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ORIGINAL PAPER

Commuting Mappings on the Hochschild Extension of an Algebra

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Abstract

In this paper, we will describe the general form of commuting mappings of Hochschild extension algebras and characterize the properness of commuting mappings on a special class of Hochschild extension algebras with the so-called p.

Keywords Commuting mapping · Hochschild extension · Triangular algebra

Mathematics Subject Classification Primary 15A78; Secondary 16W25 · 46H40

1 Introduction

Let R be a commutative ring with identity and A be a unital algebra over R. We denote $\mathcal{Z}(A)$ the center of A. Throughout this paper we shall write $[x, y]$ for the commutator $xy - yx$ of $x, y \in A$. Recall that an R-linear mapping $\theta : A \to A$ is said to be *commuting* if $[\theta(x), x] = 0$ for all $x \in A$. A commuting mapping θ of A is called *proper* if it is of the form

$$\theta(x) = \lambda x + \mu(x), \quad \forall x \in A,$$

where $\lambda \in \mathcal{Z}(A)$ and μ is an R-linear mapping from A into $\mathcal{Z}(A)$. The purpose of this paper is to identify a class of algebras on which every commuting mapping is proper.

To the best of our knowledge, the first important result on commuting mappings is the Posner's theorem [17] which says that the existence of a nonzero commut-

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ing derivation on a prime ring A implies that A is commutative. An analog of the Posner's theorem for automorphisms [13] states that if θ is a commuting automorphism on a noncommutative prime ring, then $\theta = \mathrm{Id}$. Commuting derivations and commuting automorphisms and their generalizations are successfully used in the field of *automatic continuity* [7,10,14]. In this direction, abundant results related to the noncommutative Singer–Wermer conjecture have been obtained. For example, every commuting derivation of a Banach algebra A has its range in the Jacobson radical of A [14]. An account on commuting mappings in C^*-algebras can be found in the book [1]. The extensive applications of commuting mappings in the field of analysis is one of the main motivations of our present work.

Another motivation of this paper is the connection between commuting mappings and Lie-type isomorphisms, which was initiated by Brešar [4,5] when he studied the Herstein's conjecture of Lie isomorphisms on prime rings. We encourage the reader to read the well-written survey paper [6] for a much more detailed understanding of this topic. It was Cheung [8,9] who initiated the study of additive commuting mappings on triangular algebras. He determined a class of triangular algebras, containing Hilbert space nest algebras (a type of non-selfadjoint, non-semiprime operator algebras), on which every additive commuting mapping is proper. Following the ideas of Brešar and Cheung, Benkovič-Eremita [2] and Xiao-Wei-Fošner [20] studied the Lie-type isomorphisms of triangular algebras.

In 2010, Xiao and Wei [18] identified a class of Morita context rings, named generalized matrix algebras, which contains the triangular algebras defined by Cheung in [9]. They studied the commuting linear mappings [18], the commuting traces of bilinear mappings and Lie isomorphisms [19]. In this paper, we shall identify another generalization of the triangular algebras and study the commuting linear mappings on such algebras.

Let R be a commutative ring with identity. Throughout, all the algebras and modules are assumed to be defined over R. Let A be a unital algebra over R and E be an A-bimodule. Recall that a *Hochschild 2-cocycle* is a bilinear mapping $T : A \times A \to E$ satisfying

$$aT(b, c) + T(a, bc) = T(ab, c) + T(a, b)c$$

for all $a, b, c \in A$. The R-linear space $A \oplus E$ can be equipped with an associative operation for a given Hochschild 2-cocycle as

$$\begin{bmatrix} a \\ x \end{bmatrix} \begin{bmatrix} b \\ y \end{bmatrix} := \begin{bmatrix} ab \\ ay + xb + T(a, b) \end{bmatrix}$$

for all $a, b \in A$ and $x, y \in E$. Note that we write the elements of $A \oplus E$ as column vectors, not as the usual row vectors, which is more helpful for our calculation. Then $A \oplus E$ forms an R-algebra under the vector addition and the multiplication just defined. We call this algebra the *Hochschild extension* of A from E by the 2-cocycle T, denoted by H_T. We refer the reader to the book [10] for more information on Hochschild cohomology and its applications in the field of automatic continuity, which motivated this paper and its subsequent work.

For the Hochschild extension H_T, if $T = 0$, then H_T degenerates to the trivial extension algebra. Several researchers have studied the R-linear mappings on the trivial extension algebras both in algebra [3,11,12,16] and in analysis [15,21]. Our aim is to obtain sufficient conditions on the Hochschild extension H_T so that every commuting mapping of H_T is proper.

2 Structure of Commuting Mappings

Let A be a unital algebra over R and E be an A-bimodule. For a given Hochschild 2-cocycle $T : A \times A \to E$, H_T is the Hochschild extension of A from E by the 2-cocycle T. We denote $\mathcal{Z}(H_T)$ the center of H_T. A direct computation shows

Lemma 2.1 *The center of H_T is*

$$\mathcal{Z}(H_T) = \left\{ \begin{bmatrix} a_0 \\ x_0 \end{bmatrix} : a_0 \in \mathcal{Z}(A), [a_0, x] = 0, \right.$$
$$\left. [a, x_0] = T(a_0, a) - T(a, a_0), \forall\, a \in A,\ x \in E \right\}.$$

We say that E is faithful as an A-bimodule if is is faithful as a left A-module and also as a right A-module.

