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ORIGINAL PAPER

Notes on Gorenstein Flat Modules

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Abstract
In this paper, we explore conditions under which Gorenstein flat modules are Gorenstein projective. We prove that all countably presented strongly Gorenstein flat modules are Gorenstein projective over perfect rings. Moreover, we show that if the base ring R is $\sum$-pure injective as an R-module, then the class of Gorenstein flat modules coincides with the class of Gorenstein projective modules, and hence all modules have Gorenstein projective covers. And as a corollary, we give a characterization of coherent perfect rings by Gorenstein projective and Gorenstein flat modules.

Keywords Strict Mittag–Leffler module · Gorenstein projective module · Gorenstein flat module

Mathematics Subject Classification Primary 16D40; Secondary 16E05 · 16E30

1 Introduction

As generalizations of projective and flat modules, Enochs and Jenda [6,8] extended Auslander and Bridger's ideas [2] and introduced Gorenstein projective and Gorenstein flat modules. These modules have many similar properties as projective and flat modules have, especially when the ring is Gorenstein (see [7]). But at the same time, some properties of Gorenstein projective and Gorenstein flat modules diverge from those of their homological algebra counterparts. For example, projective modules are always flat, but it is not clear whether all Gorenstein projective modules are Gorenstein flat. Similarly, it is known that finitely presented flat modules are projective, but only

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over coherent rings, do we know that all finitely presented Gorenstein flat modules are Gorenstein projective [3, Proposition 1.3], and whether it is true or not in general case still remains open. Moreover, it is well known that all flat modules are projective if and only if all modules have projective covers if and only if the base ring is perfect, however, it is not clear whether there is a suitable counterpart for Gorenstein projective and Gorenstein flat modules.

In this paper, we are mainly interested in the question when Gorenstein flat modules are Gorenstein projective. To this end, we investigate the relations between Gorenstein projective modules and Gorenstein flat modules by employing strict Mittag–Leffler conditions and modules. First, we prove that if the base ring is perfect, then all countably presented strongly Gorenstein flat modules are Gorenstein projective, which is an extention of Bennis and Mahdou's result in [3] over perfect rings. In fact, we obtain that if the base ring is perfect, a countably presented module is strongly Gorenstein projective if and only if it is strongly Gorenstein flat. Second, we show that if the base ring R is $\sum$-pure injective as an R-module, then the class of all Gorenstein flat modules coincides with the class of all Gorenstein projective modules, and hence all modules have Gorenstein projective covers. Finally, as a corollary of our main result, we give a characterization of perfect rings by Gorenstein projective and Gorenstein flat modules under the assumption that the base ring is coherent.

Throughout this paper, R is an associative ring with an identity. All modules are left R-modules unless stated otherwise. Denote by $\mathcal{P}$ and $\mathcal{F}$ the classes of projective and flat R-modules, respectively. The category of all left R-modules is denoted by R-Mod. For an R-module M, we denote by $M^+ = \mathrm{Hom}_Z(M, Q/Z)$ its character module. We assume that all direct and inverse systems are indexed by directed sets.

2 Preliminary Notions

To examine the exactness of the inverse limit functor, Grothendieck in [10] introduced the Mittag–Leffler condition for countable inverse systems. Mittag–Leffler conditions and modules were thorough and systematically studied in [1]. In the past few years, (strict) Mittag–Leffler conditions and modules were employed to solve kinds of problems in the homological algebra and the representation theory. For the definitions of (strict) Mittag–Leffler conditions, the readers are referred to [1]. Here, we presented some known facts needed in the sequel.

The proof of the following theorem is due to Angeleri Hügel and Herbera [1, Theorem 8.11]. Emmanouil gave a different proof in [4, Theorem 1.3].

Theorem 2.1 *Let M and N be R-modules. The following statements are equivalent:*

(1) *There is a direct system of finitely presented modules* $(F_\alpha, u_{\beta\alpha})_{\alpha,\beta\in I}$ *with* $M = \varinjlim F_\alpha$*, such that the inverse system*

$$(\mathrm{Hom}_R(F_\alpha, N), \mathrm{Hom}_R(u_{\beta\alpha}, N))_{\alpha,\beta\in I}$$

satisfies the strict Mittag–Leffler condition.

