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ORIGINAL PAPER

Spectral Solutions for Differential and Integral Equations with Varying Coefficients Using Classical Orthogonal Polynomials

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Abstract

Spectral methods for solving differential/integral equations are characterized by the representation of the solution by a truncated series of smooth functions. The unknowns to be determined are the expansion coefficients in such a representation. The goal of this article is to give an overview of numerical problems encountered when determining these coefficients and the rich variety of techniques proposed to solve these problems. Therefore, a series of explicit formulae expressing the derivatives, integrals and moments of a class of orthogonal polynomials of any degree and for any order in terms of the same polynomials are addressed. We restrict the current study to the orthogonal polynomials of the Hermite, generalized Laguerre, Bessel, and Jacobi (including Legendre, Chebyshev, and ultraspherical) families. Moreover, formulae expressing the coefficients of an expansion of these polynomials which have been differentiated or integrated an arbitrary number of times in terms of the coefficients of the original expansion are given. In addition, formulae for the polynomial coefficients of the moments of a general-order derivative of an infinitely differentiable function in terms of its original expanded coefficients are also presented. A simple approach to build and solve recursively for the connection coefficients between different orthogonal polynomials is established. The essential results are summarized in tables which could serve as a useful reference to numerical analysts and practitioners. Finally, applications of these results in solving differential and integral equations with varying polynomial coefficients, by reducing them to recurrence relations in the expansion coefficients of the solution, are implemented.

Keywords Orthogonal polynomials · Recurrence relations · Linear differential equations · Integral equations · Connection formulae

Mathematics Subject Classification 42C10 · 33A50 · 65L05 · 65L10

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1 Introduction

Spectral methods have been used extensively in numerical approximation of differential (integral) equations [1,2]. These methods use formulae relating the expansion coefficients of derivatives (integrals) appearing in the differential (integral) equation to those of the function itself. In fact, the coefficients of successive derivatives (integrals) of a function are related by a recurrence relation which greatly facilitates the setting up of an algebraic system to determine these unknown coefficients. The advantage of these recurrence formulae is that they possess good stability in the numerical treatment.

The traditional way to introduce a spectral method starts by approximating a solution $f(x)$ by a finite sum

$$f(x) \approx f_n(x) = \sum_{k=0}^{n} a_k \phi_k(x), \tag{1}$$

where $\{\phi_k\}_{k=0}^{\infty}$ is the set of basis functions. The main question which arises is how to choose the basis functions? Once the choice of the basis functions is made, the second question appears: how to determine the expansion coefficients a_k? A successful expansion basis meets the following requirements:

(1) The approximations $f_n(x)$ should converge rapidly to $f(x)$ as $n \to \infty$.

(2) Given coefficients $\{a_k\}_{k=0}^{n}$, it should be easy to determine another set of coefficients[1] $\left\{a_k^{(p)}\right\}_{k=0}^{n}$, $\{b_k^{p,m}\}_{k=0}^{n}$ and $\left\{b_k^{(p)}\right\}_{k=0}^{n}$ such that

$$\bullet \quad \frac{\mathrm{d}^p f(x)}{\mathrm{d}x^p} = f^{(p)}(x) = \sum_{k=0}^{n} a_k \frac{\mathrm{d}^p \phi_k(x)}{\mathrm{d}x^p} \rightsquigarrow \sum_{k=0}^{n} a_k^{(p)} \phi_k(x),$$

$$\bullet \quad x^m \frac{\mathrm{d}^p f(x)}{\mathrm{d}x^p} = \sum_{k=0}^{n} a_k^{(p)} x^m \phi_k(x) \rightsquigarrow \sum_{k=0}^{n} b_k^{p,m} \phi_k(x), \ m \geq 0, \tag{2}$$

$$\bullet \quad \mathcal{I}^p f(x) = f^{(-p)}(x) = \sum_{k=0}^{n} a_k \mathcal{I}^p \phi_k(x) \rightsquigarrow \sum_{k=0}^{n} b_k^{(p)} \phi_k(x),$$

where $\mathcal{I}^p$ is the pth integral operator.

(3) The computation of expansion coefficients $\{a_k\}_{k=0}^{n}$ from function values $\{f(x_i)\}_{i=0}^{n}$ and the reconstruction of solution values in nodes from the set of coefficients $\{a_k\}_{k=0}^{n}$ should be easy, i.e., the conversion between two data sets is algorithmically efficient

$$\{f(x_i)\}_{i=0}^{n} \ \leftrightarrows \ \{a_k\}_{k=0}^{n}.$$

The common denominator of spectral methods is to rely on high-order polynomial expansions, notably trigonometric polynomials for periodic problems, and orthogo-

[1] Here the superscript does not mean a derivative!

nal polynomials for nonperiodic boundary value problems. The computation of the expansion coefficients in (2) is the dominant part of the spectral methods. It is also the most time-consuming part of spectral tau and Galerkin methods.

The key computational task in constructing these polynomial approximations and solving differential (integral) equations with polynomial coefficients in spectral methods is the evaluation of the expansion coefficients of the derivatives (integrals) and moments of high-order derivatives of infinitely differentiable functions. This is the main issue that we address in this study, where we restrict our attention to the orthogonal polynomial expansions of the Hermite, generalized Laguerre, Bessel, and Jacobi (including Legendre, Chebyshev, and ultraspherical) families. Formulae for the expansion coefficients of a general-order derivative of an infinitely differentiable function in terms of those of the function itself are constructed for expansions in Chebyshev [3,5], Legendre [6], ultraspherical [7,8], Jacobi [9,10], generalized Laguerre [11], Hermite [12], Bessel [13] and Bernstein [14,15] polynomials. Many different algorithms for finding the recurrence relations for connection and linearization coefficients for these families are discussed and developed by many authors, see, for instance, [16–20]. It was found that the use of integral operations for constructing spectral approximations improves their rate of convergence, and allows the multiple-boundary conditions to be incorporated more efficiently [21,22]. The application of integral operators for the treatment of differential equations by orthogonal polynomials dates back to Clenshaw [23] in the late 1950's. The spectral approximation of the integration form of differential equations was put forward later in the 1960's in [24] in the spectral space and in [25] in the physical space. The reason for the success of the spectral integration approaches is basically because differentiation is inherently sensitive, as small perturbations in data can cause large changes in result, while integration is inherently stable. Phillips and Karageorghis [26] proved formulae relating the expansion coefficients of an infinitely differentiable function that has been integrated an arbitrary number of times in terms of the expansion coefficients of the function when the expansion functions are the ultraspherical polynomials. They also described how they can be used to solve two-point boundary value problems. Doha [27] proved the same formula but in a simpler way than the formula suggested by Phillips and Karageorghis. Doha proved more general formulae for Jacobi [28], Laguerre [29], Hermite [29] and Bessel [13] polynomials.

Our principal aims in this paper are:

- (i) To derive explicit formula for classical orthogonal polynomial expansion coefficients of the derivatives of an arbitrary differentiable function in terms of its original expansion coefficients.
- (ii) To present explicit expression for the derivatives of classical orthogonal polynomials of any degree and for any order in terms of the classical orthogonal polynomials themselves.
- (iii) To derive explicit formulae for classical orthogonal polynomials coefficients of the moments of a general-order derivative of an infinitely differentiable function in terms of its classical orthogonal polynomials coefficients.

(iv) To obtain explicit expression for classical orthogonal polynomials of any degree that has been integrated an arbitrary number of times in terms of the classical orthogonal polynomials themselves.
(v) To describe a simple algorithmic procedure to compute recursively the expansion coefficients in the connection problem and the expansion coefficients of associated classical orthogonal polynomials.
(vi) To show how to use these formulae for solving ordinary differential equations with polynomial coefficients by reducing them to recurrence relations in the expansion coefficients of their solutions.

It should be mentioned that one of our aims here is to emphasize the systematic character and simplicity of our algorithm, which allows one to implement it in any computer algebra (here the Mathematica symbolic language has been used).

2 Properties of Classical Orthogonal Polynomials

A family $y(x) = \phi_n(x) = k_n x^n + \ldots (n \in \{0, 1, \ldots\},\ k_n \neq 0)$ of polynomials of degree exactly n is a family of classical orthogonal polynomials if it is the solution of a differential equation of the type (see, [30–33])