Corollary 2.2 *If E is faithful as A-bimodule, then the center of H_T is*

$$\mathcal{Z}(H_T) = \left\{ \begin{bmatrix} a_0 \\ x_0 \end{bmatrix} : [a_0, x] = 0, [a, x_0] = T(a_0, a) - T(a, a_0), \forall\, a \in A,\ x \in E \right\}.$$

Proof It is sufficient to show that $a_0 \in \mathcal{Z}(A)$ if $[a_0, x] = 0$ for all $x \in E$. Indeed, for any $a \in A$ we get

$$(a_0 a - a a_0)x = a_0(ax) - a(a_0 x) = (ax)a_0 - a(x a_0) = (ax)a_0 - (ax)a_0 = 0.$$

The assumption that E is faithful as A-bimodule leads to $a_0 a - a a_0 = 0$ and hence $a_0 \in \mathcal{Z}(A)$. □

Corollary 2.3 *Suppose that E is faithful as A-bimodule and T is symmetric, then the center of H_T is*

$$\mathcal{Z}(H_T) = \left\{ \begin{bmatrix} a_0 \\ x_0 \end{bmatrix} : [a_0, x] = 0, [a, x_0] = 0, \forall\, a \in A,\ x \in E \right\}.$$

Let us recall that the natural projection $\pi_A : H_T \to A$ by

$$\pi_A : \begin{bmatrix} a \\ x \end{bmatrix} \longmapsto a.$$

Now, we immediately give the structure of commuting maps on $H_T = A \oplus E$.

Theorem 2.4 *Let θ be a commuting mapping of $H_T = A \oplus E$. Then θ can be presented as*

$$\theta\left(\begin{bmatrix} a \\ x \end{bmatrix}\right) = \begin{bmatrix} \mu_1(a) + \mu_2(x) \\ \nu_1(a) + \nu_2(x) \end{bmatrix} (a \in A, x \in E)\,, \tag{2.1}$$

where $\mu_1 : A \longrightarrow A$, $\mu_2 : E \longrightarrow \mathcal{Z}(A)$, $\nu_1 : A \longrightarrow E$, $\nu_2 : E \longrightarrow E$ are all R-linear mappings satisfying the following conditions:

(1) *μ_1 is a commuting mapping of A;*
(2) $[\mu_2(x), x] = 0$;
(3) $[\nu_1(a), a] = T(a, \mu_1(a)) - T(\mu_1(a), a)$;
(4) $[\nu_2(x), a] - [x, \mu_1(a)] = T(a, \mu_2(x)) - T(\mu_2(x), a)$ *for all $a \in A,\ x \in E$.*

Proof Assume that the linear map θ is of the form (2.1) where μ_1, μ_2 are linear mappings from A, E to A, respectively; ν_1, ν_2 are linear mappings from A, E to E, respectively. If θ is commuting, then

$$[\theta(X), X] = 0 \tag{2.2}$$

for all $X \in H_T$.

Taking $X = \begin{bmatrix} 0 \\ x \end{bmatrix}$ into (2.2) leads to

$$0 = \begin{bmatrix} 0 \\ \mu_2(x)x \end{bmatrix} - \begin{bmatrix} 0 \\ x\mu_2(x) \end{bmatrix}.$$

Thus,

$$[\mu_2(x), x] = 0$$

for all $x \in E$.

Similarly, conditions (1) and (3) follow from

$$\begin{aligned} 0 &= \left[\theta\left(\begin{bmatrix} a \\ 0 \end{bmatrix}\right), \begin{bmatrix} a \\ 0 \end{bmatrix}\right] \\ &= \begin{bmatrix} \mu_1(a)a \\ \nu_1(a)a + T(\mu_1(a), a) \end{bmatrix} - \begin{bmatrix} a\mu_1(a) \\ a\nu_1(a) + T(a, \mu_1(a)) \end{bmatrix} \end{aligned}$$

for all $a \in A$.

For a commuting mapping θ, we have

$$[\theta(X), Y] = [X, \theta(Y)]. \tag{2.3}$$

Choosing $X = \begin{bmatrix} 0 \\ x \end{bmatrix}$ and $Y = \begin{bmatrix} a \\ 0 \end{bmatrix}$ in (2.3), we compute that

$$[\theta(X), Y] = \begin{bmatrix} \mu_2(x)a - a\mu_2(x) \\ \nu_2(x)a - a\nu_2(x) + T(\mu_2(x), a) - T(a, \mu_2(x)) \end{bmatrix}$$

and

$$[X, \theta(Y)] = \begin{bmatrix} 0 \\ x\mu_1(a) - \mu_1(a)x \end{bmatrix}.$$

Thus, $\mu_2(x) \in \mathcal{Z}(A)$ and $[\nu_2(x), a] - [x, \mu_1(a)] = T(a, \mu_2(x)) - T(\mu_2(x), a)$ for all $a \in A,\ x \in E$.