(2) *For any divisible abelian group D, the natural transformation*

$$\Phi : \mathrm{Hom}_Z(N, D) \otimes_R M \longrightarrow \mathrm{Hom}_Z(\mathrm{Hom}_R(M, N), D)$$

defined by $\Phi(f \otimes m) : g \mapsto f(g(m))$, $f \in \mathrm{Hom}_Z(N, D), m \in M$ *and* $g \in \mathrm{Hom}_R(M, N)$, *is a monomorphism.*

If the equivalent conditions in the above theorem are satisfied, M is said to be a strict N-stationary module in [1] or a strict Mittag–Leffler module over N in [4]. Following Emmanouil's symbols in [4], we denote by $\mathrm{SML}(N)$ the class of strict N-stationary modules. Let $\mathcal{N}$ be a class of modules. If, for any $N \in \mathcal{N}$, M is a strict N-stationary module, we say that M is a strict $\mathcal{N}$-stationary module and denote by $M \in \mathrm{SML}(\mathcal{N})$. M is called a strict Mittag–Leffler module if $\mathcal{N} = R$-Mod.

Remark 2.2 (1) If the index set I is countable, then an inverse system satisfies the Mittag–Leffler condition if and only if it satisfies the strict Mittag–Leffler condition, see [1, Lemma 3.3].

(2) It is easy to see that the class $\mathrm{SML}(N)$ is closed under direct sums and direct summands. Furthermore, $\mathrm{SML}(N)$ is closed under pure submodules. Note that the natural transformation in Theorem 2.1(2) is an isomorphism if M is finitely presented. Thus, $\mathcal{P} \subseteq \mathrm{SML}(R\text{-Mod})$.

(3) We denote by $\mathrm{Add}(N)$ the class of all direct summands of arbitrary direct sums of copies of N. If $M \in \mathrm{SML}(N)$, then $M \in \mathrm{SML}(\mathrm{Add}(N))$ by [1, Corollary 8.5], this implies that $M \in \mathrm{SML}(N)$ if and only if $M \in \mathrm{SML}(\mathrm{Add}(N))$.

Suppose that $\cdots \to P_1 \to P_0 \to M \to 0$ is a projective resolution of an R-module M, then the natural transformation Φ in Theorem 2.1(2) induces another natural transformation

$$\Phi_M^{(n)} : \mathrm{Tor}_n^R(\mathrm{Hom}_Z(N, D), M) \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}_R^n(M, N), D).$$

It is not difficult to verify the following result using Theorem 2.1.

Proposition 2.3 [4, Proposition1.5] *The followings are equivalent for the R-modules M and N:*

(1) *The map*

$$\Phi_M^{(n)} : \mathrm{Tor}_n^R(\mathrm{Hom}_Z(N, D), M) \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}_R^n(M, N), D)$$

is monomorphic for any divisible abelian group D.

(2) *The n-th syzygy module $\Omega^n(M)$ of M is a strict N-stationary module, i.e.,* $\Omega^n(M) \in \mathrm{SML}(N)$.

3 Main Results

In this section, we explore conditions under which Gorenstein flat modules are Gorenstein projective. Following [6–8], an R-module M is called Gorenstein projective, if there exists an exact sequence

$$\mathbb{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

of projective modules with $M = \ker(P_0 \to P_{-1})$ such that $\mathrm{Hom}_R(-, P)$ leaves the sequence exact whenever P is projective. The complex $\mathbb{P}$ is called a complete projective resolution. Analogously, an R-module N is called Gorenstein flat, if there exists an exact sequence

$$\mathbb{F} = \cdots \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \cdots$$

of flat modules with $N = \ker(F_0 \to F_{-1})$ such that $I \otimes_R -$ leaves the sequence exact whenever I is an injective right R-module. The complex $\mathbb{F}$ is called a complete flat resolution.

Bennis and Mahdou introduced in [3] the notion of strongly Gorenstein projective , flat modules, which situate between projective, flat modules and Gorenstein projective, flat modules, respectively. They called an R-module M strongly Gorenstein projective if there exists an exact sequence $\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$ of projective modules with $M \cong \mathrm{Ker}(f)$ such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is projective. Strongly Gorenstein flat modules are defined similarly.

We denote by $\mathcal{GP}$, $\mathcal{GF}$, $\mathcal{SGP}$, and $\mathcal{SGF}$ the classes of Gorenstein projective, Gorenstein flat, strongly Gorenstein projective and strongly Gorenstein flat modules, respectively.

The following lemma will be useful in the sequel.

Lemma 3.1 *Let* $0 \to A_i \to B_i \to C_i \to 0$ $(i \in \mathbb{N})$ *be a countable inverse system of short exact sequences. If the inverse system* $(A_i)_{i \in \mathbb{N}}$*satisfies the Mittag–Leffler condition, then*

$$0 \to \varprojlim A_i \to \varprojlim \to B_i \to \varprojlim C_i \to 0.$$

is exact.