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n(x)y(x) = 0, \tag{3}$$

where $\sigma(x) = ax^2+bx+c$ is a polynomial of at most second degree and $\tau(x) = dx+e$ is a polynomial of first degree. Since one demands that $\phi_n(x)$ has exact degree n, then by equating the highest coefficients of x^n in (3) one gets

$$\lambda_n = -n[a(n-1) + d].$$

The solutions, $\phi_n(x)$, of Eq. (3) usually called hypergeometric-type polynomials. These polynomials satisfy the orthogonality relation

$$\int_{a'}^{b'} \phi_n(x)\phi_m(x)\rho(x)\mathrm{d}x = \delta_{nm}h_n,\ h_n > 0 \quad (n,\ m = 0, 1, \ldots), \tag{4}$$

where $\rho(x)$ is a function satisfying the so-called Pearson equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sigma(x)\rho(x)] = \tau(x)\rho(x),$$

provided that the following condition

$$\sigma(x)\rho(x)x^k\Big|_{x=a',b'} = 0, \quad \forall k \geq 0, \tag{5}$$

is satisfied. The constant h_n can be computed from the relation

$$h_n = (-1)^n n! k_n B_n \int_{a'}^{b'} \sigma^n(x)\rho(x)\mathrm{d}x,$$

where B_n is the normalization constant appearing in the Rodrigues formula

$$\phi_n(x) = \frac{B_n}{\rho(x)} D^{(n)}[\rho_n(x)], \quad D \equiv \frac{\mathrm{d}}{\mathrm{d}x} \tag{6}$$

and $\rho_n(x) = \sigma^n(x)\rho(x)$. The constants k_n and B_n are related by

$$k_n = B_n \prod_{p=0}^{n-1} [d + (n + p - 1)a].$$

An important property of classical orthogonal polynomials is that their derivatives, $\phi_n^{(m)}(x)$, form orthogonal systems. These systems are orthogonal in the interval $[a', b']$ with respect to the weight function $\rho_m(x)$, i.e.,

$$\int_{a'}^{b'} \phi_n^{(m)}(x)\phi_k^{(m)}(x)\rho_m(x)\mathrm{d}x = \delta_{nk} h_n^{(m)},\ h_n^{(m)} > 0\ (n, k = 0, 1, \ldots), \tag{7}$$

where $h_n^{(m)}$ can be expressed in terms of h_n as

$$h_n^{(m)} = (-1)^m A_{mn} h_n,$$

and the constant A_{mn} appearing in the generalization of Rodrigues' formula (6),

$$D^m \phi_n(x) = \frac{A_{mn} B_n}{\sigma^m(x)\rho(x)} D^{n-m}[\sigma^n(x)\rho(x)], \tag{8}$$

has the form

$$\begin{cases} A_{mn} = \frac{n!}{(n-m)!} \prod_{i=0}^{m-1} [d + (n + i - 1)a], & 1 \le m \le n, \\ A_{0n} = 1. \end{cases}$$

Koepf and Schmersau [34] showed that any solution $\phi_n(x)$ of (3) satisfies a recurrence relation of the type

$$x\phi_n(x) = \alpha_n\phi_{n+1}(x) + \beta_n\phi_n(x) + \gamma_n\phi_{n-1}(x), \quad n = 0, 1, \ldots,\ \phi_{-1} = 0,\ \phi_0 = 1, \tag{9}$$

where the coefficients α_n, β_n and γ_n are given by the explicit formulae

$$\begin{cases} \alpha_n = \frac{k_n}{k_{n+1}}, \\ \beta_n = \frac{2bn(an+d-a)-e(-d+2a)}{(d+2an)(d-2a+2an)}, \\ \gamma_n = -\big((an+d-2a)n(4ca-b^2) \\ \qquad +4a^2c - ab^2 + ae^2 - 4acd + db^2 - bed + d^2c\big) \\ \qquad \times \frac{(an+d-2a)n}{(d-2a+2an)^2(2an-3a+d)(2an-a+d)} \frac{k_n}{k_{n-1}}, \end{cases} \tag{10}$$

and also satisfies a structure relation of the type

$$\phi_n(x) = \bar{\alpha}_n \, D\phi_{n+1}(x) + \bar{\beta}_n \, D\phi_n(x) + \bar{\gamma}_n \, D\phi_{n-1}(x), \tag{11}$$

where the coefficients $\bar{\alpha}_n$, $\bar{\beta}_n$ and $\bar{\gamma}_n$ are given by the explicit formulae

$$\begin{cases} \bar{\alpha}_n = \frac{1}{n+1} \frac{k_n}{k_{n+1}}, \\ \bar{\beta}_n = \frac{2ea-db}{(d+2an)(d-2a+2an)}, \\ \bar{\gamma}_n = \frac{((n-1)(an+d-a)(4ac-b)+ae+dc-bed)an}{(d-2a+2an)^2(2an-3a+d)(2an-a+d)} \frac{k_n}{k_{n-1}}, \end{cases} \tag{12}$$

moreover, they proved that the power series coefficients $C_m(n)$ given by

$$\phi_n(x) = \sum_{m=0}^{n} C_m(n) x^m,$$

satisfy the recurrence relation

$$(m-n)(an+d-a+am)C_m(n) + (m+1)(bm+e)C_{m+1}(n) \\ +c(m+1)(m+2)C_{m+2}(n) = 0,$$

which carries the complete information about the hypergeometric representation of $\phi_n(x)$.

For the sake of completeness, an appendix A has been included at the end of this paper giving the expressions of $\sigma(x)$, $\tau(x)$, $\rho(x)$, λ_n, h_n, α_n, β_n, γ_n, $\bar{\alpha}_n$, $\bar{\beta}_n$, *and* $\bar{\gamma}_n$ and the hypergeometric series representation for each one of the referred classical orthogonal families (see, Tables 5, 6 in “Appendix”).

3 Expansion Coefficients of the Derivatives/Integrals of $\phi_n(x)$

The main objective of this section is to give explicit formulae for the derivatives and integrals of $\phi_n(x)$, based on the method of Sánchez-Ruiz and Dehesa [35].

Theorem 1 *In the expansion*

$$\frac{\mathrm{d}^p}{\mathrm{d}x^p}\phi_n(x) = \sum_{i=0}^{n-p} C^{+}_{p,i}(n)\phi_i(x), \quad n \geq p \geq 0, \tag{13}$$

the coefficients $C^{+}_{p,i}(n)$ *are given by*

$$C^{+}_{p,i} = \frac{(-1)^{n-p} B_i B_n A_{i+p,n}}{h_i} \int_{a'}^{b'} \rho(x)\sigma^n(x) D^{(n-p-i)}[\sigma^{-p}(x)]\mathrm{d}x, \tag{14}$$

and satisfy the recurrence relation

$$\begin{aligned} &\bar{\alpha}_{i-1}\, C^{+}_{p+1,i-1}(n) + \bar{\beta}_i\, C^{+}_{p+1,i}(n) + \bar{\gamma}_{i+1}\, C^{+}_{p+1,i+1}(n) = C^{+}_{p,i}(n), \\ &i = 1, 2, \ldots, n-p,\ n \geq p \geq 0. \end{aligned} \tag{15}$$

Proof Multiplying both sides of Eq. (13) by $\rho(x)\phi_k(x)$, and integrating between a' and b', orthogonality relation (4) immediately gives

$$C^{+}_{p,k}(n) = \frac{1}{h_k} \int_{a'}^{b'} \rho(x)\phi_k(x)\phi_n^{(p)}(x)v.$$

Using the Rodrigues representation (6) for $\phi_n(x)$, this integral can be written

$$C^{+}_{p,k}(n) = \frac{B_k}{h_k} \int_{a'}^{b'} \phi_n^{(p)}(x) \frac{d^k}{\mathrm{d}x^k}[\sigma^k(x)\rho(x)]\mathrm{d}x.$$

Integrating by parts k times, and taking into account the orthogonality condition (5), we get

$$C^{+}_{p,k}(n) = \frac{(-1)^k B_k}{h_k} \int_{a'}^{b'} \rho(x)\sigma^k(x)\phi_n^{(p+k)}(x)\mathrm{d}x.$$

Using the Rodrigues representation (8) for $\phi_n^{(p+k)}(x)$, this integral can be written

$$C^{+}_{p,k}(n) = (-1)^k \frac{B_k B_n A_{p+k,n}}{h_k} \int_{a'}^{b'} \sigma^{-p}(x) \frac{d^{n-p-k}}{\mathrm{d}x^{n-p-k}}[\sigma^n(x)\rho(x)]\mathrm{d}x.$$

Integrating by parts $(n-p-k)$ times, and taking into account the orthogonality condition (5), we get

$$C^{+}_{p,k}(n) = \frac{(-1)^{n-p} B_k B_n A_{k+p,n}}{h_k} \int_{a'}^{b'} \rho(x)\sigma^n(x) \frac{d^{n-p-k}}{\mathrm{d}x^{n-p-k}}[\sigma^{-p}(x)]\mathrm{d}x,$$

which proves (14). Let us write

$$\sum_{i=0}^{n-p-1} C^{+}_{p+1,i}(n)\phi_i(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{i=0}^{n-p} C^{+}_{p,i}(n)\phi_i(x)\right],$$

then using identity (11) leads to the recurrence relation (15), which completes the proof of the theorem. □

Note 1 *The integral (14) for the case of Bessel polynomials is evaluated by means of the residual theorem.*

The following theorem can be proved in a similar way like that of Theorem 1.