Conversely, suppose that all the conditions (1) – (4) are satisfied. It is easy to show that

$$\begin{bmatrix} (\mu_1(a) + \mu_2(x))a \\ (\mu_1(a) + \mu_2(x))x + (\nu_1(a) + \nu_2(x))a + T((\mu_1(a) + \mu_2(x)), a) \end{bmatrix} = \begin{bmatrix} a(\mu_1(a) + \mu_2(x)) \\ x(\mu_1(a) + \mu_2(x)) + a(\nu_1(a) + \nu_2(x)) + T(a, (\mu_1(a) + \mu_2(x))) \end{bmatrix}$$

for all $a \in A,\ x \in E$. That is, $\theta(X)X = X\theta(X)$ for all $X = \begin{bmatrix} a \\ x \end{bmatrix} \in H_T = A \oplus E$. □

3 The Main Theorem

Let H_T be the Hochschild extension of A from E by the 2-cocycle T. In this section, we suppose that the unital algebra A has a nontrivial idempotent p such that

$$px(1-p) = x, \forall x \in E. \tag{3.1}$$

We know that triangular algebra $\mathcal{T} = \mathcal{T}(A, M, B)$ can be regarded as the trivial extension algebra $H_0 = (A \oplus B) \oplus M$ of $A \oplus B$ from M, in which case M is considered as an $(A \oplus B)-$bimodule under the following module operations:

$$(a, b)x = ax,\ x(a, b) = xb$$

for all $(a, b) \in A \oplus B,\ x \in M$. Then $\mathcal{T}(A, M, B)$ has an idempotent p as above. Indeed, take $p = (1_A, 0)$. By a direct verification, equality (3.1) holds on $\mathcal{T}(A, M, B)$.

Set $q = 1 - p$. It follows from (3.1) that $px = xq = x,\ xp = qx = 0$ for any $x \in E$. Thus for any $x \in E$, $\begin{bmatrix} a \\ y \end{bmatrix} \in \mathcal{Z}(H_T)$, we have $[p, x] = x$ and $y = T(a, p) - T(p, a)$.

Lemma 3.1 *Suppose that the unital algebra A has a nontrivial idempotent p satisfying (3.1). Let $a_0 \in \mathcal{Z}(A)$ with $[a_0, x] = 0$ and*

$$T([a, p], a_0) = T(a_0, [a, p]), \tag{3.2}$$

for all $a \in A, x \in E$, then $a_0 \in \pi_A(\mathcal{Z}(H_T))$.

Proof By lemma 2.1, it is sufficient to show that there is $x_0 \in E$ such that

$$[a, x_0] = T(a_0, a) - T(a, a_0),\ \forall\, a \in A.$$

It follows from cocycle identity and $a_0 \in \mathcal{Z}(A)$ that

$$\begin{aligned} aT(a_0, p) &= -T(a, a_0 p) + T(aa_0, p) + T(a, a_0)p \\ &= [aT(p, a_0) - T(a, p)a_0 - T(ap, a_0)] + [a_0 T(a, p) + T(a_0, ap)] \end{aligned}$$

and

$$\begin{aligned} T(p, a_0)a &= -T(pa_0, a) + pT(a_0, a) + T(p, a_0 a) \\ &= [T(a_0, p)a - a_0 T(p, a) - T(a_0, pa)] + T(a_0, a) + \\ &\quad [-T(a, a_0) + T(p, a)a_0 + T(pa, a_0)]. \end{aligned}$$

They are

$$aT(a_0, p) = aT(p, a_0) - T(ap, a_0) + T(a_0, ap)$$

and

$$T(p, a_0)a = T(a_0, p)a - T(a_0, pa) + T(a_0, a) - T(a, a_0) + T(pa, a_0),$$

respectively, due to $[a_0, x] = 0$. Using the above two equalities we compute

$$\begin{aligned} &[a, T(a_0, p) - T(p, a_0)] \\ &\quad = aT(a_0, p) - aT(p, a_0) - T(a_0, p)a + T(p, a_0)a \\ &\quad = -T(ap, a_0) + T(a_0, ap) - T(a_0, pa) + T(a_0, a) - T(a, a_0) + T(pa, a_0) \\ &\quad = -T([a, p], a_0) + T(a_0, [a, p]) + T(a_0, a) - T(a, a_0). \end{aligned}$$

Therefore, $[a, T(a_0, p) - T(p, a_0)] = T(a_0, a) - T(a, a_0)$ by the assumption that $T([a, p], a_0) = T(a_0, [a, p])$. Take $x_0 = T(a_0, p) - T(p, a_0)$ as required. □

The next theorem is our main result which characterizes the properness of a commuting mapping on $H_T = A \oplus E$.

Theorem 3.2 *Suppose that the unital algebra A has a nontrivial idempotent p such that $px(1 - p) = x$ for all $x \in E$. Then the commuting mapping*

$$\theta\left(\begin{bmatrix} a \\ x \end{bmatrix}\right) = \begin{bmatrix} \mu_1(a) + \mu_2(x) \\ \nu_1(a) + \nu_2(x) \end{bmatrix}$$

on H_T is proper if and only if $[\mu_2(x), y] = 0$ for all $x, y \in E$ and there exist a R-linear mapping $\alpha : A \longrightarrow \pi_A(\mathcal{Z}(H_T))$ and $a_0 \in \pi_A(\mathcal{Z}(H_T))$ such that $\mu_1(a) = a_0 a + \alpha(a)$ for any $a \in A$.