Proof See [11, Lemma 5.2.11]. □

Recall that an R-module is said to be countably presented if it is the cokernel of a homomorphism between two countably generated free modules. It is known that any countably presented module can be expressed as the direct limit of a countable direct system of finitely presented modules. For countably presented Gorenstein flat modules, we have the following result.

Theorem 3.2 *Given a complete flat resolution* $\mathbb{F}$ *with* $M_n = \ker(F_n \to F_{n-1})$ *countably presented for each* $n \in \mathbb{Z}$*, then the following statements are equivalent.*

(1) $\mathbb{F}$ *is a complete projective resolution.*
(2) M_n *is a strict* R*-stationary module, i.e.,* $M_n \in \mathrm{SML}(R)$ *for any* $n \in \mathbb{Z}$*.*

Proof (1) ⇒ (2) Since $\mathbb{F}$ is a complete projective resolution, we get that each M_n is Gorenstein projective and can be viewed as the first syzygy module of M_{n-1}.Note that for any divisible abelian group D, $\mathrm{Tor}_1^R(\mathrm{Hom}_Z(R, D), M_{n-1}) = 0$. This implies that the morphism $\Phi^{(1)}_{M_{n-1}} : \mathrm{Tor}_1^R(\mathrm{Hom}_Z(R, D), M_{n-1}) \to \mathrm{Hom}_Z(\mathrm{Ext}^1_R(M_{n-1}, R), D)$ is monomorphic. By Proposition 2.3, $M_n = \Omega^1(M_{n-1})$ is a strict R-stationary module.

(2) ⇒ (1) Let M be a countably presented Gorenstein flat module and P a projective module. If $M \in \mathrm{SML}(R)$, then we will show that $\mathrm{Ext}^1_R(M, P) = 0$. In fact, since M is countably presented, there is a countable direct system

$$E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3 \to \cdots \to E_n \xrightarrow{f_n} E_{n+1} \to \cdots$$

of finitely presented modules such that $M = \varinjlim E_n$. Applying the functor $\mathrm{Hom}_R(E_n, -)$ to the pure exact sequence $0 \to P \to P^{++} \to P^{++}/P \to 0$, we get an inverse system of exact sequences

$$0 \to \mathrm{Hom}_R(E_n, P) \to \mathrm{Hom}_R(E_n, P^{++}) \to \mathrm{Hom}_R(E_n, P^{++}/P) \to 0.$$

Note that $M \in \mathrm{SML}(R)$ if and only if $M \in \mathrm{SML}(\mathcal{P})$, and so the inverse system

$$(\mathrm{Hom}_R(E_n, P), \mathrm{Hom}_R(f_n, P))$$

satisfies the strict Mittag–Leffler condition. Thus, we get an exact sequence

$$0 \to \varprojlim \mathrm{Hom}_R(E_n, P) \to \varprojlim \mathrm{Hom}_R(E_n, P^{++}) \to \varprojlim \mathrm{Hom}_R(E_n, P^{++}/P) \to 0.$$

by Lemma 3.1. And hence the sequence

$$0 \to \mathrm{Hom}_R(\varinjlim E_n, P) \to \mathrm{Hom}_R(\varinjlim E_n, P^{++}) \to \mathrm{Hom}_R(\varinjlim E_n, P^{++}/P) \to 0$$

is exact. This gives the exactness of the following sequence,

$$0 \to \mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(M, P^{++}) \to \mathrm{Hom}_R(M, P^{++}/P) \to 0$$

Thus, we get that $0 \to \mathrm{Ext}^1_R(M, P) \to \mathrm{Ext}^1_R(M, P^{++}) \cong \mathrm{Tor}_1^R(P^+, M)^+ = 0$, so $\mathrm{Ext}^1_R(M, P) = 0$. Therefore, we have that $\mathrm{Ext}^1_R(M_n, P) = 0$ for any $n \in \mathbb{Z}$. This proves that $\mathbb{F}$ remains exact by applying $\mathrm{Hom}_R(-, P)$.