Theorem 2 *In the expansion*

$$\phi_n(x) = \sum_{i=p}^{n+p} C^{-}_{p,i}(n)\phi_i^{(p)}(x), \quad p \geq 0, \tag{16}$$

the coefficients $C^{-}_{p,i}(n)$ are given by

$$C^{-}_{p,i} = \frac{(-1)^{n+p} B_i B_n A_{i-p,n}}{h_i} \int_{a'}^{b'} \rho(x)\sigma^n(x) \frac{\mathrm{d}^{n-p-i}}{dx^{n-p-i}}[\sigma^p(x)]\mathrm{d}x, \tag{17}$$

and satisfy the recurrence relation

$$\begin{aligned} &\bar{\alpha}_{i-1}\, C^{-}_{p-1,i-1}(n) + \bar{\beta}_i\, C^{-}_{p-1,i}(n) + \bar{\gamma}_{i+1}\, C^{-}_{p-1,i+1}(n) = C^{-}_{p,i}(n), \\ &\quad i = p,\ p+1,\ \ldots, n+p. \end{aligned} \tag{18}$$

Corollary 1 *It is easy to show that*

$$\mathcal{I}^p\phi_n(x) = \sum_{i=p}^{n+p} C^{-}_{p,i}(n)\phi_i(x) + \pi_{p-1,n}(x), \quad p \geq 0, \tag{19}$$

$$= \sum_{i=0}^{2p} b_{p,i}(n)\phi_{n+p-i}(x) + \tilde{\pi}_{p-1,n}(x), \quad p \geq 0, \tag{20}$$

where $\pi_{p-1,n}(x)$ and $\tilde{\pi}_{p-1,n}(x)$ are polynomials of degree at most $(p-1)$ and the coefficients $b_{p,i}(n)$ and $C^{-}_{p,i}(n)$ are related by

$$b_{p,i}(n) = C^{-}_{p,n+p-i}(n), \tag{21}$$

and moreover, the coefficients $b_{p,i}(n)$ satisfy a recurrence relation of the type

$$\begin{aligned} b_{p,i}(n) &= \bar{\alpha}_{n+p-i-1} b_{p-1,i}(n) + \bar{\beta}_{n+p-i} b_{p-1,i-1}(n) \\ &\quad + \bar{\gamma}_{n+p-i+1} b_{p-1,i-2}(n),\ i = 0,\ 1,\ \ldots, 2p, \end{aligned} \tag{22}$$

with $b_{p-1,-\ell}(j) = 0, \quad \forall \ell > 0, \quad b_{0,0} = 1, \quad b_{p-1,r}(j) = 0, \quad r = 2p-1,\ 2p.$

Proof Integration of (16) p times with respect to x gives immediately (19), which in turn may be written in the form

$$\mathcal{I}^p\phi_n(x) = \sum_{i=0}^{n} C^{-}_{p,n+p-i}(n)\phi_{n+p-i}(x) + \pi_{p-1,n}(x), \quad p \geq 0.$$

Making use of relation (17) and knowing that deg $\sigma^p(x) \leq 2p$, enables one to show that

$$b_{p,i}(n) = C^-_{p,n+p-i}(n) = 0,\ i = 2p+1,\ \ldots, n;\ n > 2p,$$

and accordingly, formula (19) may be written as formula (20). The recurrence relation (22) is a direct consequence of (18), and this completes the proof of the corollary. □

Remark 1 It is to be noted here that the expansion coefficients $C^+_{p,i}(n)$ and $C^-_{p,i}(n)$ are related by

$$C^+_{p,i}(n) = C^-_{-p,i}(n). \tag{23}$$

Corollary 2 *If the classical orthogonal polynomials $\phi_n(x)$ satisfy a recurrence relation of the type*

$$\phi_n(x) = \bar{\alpha}_n D\phi_{n+1}(x) + \bar{\gamma}_n D\phi_{n-1}(x), \tag{24}$$

then

$$\mathcal{I}^p \phi_n(x) = \sum_{i=0}^{p} b_{p,2i}(n)\phi_{n+p-2i}(x) + \tilde{\pi}_{p-1,n}(x). \tag{25}$$

Proof Making use of (22), noting that $b_{11}(n) = 0$, enables one to show that

$$b_{p,i}(n) = 0,\ i \text{ odd},$$

and accordingly formula (20) takes the form (25). □

Theorem 3 (i) *For Hermite polynomials* [12,29]

$$D^p H_n(x) = 2^p \frac{n!}{(n-p)!} H_{n-p}(x), \quad n,\ p \geq 0, \tag{26}$$

and

$$\mathcal{I}^p H_n(x) = 2^{-p} \frac{n!}{(n+p)!} H_{n+p}(x), \quad n,\ p \geq 0, \tag{27}$$

(ii) *For generalized Laguerre polynomials* [11,29]

$$D^p L^\alpha_n(x) = (-1)^p \sum_{i=0}^{n-p} \binom{n-i-1}{p-1} L^\alpha_i(x), \quad n,\ p \geq 1, \tag{28}$$

and

$$\mathcal{I}^p L^\alpha_n(x) = \sum_{i=0}^{p} (-1)^{i+p} \binom{p}{i} L^\alpha_{n+p-i}(x), \quad n,\ p \geq 1. \tag{29}$$

(iii) *For Jacobi polynomials* [9,28]

$$D^p P_n^{(\alpha,\beta)}(x) = 2^{-p}(n+\lambda)_p \times \sum_{i=0}^{n-p} C_{n-p,i}(\alpha+p, \beta+p, \alpha, \beta) P_i^{(\alpha,\beta)}(x), \quad \lambda = \alpha+\beta+1, \tag{30}$$

and

$$\mathcal{I}^p P_n^{(\alpha,\beta)}(x) = \frac{2^p}{(n-p+\lambda)_p} \sum_{i=0}^{2p} C_{n+p,n+p-i}(\alpha-p, \beta-p, \alpha, \beta) P_{n+p-i}^{(\alpha,\beta)}(x), \tag{31}$$

where

$$C_{n,i}(\gamma,\delta,\alpha,\beta) = \frac{(n+\eta)_i (i+\gamma+1)_{n-i}}{(n-i)!(\lambda+i)_i} \times {}_3F_2\left[\begin{matrix} -(n-i), n+i+\eta, i+\alpha+1 \\ i+\gamma+1, 2i+\lambda+1 \end{matrix}; 1\right],$$
$$\eta = \gamma+\delta+1.$$

(iv) *For Bessel polynomials* [13]

$$D^p Y_n^{(\alpha)}(x) = 2^{-p}(n-p+1)_p (n+\alpha+1)_p \sum_{k=0}^{n-p} M_k(\alpha+2p, \alpha, n-p) Y_k^{(\alpha)}(x), \tag{32}$$

and

$$\mathcal{I}^p Y_n^{(\alpha)}(x) = \sum_{i=0}^{2p} b_{p,i}(n) Y_{n+p-i}^{(\alpha)}(x),\ n,\ p \geq 0, \tag{33}$$

with $Y_{-r}^{(\alpha)}(x) = 0,\ r \geq 1$, *where*

$$M_i(\alpha,\beta,n) = (-1)^n (2i+\beta+1) \frac{(\alpha-\beta)_{n-i} (-n)_i (\alpha+n+1)_i}{i!(\beta+i+1)_{n+1}},$$

and

$$b_{p,i} = \binom{n+p}{i} \frac{2^p (2n+2p-2i+\alpha+1)(n+\alpha-p+1)_{n+p-i} (2p-i+1)_i}{(n+1)_p (n+\alpha-p+1)_p (n+p+\alpha-i+1)_{n+p+1}}.$$

Remark 2 For Bernstein polynomials [15]

$$D^p\, B_{i,n}(x) = \frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} B_{i-k,n-p}(x). \tag{34}$$

$$\mathcal{I}^q\, B_{i,n}(x) = \frac{n!}{(n+p)!} \sum_{k=i+q}^{n+q} \binom{j-i-1}{q-1} B_{j,n+q}(x). \tag{35}$$

4 The Coefficients of Differentiated/Integrated Expansions of $\phi_n(x)$

The main results of this section are two explicit formulae which express the expansion coefficients of a general-order derivative (integral) of an infinitely differentiable function in terms of its original expansion coefficients.

Theorem 4 *Suppose we are given a regular function $f(x)$ which is formally expanded in the infinite series*

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x), \tag{36}$$

and for the pth derivatives of $f(x)$,

$$f^{(p)}(x) = \sum_{n=0}^{\infty} a_n^{(p)} \phi_n(x), \quad a_n^{(0)} = a_n, \tag{37}$$

then

$$a_n^{(p)} = \sum_{i=0}^{\infty} C^+_{p,n}(n+p+i) a_{n+p+i}, \quad n \geq 0, \tag{38}$$

where the coefficients $C^+_{p,i}(n)$ are given by (14). Moreover, the coefficients $a_n^{(p)}$ satisfy the recurrence relation

$$\bar{\alpha}_{n-1} a_{n-1}^{(p+1)} + \bar{\beta}_n a_n^{(p+1)} + \bar{\gamma}_{n+1} a_{n+1}^{(p+1)} = a_n^{(p)}, \quad p \geq 0,\ n \geq 1. \tag{39}$$

Proof By differentiating Eq. (36) p times and using (13), we obtain

$$f^{(p)}(x) = \sum_{n=p}^{\infty} a_n \sum_{i=0}^{n-p} C^+_{p,i}(n) \phi_i(x). \tag{40}$$

Expanding (40) and collecting similar terms, we obtain

$$f^{(p)}(x) = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{\infty} C^+_{p,n}(n+p+i) a_{n+p+i} \right] \phi_n(x). \tag{41}$$

Identifying (37) with (41) gives immediately

$$a_n^{(p)} = \sum_{i=0}^{\infty} C^{+}_{p,n}(n+p+i)a_{n+p+i}.$$

Now, let us write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{i=0}^{\infty} a_n^{(p-1)}\phi_n(x)\right] = \sum_{n=0}^{\infty} a_n^{(p)}\phi_n(x),$$

then use of identity (11) leads to the recurrence relation (39), which completes the proof of the theorem □

Theorem 5 *Suppose that $f^{(-m)}(x) = D^{-m}f(x)$ for some $m \geq 1$ is an infinitely differentiable function, and $f(x)$ is formally expanded as in (36). Let $b_n^{(p)}$, $m \geq p \geq 1$, denote the expansion coefficients of $f^{(-p)}(x)$ in the expansion*

$$f^{(-p)}(x) = \sum_{n=0}^{\infty} b_n^{(p)}\phi_n(x) + \pi_{p-1}(x), \quad b_n^{(0)} = a_n, \tag{42}$$

where $\pi_{p-1}(x)$ is a polynomial of degree at most $(p-1)$, then the coefficients $b_n^{(p)}$ are related to a_n by

$$b_n^{(p)} = \sum_{i=0}^{2p} b_{pi}(n-p+i)a_{n-p+i}, \quad n \geq p, \tag{43}$$

and they satisfy a recurrence relation of the type

$$b_n^{(p)} = \bar{\alpha}_{n-1}b_{n-1}^{(p-1)} + \bar{\beta}_n b_n^{(p-1)} + \bar{\gamma}_{n+1}b_{n+1}^{(p-1)}, \quad n \geq p. \tag{44}$$