Proof Suppose that θ is proper. There exist $C = \begin{bmatrix} a_0 \\ x_0 \end{bmatrix} \in \mathcal{Z}(H_T)$ and a R-linear mapping $\Omega : H_T \longrightarrow \mathcal{Z}(H_T)$ such that $\theta(X) = XC + \Omega(X)$ for all $X \in H_T$. Let us choose $X = \begin{bmatrix} a \\ 0 \end{bmatrix} \in H_T$. We have

$$\theta(X) = \begin{bmatrix} \mu_1(a) \\ \nu_1(a) \end{bmatrix}.$$

On the other hand,

$$\theta(X) = XC + \Omega(X) = \begin{bmatrix} a_0 a \\ x_0 a + T(a_0, a) \end{bmatrix} + \Omega\left(\begin{bmatrix} a \\ 0 \end{bmatrix}\right) \tag{3.3}$$

for all $a \in A$. Hence, $\begin{bmatrix} \mu_1(a) - a_0 a \\ \nu_1(a) - x_0 a - T(a_0, a) \end{bmatrix} = \Omega\left(\begin{bmatrix} a \\ 0 \end{bmatrix}\right) \in \mathcal{Z}(H_T)$. For any $a \in A$, define $\alpha(a) = \mu_1(a) - a_0 a$ which is required. Similarly, take $X = \begin{bmatrix} 0 \\ x \end{bmatrix} \in H_T$. Then by the same computational procedure, $\mu_2(x) \in \pi_A(\mathcal{Z}(H_T))$ for all $x \in E$. Thus, $[\mu_2(x), y] = 0$ for all $x, y \in E$.

Conversely, assume that there exist an element a_0 and a mapping α satisfying the conditions as the theorem shows. Let $\begin{bmatrix} a_0 \\ x_0 \end{bmatrix} \in \mathcal{Z}(H_T)$. Then,

$$x_0 = T(a_0, p) - T(p, a_0). \tag{3.4}$$

By condition (4) of Theorem 2.4, we obtain

$$\begin{aligned} [a, \nu_2(x) - a_0 x] &= [a, \nu_2(x)] - a_0[a, x] \\ &= [\mu_1(a), x] - T(a, \mu_2(x)) + T(\mu_2(x), a) - a_0[a, x] \\ &= [\alpha(a), x] + T(\mu_2(x), a) - T(a, \mu_2(x)) \\ &= T(\mu_2(x), a) - T(a, \mu_2(x)) \end{aligned}$$

for all $a \in A$. Combining with $[\mu_2(x), y] = 0$ yields

$$\begin{bmatrix} \mu_2(x) \\ \nu_2(x) - a_0 x \end{bmatrix} \in \mathcal{Z}(H_T) \ \ (x \in E). \tag{3.5}$$

Putting $a = p$ in condition (3) of Theorem 2.4 leads to

$$\nu_1(p) = T(\mu_1(p), p) - T(p, \mu_1(p)). \tag{3.6}$$

Besides, linearizing the condition (3) gives that

$$[a, \nu_1(b)] = [\nu_1(a), b] - T(b, \mu_1(a)) + T(\mu_1(a), b) + T(\mu_1(b), a) - T(a, \mu_1(b))$$

for all $a,\ b \in A$. This implies that

$$\nu_1(b) = [\nu_1(p), b] - T(b, \mu_1(p)) + T(\mu_1(p), b) + T(\mu_1(b), p) - T(p, \mu_1(b)) \quad (3.7)$$

for all $b \in A$. Combining (3.4) with (3.7), we get

$$\begin{aligned}\nu_1(a) - x_0 a - T(a_0, a) = [\nu_1(p), a] - T(a, \mu_1(p)) + T(\mu_1(p), a) + T(\mu_1(a), p)\\ -T(p, \mu_1(a)) - T(a_0, a) - [T(a_0, p) - T(p, a_0)]a\end{aligned} \quad (3.8)$$

for all $a \in A$. It follows from (3.6) that

$$\begin{aligned}[\nu_1(p), a] &= [T(\mu_1(p), p) - T(p, \mu_1(p)), a]\\ &= [T(a_0 p, p) - T(p, a_0 p) + T(\alpha(p), p) - T(p, \alpha(p)), a]\end{aligned}$$

for all $a \in A$. Note that $T(p, a_0 p) + pT(a_0, p) = T(p, a_0)p + T(a_0 p, p) = T(a_0 p, p)$. Then the above equality becomes

$$[\nu_1(p), a] = [T(a_0, p), a] + [T(\alpha(p), p) - T(p, \alpha(p)), a]. \quad (3.9)$$