Now we prove that each term of $\mathbb{F}$ is projective. Since $M_n = \ker(F_n \to F_{n-1})$ is countably presented for each $n \in \mathbb{Z}$, the exact sequence $0 \to M_n \to F_n \to M_{n-1} \to 0$ gives that F_n is countably generated flat. Note that the exact sequence $0 \to M_n \to F_n \to M_{n-1} \to 0$ remains exact by applying $\mathrm{Hom}_R(-, R)$ and $\mathrm{Hom}_Z(R, D) \otimes -$ for any divisible abelian group D, so we obtain the following commutative diagram:

$$\begin{array}{ccccc}
0 \to \mathrm{Hom}_Z(R, D) \otimes_R M_n & \longrightarrow & \mathrm{Hom}_Z(R, D) \otimes_R F_n & \longrightarrow & \mathrm{Hom}_Z(R, D) \otimes_R M_{n-1} \\
\downarrow \Phi_{M_n} & & \downarrow \Phi_{F_n} & & \downarrow \Phi_{M_{n-1}} \\
0 \to \mathrm{Hom}_Z(\mathrm{Hom}_R(M_n, R), D) & \longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(F_n, R), D) & \longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(M_{n-1}, R), D).
\end{array}$$

Since M_n and M_{n-1} are both strict R-stationary, we get that Φ_{M_n} and $\Phi_{M_{n-1}}$ are both monomorphic by Theorem 2.1. Thus, Φ_{F_n} is monomorphic by diagram chasing, and hence F_n is a strict R-stationary module. Following [1, Corollary 5.5], we have that F_n is projective. So each term of $\mathbb{F}$ is projective. Therefore, $\mathbb{F}$ is a complete projective resolution. □

Remark 3.3 (1) Note that the assumption that each M_n is countably presented is not necessary in the proof of (1) ⇒ (2) in the above theorem. In fact, if the complete flat resolution $\mathbb{F}$ with $M_n = \ker(F_n \to F_{n-1}), n \in \mathbb{Z}$, is also a complete projective resolution, then each M_n is a strict R-stationary module.

(2) Note that every flat module is a pure quotient of a suitable projective module, so [1, Corollary 8.5] gives that $M_n \in \mathrm{SML}(R)$ if and only if $M_n \in \mathrm{SML}(\mathcal{P})$ if and only if $M_n \in \mathrm{SML}(\mathcal{F})$. This shows that if each M_n satisfies the conditions in the above theorem, then the complex $\mathbb{F}$ remains exact by applying $\mathrm{Hom}_R(-, F)$ for any $F \in \mathcal{F}$.

Let R be a left perfect ring and M a Gorenstein flat module, then there is an exact sequence $0 \to M \to P \to N \to 0$ with P projective and N Gorenstein flat. Note that for any divisible abelian group D, $\mathrm{Tor}_R^1(\mathrm{Hom}_Z(R, D), N) = 0$, it follows that $M = \Omega^1(N)$ is a strict R-stationary module using Proposition 2.3. This shows that all Gorenstein flat modules are strict R-stationary modules over perfect rings. The following result is an immediate consequence of Theorem 3.2.

Corollary 3.4 *Let R be a left perfect ring, Then every countably presented strongly Gorenstein flat module is Gorenstein projective.*

We proved in [14] that countably presented strongly Gorenstein projective modules are strongly Gorenstein flat. This shows that if the base ring R is left perfect, then a countably presented module is strongly Gorenstein flat if and only if it is strongly Gorenstein projective.

We now give conditions under which all Gorenstein flat modules are Gorenstein projective.

Theorem 3.5 *If the ring R is $\sum$-pure injective as a left R-module, then $\mathcal{GP} = \mathcal{GF}$.*

Proof If R is $\sum$-pure injective as an R-module, then R-Mod $= \mathrm{SML}(R)$ by [15, Theorem 3.8]. Thus, all Gorenstein projective modules are strict R-stationary modules. Therefore, $\mathcal{GP} \subseteq \mathcal{GF}$ by [5, Theorem 2.2].

Now we prove that $\mathcal{GF} \subseteq \mathcal{GP}$. Let $\mathbb{F} = \cdots \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \cdots$ be a complete flat resolution with $M_n = \ker(F_n \to F_{n-1}), n \in \mathbb{Z}$, then for any projective left R-module P, we have that $\mathrm{Ext}_R^1(M_n, P^{++}) \cong \mathrm{Tor}_1^R(P^+, M_n)^+ = 0$. Note that every projective R-module is a direct summand of $R^{(I)}$ for some index set I. Thus, all projective modules are pure injective. This implies that the pure embedding $0 \to P \to P^{++}$ is splitting, and hence $\mathrm{Ext}_R^1(M_n, P) = 0$. So the complex $\mathbb{F}$ remains exact by applying $\mathrm{Hom}_R(-, P)$. By the assumption on the base ring, we have that $\mathcal{GF} \subseteq \mathrm{SML}(R)$, and so R is left perfect by [13, Lemma 4.5]. Thus, $\mathbb{F}$ is a complete projective resolution. Therefore, we have that $\mathcal{GF} \subseteq \mathcal{GP}$. □

It is known that all modules have Gorenstein flat precovers [12], and we have proven in [13, Theorem 4.7] that if R is $\sum$-pure injective, then $\mathcal{GP}$ is closed under direct limits. So Theorem 3.5 gives the following result.