Proof By integrating Eq.(36) p times and using (20), we obtain

$$\mathcal{I}^p f(x) = \sum_{n=0}^{\infty} a_n \sum_{i=0}^{2p} b_{pi}\phi_{n+p-i}(x) + \bar{\pi}_{p-1}(x). \tag{45}$$

Expanding (45), collecting similar terms and noting that $\phi_{-r}(x) = 0$ for $r > 0$, we obtain

$$\mathcal{I}^p f(x) = \sum_{n=p}^{\infty}\left[\sum_{i=0}^{2p} b_{pi}(n-p+i)a_{n-p+i}\right]\phi_n(x) + \pi_{p-1}(x), \tag{46}$$

where $\pi_{p-1}(x)$ is a polynomial of degree at most $(p-1)$. Identifying (46) with (42) gives immediately

$$b_n^{(p)} = \sum_{i=0}^{2p} b_{pi}(n-p+i)a_{n-p+i}, \quad n \geq p.$$

Now, let us write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=p}^{\infty} b_n^{(p)}\phi_n(x) + \pi_{p-1}(x)\right] = \sum_{n=p-1}^{\infty} b_n^{(p-1)}\phi_n(x) + \pi_{p-2}(x),$$

then making use of identity (11) leads to the recurrence relation (44), which completes the proof of the theorem. □

In view of Corollary 3.2, we obtain the following result:

Corollary 3 *If the classical orthogonal polynomials $\phi_n(x)$ satisfy a recurrence relation of the type* (24)*, then*

$$b_n^{(p)} = \sum_{i=0}^{2p} b_{p,2i}(n-p+2i)a_{n-p+2i}, \quad n \geq p. \tag{47}$$

In the following, we give explicit expressions that relate the expansion coefficients $a_n^{(p)}$ and $b_n^{(p)}$ with a_n for different expansion basis (Hermite, generalized Laguerre, Jacobi, Bessel and Bernstein):

Theorem 6 (i) *For Hermite polynomials* [12,29]

$$a_n^{(p)} = 2^p \frac{(n+p)!}{n!} a_{n+p}, \tag{48}$$

and

$$b_n^{(p)} = 2^{-p} \frac{(n-p)!}{n!} a_{n-p}, \quad n \geq p. \tag{49}$$

(ii) *For generalized Laguerre polynomials* [11,29]

$$a_n^{(p)} = (-1)^p \sum_{i=0}^{\infty} \binom{p+i-1}{p-1} a_{n+i+p}, \tag{50}$$

and

$$b_n^{(p)} = \sum_{i=0}^{\infty} (-1)^{p+i} \binom{p}{i} a_{n+i-p}, \quad n \geq p. \tag{51}$$

(iii) *For Jacobi polynomials* [28]

$$a_n^{(p)} = 2^{-p}\sum_{i=0}^{\infty}(n+i+p+\lambda)_p C_{n+1,n}(\alpha+p,\beta+p,\alpha,\beta)a_{n+i+p},\quad n,p\geq 0, \tag{52}$$

and

$$b_n^{(p)} = \sum_{i=0}^{2p}\frac{2^p}{(i+n-2p+\lambda)_p}C_{n+i,n}(\alpha-p,\beta-p,\alpha,\beta)a_{n+i-p},\quad n\geq p. \tag{53}$$

(iv) *For Bessel polynomials* [13]

$$a_n^{(p)} = 2^{-p}\sum_{i=0}^{\infty}(n+i+1)_p(n+p+i+\alpha+1)_p M_n(\alpha+2p,\alpha,n+i),\quad n\geq 0,\ p\geq 1, \tag{54}$$

and

$$b_n^{(p)} = \sum_{i=0}^{2p} b_{p,i}(n-p+i)a_{n-p+i},\quad n\geq p. \tag{55}$$

Remark 3 For Bernstein polynomials [14]

$$a_{i,n}^{(q)} = \sum_{k=-q}^{q} C_k(i,n,q)a_{i-k,n},$$

where

$$C_k(i,n,q) = q!\sum_{m=0}^{q}\binom{q}{m}\binom{i}{m+k}\binom{n-i}{q-m-k}.$$

5 Products of Powers and Orthogonal Polynomials

For the evaluation of the expansion coefficients of $x^m f^{(p)}(x)$ in a series of $\phi_n(x)$, the following theorem is needed.

Theorem 7 ([35], p. 163)

$$x^m\phi_j(x) = \sum_{n=0}^{m+j} a_{m,j+m-n}(j)\phi_n(x), \tag{56}$$

where

$$a_{m,j+m-n}(j) = \frac{(-1)^n B_n B_j m!}{h_n} \times \sum_{k=k-}^{k+} \binom{n}{k} \frac{A_{kj}}{(m-n+k)!} \times \int_{a'}^{b'} x^{m-n+k} \sigma^{n-k}(x) D^{j-k}[\sigma^j(x)\rho(x)]dx, \tag{57}$$

$k_- = \max(0, n-m)$, $k_+ = \min(n, j)$.

The explicit expressions of (56) when $\{\phi_n(x)\}$ is one of the classical families of Hermite, Laguerre, Jacobi and Bessel polynomials are given in [35] formulae (3.10), (3.17), (3.22) and (3.31), pp. 164–168.

Lemma 1 *The expansion* (56) *may be written in the form*

$$x^m \phi_j(x) = \sum_{n=0}^{2m} a_{m,n}(j) \phi_{j+m-n}(x). \tag{58}$$

Proof The expansion (56) can be written as

$$x^m \phi_j(x) = \sum_{n=0}^{m+j} a_{m,n}(j) \phi_{j+m-n}(x), \tag{59}$$

then multiplying both sides by $\rho(x)\phi_{j+m-k}(x)$, integrating between a' and b' and making use of relation (4), yield

$$h_{j+m-n}\, a_{m,n}(j) = \int_{a'}^{b'} \rho(x) x^m \phi_j(x) \phi_{j+m-n}(x) \mathrm{d}x, \quad n = 0,\ 1,\ 2,\ \ldots,\ j+m. \tag{60}$$

Now, applying the orthogonality properties of ϕ_n for $n = 2m+1, \ \ldots, \ j+m$, we obtain

$$a_{m,n}(j) = 0, \quad n = 2m+1,\ \ldots,\ j+m, \quad j > m,$$

and by noting that $\phi_{-r}(x) = 0$ for $r > 0$, we get

$$\phi_{j+m-n} = 0, \quad n = j+m+1,\ \ldots,\ 2m, \quad j < m,$$

and accordingly, Eq. (59) takes the form of (58), which completes the proof of Lemma 1. ◻

Corollary 4 *If the classical orthogonal polynomials* $\phi_j(x)$ *satisfy a recurrence relation of the type*

$$x\phi_j(x) = \alpha_j \phi_{j+1}(x) + \gamma_j \phi_{j-1}(x), \quad j = 0,\ 1,\ 2,\ \ldots,\ \phi_{-1} \equiv 0, \quad \phi_0 \equiv 1, \tag{61}$$

then

$$x^m \phi_j(x) = \sum_{n=0}^{m} a_{m,2n}(j)\phi_{j+m-2n}(x). \tag{62}$$

Proof Using recurrence relation (59) and putting $a_{11}(j) = 0$, then it is not difficult to show that $a_{m,n}(j) = 0$, for n odd, and accordingly formula (58) takes the form (62). □

The explicit expressions of (58) when $\{\phi_n(x)\}$ is one of the classical orthogonal polynomials (Hermite, generalized Laguerre, Jacobi, Bessel) are given in the following theorem.

Theorem 8 (i) *For Hermite polynomials* [12]

$$x^m H_j(x) = \sum_{n=0}^{m} a_{m,2n}(j)H_{j+m-2n}(x), \quad m,\ n \geq 0, \tag{63}$$

where

$$a_{m,2n}(j) = \frac{2^{j-m} m!j!}{(j+m-2n)!} \sum_{k=\max(0,j-2n)}^{\min(j+m-2n,j)} \binom{j+m-2n}{k} \frac{1}{2^k (j-k)!(n+k-j)!}. \tag{64}$$

(ii) *For generalized Laguerre polynomials* [11]

$$x^m L_j^{\alpha}(x) = \sum_{n=0}^{2m} a_{m,n}(j)L_{j+m-n}^{\alpha}(x), \quad m,\ n \geq 0, \tag{65}$$

where

$$a_{m,n}(j) = \frac{(-1)^{m-n}(m!)^2}{\Gamma(j+m-n+\alpha+1)} \times \sum_{k=\max(0,j-n)}^{\min(j+m-n,j)} \binom{j+m-n}{k} \frac{\Gamma(m+k+\alpha+1)}{(j-k)!(n-j+k)!(m-j+k)!}. \tag{66}$$