Since

$$T(a_0 p, a) + T(p, a_0)a = T(p, a_0 a) + T(a_0, a)$$

and

$$T(a, a_0 p) + aT(a_0, p) = T(a_0 a, p),$$

we have

$$\begin{aligned}T(\mu_1(p), a) - T(a, \mu_1(p)) &= T(a_0 p, a) - T(a, a_0 p) + T(\alpha(p), a) - T(a, \alpha(p))\\ &= T(p, a_0 a) + T(a_0, a) - T(a_0 a, p) + aT(a_0, p)\\ &\quad -T(p, a_0)a + T(\alpha(p), a) - T(a, \alpha(p)).\end{aligned} \quad (3.10)$$

Similarly,

$$T(\mu_1(a), p) - T(p, \mu_1(a)) = T(a_0 a, p) - T(p, a_0 a) + T(\alpha(a), p) - T(p, \alpha(a)). \quad (3.11)$$

Combine the equalities from (3.8) to (3.11) and note that

$$[T(\alpha(p), p) - T(p, \alpha(p)), a] = T(a, \alpha(p)) - T(\alpha(p), a).$$

We compute that

$$\nu_1(a) - x_0 a - T(a_0, a) = T(\alpha(a), p) - T(p, \alpha(a))$$

for all $a \in A$, which implies that

$$\begin{bmatrix} \alpha(a) \\ \nu_1(a) - x_0 a - T(a_0, a) \end{bmatrix} \in \mathcal{Z}(H_T). \tag{3.12}$$

Taking into account the formulas (3.5) and (3.12), we obtain

$$\begin{bmatrix} \mu_1(a) + \mu_2(x) \\ \nu_1(a) + \nu_2(x) \end{bmatrix} - \begin{bmatrix} a \\ x \end{bmatrix} \begin{bmatrix} a_0 \\ x_0 \end{bmatrix} \in \mathcal{Z}((H_T)$$

for all $\begin{bmatrix} a \\ x \end{bmatrix} \in H_T$ which shows that θ is proper.

4 Applications

In this section, we apply Theorem 3.2 to a special class of algebras containing triangular algebras. Suppose that $\mathcal{A}, \mathcal{B}$ are unital algebras over a commutative ring R and $\mathcal{M}$, an $(\mathcal{A},\ \mathcal{B})-$bimodule which is faithful as a left $\mathcal{A}-$module as well as a right $\mathcal{B}-$module. The element in $\mathcal{A} \oplus \mathcal{B}$ is denoted by $a \oplus b\, (a \in \mathcal{A},\ b \in \mathcal{B})$, where $\mathcal{A} \oplus \mathcal{B}$ is the direct product of $\mathcal{A}$ and $\mathcal{B}$ which has its usual pairwise operations. Like in triangular algebras, $\mathcal{M}$ is an $\mathcal{A} \oplus \mathcal{B}-$bimodule equipped with the module operations

$$(a \oplus b)x = ax,\ x(a \oplus b) = xb$$

for all $a \oplus b \in \mathcal{A} \oplus \mathcal{B},\ x \in \mathcal{M}$. We will consider the properness of commuting maps on Hochschild extension algebras $H_T = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{M}$ of $\mathcal{A} \oplus \mathcal{B}$ from $\mathcal{M}$ by the Hochschild 2-cocycle T. Recall that the multiplication on $H_T = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{M}$ is

$$\begin{bmatrix} a_1 \oplus b_1 \\ x_1 \end{bmatrix} \begin{bmatrix} a_2 \oplus b_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 \oplus b_1 b_2 \\ a_1 x_2 + x_1 b_2 + T(a_1 \oplus b_1, a_2 \oplus b_2) \end{bmatrix}$$

and its center is

$$\begin{aligned} \mathcal{Z}(H_T) = \Bigg\{ \begin{bmatrix} a_0 \oplus b_0 \\ x_0 \end{bmatrix} &: a_0 x = x b_0, \\ & ax_0 - x_0 b = T(a_0 \oplus b_0, a \oplus b) - T(a \oplus b, a_0 \oplus b_0), \\ & \forall\, a \oplus b \in \mathcal{A} \oplus \mathcal{B},\ x \in \mathcal{M} \Bigg\}. \end{aligned}$$

It should be mentioned that the extension algebra $H_T = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{M}$ has a non-trivial idempotent $p = 1_A \oplus 0$ satisfying conditions (3.1) and (3.2). By Lemma 3.1, $a_0 \oplus b_0 \in \pi_{\mathcal{A} \oplus \mathcal{B}}(\mathcal{Z}(H_T))$ exactly when $a_0 x = x b_0$ for all $x \in \mathcal{M}$. Applying [9, proposition 3], $\mathcal{Z}(\mathcal{S})$ is determined by $\pi_{\mathcal{A}}(\mathcal{Z}(H_T))$ or $\pi_{\mathcal{B}}(\mathcal{Z}(H_T))$ due to the faithfulness of $\mathcal{M}$. Concretely, there exists a unique algebraic isomorphism φ from $\pi_{\mathcal{A}}(\mathcal{Z}(H_T))$ to $\pi_{\mathcal{B}}(\mathcal{Z}(H_T))$ such that $ax = x\varphi(a)$ for all $x \in \mathcal{M}$.

If $T = 0$, then the trivial extension algebra $H_T = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{M}$ is just the triangular algebra $\mathcal{T} = \mathcal{T}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. As consequences of Theorem 2.4 and Theorem 3.2, we have the next two theorems which can derive the results of Cheung [9].