Corollary 3.6 *If R is $\sum$-pure injective as a left R-module, then all modules have Gorenstein projective covers .*

It is well known that a ring R is perfect if and only if all flat modules are projective. Assume further that R is coherent, we can extend this characterization of perfect rings to the Gorenstein case.

Corollary 3.7 *Let R be a right coherent ring, then the following two statements are equivalent:*

(1) *R is a left perfect ring.*
(2) $\mathcal{GP} = \mathcal{GF}$.

Proof (1) $\Rightarrow$ (2). Let R be right coherent and left perfect, then $\mathcal{P}(R)$ is closed under products. and so R is $\sum$-pure injective by [11, Lemma 1.2.23]. Theorem 3.5 applies.

(2) $\Rightarrow$ (1) By [5, Theorem 2.2], $\mathcal{GP} = \mathcal{GF}$ yields that all Gorenstein flat modules are contained in $\mathrm{SML}(R)$, and so R is left perfect by [13, Lemma 4.5]. □

At the end of this paper, we follow the idea of corollary 3.7, investigating the relations between Gorenstein flat and Gorenstein projective modules over coherent rings. Recall that an R-module M is called a cotorsion module if $\mathrm{Ext}^1_R(F, M) = 0$ for any flat module F. The cotorsion dimension of M is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}_R(F, M) = 0$ for any flat module F. The following result generalizes the above corollary to some extent.

Proposition 3.8 *If R is a right coherent ring such that all flat modules have finite cotorsion dimensions, then every complete flat resolution of projective modules is a complete projective resolution.*

Proof Given a flat left R-module F, we assume that F has cotorsion dimension $\leq n$. Note that all pure injective modules are cotorsion modules, we consider a pure injective resolution $0 \to F \to F_0^{++} \to F_1^{++} \to \cdots \to F_n^{++} \to F_{n+1}^{++} \to \cdots$ where $F_0 = F$ and $F_{n+1} = F_n^{++}/F_n$ for each $n \geq 0$. It is easy to see that each term of the above sequence is flat, and hence each F_n is flat, since R is right coherent and the above sequence is pure exact. For any flat module F', by dimension shifting, we get that $\mathrm{Ext}^1_R(F', F_n) \cong \mathrm{Ext}^{n+1}_R(F', F) = 0$, so $\mathrm{Ext}^1_R(F_{n+1}, F_n) = 0$. This implies that the short exact sequence $0 \to F_n \to F_n^{++} \to F_{n+1} \to 0$ splits. Thus, we obtain a long exact sequence $0 \to F \to F_0^{++} \to F_1^{++} \to \cdots \to F_{n-1}^{++} \to F_n \to 0$ where F_n is a direct summand of F_n^{++}. Note that for any Gorenstein flat left R-module M, $\mathrm{Ext}^i_R(M, F_j^{++}) \cong (\mathrm{Tor}^R_i(F_j^+, M))^+ = 0$ for any $i \geq 1$ and $j \geq 0$. So $\mathrm{Ext}^i_R(M, F_n) = 0$ for any $i \geq 1$. This implies that $\mathrm{Ext}^{n+i}_R(M, F) \cong \mathrm{Ext}^i_R(M, F_n) = 0$.

Let $\mathbb{F}$ be a complete flat resolution of projective modules. We assume that N is a Gorenstein flat module which appears as the kernel of $\mathbb{F}$. Then we conclude that $\mathrm{Ext}^i_R(N, F) = 0$ for any $F \in \mathcal{F}$ and each $i \geq 1$. Therefore, $\mathrm{Hom}_R(\mathbb{F}, F)$ is exact, and $\mathbb{F}$ is a complete projective resolution. □

Remark 3.9 (1) Let R be a right coherent and left perfect ring, then all flat modules are projective, and hence all modules are cotorsion modules. Thus, the conditions in Proposition 3.8 are satisfied and therefore $\mathcal{GF} \subseteq \mathcal{GP}$.

(2) Recall that a ring R is called left n-perfect if every flat module has projective dimension $\leq n$. Note that over a left n-perfect ring, all modules have cotorsion dimension $\leq n$. Let R be right coherent and left n-perfect, then the hypothesis of Proposition 3.8 is clearly satisfied. Therefore, a complete flat resolution consisting of projective modules is a complete projective resolution. This has been proven by Iacob in [9, Corollary 1].

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