,n+4,3}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_n,e_{n+2}] = e_{n+3}$, $[e_1,e_3,\dots,e_n,e_{n+2}] = e_{n+4}$
$A_{n,n+5,1}$	$[e_1,\dots,e_n] = e_{n+4}, [e_2,\dots,e_{n+1}] = e_{n+5}$
$A_{n,n+5,2}$	$[e_1,\dots,e_n] = e_{n+4}, [e_3,\dots,e_{n+2}] = e_{n+5}$ $(n \ge 3)$
$A_{n,n+5,3}$	$[e_1,\dots,e_n] = e_{n+5}, [e_4,\dots,e_{n+3}] = e_{n+4}$ $(n \ge 3)$
$A_{n,n+5,4}$	$[e_1,\dots,e_n] = e_{n+3}, [e_2,\dots,e_{n+1}] = e_{n+4}$, $[e_1,e_3,\dots,e_{n+1}] = e_{n+5}$
$A_{n,n+5,5}$	$[e_1,\dots,e_n] = e_{n+3}, [e_2,\dots,e_{n+1}] = e_{n+4}$, $[e_2,\dots,e_n,e_{n+2}] = e_{n+5}$
$A_{n,n+5,6}$	$[e_1,\dots,e_n] = e_{n+3}, [e_2,\dots,e_{n+1}] = e_{n+4}$, $[e_3,\dots,e_{n+2}] = e_{n+5}$
$A_{n,n+5,7}$	$[e_1,\dots,e_n] = e_{n+1}, [e_1,e_2,e_4,\dots,e_n,e_{n+2}] = e_{n+5}$, $[e_1,e_3,\dots,e_n,e_{n+2}] = e_{n+4}, [e_2,\dots,e_n,e_{n+2}] = e_{n+3}$
$L_{7,1}$	$[e_1,e_2] = e_4, [e_1,e_3] = e_5$
$L_{7,2}$	$[e_1,e_2] = e_4, [e_1,e_3] = e_5, [e_2,e_3] = e_6$
$L_{7,3}$	$[e_1,e_2] = [e_3,e_4] = e_5, [e_3,e_4] = e_6$
$L_{7,4}$	$[e_1,e_2] = e_5, [e_3,e_4] = e_6$
$L_{7,5}$	$[e_1,e_2] = e_5, [e_2,e_3] = e_6, [e_2,e_4] = e_7$
$L_{7,6}$	$[e_1,e_2] = e_5, [e_2,e_3] = e_6, [e_3,e_4] = e_7$
$L_{7,7}$	$[e_1,e_2] = [e_3,e_4] = e_5, [e_2,e_3] = e_6, [e_2,e_4] = e_7$
$L_{7,8}$	$[e_1,e_2] = [e_3,e_4] = e_5, [e_1,e_3] = e_6, [e_2,e_4] = e_7$
$L_{7,9}$	$[e_1,e_5] = [e_3,e_4] = e_6, [e_2,e_5] = e_7$
$L_{7,10}$	$[e_1,e_2] = [e_3,e_4] = e_6, [e_1,e_5] = [e_2,e_3] = e_7$

In Table 1 we list the algebras which are presented in this article.

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