Theorem 4.1 *Let θ be a commuting mapping of $H_T = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{M}$ of the form:*

$$\theta\left(\begin{bmatrix} a \oplus b \\ x \end{bmatrix}\right) = \begin{bmatrix} (f_{AA}(a) + f_{AB}(b) + g_A(x)) \oplus (f_{BA}(a) + f_{BB}(b) + g_B(x)) \\ h_A(a) + h_B(b) + \nu_2(x) \end{bmatrix},$$

where $f_{AA} : \mathcal{A} \longrightarrow \mathcal{A}$, $f_{AB} : \mathcal{B} \longrightarrow \mathcal{A}$, $f_{BA} : \mathcal{A} \longrightarrow \mathcal{B}$, $f_{BB} : \mathcal{B} \longrightarrow \mathcal{B}$, $g_A : \mathcal{M} \longrightarrow \mathcal{A}$, $g_B : \mathcal{M} \longrightarrow \mathcal{B}$, $h_A : \mathcal{A} \longrightarrow \mathcal{M}$, $h_B : \mathcal{B} \longrightarrow \mathcal{M}$, $\nu_2 : \mathcal{M} \longrightarrow \mathcal{M}$ are all R-linearing mappings. Then the following statements are equivalent:

(1) *θ is proper: i.e., $\theta(X) = XC + \Omega(X)$ for all $X \in H_T$, where $C \in \mathcal{Z}(H_T)$ and Ω maps H_T into $\mathcal{Z}(\mathcal{H}_\mathcal{T})$.*
(2) *$f_{AB}(\mathcal{B}) \subseteq \pi_\mathcal{A}(\mathcal{Z}(H_T))$, $f_{BA}(\mathcal{A}) \subseteq \pi_\mathcal{B}(\mathcal{Z}(H_T))$, and $g_A(x) \oplus g_B(x) \in \pi_{\mathcal{A}\oplus\mathcal{B}}(\mathcal{Z}(H_T))$ for all $x \in \mathcal{M}$.*
(3) *$f_{AA}(1_\mathcal{A}) \in \pi_\mathcal{A}(\mathcal{Z}(H_T))$, $f_{BA}(1_\mathcal{A}) \in \pi_\mathcal{B}(\mathcal{Z}(H_T))$, and $g_A(x) \oplus g_B(x) \in \pi_{\mathcal{A}\oplus\mathcal{B}}(\mathcal{Z}(H_T))$ for all $x \in \mathcal{M}$.*

Proof Let μ_1, μ_2, ν_1 be as in Theorem 3.2. Then, $\mu_1(a \oplus b) = (f_{AA}(a) + f_{AB}(b)) \oplus (f_{BA}(a) + f_{BB}(b))$, $\mu_2(x) = g_A(x) \oplus g_B(x)$, $\nu_1(a \oplus b) = h_A(a) + h_B(b)$. First, we will get some information from the fact that θ is a commuting mapping.

By Theorem 2.4(1), μ_1 is commuting. Hence,

$$0 = [\mu_1(a \oplus 0), a \oplus 0] = [f_{AA}(a) \oplus f_{BA}(a), a \oplus 0] = (f_{AA}(a)a - af_{AA}(a)) \oplus 0,$$

which implies that f_{AA} is a commuting map on $\mathcal{A}$. Similarly, f_{BB} is also a commuting map on $\mathcal{B}$. Again, combining with $0 = [\mu_1(a \oplus b), a \oplus b]$ for all $a \oplus b \in \mathcal{A} \oplus \mathcal{B}$. we have that the image of f_{AB} and f_{BA} is in $\mathcal{Z}(\mathcal{A})$ and $\mathcal{Z}(\mathcal{B})$, respectively.

By Theorem 2.4(2), $0 = [\mu_2(x), x] = [g_A(x) \oplus g_B(x), x]$, i.e.,

$$g_A(x)x = xg_B(x) \tag{4.1}$$

for all $x \in \mathcal{M}$. It follows from Theorem 2.4(4) that $[\nu_2(x), p] - [x, \mu_1(p)] = T(p, \mu_2(x)) - T(\mu_2(x), p)$. Therefore,

$$\begin{aligned}\nu_2(x) = f_{AA}(1_\mathcal{A})x - xf_{BA}(1_\mathcal{A}) + T(g_A(x) \oplus g_B(x),\\ 1_\mathcal{A} \oplus 0) - T(1_\mathcal{A} \oplus 0, g_A(x) \oplus g_B(x)).\end{aligned} \tag{4.2}$$

Combining equality (4.2) with

$$[\nu_2(x), a \oplus 0] - [x, \mu_1(a \oplus 0)] = T(a \oplus 0, \mu_2(x)) - T(\mu_2(x), a \oplus 0)$$

leads to

$$\begin{aligned} &a(xf_{BA}(1_{\mathcal{A}}) - f_{AA}(1_{\mathcal{A}})x) \\ &\quad = xf_{BA}(a) - f_{AA}(a)x + T(a \oplus 0, g_A(x) \oplus g_B(x)) - T(g_A(x) \oplus g_B(x), a \oplus 0) \\ &\qquad + aT(g_A(x) \oplus g_B(x), 1_{\mathcal{A}} \oplus 0) - aT(1_{\mathcal{A}} \oplus 0, g_A(x) \oplus g_B(x)). \end{aligned} \tag{4.3}$$