(iii) *For Jacobi polynomials* [10,28]

$$x^m P_j^{\alpha,\beta}(x) = \sum_{n=0}^{2m} a_{m,n}(j)P_{j+m-n}^{\alpha,\beta}(x), \tag{67}$$

where

$$a_{m,n}(j) = \frac{(-1)^n 2^{j+m-n} m!(2j+2m-2n+\lambda)(j+\lambda)_{m-n}}{(j+\alpha+1)_{m-n}(j+\beta+1)_{m-n}}$$

$$\times \sum_{k=\max(0,j-n)}^{\min(j+m-n,j)} \frac{\binom{j+m-n}{k}\Gamma(j+k+\lambda)}{2^k(n-j+k)!\Gamma(3j+2m-2n-k+\lambda+1)}$$
$$\times \sum_{\ell=0}^{j-k} \frac{(-1)^\ell \Gamma(2j+m-n-k-\ell+\alpha+1)\Gamma(j+m+\ell-n+\beta+1)}{\ell!(j-k-\ell)!\Gamma(j-\ell+\alpha+1)\Gamma(k+\ell+\beta+1)}$$
$$\times {}_2F_1\left[\left.\begin{matrix} j-n-k,\, j+m-n+\beta+\ell+1 \\ 3j+2m-2n-k+\lambda+1 \end{matrix}\right| 2\right], \quad \lambda=\alpha+\beta+1. \tag{68}$$

(iv) *For Bessel polynomials* [13]

$$x^m Y_j^{(\alpha)}(x) = \sum_{n=0}^{2m} a_{m,n}(j) Y_{j+m-n}^{(\alpha)}(x), \quad m \geq 0, \ j \geq 0, \tag{69}$$

where

$$a_{m,n}(j) = \frac{(-1)^{j-n} 2^m m! j! (2j+2m-2n+\alpha+1)\Gamma(j+m-n+\alpha+1)}{(j+m-n)!(2m-n)!\Gamma(j+\alpha+1)\Gamma(2j+2m-n+\alpha+2)}$$
$$\times \sum_{k=\max(0,j-n)}^{\min(j+m-n,j)} \binom{j+m-n}{k}$$
$$\times \frac{(-1)^k \Gamma(j+k+\alpha+1)\Gamma(j+2m-n-k+1)}{(j-k)!(n+k-j)!}. \tag{70}$$

Note 2 *Doha* [11,12,28] *and Doha and Ahmed* [13] *proved that the explicit expressions of* (64), (66), (68) *and* (70), *when* $\{\phi(x)\}$ *is one of the classical orthogonal polynomials (Hermite, generalized Laguerre, Jacobi, Bessel), respectively, satisfy the recurrence relation*

$$a_{m,n}(j) = \alpha_{j+m-n-1} a_{m-1,n}(j) + \beta_{j+m-n} a_{m-1,n-1}(j) + \gamma_{j+m-n+1} a_{m-1,n-2}(j),$$
$$n = 0,\ 1,\ \ldots, 2m, \tag{71}$$

with $a_{m-1,-\ell}(j) = 0, \quad \forall \ell > 0, \quad a_{0,0}(j) = 1, \quad a_{m-1,r}(j) = 0, \quad r = 2m-1,\ 2m.$

6 The Expansion Coefficients of the Moments of a General-Order Derivative of an Infinitely Differentiable Function

In this section, we state and prove a theorem which relates the expansion coefficients of $x^\ell f^{(p)}$ in terms of $a_i^{(p)}$.

Theorem 9 *Assume that $f(x)$, $f^{(p)}(x)$ and $x^{\ell}\phi_j(x)$ have the expansions (36), (37) and (58), respectively, and assume also that*

$$x^{\ell}\left(\sum_{i=0}^{\infty} a_i^{(p)}\phi_i(x)\right) = \sum_{i=0}^{\infty} b_i^{p,\ell}\phi_i(x) = I^{p,\ell}, \tag{72}$$

then the expansion coefficients $b_i^{p,\ell}$ are given by

$$\begin{cases} \sum_{k=0}^{\ell-1} a_{\ell,k+\ell-i}(k)a_k^{(p)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(p)}, & 0 \le i \le \ell, \\ \sum_{k=i-\ell}^{\ell-1} a_{\ell,k+\ell-i}(k)a_k^{(p)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(p)}, & \ell+1 \le i \le 2\ell-1, \\ \sum_{k=i-2\ell}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(p)}, & i \ge 2\ell. \end{cases} \tag{73}$$

Proof Equations (58) and (72) give

$$I^{p,\ell} = \sum_{k=0}^{\infty} a_k^{(p)} \sum_{j=0}^{2\ell} a_{\ell,j}(k)\phi_{k+\ell-j}(x). \tag{74}$$

By letting $i = k + \ell - j$, then (74) may be written in the form

$$\begin{aligned} I^{p,\ell} &= \sum_{k=0}^{\ell-1} a_k^{(p)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k)\phi_i(x) + \sum_{k=\ell}^{\infty} a_k^{(p)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k)\phi_i(x) \\ &= \sum_1 + \sum_2, \end{aligned} \tag{75}$$

where

$$\sum_1 = \sum_{k=0}^{\ell-1} a_k^{(p)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k)\phi_i(x),$$

$$\sum_2 = \sum_{k=\ell}^{\infty} a_k^{(p)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k)\phi_i(x).$$

By noting that $\phi_{-i}(x) = 0$, for $i \ge 1$, then it can be easily shown that

$$\sum_{1} = \sum_{k=0}^{\ell-1} a_k^{(p)} \sum_{i=0}^{k+\ell} a_{\ell,k+\ell-i}(k)\phi_i(x)$$
$$= \sum_{i=0}^{\ell}\sum_{k=0}^{\ell-1} a_k^{(p)} a_{\ell,k+\ell-i}(k)\phi_i(x) + \sum_{i=\ell+1}^{2\ell-1}\sum_{k=i-\ell}^{\ell-1} a_k^{(p)} a_{\ell,k+\ell-i}(k)\phi_i(x),$$

hence

$$\sum_{1} = \sum_{i=0}^{2\ell-1}\sum_{k=\max(0,i-\ell)}^{\ell-1} a_k^{(p)}\, a_{\ell,k+\ell-i}(k)\phi_i(x). \tag{76}$$

If when considering $\sum_{2}$, one takes $k+\ell$ instead of k, then it is not difficult to show that

$$\sum_{2} = \sum_{i=0}^{\infty}\sum_{k=\max(0,i-2\ell)}^{i} a_{k+\ell}^{(p)}\, a_{\ell,k+2\ell-i}(k+\ell)\phi_i(x). \tag{77}$$

Substitution of (76) and (77) into (75) gives the required results of (73) and completes the proof of the theorem. □

7 Connection Coefficients Between Different Classical Orthogonal Polynomial Systems

In this section, we consider the problem of determining the connection coefficients between different orthogonal polynomial systems. An interesting question is how to transform the Fourier coefficients of a given polynomial corresponding to an assigned orthogonal basis, into the coefficients of another basis orthogonal with respect to a different weight function. The aim is to determine the so-called connection coefficients of the expansion of any element of the first basis in terms of the elements of the second basis.

Suppose V is a vector space of all polynomials over the real or complex numbers and V_m is the subspace of polynomials of degree less or equal to m. Suppose $p_0(x), p_1(x), p_2(x), \ldots$ is a sequence of polynomials such that $p_n(x)$ is of exact degree n; let $q_0(x), q_1(x), q_2(x), \ldots$ be another such sequence. Clearly, these sequences form a basis for V. It is also evident that $p_0(x), p_1(x), \ldots, p_m(x)$ and $q_0(x), q_1(x), \ldots, q_m(x)$ give two bases for V_m. While working with finite-dimensional vector spaces, it is often necessary to find the matrix that transforms a basis of a given space to another basis. This means that one is interested in the connection coefficients $a_i(n)$ that satisfy

$$\psi_n(\bar{a}x+\bar{b}) = \sum_{i=0}^{n} a_i(n)\phi_i(x), \tag{78}$$

where $\bar{a}$ and $\bar{b}$ are constants.

The connection coefficients between many of the classical orthogonal polynomial systems have been determined by different kinds of methods, see e.g., [36–38]. The aim of this section is to describe a simple procedure (based on the results of Theorem 9) to find recurrence relations between the coefficients $a_i(n)$ when $p_i(x)$ and $q_i(x)$ belong to the class of classical orthogonal polynomials. This gives an alternative and different way to be compared to the approaches of Askey and Gasper [39], Ronveaux et al. [40, 41], Area et al. [16], Koepf and Schmersau [34], Lewanowicz [19,20,42], Lewanowicz et al. [17], and Sánchez-Ruiz and Dehesa [35].

The differential equation satisfied by $\psi_n(\bar{a}x+\bar{b})$ for the cases of Hermite, generalized Laguerre, Jacobi and Bessel polynomials has the form

$$[\sigma(\bar{a}x+\bar{b})D^2+\bar{a}\tau(\bar{a}x+\bar{b})D+\bar{a}^2\lambda_n]y(x)=0, \tag{79}$$

which may be written in the form

$$[(b_2x^2+b_1x+b_0)D^2+(c_1x+c_0)D+\mu]y(x)=0, \tag{80}$$

where $b_2=a\,\bar{a}^2,\ b_1=2a\bar{a}\bar{b}+\bar{a}b,\ b_0=a\bar{b}^2+b\bar{b}+c,\ c_1=d\bar{a},\ c_0=\bar{a}(d\bar{b}+e)$ and $\mu=\bar{a}^2\lambda_n$.