Applying the cocycle identity, we compute

$$\begin{aligned} &T(a \oplus 0, g_A(x) \oplus g_B(x)) = T((a \oplus 0)(1_{\mathcal{A}} \oplus 0), g_A(x) \oplus g_B(x)) \\ &\quad = -T(a \oplus 0, 1_{\mathcal{A}} \oplus 0)g_B(x) + aT(1_{\mathcal{A}} \oplus 0, g_A(x) \oplus g_B(x)) + T(a \oplus 0, g_A(x) \oplus 0); \end{aligned} \tag{4.4}$$

$$\begin{aligned} &T(g_A(x) \oplus g_B(x), a \oplus 0) = T(g_A(x) \oplus g_B(x), (a \oplus 0)(1_{\mathcal{A}} \oplus 0)) \\ &\quad = -g_A(x)T(a \oplus 0, 1_{\mathcal{A}} \oplus 0) + T(ag_A(x) \oplus 0, 1_{\mathcal{A}} \oplus 0) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &aT(g_A(x) \oplus g_B(x), 1_{\mathcal{A}} \oplus 0) = (a \oplus 0)T(g_A(x) \oplus g_B(x), 1_{\mathcal{A}} \oplus 0) \\ &\quad = -T(a \oplus 0, g_A(x) \oplus 0) + T(ag_A(x) \oplus 0, 1_{\mathcal{A}} \oplus 0) \end{aligned}. \tag{4.6}$$

Taking these equalities from (4.3)to (4.6) into account, we get

$$\begin{aligned} &a(xf_{BA}(1_{\mathcal{A}}) - f_{AA}(1_{\mathcal{A}})x) \\ &\quad = xf_{BA}(a) - f_{AA}(a)x + g_A(x)T(a \oplus 0, 1_{\mathcal{A}} \oplus 0) - T(a \oplus 0, 1_{\mathcal{A}} \oplus 0)g_B(x) \end{aligned} \tag{4.7}$$

for all $a \in \mathcal{A}$, $x \in \mathcal{M}$. Similarly, using the same computational skills and

$$[\nu_2(x), 0 \oplus b] - [x, \mu_1(0 \oplus b)] = T(0 \oplus b, \mu_2(x)) - T(\mu_2(x), 0 \oplus b),$$

we can prove that

$$\begin{aligned} &(f_{AA}(1_{\mathcal{A}})x - xf_{BA}(1_{\mathcal{A}}))b \\ &\quad = xf_{BB}(b) - f_{AB}(b)x - g_A(x)T(1_{\mathcal{A}} \oplus 0, 0 \oplus b) + T(1_{\mathcal{A}} \oplus 0, 0 \oplus b)g_B(x) \end{aligned} \tag{4.8}$$

for all $b \in \mathcal{B}$, $x \in \mathcal{M}$.

Now, (1) $\Longrightarrow$ (2). Suppose that the commuting mapping θ on $H_T = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{M}$ is proper. Then applying Theorem 3.2, $0 = [\mu_2(x), y] = [g_A(x) \oplus g_B(x), y] = g_A(x)y - yg_B(x)$ which shows that $g_A(x) \oplus g_B(x) \in \pi_{\mathcal{A}\oplus\mathcal{B}}(\mathcal{Z}(H_T))$ for all $x \in \mathcal{M}$ by Lemma 3.1. Also, $\mu_1(a \oplus b) = (a_0 \oplus b_0)(a \oplus b) + \alpha(a \oplus b)$, where $a_0 \oplus b_0 \in \pi_{\mathcal{A}\oplus\mathcal{B}}(\mathcal{Z}(H_T))$ and $\alpha : \mathcal{A} \oplus \mathcal{B} \longrightarrow \pi_{\mathcal{A}\oplus\mathcal{B}}(\mathcal{Z}(H_T))$ is a linear mapping. Define $\alpha(a \oplus b) = (\alpha_{AA}(a) + \alpha_{AB}(b)) \oplus (\alpha_{BA}(a) + \alpha_{BB}(b))$. Then,

$$\mu_1(a \oplus b) = [a_0 a + \alpha_{AA}(a) + \alpha_{AB}(b)] \oplus [b_0 b + \alpha_{BA}(a) + \alpha_{BB}(b)].$$

On the other hand,

$$\mu_1(a \oplus b) = (f_{AA}(a) + f_{AB}(b)) \oplus (f_{BA}(a) + f_{BB}(b)).$$

Combining with these two equalities leads to

$$f_{AB}(b) = \alpha_{AB}(b) \in \pi_{\mathcal{A}}(\mathcal{Z}(H_T))$$

and

$$f_{BA}(a) = \alpha_{BA}(a) \in \pi_{\mathcal{B}}(\mathcal{Z}(H_T))$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, which is the desired result.