By substituting (78) and by virtue of formulae (72), Eq. (79) takes the form

$$b_2b_i^{2,2}+b_1b_i^{2,1}+b_0b_i^{2,0}+c_1b_i^{1,1}+c_0b_i^{1,0}+\mu b_i^{0,0}=0,$$

and by making use of (71) and (73), we obtain

$$\sum_{m=0}^{2}\sum_{k=-m}^{m}\gamma_k^{(m)}(i)a_{i+k}^{(m)}=0,\quad i\geq 0, \tag{81}$$

which is of order 4, where

$$\begin{aligned}
&\gamma_0^{(0)}(i)=\lambda,\ \gamma_{-1}^{(1)}(i)=c_1\alpha_{i-1},\ \gamma_0^{(1)}(i)=c_0+c_1\beta_i,\ \gamma_1^{(1)}(i)=c_1\gamma_{i+1},\\
&\gamma_{-2}^{(2)}(i)=b_2\alpha_{i-2}\alpha_{i-1},\\
&\gamma_{-1}^{(2)}(i)=\alpha_{i-1}[b_1+b_2(\beta_{i-1}+\beta_i)],\\
&\gamma_0^{(2)}(i)=b_0+b_1\beta_i+b_2(\beta_i^2+\alpha_{i-1}\gamma_i+\alpha_i\gamma_{i+1}),\\
&\gamma_1^{(2)}(i)=\gamma_{i+1}[b_1+b_2(\beta_i+\beta_{i+1})],\ \gamma_2^{(2)}(i)=b_2\gamma_{i+1}\gamma_{i+2}.
\end{aligned}$$

Following the same procedure as in Example 1 of Sect. 8.1, we get recurrence relations satisfied by $a_i(n)$ when $\{\phi_i(x)\}$ is one of the classical families of Hermite, generalized Laguerre, Jacobi and Bessel.

7.1 The $\psi_n(\bar{a}x+\bar{b})$-Hermite Connection Problem

In this problem

$$\psi_n(\bar{a}x+\bar{b})=\sum_{i=0}^{n}a_i(n)H_i(x), \tag{82}$$

the coefficients $a_i(n)$ satisfy the recurrence relation

Table 1 The $\psi_n(\bar{a}x+\bar{b})$-Hermite connection coefficients

$\psi_n(\bar{a}x+\bar{b})$	$a_i(n) \quad (0 \le i \le n)$
$H_n(\bar{a}x+\bar{b})$	$\frac{(\bar{a}/2)^i n!}{i!} \sum_{k=0}^{[\frac{n-i}{2}]} \frac{(-1)^k 2^{n-2k} \bar{b}^{n-2k-i}}{k!(n-2k-i)!} f_{n-i-2k}(i, \frac{\bar{a}^2}{\bar{b}^2})$
$L_n^{(\gamma)}(\bar{a}x+\bar{b})$	$\frac{(\bar{a}/2)^i}{i!n!} \frac{(-n)_i(\gamma+1)_n}{(\gamma+1)_i} \sum_{k=0}^{n-i} \frac{\bar{b}^k}{k!} \frac{(-(n-i))_k}{(\gamma+i+1)_k} f_k(i, \frac{\bar{a}^2}{\bar{b}^2})$
$P_n^{(\gamma,\delta)}(\bar{a}x+\bar{b})$	$\binom{n+\gamma}{n} \frac{(-\bar{a}/4)^i}{i!} \frac{(-n)_i(n+\eta)_i}{(\gamma+1)_i} \sum_{k=0}^{n-i} \frac{(1-\bar{b})^k}{2^k k!} \frac{(-(n+i))_k(n+\eta+i)_k}{(\gamma+i+1)_k} f_k(i, \frac{\bar{a}^2}{(1-\bar{b})^2})$, $\eta=\gamma+\delta+1$
$Y_n^{(\gamma)}(\bar{a}x+\bar{b})$	$\frac{(-\bar{a}/4)^i}{i!}(-n)_i(n+\gamma+1)_i \sum_{k=0}^{n-i} \frac{(-\bar{b}/2)^k(-n+i)_k(n+\eta+i+1)_k}{k!} f_k(i, \frac{\bar{a}^2}{\bar{b}^2})$
	where $f_k(i,z) = {}_2F_0[-k/2, -(k-1)/2; -; z]$.

$$\begin{aligned} &\delta_{i0}a_i(n) + \delta_{i1}a_{i+1}(n) + \delta_{i2}a_{i+2}(n) + \delta_{i3}a_{i+3}(n) + \delta_{i4}a_{i+4}(n) = 0, \\ &\quad i = n-1, n-2, \dots, 0, \end{aligned} \tag{83}$$

which is of order 4, where

$$\begin{aligned} \delta_{i0} &= (\mu + b_2 i(i-1) + c_1 i), \ \delta_{i1} = 2(i+1)(b_1 i + c_0), \\ \delta_{i2} &= 2(i+1)_2(2b_0 + b_2(2i+1) + c_1), \\ \delta_{i3} &= 4b_1(i+1)_3, \ \delta_{i4} = 4b_2(i+1)_4. \end{aligned}$$

It is to be noted here that the fourth-order recurrence relation (83) generates the coefficients $a_i(n)$ by recurring backwards with the initial conditions given by $a_{n+s}(n) = 0, \ s = 1, 2, 3,$ and $a_n(n) = 2^{-n} \times$ Leading coefficient of $\psi_n(\bar{a}x+\bar{b})$. Table 1 summarizes the $\psi_n(\bar{a}x+\bar{b})$-Hermite connection coefficients.

7.2 The $\psi_n(\bar{a}x+\bar{b})$-Generalized Laguerre Connection Problem

In this problem

$$\psi_n(\bar{a}x+\bar{b}) = \sum_{i=0}^{n} a_i(n) L_i^{(\alpha)}(x), \tag{84}$$

the coefficients $a_i(n)$ satisfy the recurrence relation

$$\begin{aligned} &\delta_{i0}a_i(n) + \delta_{i1}a_{i+1}(n) + \delta_{i2}a_{i+2}(n) + \delta_{i3}a_{i+3}(n) + \delta_{i4}a_{i+4}(n) = 0, \\ &\qquad i = n-1, n-2, \dots, 0, \end{aligned} \tag{85}$$

where

$$\begin{aligned} \delta_{i0} &= (\mu + b_2 i(i-1) + c_1 i), \ \delta_{i1} = -(b_1 i + 2b_2(2i+\alpha+2)i \\ &\quad + c_1(3i+\alpha+3) - 2\mu + c_0), \end{aligned}$$

Table 2 The $\psi_n(\bar{a}x+\bar{b})$-generalized Laguerre connection coefficients

$\psi_n(\bar{a}x+\bar{b})$	$a_i(n)$ $(0 \le i \le n)$
$H_n(\bar{a}x+\bar{b})$	$(-\bar{a})^i n! \sum_{k=0}^{[\frac{n-i}{2}]} \frac{(-1)^k 2^{n-2k} \bar{b}^{n-2k-i}}{k!(n-2k-i)!} f_{n-i-2k}(i,\alpha,\frac{-\bar{a}}{\bar{b}})$
$L_n^{(\gamma)}(\bar{a}x+\bar{b})$	$\frac{(-\bar{a})^i}{n!} \frac{(-n)_i(\gamma+1)_n}{(\gamma+1)_i} \sum_{k=0}^{n-i} \frac{\bar{b}^k}{k!} \frac{(-(n-i))_k}{(\gamma+i+1)_k} f_k(i,\alpha,\frac{-\bar{a}}{\bar{b}})$
$P_n^{(\gamma,\delta)}(\bar{a}x+\bar{b})$	$\binom{n+\gamma}{n}(\bar{a}/2)^i \frac{(-n)_i(n+\eta)_i}{(\gamma+1)_i} \sum_{k=0}^{n-i} \frac{(1-\bar{b})^k(-(n+i))_k(n+\eta+i)_k}{2^k k!(\gamma+i+1)_k} f_k(i,\alpha,\frac{\bar{a}}{1-\bar{b}})$
$Y_n^{(\gamma)}(\bar{a}x+\bar{b})$	$(\bar{a}/2)^i(-n)_i(n+\gamma+1)_i \sum_{k=0}^{n-i} \frac{(-\bar{b}/2)^k(-n+i)_k(n+\eta+i+1)_k}{k!} f_k(i,\alpha,\frac{-\bar{a}}{\bar{b}})$
	where $f_k(i,\alpha,z) = {}_2F_0[-k, i+\alpha+1; -; z]$.

$$\begin{aligned}
\delta_{i2} &= b_1(3+2i+\alpha) \\
&\quad + b_2(6(i+1)_2 + (6i+\alpha+7)\alpha) + c_1(3i+2\alpha+6) + b_0 + c_0 + \mu, \\
\delta_{i3} &= (2b_2(2i+\alpha+4) + b_1 + c_1)(i+\alpha+3), \; \delta_{i4} = b_2(i+\alpha+3)_2.
\end{aligned}$$

with $a_{n+s}(n) = 0,\ s = 1,2,3,$ and $a_n(n) = (-1)^n n! \times$ Leading coefficient of $\psi_n(\bar{a}x+\bar{b})$. Table 2 summarizes the $\psi_n(\bar{a}x+\bar{b})$-generalized Laguerre connection coefficients.