(2) $\Longrightarrow$ (3). $f_{BA}(1_{\mathcal{A}}) \in f_{BA}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(\mathcal{Z}(H_T))$. By the assumption, we have

$$g_A(x)y - yg_B(x) = 0, \forall x, y \in \mathcal{M}$$

which makes (4.7) and (4.8) become

$$a(xf_{BA}(1_{\mathcal{A}}) - f_{AA}(1_{\mathcal{A}})x) = xf_{BA}(a) - f_{AA}(a)x \tag{4.9}$$

and

$$(f_{AA}(1_{\mathcal{A}})x - xf_{BA}(1_{\mathcal{A}}))b = xf_{BB}(b) - f_{AB}(b)x, \tag{4.10}$$

respectively. Hence,

$$\begin{aligned} f_{AA}(1_{\mathcal{A}})x &= xf_{BA}(1_{\mathcal{A}}) + xf_{BB}(1_{\mathcal{B}}) - f_{AB}(1_{\mathcal{B}})x \\ &= x(f_{BA}(1_{\mathcal{A}}) + f_{BB}(1_{\mathcal{B}}) - \varphi(f_{AB}(1_{\mathcal{B}})). \end{aligned}$$

This implies that $f_{AA}(1_{\mathcal{A}}) \in \pi_{\mathcal{A}}(\mathcal{Z}(H_T))$.

(3) $\Longrightarrow$ (1). Set $a_0 \oplus b_0 = [f_{AA}(1_{\mathcal{A}}) - \varphi^{-1}(f_{BA}(1_{\mathcal{A}}))] \oplus [\varphi(f_{AA}(1_{\mathcal{A}})) - f_{BA}(1_{\mathcal{A}})] \in \pi_{\mathcal{A}\oplus\mathcal{B}}(\mathcal{Z}(H_T))$. We will show that $\mu_1(a \oplus b) - (a_0 \oplus b_0)(a \oplus b) \in \pi_{\mathcal{A}\oplus\mathcal{B}}(\mathcal{Z}(H_T))$ for all $a \oplus b \in \mathcal{A} \oplus \mathcal{B}$. Then by Theorem 3.2, we immediately obtain that θ is proper. Indeed,

$$\begin{aligned} \mu_1(a \oplus b) - (a_0 \oplus b_0)(a \oplus b) = {} & [f_{AA}(a) - f_{AA}(1_{\mathcal{A}})a \\ & + \varphi^{-1}(f_{BA}(1_{\mathcal{A}}))a + f_{AB}(b)] \oplus \\ & [f_{BA}(a) + f_{BB}(b) - \varphi(f_{AA}(1_{\mathcal{A}}))b + f_{BA}(1_{\mathcal{A}})b]. \end{aligned}$$

By (4.9), (4.10) and condition (3), it follows that

$$\begin{aligned} & [f_{AA}(a) - f_{AA}(1_{\mathcal{A}})a + \varphi^{-1}(f_{BA}(1_{\mathcal{A}}))a + f_{AB}(b)]x \\ & = xf_{BA}(a) - axf_{BA}(1_{\mathcal{A}}) + axf_{BA}(1_{\mathcal{A}}) + f_{AB}(b)x \\ & = xf_{BA}(a) + f_{AB}(b)x \\ & = xf_{BA}(a) + xf_{BB}(b) - (f_{AA}(1_{\mathcal{A}})x - xf_{BA}(1_{\mathcal{A}}))b \\ & = x[f_{BA}(a) + f_{BB}(b) - \varphi(f_{AA}(1_{\mathcal{A}}))b + f_{BA}(1_{\mathcal{A}})b] \end{aligned}$$

for all $x \in \mathcal{M}$. □

Now, sufficient conditions are given such that every commuting mapping of $H_T = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{M}$ is proper.

Theorem 4.2 *Let θ be a commuting mapping of $H_T = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{M}$. If the following three conditions are satisfied:*

(1) $\mathcal{Z}(\mathcal{B}) = \pi_{\mathcal{B}}(\mathcal{Z}(H_T))$, *or* $\mathcal{A} = [\mathcal{A}, \mathcal{A}]$;
(2) $\mathcal{Z}(\mathcal{A}) = \pi_{\mathcal{A}}(\mathcal{Z}(H_T))$, *or* $\mathcal{B} = [\mathcal{B}, \mathcal{B}]$;
(3) *there exists* $x_0 \in \mathcal{M}$ *such that*

$$\pi_{\mathcal{A}\oplus\mathcal{B}}(\mathcal{Z}(H_T)) = \{a \oplus b : a \in \mathcal{Z}(\mathcal{A}), b \in \mathcal{Z}(\mathcal{B}), ax_0 = x_0 b\},$$

then θ is proper.

The proof of this theorem is the same as that of the main theorem of Cheung [9]. We do not want to give the details here, just remark that the condition (3) implies $g_A(x) \oplus g_B(x) \in \pi_{\mathcal{A}\oplus\mathcal{B}}(\mathcal{Z}(H_T))$ from which it follows that (4.9) and (4.10), two useful equalities when proving $[\mathcal{A}, \mathcal{A}] \subseteq f_{BA}^{-1}(\pi_{\mathcal{B}}(\mathcal{Z}(H_T)))$ and $[\mathcal{B}, \mathcal{B}] \subseteq f_{AB}^{-1}(\pi_{\mathcal{A}}(\mathcal{Z}(H_T)))$.

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