7.3 The $\psi_n(\bar{a}x+\bar{b})$-Jacobi Connection Problem

In this problem

$$\psi_n(\bar{a}x+\bar{b}) = \sum_{i=0}^{n} a_i(n) P_i^{(\alpha,\beta)}(x), \tag{86}$$

the coefficients $a_i(n)$ satisfy the recurrence relation

$$\begin{aligned}
&\delta_{i0}a_i(n) + \delta_{i1}a_{i+1}(n) + \delta_{i2}a_{i+2}(n) + \delta_{i3}a_{i+3}(n) + \delta_{i4}a_{i+4}(n) = 0, \\
&\quad i = n-1, n-2, \ldots, 0,
\end{aligned} \tag{87}$$

where

$$\begin{aligned}
\delta_{i0} &= [(2i+\lambda)_4]^{-1}(i+\lambda)_4(\mu + b_2(i-1)i + c_1 i), \\
\delta_{i1} &= [2(2i+\lambda+1)_3(2i+\lambda+5)]^{-1}(i+\lambda+1)_3 \\
&\quad \times [(\alpha-\beta)(4\mu - 2b_2(\lambda+3)i + c_1(2i-\lambda-1)) \\
&\quad + (b_1 i + c_0)(2i+\lambda+1)(2i+\lambda+5)], \\
\delta_{i2} &= \frac{1}{4}(i+\lambda+1)_3\{b_0 + [(2i+\lambda+2)_3(2i+\lambda+5)_2]^{-1} \\
&\quad \times [(-4\mu + 2c_1(\lambda+2))(2(i+2)(i+\lambda+2) + \alpha(1-\alpha) + \beta(1-\beta) + 4\alpha\beta) \\
&\quad + b_2(8i^3(i+2(\lambda+4)) + (\lambda+2)_2(6+(\alpha+\beta)+(\alpha-\beta)^2)
\end{aligned}$$

Table 3 The $\psi_n(\bar{a}x+\bar{b})$-Jacobi connection coefficients

$\psi_n(\bar{a}x+\bar{b})$	$a_i(n) \quad (0 \le i \le n)$
$H_n(\bar{a}x+\bar{b})$	$\frac{2^{n+i}(\bar{a})^i n!}{(n-i)!(i+\lambda)_i} \sum_{k=0}^{[\frac{n-i}{2}]} \frac{(-1)^k(\bar{a}+\bar{b})^{n-2k-i}(\frac{i-n}{2})_k(\frac{i-n+1}{2})_k}{k!} f_{n-i-2k}(i,\alpha,\beta,\frac{2\bar{a}}{\bar{a}+\bar{b}})$
$L_n^{(\gamma)}(\bar{a}x+\bar{b})$	$\frac{(-2\bar{a})^i}{(n-i)!}\frac{(\gamma+i+1)_{n-i}}{(i+\lambda)_i} \sum_{k=0}^{n-i} \frac{(\bar{a}+\bar{b})^k}{k!}\frac{(-(n-i))_k}{(\gamma+i+1)_k} f_k(i,\alpha,\beta,\frac{2\bar{a}}{\bar{a}+\bar{b}})$
$P_n^{(\gamma,\delta)}(\bar{a}x+\bar{b})$	$\binom{n+\gamma}{n}\frac{(-\bar{a})^i}{(i+\lambda)!}\frac{(-n)_i(n+\eta)_i}{(\gamma+1)_i} \sum_{k=0}^{n-i} \frac{(1-\bar{a}-\bar{b})^k(-n+i)_k(n+\eta+i)_k}{2^k k!(\gamma+i+1)_k} f_k(i,\alpha,\beta,\frac{2\bar{a}}{\bar{a}+\bar{b}-1})$
$Y_n^{(\gamma)}(\bar{a}x+\bar{b})$	$\frac{(-\bar{a})^i(-n)_i(n+\gamma+1)_i}{(i+\lambda)!} \sum_{k=0}^{n-i} \frac{(-1)^k(\bar{a}+\bar{b})^k(-(n-i))_k(n+\gamma+i+1)_k}{2^k k!} f_k(i,\alpha,\beta,\frac{2\bar{a}}{\bar{a}+\bar{b}})$
	where $f_k(i,\alpha,\beta,z) = {}_2F_1[-k, i+\alpha+1; 2i+\lambda+1; z]$.

$$
\begin{aligned}
&+ 4i(\lambda+4)(6\beta+2\alpha(\beta+3)+15) + 4i^2(65+2\alpha^2+2\beta(\beta+13) \\
&+ \alpha(26+6\beta))) \\
&+ (\alpha-\beta)((2i+\lambda+2)(2i+\lambda+5)(2c_0-b_1(\lambda+3))\}, \\
\delta_{i3} \;=\;& [2(2i+\lambda+3)(2i+\lambda+5)_3]^{-1}(i+\alpha+3)(i+\beta+3)(i+\lambda+3) \\
&\times [b_1(i+\lambda+4)(2i+\lambda+3)(2i+\lambda+7) - c_0(2i+\lambda+5)^2 \\
&+ 4(c_0+(\beta-\alpha)\mu) \\
&+ (\alpha-\beta)(c_1(2i+3(\lambda+3)) - 2b_2(\lambda+3)(i+\lambda+4))], \\
\delta_{i4} \;=\;& (i+\alpha+3)_2(i+\beta+3)_2((i+\lambda+4)(b_2(i+\lambda+5)-c_1)+\mu) \\
&\times [(2i+\lambda+5)_4]^{-1}, \\
\lambda \;=\;& \alpha+\beta+1,
\end{aligned}
$$

with $a_{n+s}(n) = 0,\ s = 1, 2, 3$, and $a_n(n) = \frac{2^n n!}{(n+\lambda)_n} \times$ Leading coefficient of $\psi_n(\bar{a}x+\bar{b})$. Table 3 summarizes the $\psi_n(\bar{a}x+\bar{b})$-Jacobi connection coefficients.

7.4 The $\psi_n(\bar{a}x+\bar{b})$-Bessel Connection Problem

In this problem

$$\psi_n(\bar{a}x+\bar{b}) = \sum_{i=0}^{n} a_i(n) Y_i^{(\alpha)}(x), \tag{88}$$

the coefficients $a_i(n)$ satisfy the recurrence relation

$$
\begin{aligned}
&\delta_{i0}a_i(n) + \delta_{i1}a_{i+1}(n) + \delta_{i2}a_{i+2}(n) + \delta_{i3}a_{i+3}(n) + \delta_{i4}a_{i+4}(n) = 0, \\
&\quad i = n-1, n-2, \ldots, 0,
\end{aligned} \tag{89}
$$

where

$$\delta_{i0} \;=\; [(2i+\alpha+1)_4]^{-1}(i+\alpha+1)_4(\mu+b_2(i-1)i+c_1 i),$$

Table 4 The $\psi_n(\bar{a}x+\bar{b})$-Bessel connection coefficients

$\psi_n(\bar{a}x+\bar{b})$	$a_i(n) \quad (0 \le i \le n)$
$H_n(\bar{a}x+\bar{b})$	$\frac{n!}{(i+\alpha+1)_i\, i!}\sum_{k=0}^{[\frac{n-i}{2}]}\frac{(-1)^k(4\bar{a})^{n-2k}}{k!(n-2k-i)!(2i+\alpha+2)_{n-i-2k}}\, g_{n-i-2k}(i,\alpha,-\frac{\bar{b}}{2\bar{a}})$
$L_n^{(\gamma)}(\bar{a}x+\bar{b})$	$\frac{(2\bar{a})^i}{(i+\alpha+1)_i\, i!}\,\frac{(-n)_i(\gamma+1)_n}{n!(\gamma+1)_i}\sum_{k=0}^{n-i}\frac{(-2\bar{a})^k}{k!(2i+\alpha+2)_k}\,\frac{(-(n-i))_k}{(\gamma+i+1)_k}\, g_k(i,\alpha,-\frac{\bar{b}}{2\bar{a}})$
$P_n^{(\gamma,\delta)}(\bar{a}x+\bar{b})$	$\frac{(\gamma+1)_n(-\bar{a})^i}{(i+\alpha+1)_i\, i!n!}\,\frac{(-n)_i(n+\eta)_i}{(\gamma+1)_i}\sum_{k=0}^{n-i}\frac{\bar{a}^k(-n+i)_k(n+\eta+i)_k}{k!(\gamma+i+1)_k(2i+\alpha+2)_k}\, g_k(i,\alpha,\frac{1-\bar{b}}{2\bar{a}})$
$Y_n^{(\gamma)}(\bar{a}x+\bar{b})$	$\frac{(-\bar{a})^i(-n)_i(n+\gamma+1)_i}{(i+\alpha+1)_i\, i!}\sum_{k=0}^{n-i}\frac{\bar{a}^k(-n+i)_k(n+\gamma+i+1)_k}{k!(2i+\alpha+2)_k}\, g_k(i,\alpha,-\frac{\bar{b}}{2\bar{a}})$
	where $g_k(i,\alpha,z) = {}_2F_0[-k,-2i-k-\alpha-1;-;z]$.

$$
\begin{aligned}
\delta_{i1} &= [2(2i+\alpha+2)_3(2i+\alpha+6)]^{-1}(i+\alpha+2)_3(i+1)\\
&\quad\times[8\mu-4b_2 i(\alpha+4)+2c_1(2i-\alpha-2)\\
&\quad+(b_1 i+c_0)(2i+\alpha+2)(2i+\alpha+6)],\\
\delta_{i2} &= \frac{1}{4}(i+\alpha+3)_2(i+1)_2\{b_0-2[(2i+\alpha+3)_3(2i+\alpha+6)_2]^{-1}\\
&\quad\times[-12\mu+((\alpha+4)b_1-2c_0)(2i+\alpha+3)(2i+\alpha+7)+6(\alpha+3)c_1\\
&\quad+2(2i^2-(\alpha+3)_2+2i(\alpha+5))b_2]\},\\
\delta_{i3} &= -[2(2i+\alpha+4)(2i+\alpha+6)_3]^{-1}(i+\alpha+4)(i+1)_3\\
&\quad\times[b_1(i+\alpha+5)(2i+\alpha+4)(2i+\alpha+8)-c_0(2i+\alpha+4)(2i+\alpha+8)\\
&\quad-8(\mu-3c_1)+2c_1(2i+3\alpha)-4b_2(\alpha+4)(i+\alpha+5)],\\
\delta_{i4} &= [(2i+\alpha+6)_4]^{-1}[(i+\alpha+5)(b_2(i+\alpha+6)-c_1)+\mu],
\end{aligned}
$$

with $a_{n+s}(n) = 0,\ s = 1,2,3$, and $a_n(n) = \frac{2^n}{(n+\alpha+1)_n}\times$ Leading coefficient of $\psi_n(\bar{a}x+\bar{b})$. Table 4 summarizes the $\psi_n(\bar{a}x+\bar{b})$-Bessel connection coefficients.

8 Applications

8.1 Ordinary Differential Equations with Varying Coefficients

Let $f(x)$ has the expansion (36), and assume that it satisfies the linear nonhomogeneous differential equation of order m

$$\sum_{i=0}^{m} p_i(x) f^{(i)}(x) = p(x), \tag{90}$$

where $p_0, p_1, \ldots, (p_m \neq 0)$ are polynomials in x, and the expansion coefficients of the function $p(x)$ in terms of $\phi_n(x)$ are known, then formulae (38), (58) and (73), enable us to construct in view of Eq. (90) the linear recurrence relation of order r,

$$\sum_{j=0}^{r} \alpha_j(k) a_{k+j} = \beta(k), \quad k \geq 0,$$

where $\alpha_0, \alpha_1, \ldots, \alpha_r$ ($\alpha_0 \neq 0, \alpha_r \neq 0$) are polynomials of the variable k.

An example dealing with nonhomogeneous differential equation is considered to clarify application of the results obtained.

Example 1 Consider the nonhomogeneous differential equation

$$2xf''(x) + (1+4x)f'(x) + (1+2x)f(x) = e^{-x}, \quad f(0) = 0, \; f'(0) = 1. \quad (91)$$

If $f(x)$ and e^{-x} are expanded in terms of Hermite polynomials, $H_i(x)$, in the forms

$$f(x) \sum_{i=0}^{\infty} a_i H_i(x),$$

and

$$e^{-x} = \sum_{i=0}^{\infty} \frac{e^{\frac{1}{4}}(-1)^i}{2^i i!} H_i(x),$$

then by virtue of formulae (72), Eq. (91) takes the form

$$2b_i^{2,1} + 4b_i^{1,1} + b_i^{1,0} + 2b_i^{0,1} + b_i^{0,0} = \frac{e^{\frac{1}{4}}(-1)^i}{2^i i!}, \quad i \geq 0.$$

By making use of (63) and (73) for the Hermite case, we obtain

$$\begin{aligned} &a_{i-1} + a_i + 2(1+i)a_{i+1} + 2a_i^{(1)} - 1 + a_i^{(1)} + 4(i+1)a_{i+1}^{(1)} \\ &\quad + a_{i-1}^{(2)} + 2(i+1)a_{i+1}^{(2)} = \frac{e^{\frac{1}{4}}(-1)^i}{2^i i!}, \quad i \geq 0. \end{aligned} \quad (92)$$

Using formula (48) with (92) yields

$$\begin{aligned} &a_i + (4i+5)a_{i+1} + 4(2+i)^2 a_{i+2} + 8(2+i)(3+i)a_{i+3} \\ &\quad + 8(2+i)(3+i)(4+i)a_{i+4} = \frac{e^{\frac{1}{4}}(-1)^{i+1}}{2^{i+1}(i+1)!}, \quad i \geq 0. \end{aligned} \quad (93)$$

The complete solution for Example 1 may be obtained by solving the recurrence relation (93). What is worthy noting that the analytical solution for this recurrence relation is given explicitly by

$$a_i = \frac{e^{\frac{1}{4}}(-1)^{i+1}}{2^{i+1}(i)!}(2i+1), \quad i \geq 0. \quad (94)$$

Analytical solution like (94), is not generally easy to obtain. The alternative approach for solving (93) can be obtained using the modification of Miller's recurrence algorithm, see [43,44].

8.2 The Integrated System of Ordinary Differential Equations with Polynomial Coefficients

Let $f(x)$ has the expansion (36), and assume that it satisfies the linear nonhomogeneous differential equation (90). The integration of Eq. (90) m times with respect to x, gives

$$\sum_{i=0}^{m} D^{-m}[p_i(x) f^{(i)}(x)] = D^{-m} p(x) + \sum_{i=0}^{m-1} e_i \phi_i(x), \tag{95}$$

where $e_0,\ e_1,\ \ldots,\ e_{m-1}$ are constants of integration. It can be easily shown that

$$D^{-m}[p_i(x) f^{(i)}(x)] = \sum_{j=0}^{i} (-1)^i \binom{i}{j} D^{-(j+m-i)}[p_i^{(j)} f(x)], \quad i = 0, 1, \ldots, m. \tag{96}$$

Substitution of (96) into (95) and collecting similar terms containing the same number of repeated integrations yield

$$\sum_{i=0}^{m} \sum_{j=i}^{m} (-1)^{j-i} \binom{j}{i} D^{-(m-i)}[p_j^{(j-i)} f(x)] = D^{-m} p(x) + \sum_{i=0}^{m-1} e_i \phi_i(x). \tag{97}$$

Equation (97) may be written in the form

$$\sum_{i=0}^{m} D^{-(m-i)}[Q_i(x) f(x)] = D^{-m} p(x) + \sum_{i=0}^{m-1} e_i \phi_i(x), \tag{98}$$

where

$$Q_i(x) = \sum_{j=i}^{m} (-1)^{j-i} \binom{j}{i} p_j^{(j-i)}, \quad i \geq 0.$$

If the expansion (36) is substituted into (98), and a linearization of $Q_i(x)\, \phi_n(x)$ as a linear combination of suitable $\phi_n(x)$ is made, and if $D^{-m} p(x)$ is expanded into a series of $\phi_n(x)$, then making use of (20) enables us to obtain a recurrence relation for the expansion coefficients a_n of the form

$$\sum_{j=0}^{s} \tilde{\alpha}_j(k) a_{k+j} = \tilde{\beta}(k), \quad k \geq 0. \tag{99}$$

where $\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ $(\tilde{\alpha}_0 \neq 0, \tilde{\alpha}_s \neq 0)$ are polynomials of the variable k.

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Appendix A

See Tables 5 and 6.

Table 5 Basic data of four families of classical orthogonal polynomials

$y_n(x)$	Hermite	Modified Laguerre	Jacobi	Bessel
I	$(-\infty,\infty)$	$(0,\infty)$	$(-1,1)$	$\{x:\lvert x\rvert=1\}$
$\sigma(x)$	1	x	$1-x^2$	x^2
$\tau(x)$	$-2x$	$1+\alpha-x$	$\beta-\alpha-(\alpha+\beta+2)x$	$(\alpha+2)x+2$
$\rho(x)$	e^{-x^2}	$x^\alpha e^{-x}$	$(1-x)^\alpha(1+x)^\beta$	$\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+2)}{\Gamma(k+\alpha+1)}\left(\frac{-2}{x}\right)^k$
α_n	$\frac{1}{2}$	$-(n+1)$	$\frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)_2}$	$\frac{2(n+\alpha+1)}{(2n+\alpha+1)_2}$
β_n	0	$2n+\alpha+1$	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$	$\frac{-2\alpha}{(2n+\alpha)(2n+\alpha+2)}$
γ_n	n	$-(n+\alpha)$	$\frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)_2}$	$\frac{-2n}{(2n+\alpha)_2}$
$\bar{\alpha}_n$	$\frac{1}{2(n+1)}$	-1	$\frac{2(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)_2}$	$\frac{2(n+\alpha+1)}{(n+1)(2n+\alpha+1)_2}$
$\bar{\beta}_n$	0	1	$\frac{2(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$	$\frac{4}{(2n+\alpha)(2n+\alpha+2)}$
$\bar{\gamma}_n$	0	0	$\frac{-2(n+\alpha)(n+\beta)}{(n+\alpha+\beta)(2n+\alpha+\beta)_2}$	$\frac{2n}{(n+\alpha)(2n+\alpha)_2}$

Table 6 Basic data of four families of classical orthogonal polynomials

$\phi_n(x)$	Hypergeometric representation
Hermite	$H_n(x)=(2x)^n{}_2F_0[-n/2;-(n-1)/2;-1/x^2]$.
Modified Laguerre	$L_n^{(\alpha)}(x)=\frac{(\alpha+1)_n}{n!}{}_1F_1[-n;\alpha+1;x],\quad \alpha>-1$.
Jacobi	$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_n}{n!}{}_2F_1[-n;n+\lambda;\alpha+1;(1-x)/2],\quad (\alpha,\beta>-1)$, where $\lambda=\alpha+\beta+1$, and $(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol.
Ultraspherical	It is convenient to weight the ultraspherical polynomials so that $C_n^{(\alpha)}(x\frac{n!}{(\alpha+1/2)_2}P_n^{(\alpha-1/2,\alpha-1/2)}(x)$.
Legendre	$P_n(x)=C_n^{(1/2)}(x)$.
Chebyshev of first kind	$T_n(x)=C_n^{(0)}(x)$.
Chebyshev of second kind	$U_n(x)=(n+1)C_n^{(1)}(x)$.
Bessel	$Y_n^{(\alpha)}(x)={}_2F_0[-n;n+\alpha+1;-x/2]$, where $x\neq0$ and $\alpha\neq-2,\ -3,4,\ldots,$ are complex numbers.

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